On a coverage model in communications and its relations to a Poisson-Dirichlet process

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OUTLINE

Today:

- “Germ-grain” coverage models in stochastic geometry,
- SINR (or shot-noise) coverage model,
- Palm and stationary coverage characteristics.

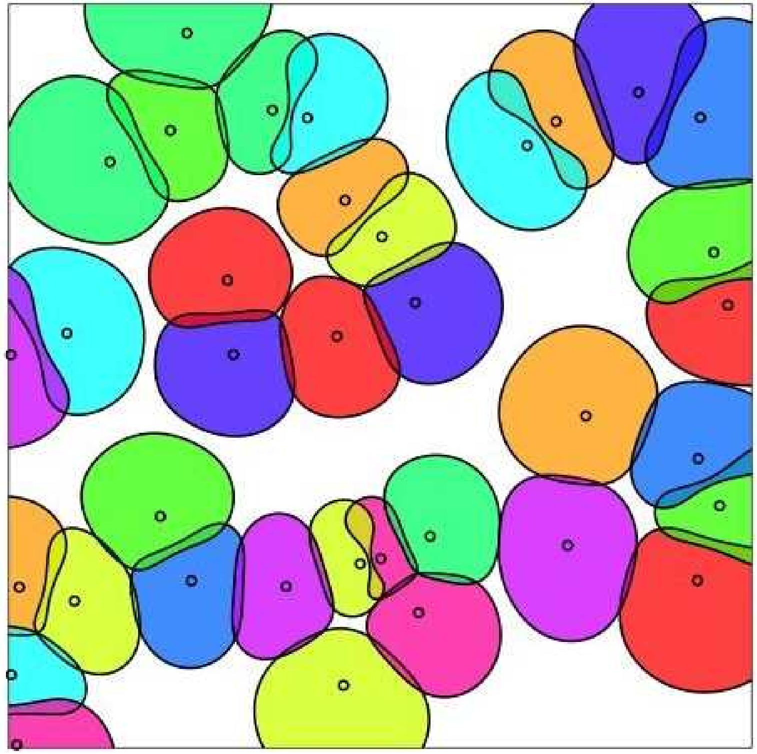
Tomorrow:

- Poisson-Dirichlet processes,
- Relations to SINR coverage.

“Germ-grain” coverage models in stochastic geometry

General “germ-grain” coverage model

Consider a general germ-grain (GG) coverage model $\{(X_i, \mathcal{C}_i)\}$, where $\{X_i\}$ are “germs” forming a point process Φ on $\mathbb{R}^d$, and $\mathcal{C}_i = \mathcal{C}_i(X_i, \Phi)$ are, possibly dependent, random closed subsets of $\mathbb{R}^d$, called “grains”.



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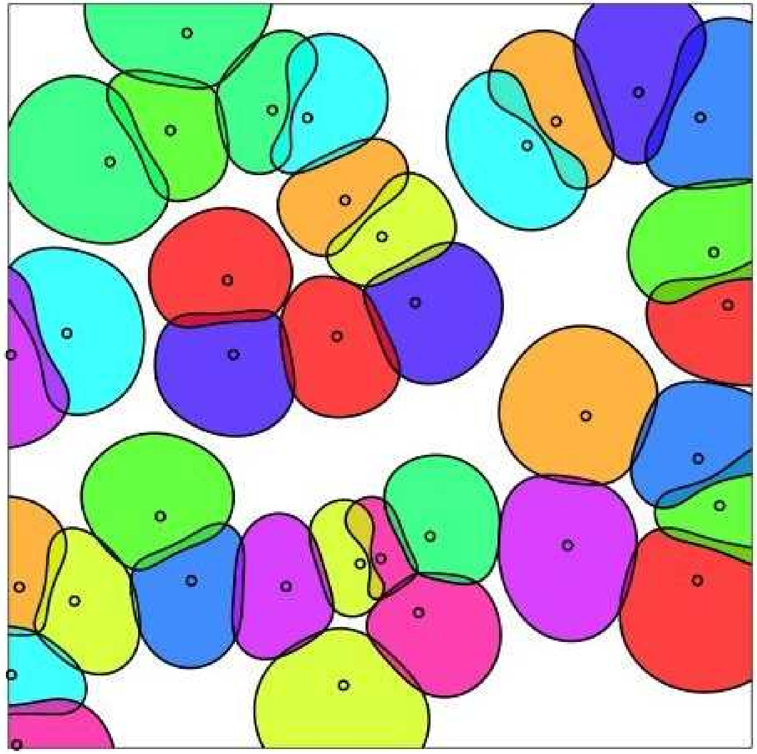
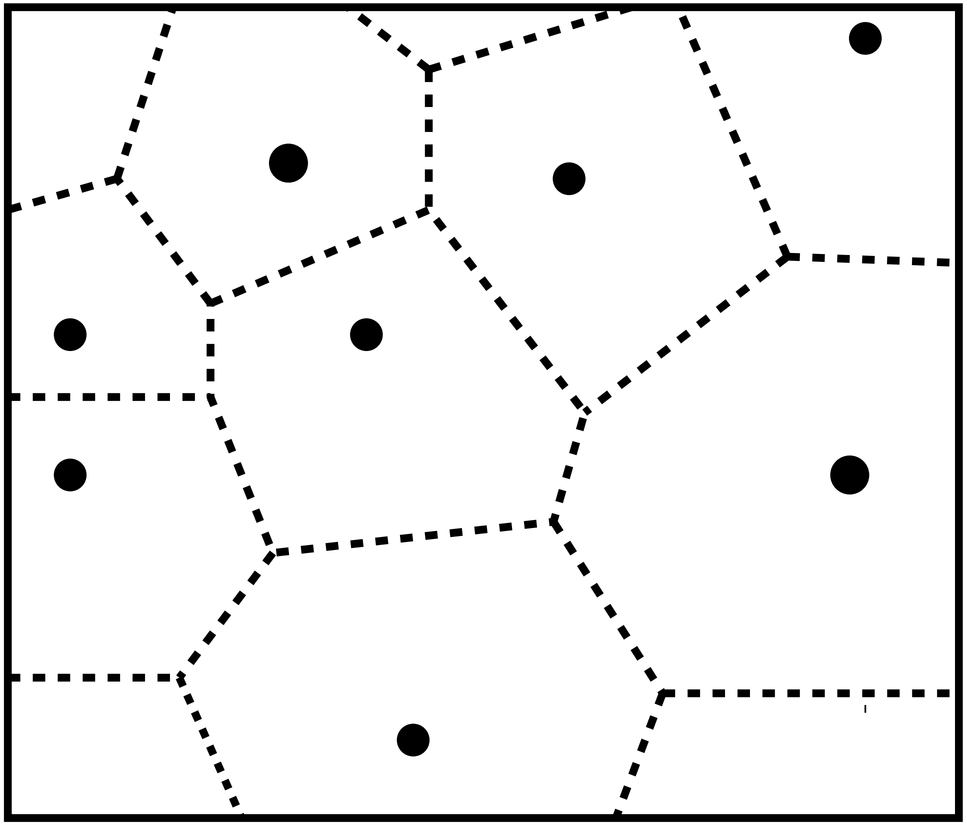


Voronoi tessellation and Boolean Model are special cases of GG coverage model.

Voronoi tessellation (VT)

