Clustering, percolation and directionally convex ordering of point processes

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joint work with D. Yogeshwaran

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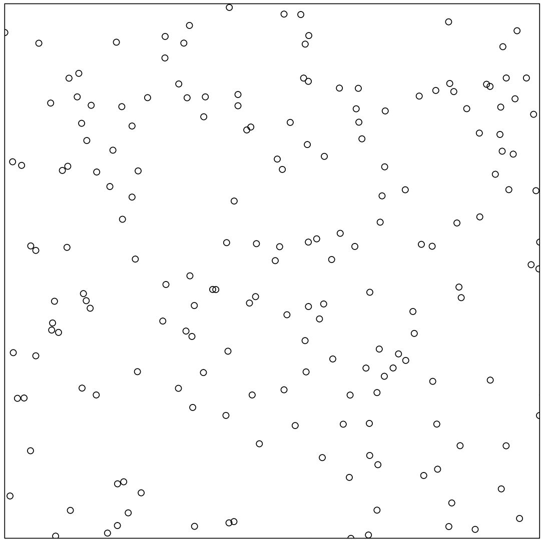
Point process

Point process: random, locally finite, “pattern of points” $\mathbf{\Phi}$ in some space $\mathbb{E}$.



A realization of $\mathbf{\Phi}$ on $\mathbb{E} = \mathbb{R}^{\mathbf{2}}$.

Point process; cont'd

Usual probabilistic formalism:

- Φ is a measurable mapping from a probability space $(\Omega, \mathcal{A}, \mathsf{P})$ to a measurable space $\mathbb{M}$ "of point patterns", say, on Euclidean space $\mathbb{E} = \mathbb{R}^d$ of dimension $d \geq 1$.

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- A point pattern is considered as a counting measure; its points are atoms of this measure. Hence

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- Mean measure of Φ:

$$\mathbf{E}(\Phi(B)) = \text{expected number of points of } \Phi \text{ in } B\,.$$

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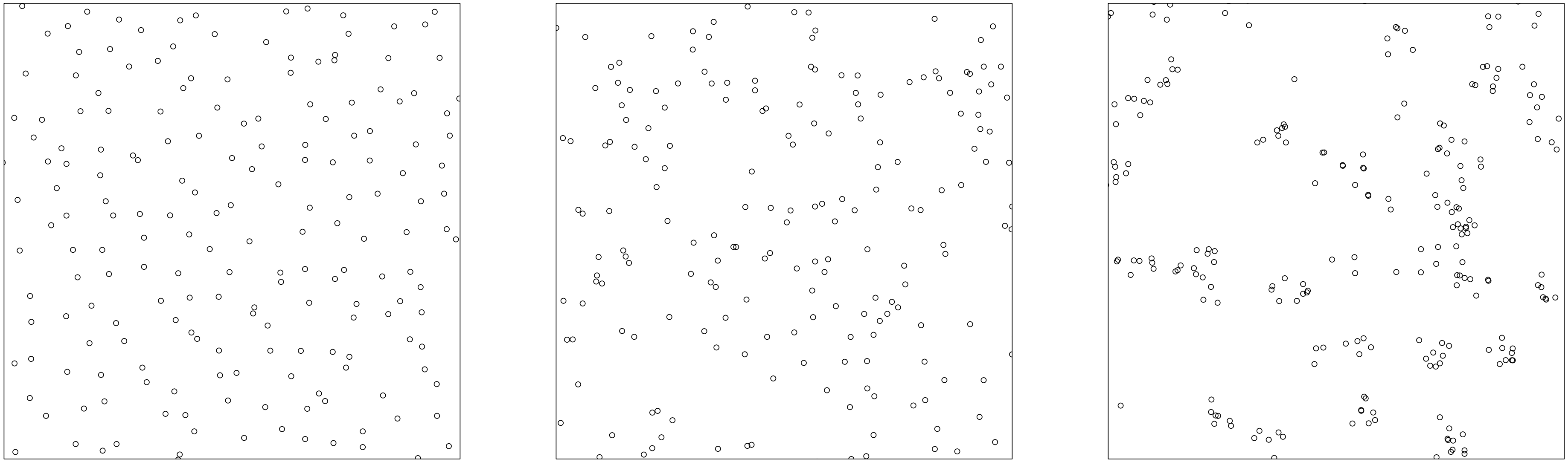