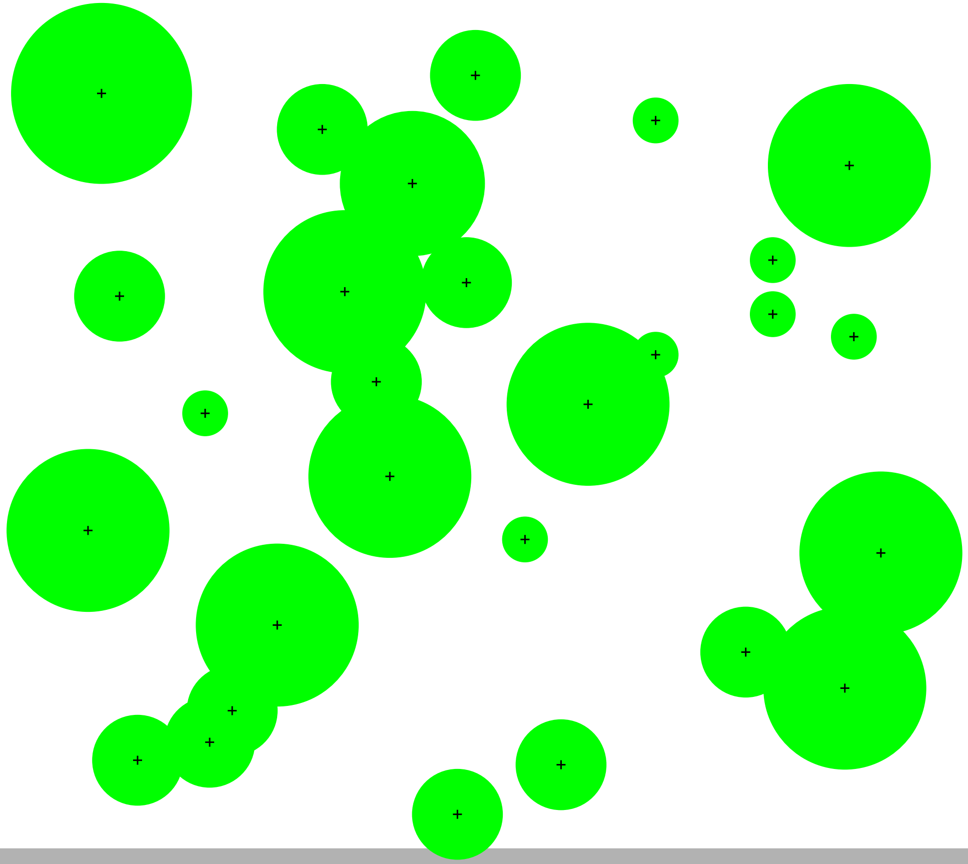
$$\mathcal{C}_i = \{y \in \mathbb{R}^d : |y - x| \leq |y - X_i| \ \forall X_i \in \Phi\}$$



Boolean model (BM)

$$\mathcal{C}_i = X_i \oplus G_i = \{X_i + y : y \in G_i\},$$

where, given $\Phi = \{X_i\}$, G_i are i.i.d. random closed (compact) sets in $\mathbb{R}^d$.



Coverage probabilities

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Coverage by the typical grain
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Stationary coverage
$p := \mathsf{P}\left\{0 \in \bigcup_i \mathcal{C}_i\right\}$ arbitrary location 0 covered by the union.

Stationary coverage number

More generally, denote by $\mathcal{N}$, the number of grains covering the origin $\mathbf{0}$

$$\mathcal{N} := \sum_i \mathbf{1}(\mathbf{0} \in \mathcal{C}_i)$$

and its (stationary) distribution by

$$p_k := \mathsf{P}\{\mathcal{N} \geq k\}\,.$$

p_k is called stationary k-coverage probability

Obviously, $p = p_1 = \mathsf{P}\{0 \in \bigcup_i \mathcal{C}_i\}$ stationary coverage probability.

Exercise: coverage in Poisson-VT

Typical cell coverage

$$
\begin{aligned}
p(x) &:= \mathsf{P}^0\Big\{|x - 0| \le |x - X_i| \ \forall 0 \ne X_i \in \Phi\Big\} \\
\text{Slivnyak} &= \mathsf{P}\{\Phi(B_x(|x|)) = 0\} \\
\text{Poisson definition} &= e^{-\lambda \kappa_d |x|^d},
\end{aligned}
$$

where $B_a(r) = \{y : |y - a| \le r\}$ and $\kappa_d = |B_0(1)|$ and λ is the intensity of Poisson Φ.

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Stationary coverage: (Almost) trivially
$p_k := \mathsf{P}\Big\{\#\{i : 0 \in \mathcal{V}_i\} \ge k\Big\} = 1$ for $k = 1$ and 0 for $k \ge 2$.
Indeed, VT is a partition of $\mathbb{R}^d$ modulo boundaries of the cells, on which 0 lies with probability $\mathsf{P} = 0$.

Exercise: coverage in (Poisson-) BM

Typical grain coverage

By the Slivnyk's theorem and the independence of grains $\boldsymbol{G_i}$ $\boldsymbol{p(x) := \mathsf{P}^0\{x \in 0 \oplus G_0\} = \mathsf{P}\{x \in G_0\}}$ is given directly by the generic grain $\boldsymbol{G}$ distribution.

Exercise: coverage in (Poisson-) BM

Stationary coverage: $\mathcal{N}$ is Poisson$(\lambda \mathsf{E}[|G|])$.

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$$p_k := \mathsf{P}\Big\{\#\{i : 0 \in X_i \oplus G_i\} \geq k\Big\}$$
$$= \mathsf{P}\{\Phi'(\mathbb{R}^d) \geq k\}\,,$$

where $\Phi' = \sum_{X_i \in \Phi} 1(0 \in X_i \oplus G_i)\delta_{X_i}$ is an independent thinning of points of Φ.

Exercise: coverage in (Poisson-) BM

Stationary coverage: $\mathcal{N}$ is Poisson(λE$[|G|]$). Indeed:

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$\Lambda'(\mathrm{d}x) := \mathsf{E}[\Phi'(\mathrm{d}x)] = \mathsf{P}\{0 \in x \oplus G\}\,\lambda \mathrm{d}x = \mathsf{P}\{x \in \check{G}\}\,\lambda \mathrm{d}x$, where $\check{G} = \{-y : y \in G\}$.

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Stationary coverage: $\mathcal{N}$ is Poisson($\lambda\mathsf{E}[|G|]$). Indeed:

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$\Lambda'(\mathrm{d}x) := \mathsf{E}[\Phi'(\mathrm{d}x)] = \mathsf{P}\{0 \in x \oplus G\}\,\lambda\mathrm{d}x = \mathsf{P}\{x \in \check{G}\}\,\lambda\mathrm{d}x$, where $\check{G} = \{-y : y \in G\}$. Consequently
$p_k = \sum_{n=k}^{\infty} e^{-\Lambda'}\Lambda'^n/n!$ where $\Lambda' := \Lambda'(\mathbb{R}^d) = \lambda\mathsf{E}[|G|]$.
In particular

$$p_0 = e^{-\lambda\mathsf{E}[|\check{G}|]}\,.$$

Factorial moments of $\mathcal{N}$

For $n \geq 1$, the k-th factorial moment of (an integer valued rv) $\mathcal{N}$ is defined as

$$\mathsf{E}[\mathcal{N}^{(k)}] := \mathsf{E}\left[\mathcal{N}\,(\mathcal{N}-1)^{+}\ldots(\mathcal{N}-k+1)^{+}\right].$$

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FACT Factorial moments characterize the distribution of the random variable. In particular, for $k \geq 1$

$$\begin{aligned}
p_k &= \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n-1}{k-1} n! \mathsf{E}[\mathcal{N}^{(n)}] \,, \\
\mathsf{P}\{\mathcal{N} = k\} &= \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n}{k} n! \mathsf{E}[\mathcal{N}^{(n)}] \,, \\
\mathsf{E}[z^{\mathcal{N}}] &= \sum_{n=0}^{\infty} (z-1)^{n} n! \mathsf{E}[\mathcal{N}^{(n)}] \,, \quad z \in [0,1] \,.
\end{aligned}$$

Little's law (or a mass transport principle)

$$
\begin{aligned}
\mathsf{E}[\mathcal{N}^{(1)}] &= \mathsf{E}[\mathcal{N}] \\
&= \mathsf{E}\Big[\sum_{X_i \in \Phi} 1(0 \in C_i)\Big] \\
\text{Campbell} \quad &= \int_{\mathbb{R}^d} \mathsf{P}^x\{0 \in C_x\}\, \lambda \mathrm{d}x \\
\text{symmetry} \quad &= \int_{\mathbb{R}^d} \mathsf{P}^0\{x \in C_0\}\, \lambda \mathrm{d}x \\
&= \int_{\mathbb{R}^d} p(x)\, \mathrm{d}x = \lambda \mathsf{E}^0[|\mathcal{C}_0|]\,,
\end{aligned}
$$

where $p(x)$ is the typical grain coverage probability.

Higher-order extensions

For $n \geq 1$, quite similarly

$$\mathsf{E}[\mathcal{N}^{(n)}] = \mathsf{E}\Big[\sum_{\substack{X_{i_1}, X_{i_2}, \ldots, X_{i_n} \in \Phi \\ \text{distinct}}} \mathbf{1}\Big(0 \in \bigcap_{j=1}^{n} C_{i_j}\Big)\Big]$$

higher-order Campbell $$= \int_{\mathbb{R}^d} \mathsf{P}^{x_1,\ldots,x_n}\Big(0 \in \bigcap_{j=1}^{n} C_x\Big)\, \lambda^{(n)}(\mathrm{d}(x_1 \ldots x_n))$$

where $\mathsf{P}^{x_1,\ldots,x_n}$ is n-fold Palm distribution of Φ and $\lambda^{(n)}(\cdot)$ is n-fold factorial moment measure of Φ.