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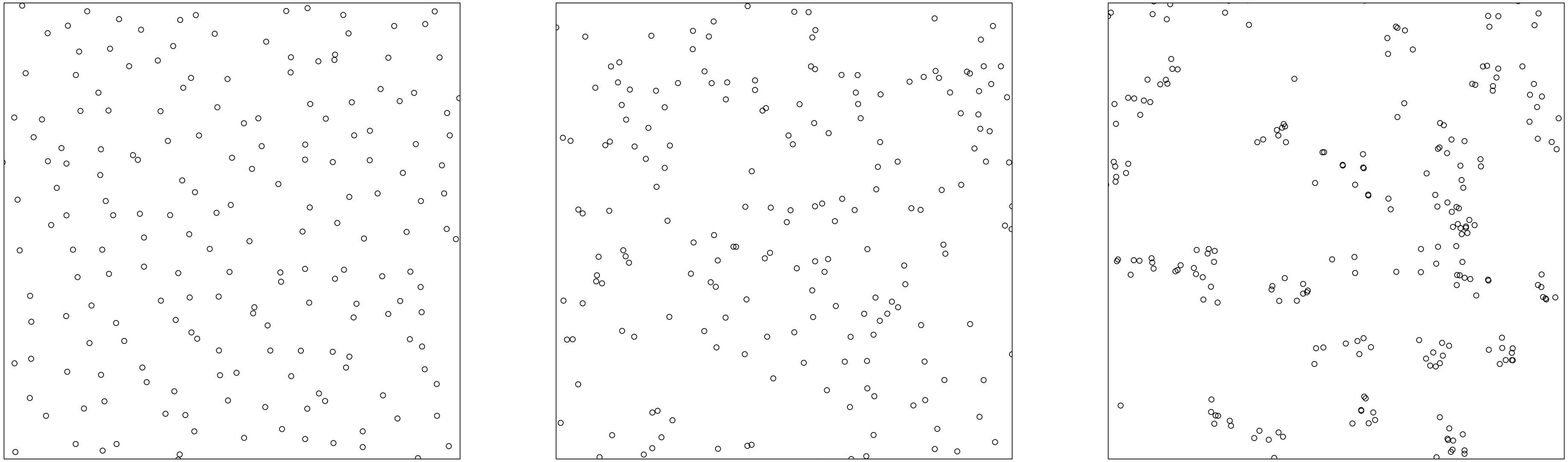

How to compare clustering properties of two point processes (pp) $\mathbf{\Phi_1}$, $\mathbf{\Phi_2}$ having “on average” the same number of points per unit of space?

More precisely, having the same mean measure: $\mathbf{E}(\Phi_1(B)) = \mathbf{E}(\Phi_2(B))$ for all $B \subset \mathbb{E}$.

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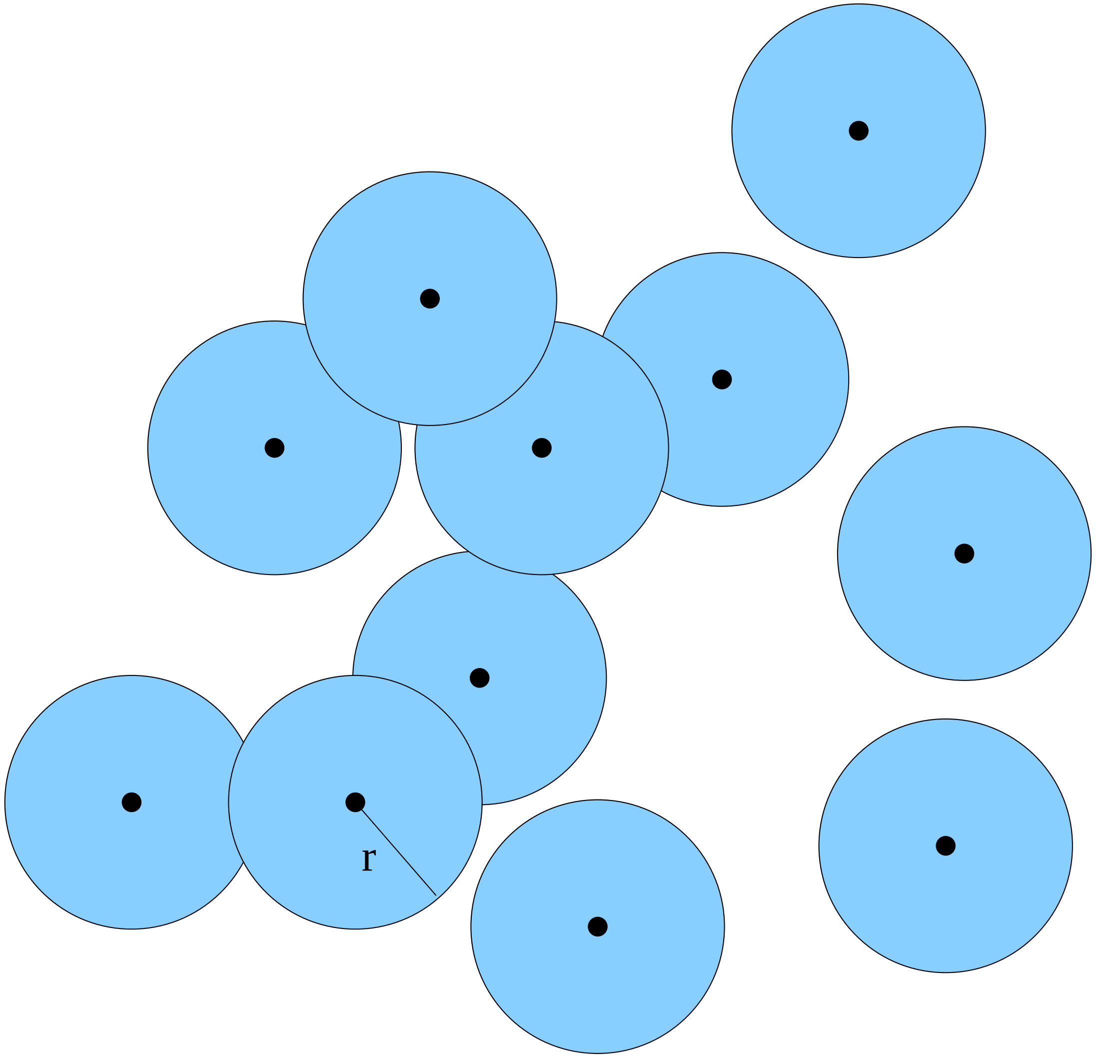
Should be consistent with statistical descriptors of clustering (to be explained).