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where $\mathsf{P}^{x_1,\ldots,x_n}$ is n-fold Palm distribution of Φ and $\lambda^{(n)}(\cdot)$ is n-fold factorial moment measure of Φ.

In case of Poisson Φ of intensity$\lambda(\cdot)$,

$$\mathsf{P}_{\Phi}^{x_1,\ldots,x_n} = \mathsf{P}_{\Phi + \sum_{j=1}^{n} \delta_{x_j}} \qquad \text{(Slivnyak's Thm)}$$

and $\qquad \lambda^{(n)}(\mathrm{d}(x_1 \ldots x_n)) = \lambda(\mathrm{d}x_1) \ldots \lambda(\mathrm{d}x_n).$

Stationary coverage via moment expansion

COR

$$p_k = \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n-1}{k-1} n! \int_{\mathbb{R}^d} \mathsf{P}^{x_1,\ldots,x_n}\Big(0 \in \bigcap_{j=1}^{n} C_x\Big) \times \lambda^{(n)}(\mathrm{d}(x_1 \ldots x_n))$$

and similarly for $\mathsf{P}\{\mathcal{N} = k\}$, $\mathsf{E}[z^{\mathcal{N}}]$.

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and similarly for $\mathsf{P}\{\mathcal{N} = k\}$, $\mathsf{E}[z^{\mathcal{N}}]$.

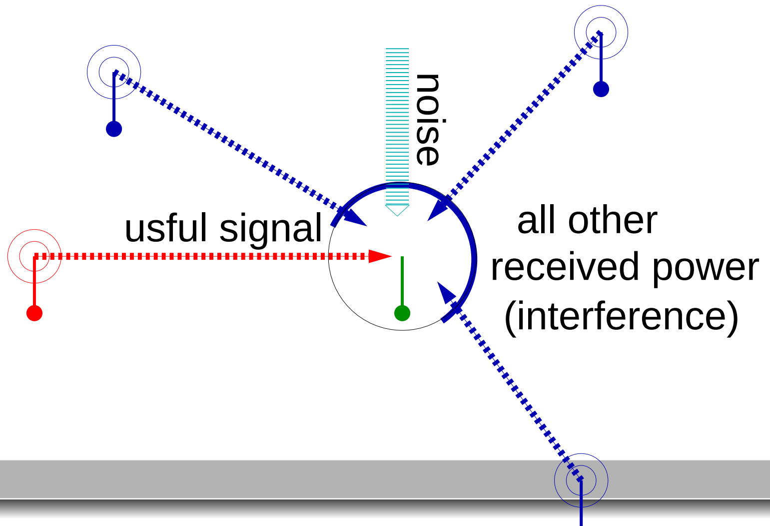
Factorial moment expansions exist for more general characteristics of the point process. BB (1995).

Coverage model for communications

SINR=Signal-to-Interference-and-Noise Ratio

$$\text{SINR} = \frac{\text{USEFUL SIGNAL RECEIVED POWER}}{\text{ALL OTHER SIGNALS RECEIVED POWER (and/or) NOISE}}$$

SINR characterizes the throughput of the communication channel; i.e., the number of bits/second that can be reliably sent in this channel.
Formalization on the ground of information theory.



SINR coverage model

In what follows, we will consider a GG coverage model, where

- germs represent locations of wireless transmitters
- grains are regions where the SINR with respect to respective transmitter is large enough.

SINR cell

SINR grain, or cell:

$$C_i = \left\{ y \in \mathbb{R}^2 : \frac{S_i/\ell(|y - X_i|)}{W + \gamma \sum_{j \neq i} S_j/\ell(|y - X_j|)} \geq \tau \right\}$$

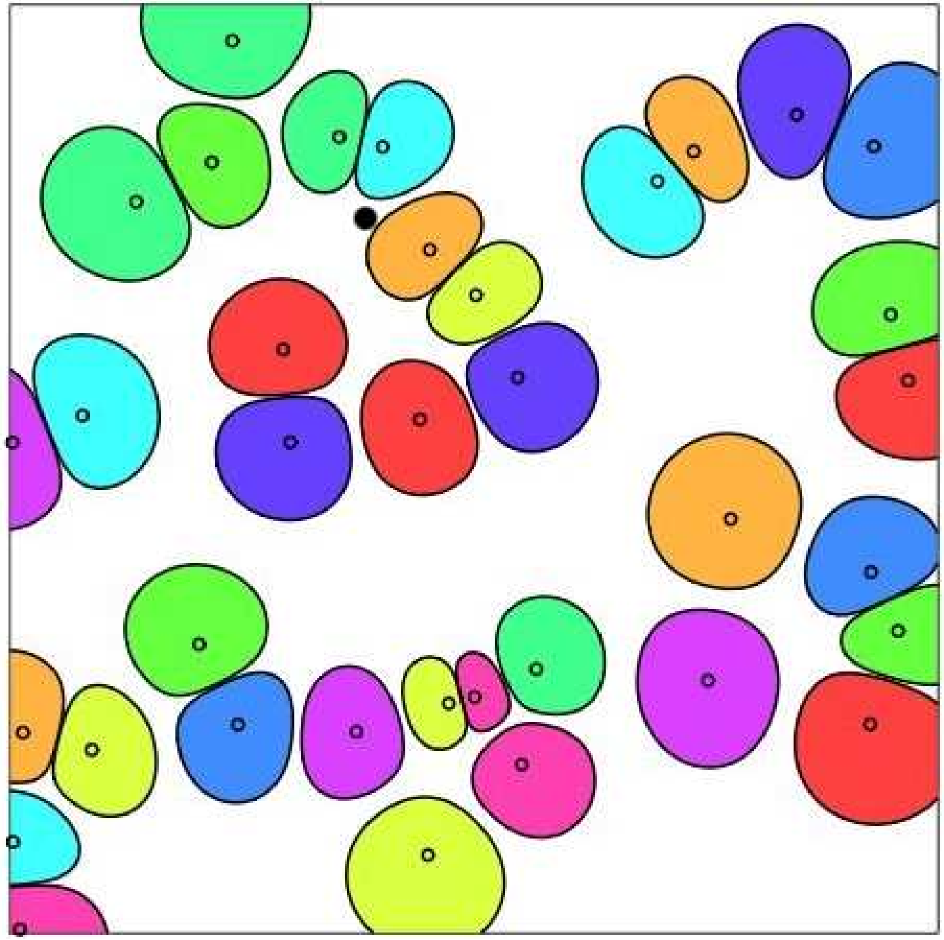
- $\Phi = \{X_i\}$ hom. Poisson p.p. on $\mathbb{R}^2$ of int. λ; locations of wireless transmitters (extension to $\mathbb{R}^d$ straightforward)
- $\tilde{\Phi} = \{(X_i, S_i)\}$ independently marked Φ, $S_i \sim S \geq 0$, $\mathsf{E}[S^{2/\beta}] < \infty$; random signal propagation effects, “shadowing”, “fading”
- $W \geq 0$, r.v. independent of $\tilde{\Phi}$; “noise” power
- $\ell(r) = (Kr)^\beta$, $(K \geq 0$, $\beta > 2)$ “path-loss” function,
- $\tau, \gamma \geq 0$ parameters.

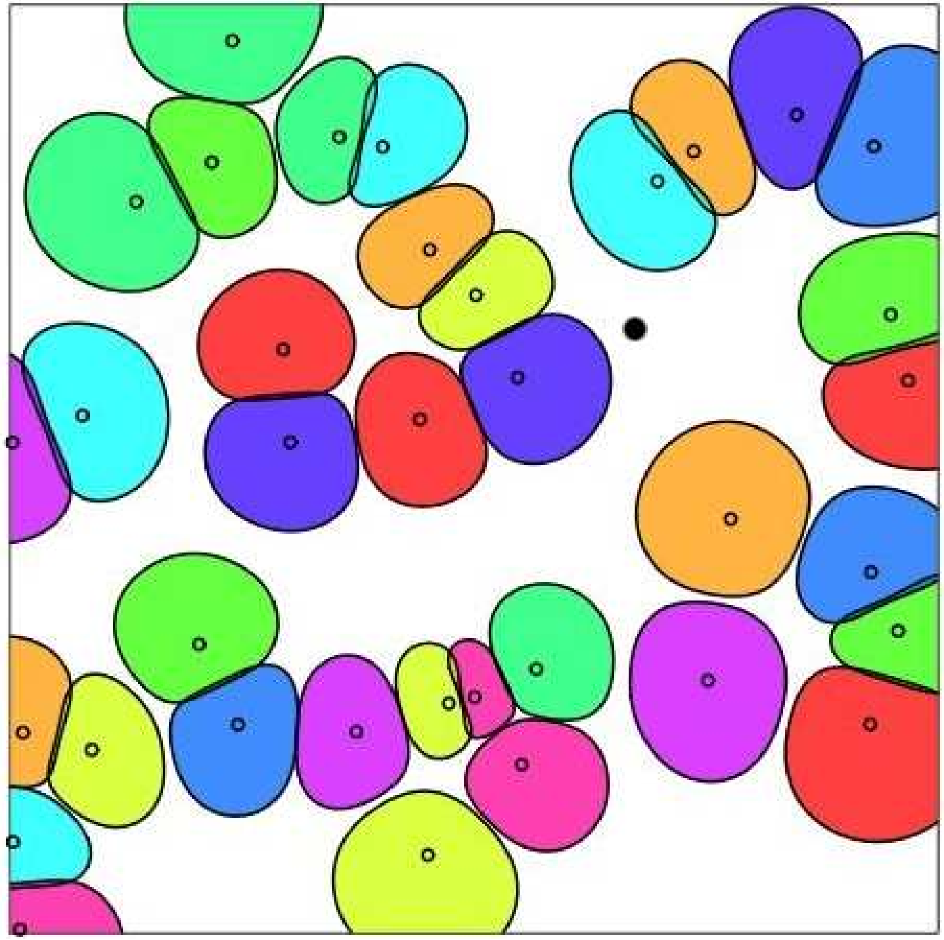
SINR coverage model

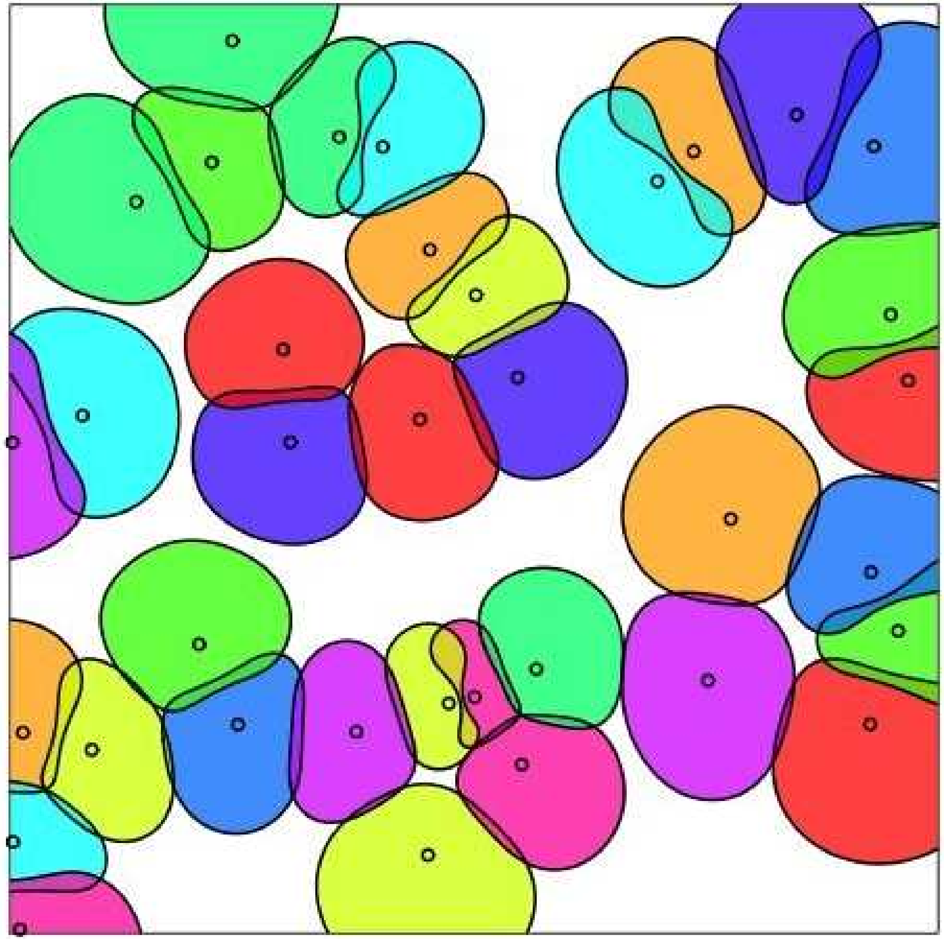
$$\{(X_i, C_i)\}$$

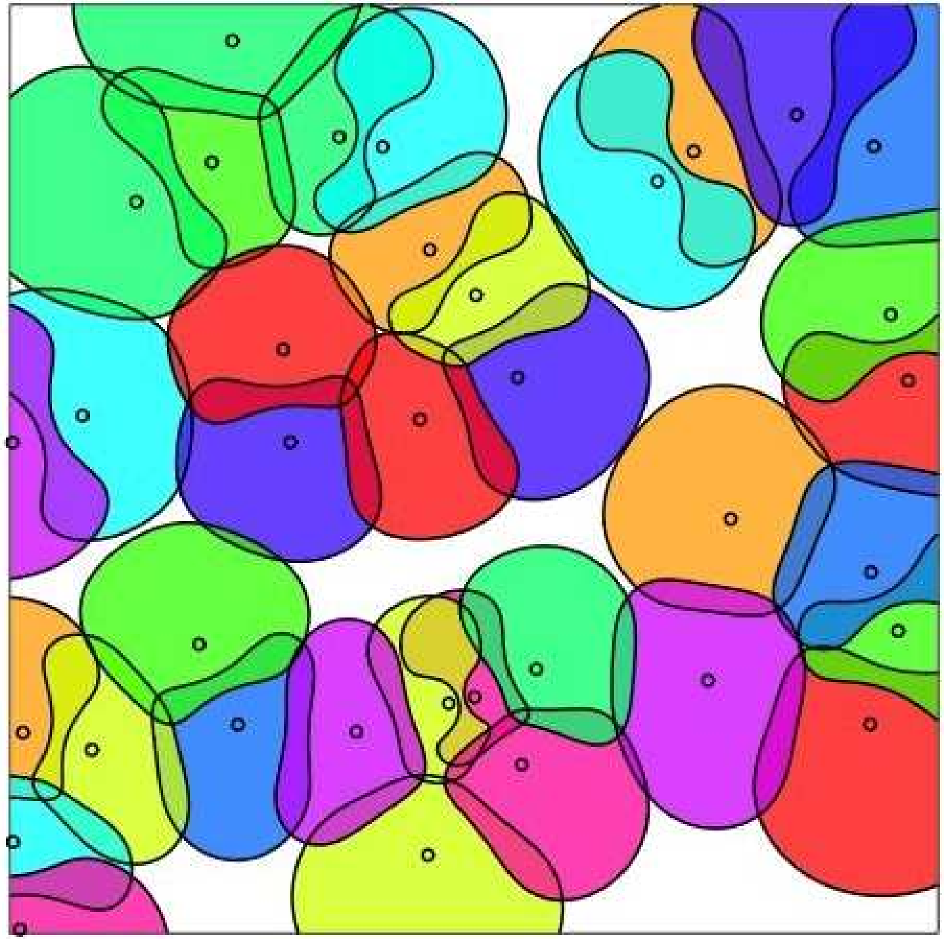
clearly is an example of a GG model with dependent grains.