Continuum percolation

Boolean model $C(\Phi, r)$:
germs in Φ,
spherical grains of given radius r.



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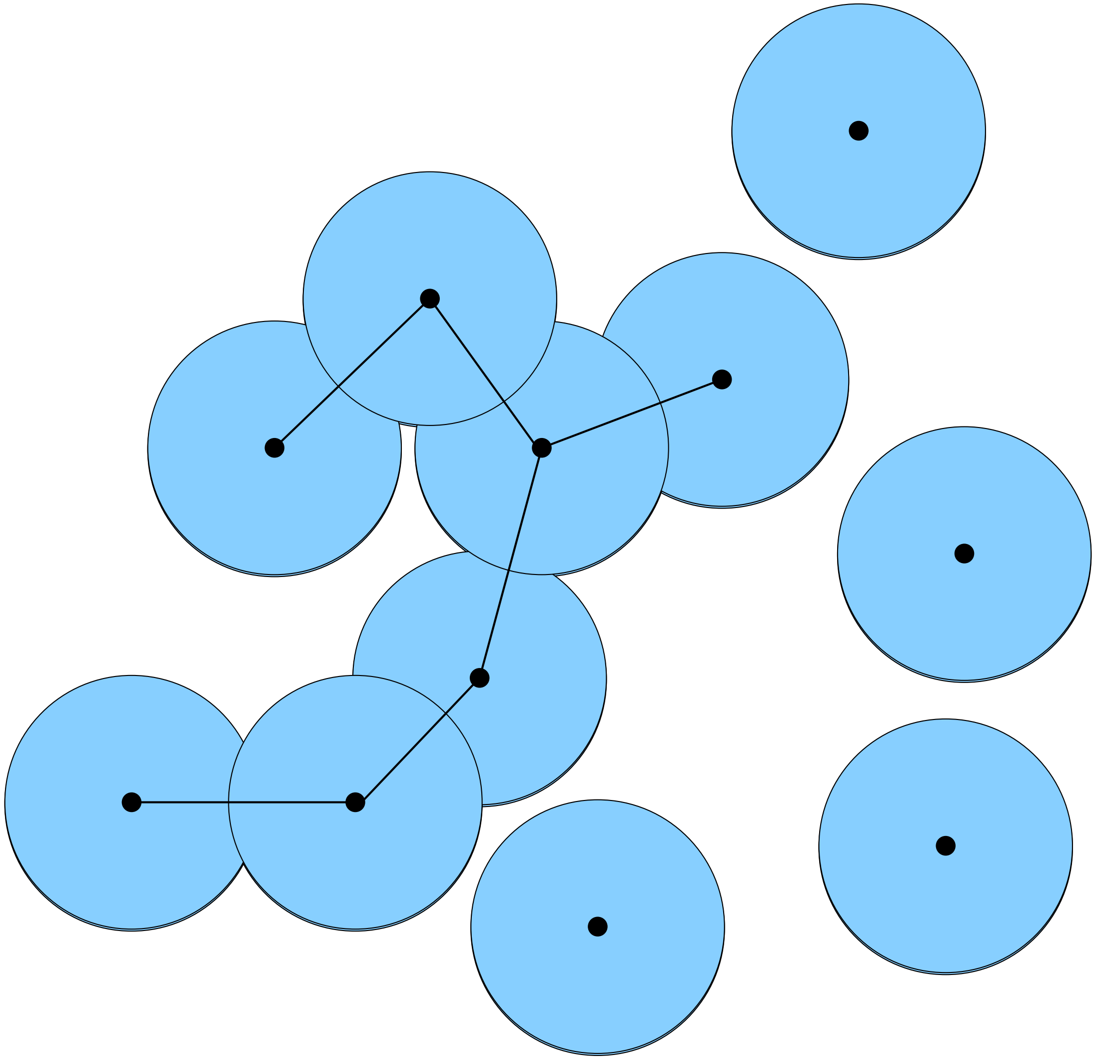

Joining germs whose grains intersect one gets Random Geometric Graph (RGG).