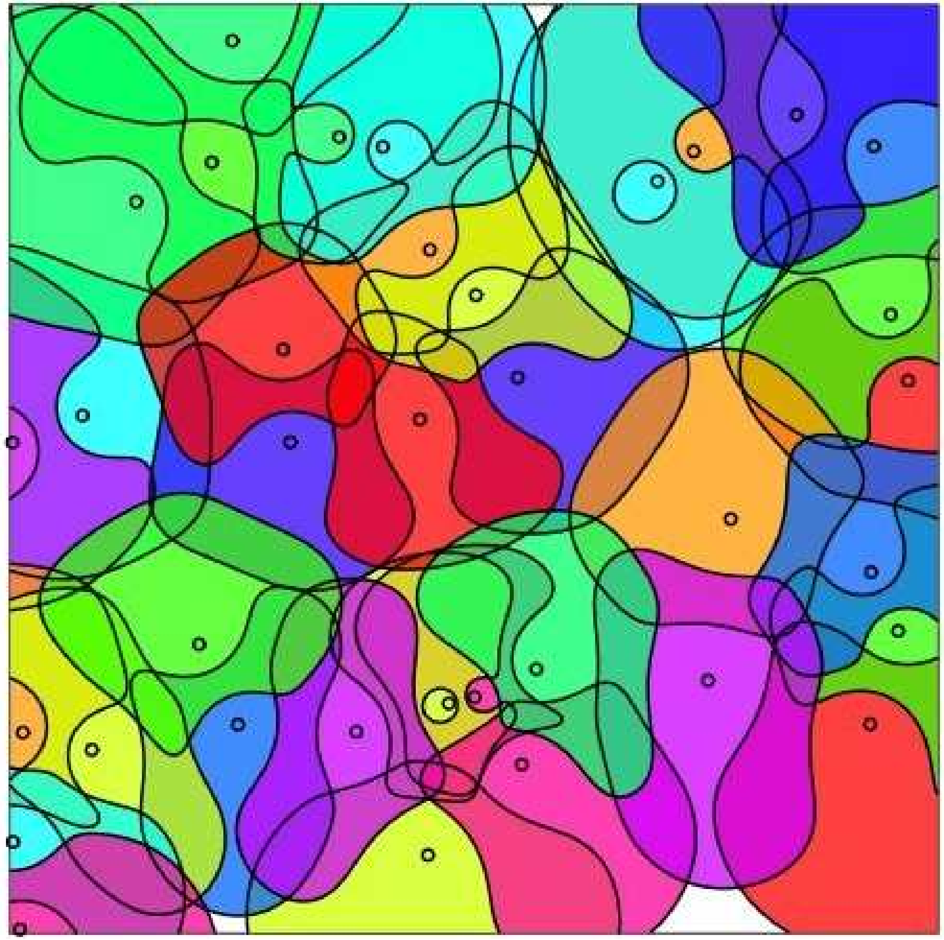
- Introduced (in a bit more general setting) in Baccelli, BB (2001).
- Studied since then in many many variants and aspects.
- Recently called shot-noise coverage model in Chiu, Stoyan, Kendall, Mecke (2013); (interference modeled by a shot-noise field).







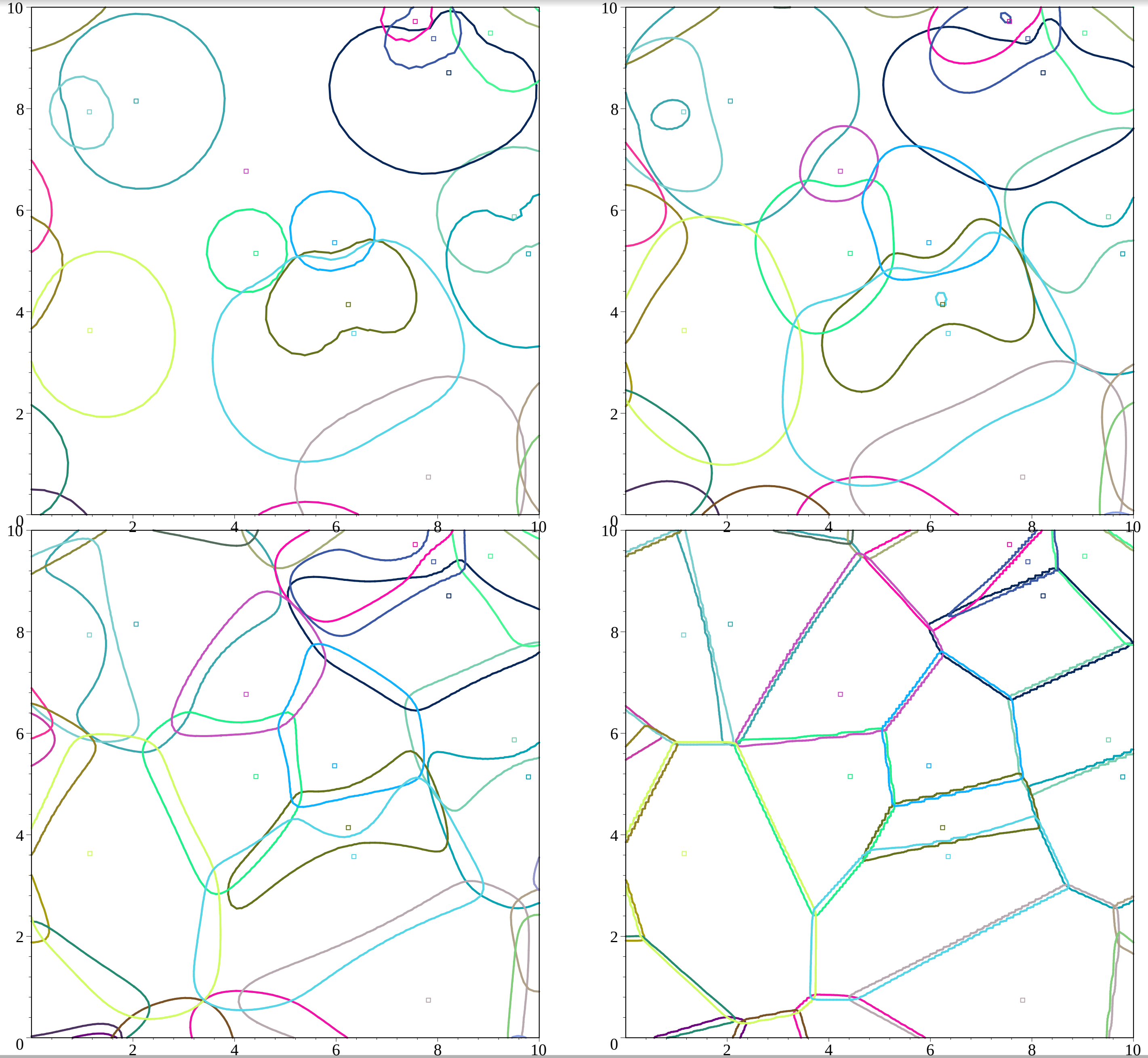




“In-between” MB and VT

- When $\gamma = \mathbf{0}$ (no interference) SINR grains (cells) are independent; Boolean Model approximations,
- When $\boldsymbol{W} = \mathbf{0}$ (no noise) and $\boldsymbol{\beta} \to \infty$ (“strong path-loss”) SINR cells converge to Voronoi cells,
- Playing with $\boldsymbol{W} \to \mathbf{0}$ and $\boldsymbol{\beta} \to \infty$ SINR becomes Johnson-Mehl.

SINR coverage model; examples



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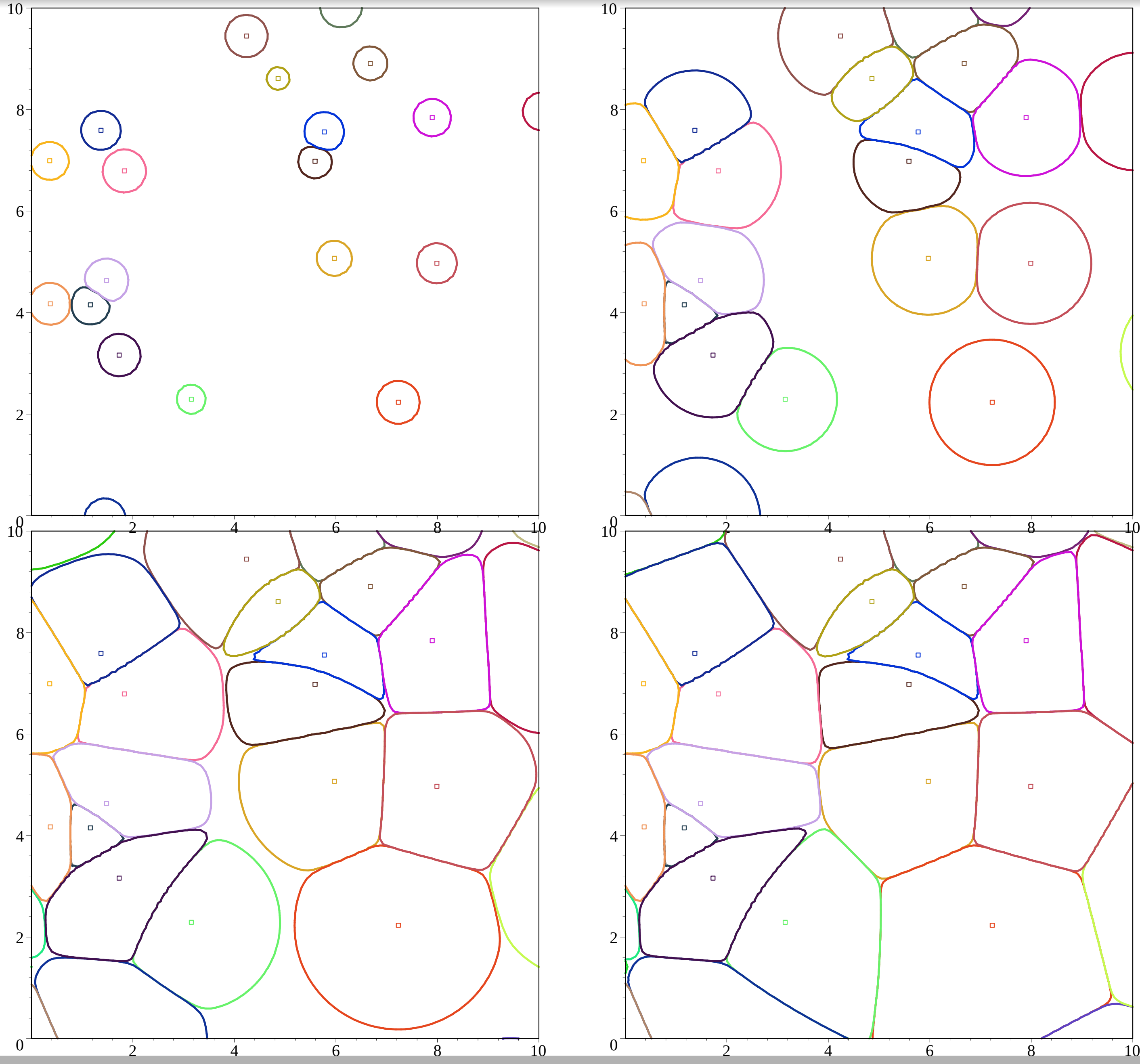

Qualitatively different from BM and VT

Locally:

- Maximal overlapping phenomenon: $\mathcal{N}$ has a finite support (unlike in BM, where $\mathcal{N}$ $\sim$Poisson); to be explained...

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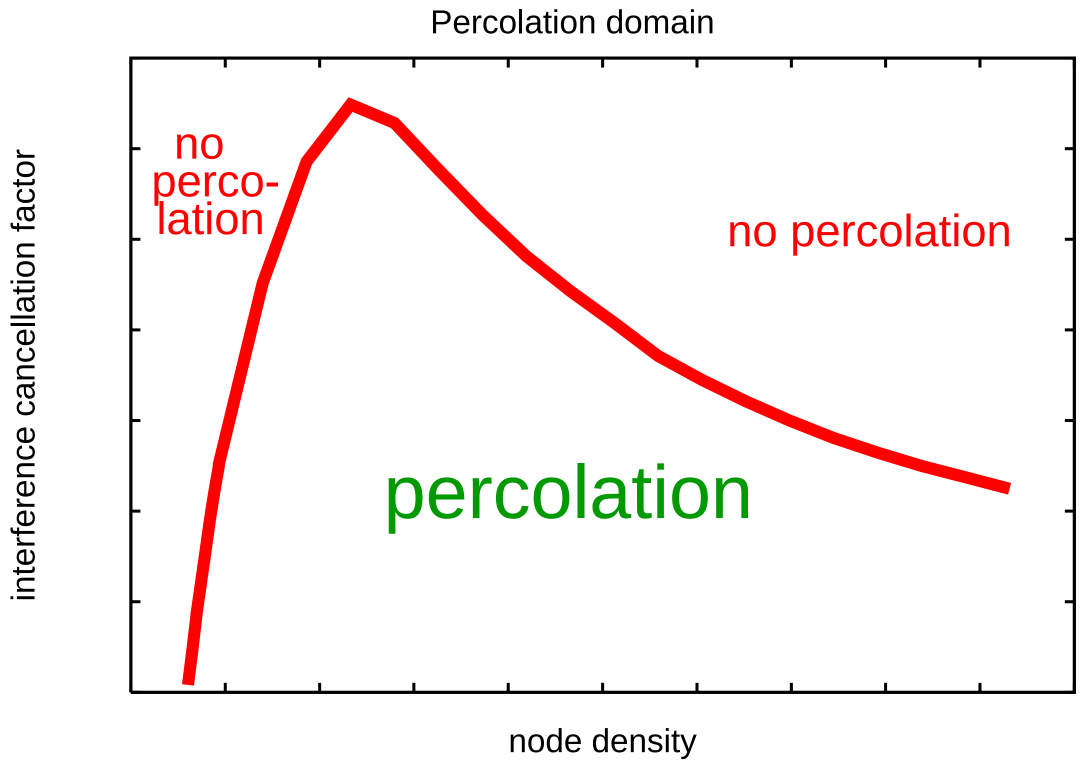
Globally:

- Bounded super-critical percolation regime: Increasing node density may destroy infinite components.

Percolation in SINR coverage model

Dousse, F. Baccelli, and P Thiran (2003),

Dousse, Franceschetti, Macris, Meester, Thiran (2006)



Increasing node density may destroy infinite component(s)!

Coverage characteristics

Coverage by the typical cell

Without loss of generality $\gamma = 1$.

Under Palm P^0, cell C_0 of $X_0 = 0$, $x \in \mathbb{R}^2$, $|x| = r$,

$$\mathsf{P}^0\{x \in C_0\} = \mathsf{P}^0\left\{S_0 \geq \tau W \ell(r) + \tau \ell(r) \sum_{i \neq 0} \frac{S_i}{\ell(|y - X_i|)}\right\}$$

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By the Slivnyak's theorem

$$\mathsf{P}^{\mathbf{0}}\left\{ S_0 \geq \tau W \ell(r) + \tau \ell(r) \sum_{i \neq 0} \frac{S_i}{\ell(|y - X_i|)} \right\}$$

$$= \mathsf{P}\left\{ S \geq \tau W \ell(r) + \tau \ell(r) \sum_{i} \frac{S_i}{\ell(|y - X_i|)} \right\}$$

with S, W and $\sum_i(\ldots)$ independent under P.

Shot-noise functional

The linear functional

$$I = \sum_i f(X_i) = \int f(x)\Phi(\mathrm{d}x)$$

of a point process $\Phi = \{X_i\}$ is called in SG shot-noise (SN) of Φ with the response function f.

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The Laplace transform $\mathcal{L}_I$ of the SN I can be directly expressed by the Laplace transform of the point process Φ

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Consequently, for Poisson point process Φ of intensity $\Lambda(\mathrm{d}x)$

$$\mathcal{L}_I(\xi) = e^{-\int (1-e^{-\xi f(x)})\,\Lambda(\mathrm{d}x)}\,.$$

Back to the typical cell coverage

The Laplace transform of $I = \sum_i \frac{S_i}{\ell(|y - X_i|)}$ with $\ell(r) = (Kr)^\beta$ is equal to

$$\mathcal{L}_I(\xi) = e^{-\lambda K^{-2}\xi^{2/\beta}\pi\Gamma(1-2/\beta)\mathsf{E}[S^{\frac{2}{\beta}}]}$$

where $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,\mathrm{d}t$ (gamma function).

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where $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \, \mathrm{d}t$ (gamma function).

Calculation of $\mathsf{P}^0\{x \in C_0\}$ is reduced to the problem of calculating the probabilities

$$\mathsf{P}\Big\{\tau W \ell(r) + \tau \ell(r) I - S \geq 0\Big\},$$

where S and W and I are independent with known Laplace transforms $\mathcal{L}_W, \mathcal{L}_I$ and $\mathcal{L}_S$, respectively.

A Riemann boundary problem (RBP)

For a given $\Psi(z)$ defined for z on the imaginary axis $\mathcal{I}$, find $\Psi^+(z)$ and $\Psi^-(z)$ defined and analytic on $\mathrm{Re}(z) \geq 0$ and $\mathrm{Re}(z) \leq 0$, respectively, satisfying

$$\Psi(z) = \Psi^+(z) + \Psi^-(z) \quad \text{for } z \in \mathcal{I}.$$

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Sokhotski's solution: unique

$$\Psi^\pm(z) = \frac{\Psi(z)}{2} 1(z \in \mathcal{I}) \mp \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Psi(\xi)}{\xi - z} \mathrm{d}\xi,$$

where, for $z \in \mathcal{I}$, the singular at z integral is understood in the principal value sense (limit of the integral over $(-\infty, z-\epsilon] \cup [z+\epsilon, \infty)$ with $\epsilon \to 0$), provided $\Psi(z)$ is Hölder and integrable on the imaginary axis with $|\Psi(z)| \leq A/|z|$ for some A and large $|z|$.