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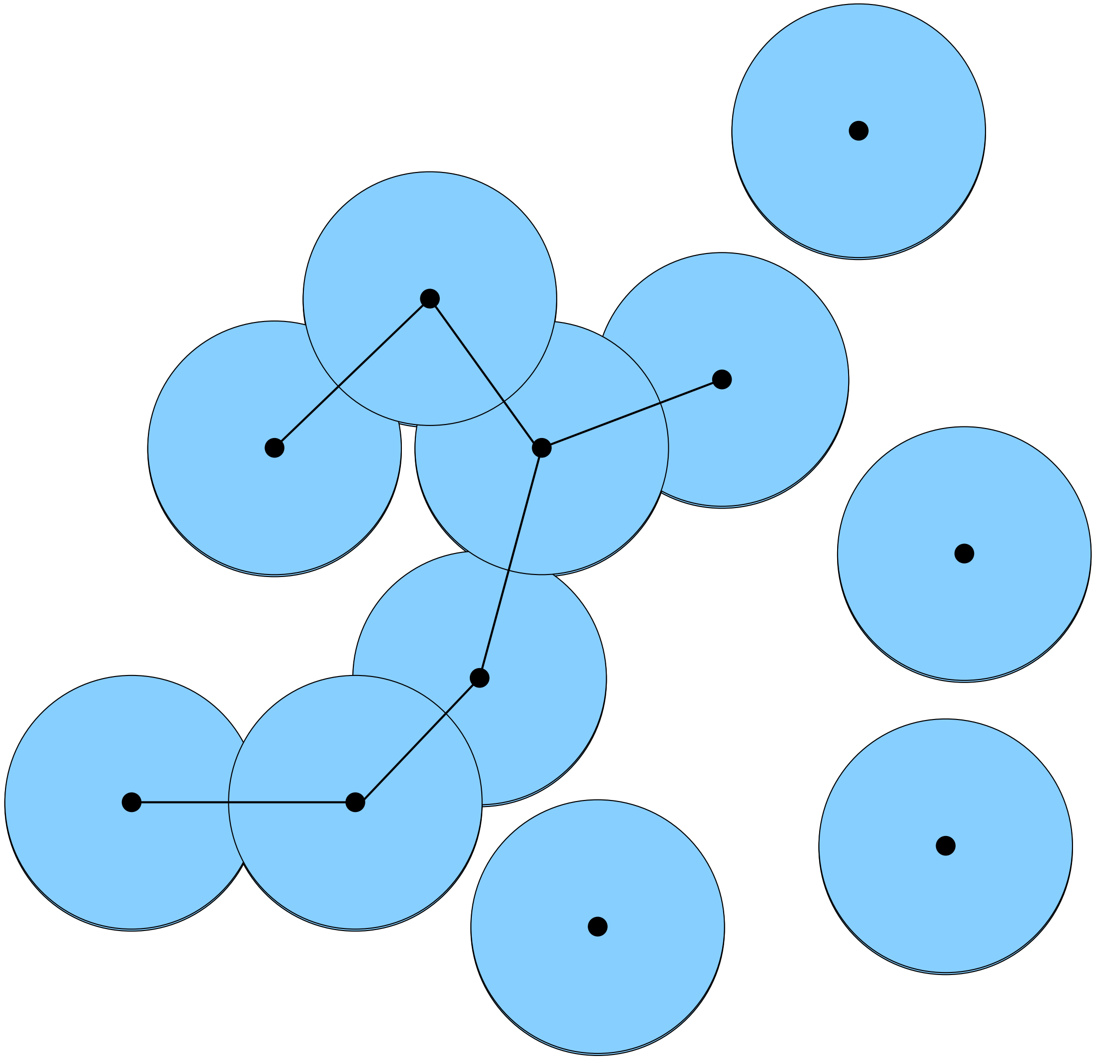

percolation $\equiv$ existence of an infinite connected subset (component).

Critical radius for percolation

Critical radius for the percolation in the Boolean Model with germs in $\boldsymbol{\Phi}$