Probabilities via the RBP

FACT: Consider random variable $\boldsymbol{Y}$ having a density and denote by $\boldsymbol{\mathcal{L}_Y}$ its Laplace transform. Then

$$\mathsf{P}(\boldsymbol{Y} \geq \mathbf{0}) = \frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi i}} \int_{-\infty}^{\infty} \frac{\boldsymbol{\mathcal{L}_Y(i\xi)}}{\boldsymbol{\xi}} \, \mathrm{d}\boldsymbol{\xi},$$

where $\boldsymbol{i}$ is the imaginary unit and the singular at 0 integral is understood in the principal value sense.

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where i is the imaginary unit and the singular at 0 integral is understood in the principal value sense.

proof: Denote by $f(x)$ the density of Y and define $f^+(x) = f(x)1(x \geq 0)$, $f^-(x) = f(x)1(x \leq 0)$. Consider $\Psi(z) = \int_{-\infty}^{\infty} e^{-zx} f(x)\, dx$ and $\Psi^{\pm}(z) = \int_{-\infty}^{\infty} e^{-zx} f^{\pm}(x)\, dx$. Ψ and $\Psi^{\pm}$ satisfy the Rieman boundary problem having the unique solution. Thus $\mathsf{P}\{Y \geq 0\}$ must be equal to $\Psi^+(0)$ where $\Psi^{\pm}(z)$ is the Sokhotski's solution of the problem.

Typical cell coverage via the RBP

COR.: I has density provided $\mathsf{P}\{S > 0\} > 0$ and

$$\mathsf{P}^0\{x \in C_0\}$$
$$= \frac{1}{2} - \frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\mathcal{L}_W(-i\xi\tau\ell(r))\,\mathcal{L}_I(-i\xi\tau\ell(r))\,\mathcal{L}_S(i\xi)}{\xi}\,\mathrm{d}\xi\,.$$

with the singular at 0 integral understood in the principal value sense.

Baccelli, BB (2001)

Plancherel-Parseval theorem

FACT: For all square integrable functions f and g,

$$\int_{-\infty}^{\infty} f(t)g(t)\mathrm{d}t = \int_{-\infty}^{\infty} \widehat{f}(s)\overline{\widehat{g}(s)}\mathrm{d}s,$$

where $\widehat{f}(s) = \int_{\mathbb{R}} e^{-2i\pi ts} f(t)\mathrm{d}t$, denotes Fourier transform and $\overline{\widehat{g}(s)}$ is the complex conjugate of $\widehat{g}(s)$.

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$$\int_a^b f(t)\mathrm{d}t = \int_{-\infty}^{\infty} \widehat{f}(s)\frac{e^{2i\pi bs} - e^{2i\pi as}}{2i\pi s}\mathrm{d}s.$$

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COR.:Assume $\mathsf{P}\{S > 0\} > 0$. Then $\mathsf{P}^0\{x \in C_0\} =$

$$\int_{-\infty}^{\infty} \mathcal{L}_I\left(2i\pi\ell(r)Ts\right)\mathcal{L}_W\left(2i\pi\ell(r)Ts\right)\frac{\mathcal{L}_S(-2i\pi s) - 1}{2i\pi s}\mathrm{d}s\,.$$

Case exponential S

Assume S exponential (mean 1 without loss of generality).
With $|x| = r$

$$\mathsf{P}^0\{x \in C_0\} = \mathsf{P}\Big\{S \geq \tau W \ell(r) + \tau \ell(r) I\Big\}$$

$$= \mathsf{E}\Big[e^{-\tau W \ell(r) - \tau \ell(r) I}\Big]$$

$$= \mathcal{L}_W\Big(\tau \ell(r)\Big) \times \mathcal{L}_I\Big(\tau \ell(r)\Big)$$

$$= \mathcal{L}_W\Big(\tau (K r)^\beta\Big) \times \exp\Big\{-\lambda r^2 \tau^{2/\beta} \pi \Gamma(1 - 2/\beta)\Gamma(1 + 2\beta)/\beta\Big\}$$

Explicit expression!

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Explicit expression!
Exponential distribution of S corresponds to wireless channels with the so called Rayleigh fading.
So it is not merely for mathematical convenience!

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Baccelli, BB (2003), cf also Zorzi, Pupolin (1994) for an early idea (with “doubly-stochastic” exponential S)
This very simple observation inspired amazing amount of subsequent works in the engineering literature...

Stationary coverage

Denote the number of cells covering the origin $\mathbf{0}$ by

$$\mathcal{N} = \sum_i \mathbf{1}(0 \in C_i)\,.$$

We are interested in the distribution of $\mathcal{N}$

$$p_k := \mathsf{P}\{\mathcal{N} \geq k\}\,.$$

p_k is called stationary k-coverage probability and
$p := p_1 = \mathsf{P}\{0 \in \bigcup_i C_i\}$ stationary coverage probability.

Bounded support of $\mathcal{N}$

FACT:

$$\mathcal{N} \leq \lceil 1/\tau \rceil \quad \text{P-a.s.},$$

where $\lceil x \rceil$ is the ceiling of x (the smallest integer not less than x). In other words

$$p_k = 0 \quad \text{for } k \geq 1 + \frac{1}{\tau}.$$

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Recall,
for VT $p_1 = 1$ and $p_k = 0$ for $k \geq 2$,
and for BM $\mathcal{N}$ is a Poisson variable, thus $p_k > 0$ for all $k \geq 0$.

Bounded support of $\mathcal{N}$

Proof: $y \in C_{i_j}$ for $j = 1, \ldots, n$ means

$$\text{SINR}_{i_j} := \frac{S_{i_j}/\ell(|y - X_{i_j}|)}{W + \sum_{k \neq i_j} S_k/\ell(|y - X_k|)} \geq \tau \quad j = 1, \ldots, n.$$

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Equivalently, for STINR (TI=Total Interference):

$$\mathrm{STINR}_{i_j} : \frac{S_{i_j}/\ell(|y - X_{i_j}|)}{W + \sum_{k} S_k/\ell(|y - X_k|)} \geq \frac{\tau}{1+\tau} \quad j = 1, \ldots, n.$$

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Consequently

$$1 > \sum_{j=1}^{n} \frac{S_{i_j}/\ell(|y - X_{i_j}|)}{W + \sum_k S_k/\ell(|y - X_k|)} \geq \frac{n\tau}{1+\tau}$$

and thus $n < 1 + 1/\tau$.

Finite factorial expansions

COR $\mathsf{E}[\mathcal{N}^{(k)}] := \mathsf{E}\left[\mathcal{N}(\mathcal{N}-1)^+ \dots (\mathcal{N}-k+1)^+\right] = 0$ for $k \geq 1 + \frac{1}{\tau}$ and thus the usual expansions are in fact finite sums: for $k \geq 1$

$$\begin{aligned}
p_k &= \sum_{n=k}^{\lceil 1/\tau \rceil} (-1)^{n-k} \binom{n-1}{k-1} n! \mathsf{E}[\mathcal{N}^{(n)}], \\
\mathsf{P}\{\mathcal{N} = k\} &= \sum_{n=k}^{\lceil 1/\tau \rceil} (-1)^{n-k} \binom{n}{k} n! \mathsf{E}[\mathcal{N}^{(n)}], \\
\mathsf{E}[z^{\mathcal{N}}] &= \sum_{n=0}^{\lceil 1/\tau \rceil} (z-1)^n n! \mathsf{E}[\mathcal{N}^{(n)}], \quad z \in [0,1].
\end{aligned}$$

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$$\mathsf{P}\{\mathcal{N}=k\} = \sum_{n=k}^{\lceil 1/\tau \rceil} (-1)^{n-k} \binom{n}{k} n! \mathsf{E}[\mathcal{N}^{(n)}],$$

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As for U-statistics, i.e. functionals of the form $\sum_{(X_{i_j}:j)\in\Phi^{(n)}} \xi(X_{i_j}:j)$; cf. Reitzner, Schulte (2013).

Invariance of the distribution of $\mathcal{N}$

Denote $\boldsymbol{\Theta} := \left\{ \boldsymbol{Y_i} := \frac{\boldsymbol{S_i}}{\boldsymbol{\ell(|X_i|)}}, \boldsymbol{X_i} \in \boldsymbol{\Phi} \right\}$ (user path-gain process)

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LEM.: Θ is inhomogeneous Poisson pp on $(0, \infty)$ with intensity measure $2a/\beta\, t^{-1-2/\beta}\, \mathrm{d}t$, where $a := \frac{\lambda \pi \mathsf{E}[S^{\frac{2}{\beta}}]}{K^2}$.

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Helpful in proofs, where for mathematical convenience S can be assumed exponential with the results generalized to arbitrary S by appropriate modification of a.

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LEM.: $\boldsymbol{\Theta}$ is inhomogeneous Poisson pp on $(\mathbf{0}, \infty)$ with intensity measure $\boldsymbol{2a/\beta\, t^{-1-2/\beta}}\,\mathrm{d}\boldsymbol{t}$, where $\boldsymbol{a} := \frac{\boldsymbol{\lambda\pi}\mathsf{E}[\boldsymbol{S}^{\frac{2}{\beta}}]}{\boldsymbol{K^2}}$.

COR.: The $\boldsymbol{k}$-coverage probabilities (and all functionals of $\boldsymbol{\Theta}$ and $\boldsymbol{W}$) depend only on $\boldsymbol{\beta}$ and $\boldsymbol{a}\ \boldsymbol{W}$, but are invariant w.r.t. the distribution of $\boldsymbol{S}$.

Helpful in proofs, where for mathematical convenience $\boldsymbol{S}$ can be assumed exponential with the results generalized to arbitrary $\boldsymbol{S}$ by appropriate modification of $\boldsymbol{a}$.

Credits: shot-noise equiv. Gilbert, Pollak (1960), Lowen, Teich (1990),
in physics (spin glasses) Bolthausen, Sznitman (1998),
in the SINR context BB, Karray, Klepper (2010)
in secrecy graphs, Pinto, Barros, Win (2012).

Invariance, cont'd

Proof: By the displacement theorem, Θ is Poisson point process on $(0, \infty)$.

Invariance, cont'd

Proof: By the displacement theorem, Θ is Poisson point process on $(0, \infty)$. We calculate its intensity measure:

$$
\begin{aligned}
\Lambda([s,\infty)) :=& \mathsf{E}[\Theta([s,\infty))] \\
=& \lambda \int_{\mathbb{R}^2} \mathsf{P}\{\, S/\ell(|z|) \geq s \,\}\,\mathrm{d}z \\
=& 2\pi\lambda \int_0^\infty r\mathsf{P}\{\, S/\ell(r) \geq s \,\}\,\mathrm{d}r \\
=& 2\pi\lambda \int_0^\infty r\mathsf{E}\left[1\left(r \leq (sS)^{1/\beta}/K\right)\right]\mathrm{d}r \\
=& 2\pi\lambda\mathsf{E}\left[\int_0^{(sS)^{1/\beta}/K} r\,dr\right] = \frac{\lambda s^{2/\beta}\pi}{K^2}\mathsf{E}\left[S^{\frac{2}{\beta}}\right].
\end{aligned}
$$

Some special functions for $\mathbf{E}[\mathcal{N}^{(n)}]$

For $n \geq 1$, define some functions of $x \geq 0$

$$\mathcal{I}_{n,\beta}(x) = \frac{2^n \int_0^\infty u^{2n-1} e^{-u^2 - u^\beta x \Gamma(1-2/\beta)^{-\beta/2}} du}{\beta^{n-1} (\Gamma(1-2/\beta)\Gamma(1+2/\beta))^n (n-1)!}.$$

In particular

$$\mathcal{I}_{n,\beta}(0) = \frac{2^{n-1}}{\beta^{n-1}(C'(\beta))^n},$$

where $C'(\beta) = \Gamma(1-2/\beta)\Gamma(1+2/\beta)$.

Another special functions for $\mathbf{E}[\mathcal{N}^{(n)}]$

For $n \geq 1$, define also functions of $(x_1, \ldots, x_i) \geq 0$

$$\mathcal{J}_{n,\beta}(x_1, \ldots, x_n)$$

$$= \frac{(1 + \sum_{j=1}^{n} x_j)}{n} \int_{[0,1]^{n-1}} \frac{\prod_{i=1}^{n-1} v_i^{i(2/\beta+1)-1} (1 - v_i)^{2/\beta}}{\prod_{i=1}^{n} (x_i + \eta_i)} dv_1 \ldots dv_{n-1},$$

where

$$\begin{cases} \eta_1 & = v_1 v_2 \ldots v_{n-1} \\ \eta_2 & = (1 - v_1) v_2 \ldots v_{n-1} \\ \eta_3 & = (1 - v_2) v_3 \ldots v_{n-1} \\ & \ldots \\ \eta_n & = 1 - v_{n-1}. \end{cases}$$

Factorial moments of $\mathcal{N}$

PROP.: Assume $\mathsf{E}(S^{2/\beta}) < \infty$ and (for simplicity) deterministic W. Then for $n \geq 1$

$$\mathsf{E}[\mathcal{N}^{(n)}] = \begin{cases} \tau_n^{-2n/\beta} \mathcal{I}_{n,\beta}(Wa^{-\beta/2}) \mathcal{J}_{n,\beta}(\tau_n) & \text{for } 0 < \tau < \frac{1}{n-1} \\ 0 & \text{otherwise,} \end{cases}$$

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$$\tau_n = \frac{\tau}{1-(n-1)\tau}\,.$$

Keeler, BB, Karray (2013).

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where

$$\tau_n = \frac{\tau}{1-(n-1)\tau}.$$

Keeler, BB, Karray (2013).

Remark, the smaller τ the larger maximal non-null moments. Also, $\mathsf{E}[\mathcal{N}^{(n)}]$ depends on W only via $\mathcal{I}_{n,\beta}(W a^{-\beta/2})$; (factorization of $\mathsf{E}[\mathcal{N}^{(n)}]$ with respect to W, similar to the factorization of $p(x)$ for exponential S).

Stationary coverage distribution

COR.: For arbitrary distribution of S with $\mathsf{E}(S^{2/\beta}) < \infty$ and deterministic W

$$p_k = \sum_{n=k}^{\lceil 1/\tau \rceil} (-1)^{n-k} \binom{n-1}{k-1} \tau_n^{-2n/\beta} \mathcal{I}_{n,\beta}(W a^{-\beta/2}) \mathcal{J}_{n,\beta}(\tau_n) \, .$$

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Keeler, BB, Karray (2013).

In particular for $\tau \geq 1$ we have $\lceil 1/\tau \rceil = 1$ and thus $p_k = 0$ for all $k \geq 2$ (like VT, one-coverage only!) and

$$p = p_1 = \frac{2\tau^{-2/\beta}}{\Gamma(1+\frac{2}{\beta})} \int_0^\infty u e^{-u^2 \Gamma(1-2/\beta)} \mathcal{L}_W\left(a^{-\beta/2} u^\beta\right) du \,.$$

Dhillon, Ganti, Baccelli, Andrews (2012).

Proof idea

For $n = 1$: by the Little's law

$$\mathsf{E}[\mathcal{N}^{(1)}] = \mathsf{E}[\mathcal{N}] = \int_{\mathbb{R}^2} p(x)\,\lambda \mathrm{d}x\,,$$

where $p(x)$ is the typical cell coverage probability.

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This can be done without loss of generality by the invariance property of $\mathcal{N}$.

The proof ($n = 1$) follows by direct calculations with exponential S.

Proof idea, cont'd

For $n \geq 1$, quite similarly

$$\mathsf{E}[\mathcal{N}^{(n)}] = \mathsf{E}\Big[\sum_{\substack{X_{i_1}, X_{i_2}, \ldots, X_{i_n} \in \Phi \\ \text{distinct}}} \mathbf{1}\Big(0 \in \bigcap_{j=1}^{n} C_{i_j} \Big) \Big]$$

$$\overset{\text{higher-order Campbell}}{=} \int_{\mathbb{R}^2} \mathsf{P}^{x_1, \ldots, x_n}\Big(0 \in \bigcap_{j=1}^{n} C_x \Big) \lambda^n \mathrm{d}x_1 \ldots \mathrm{d}x_n \,.$$

Proof idea, cont'd

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The probabilities $\mathsf{P}^{x_1,\dots,x_n}\Big(0 \in \bigcap_{j=1}^{n} C_x\Big)$ can be evaluated explicitly assuming (without loss of generality!) exponential S and using (higher-order) Slivnyak's theorem

$$\mathsf{P}_{\Phi}^{x_1,\dots,x_n} = \mathsf{P}_{\Phi+\sum_{j=1}^{n}\delta_{x_j}}\,.$$

Proof idea, cont'd

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The proof follows by direct calculations with exponential S.

Thank you for today.
Tomorrow: Relations to Poisson-Dirichlet processes