$$r_c(\Phi) = \inf\{r > 0 : \mathsf{P}(C(\Phi, r)\text{percolates}) > 0\}$$

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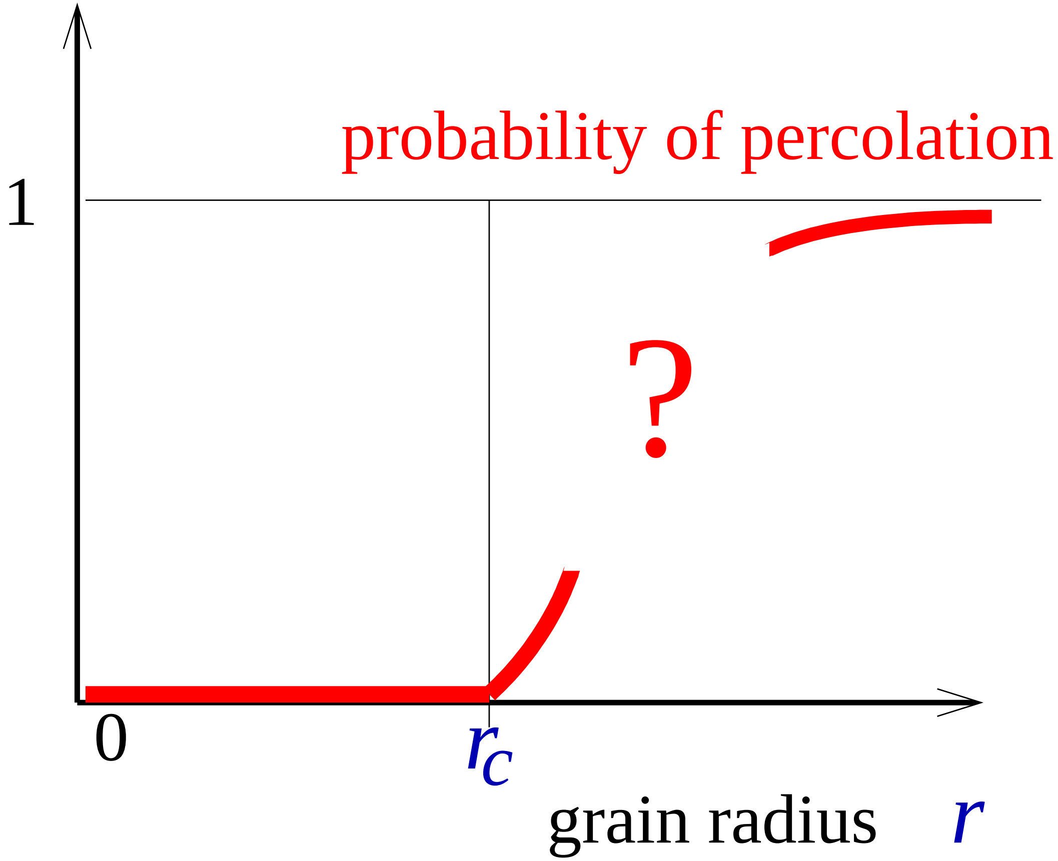
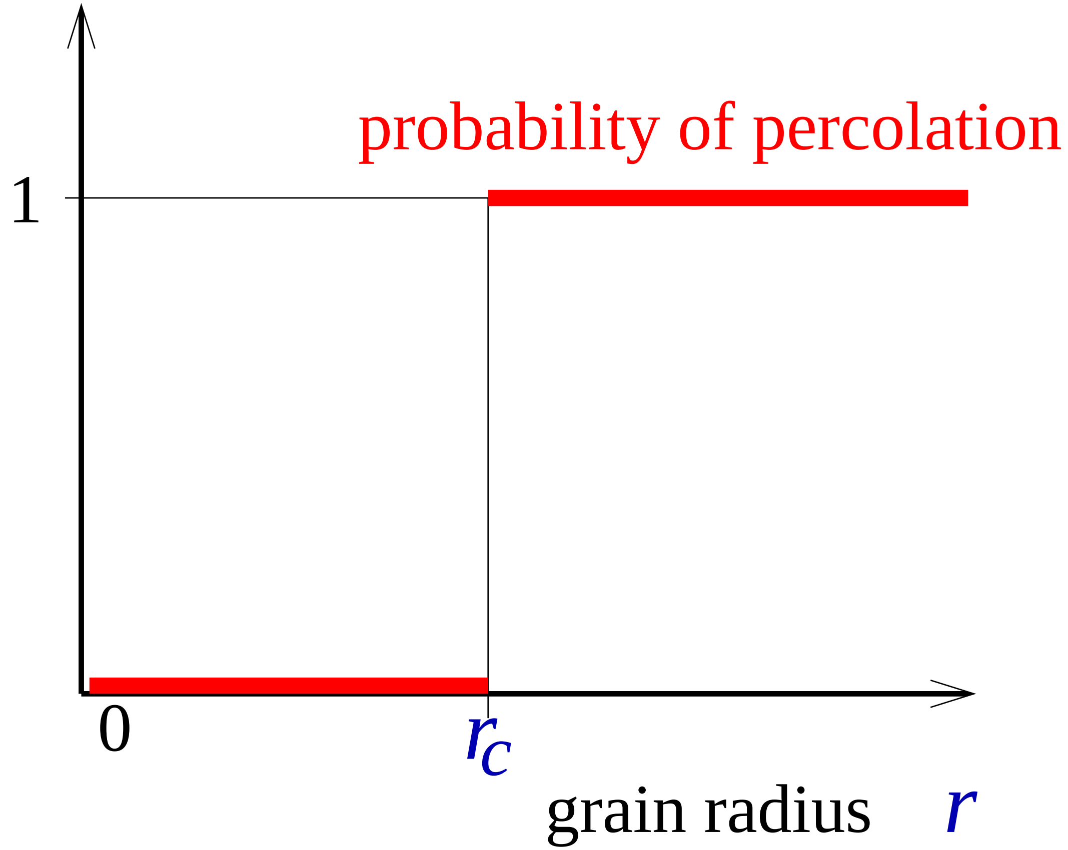


Phase transition in ergodic case

In the case when Φ is stationary and ergodic



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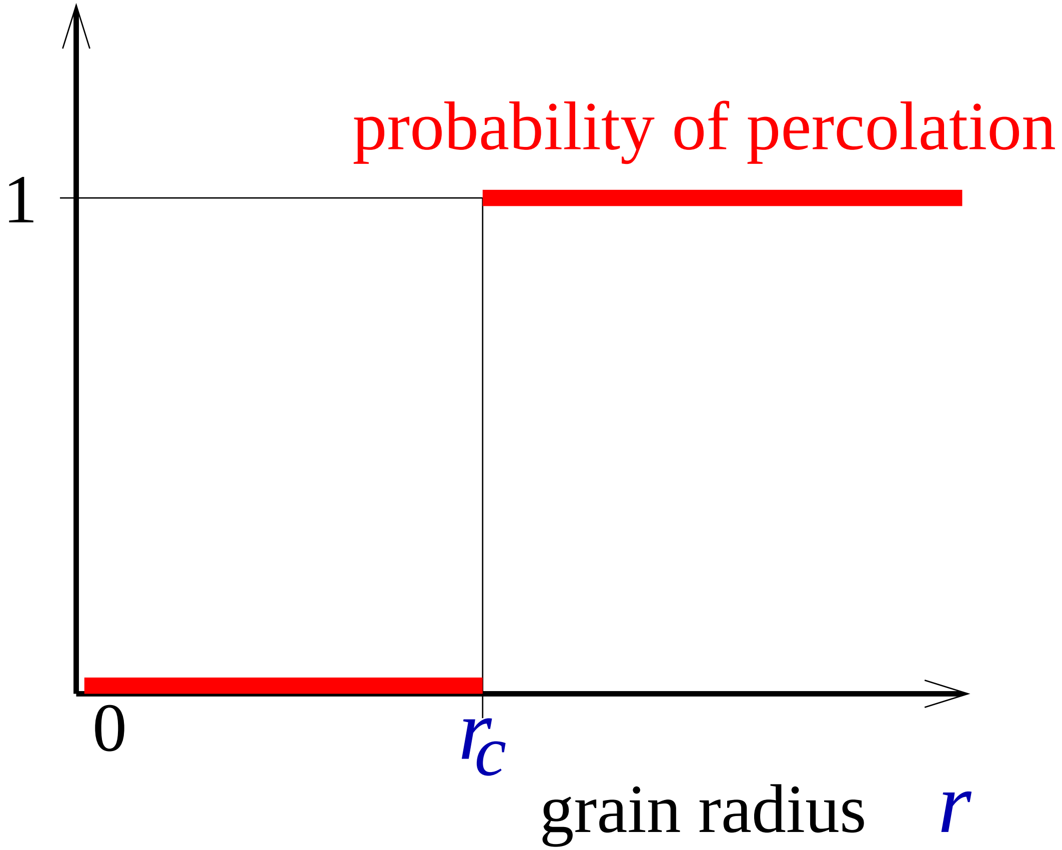

If $0 < r_c < \infty$ we say that the phase transition is non-trivial.

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Point processes exhibiting more clustering of points should have larger critical radius r_c for the percolation of their continuum percolation models.

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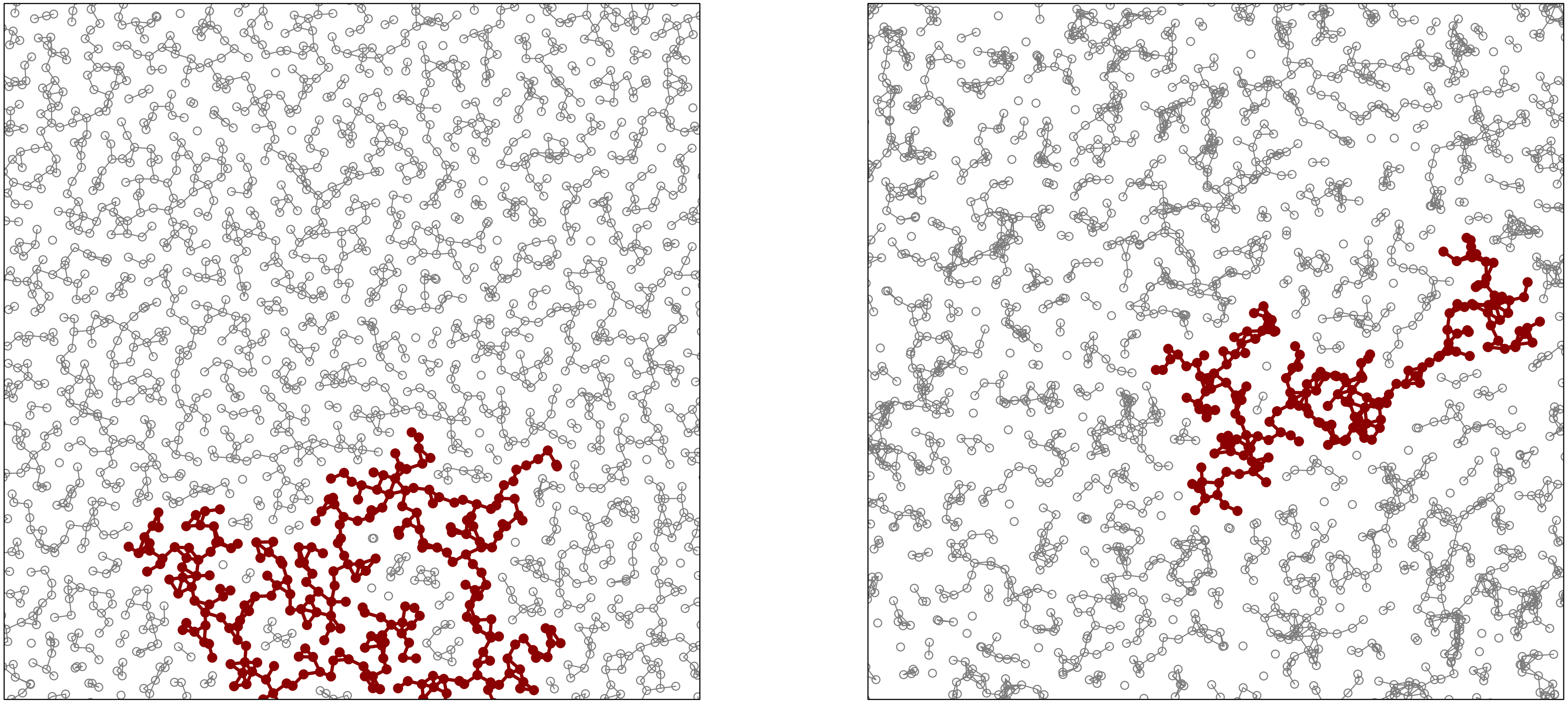
Indeed, points lying in the same cluster of will be connected by edges for some smaller r *but points in different clusters need a relatively higher* r *for having edges between them, and percolation cannot be achieved without edges between some points of different clusters. Spreading points from clusters of ”more homogeneously” in the space should result in a decrease of the radius r for which the percolation takes place.*

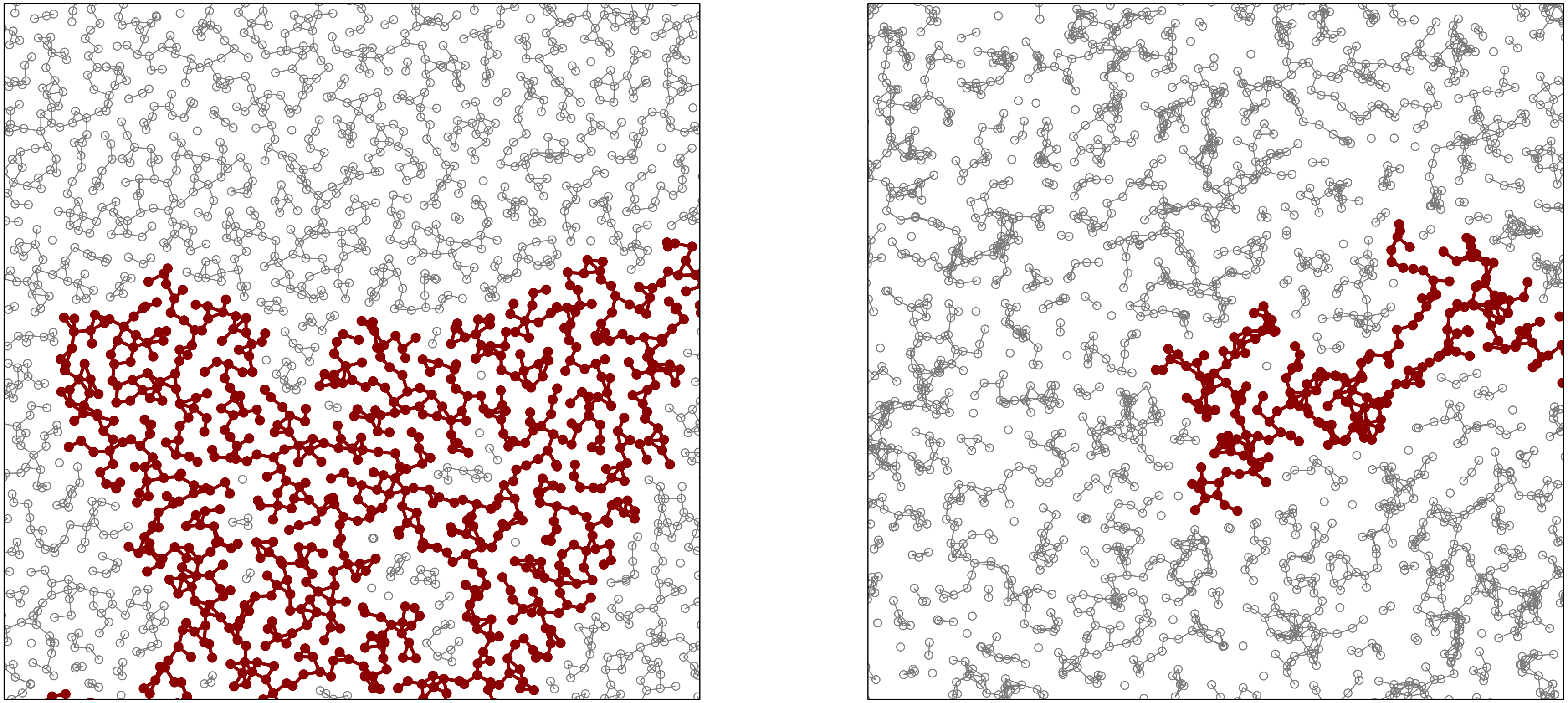
Clustering and percolation



RGG with $r = 98$.

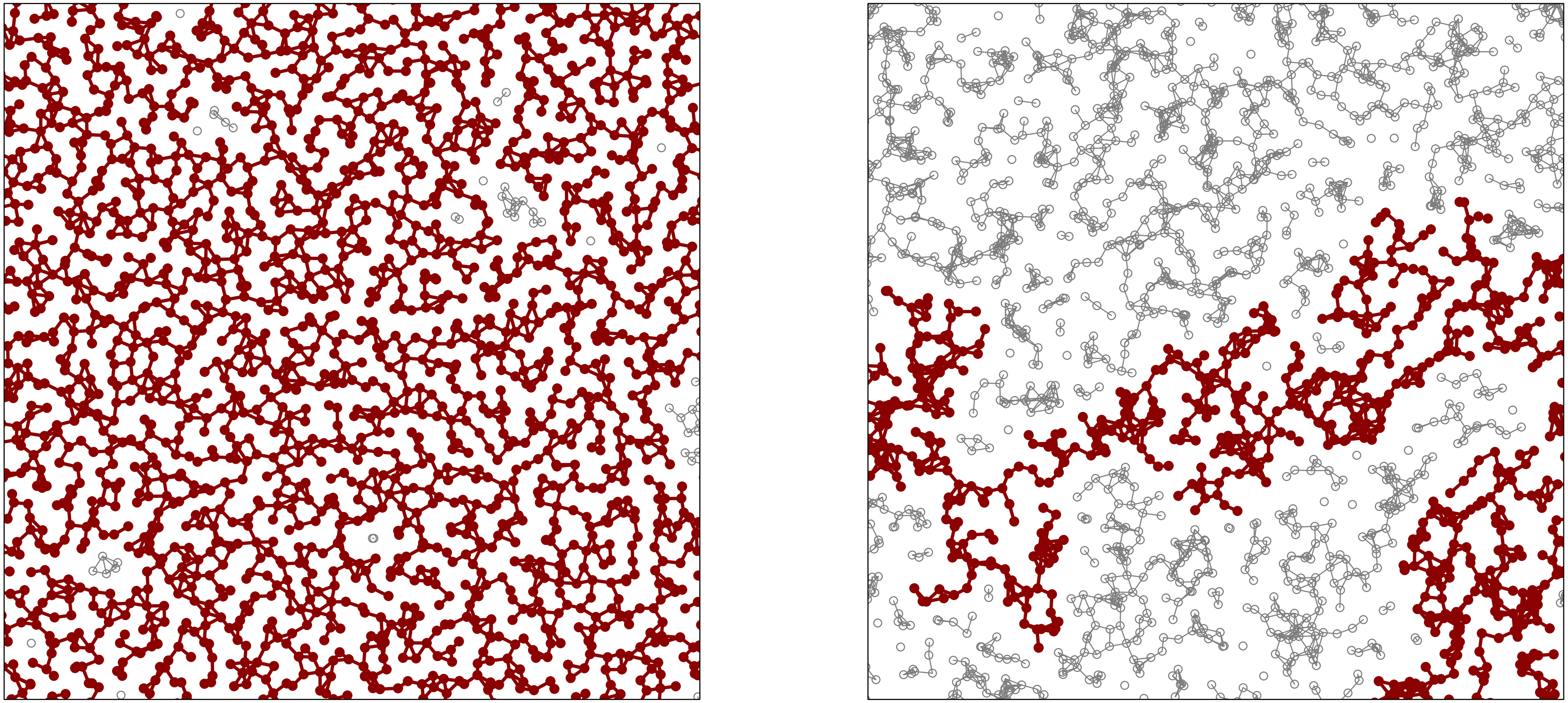
The largest component in the window is highlighted.

Clustering and percolation



$$r = 100$$

Clustering and percolation



$r = 108$

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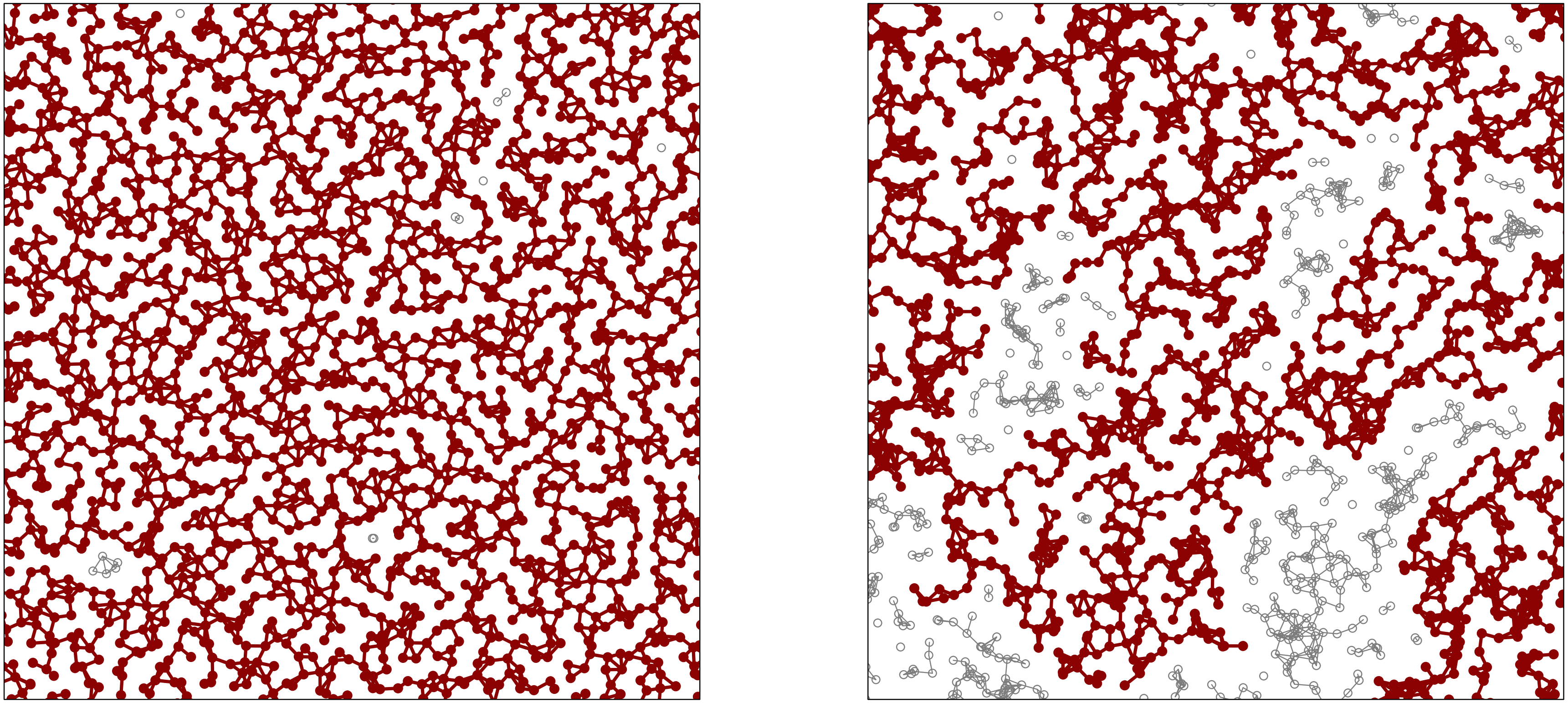
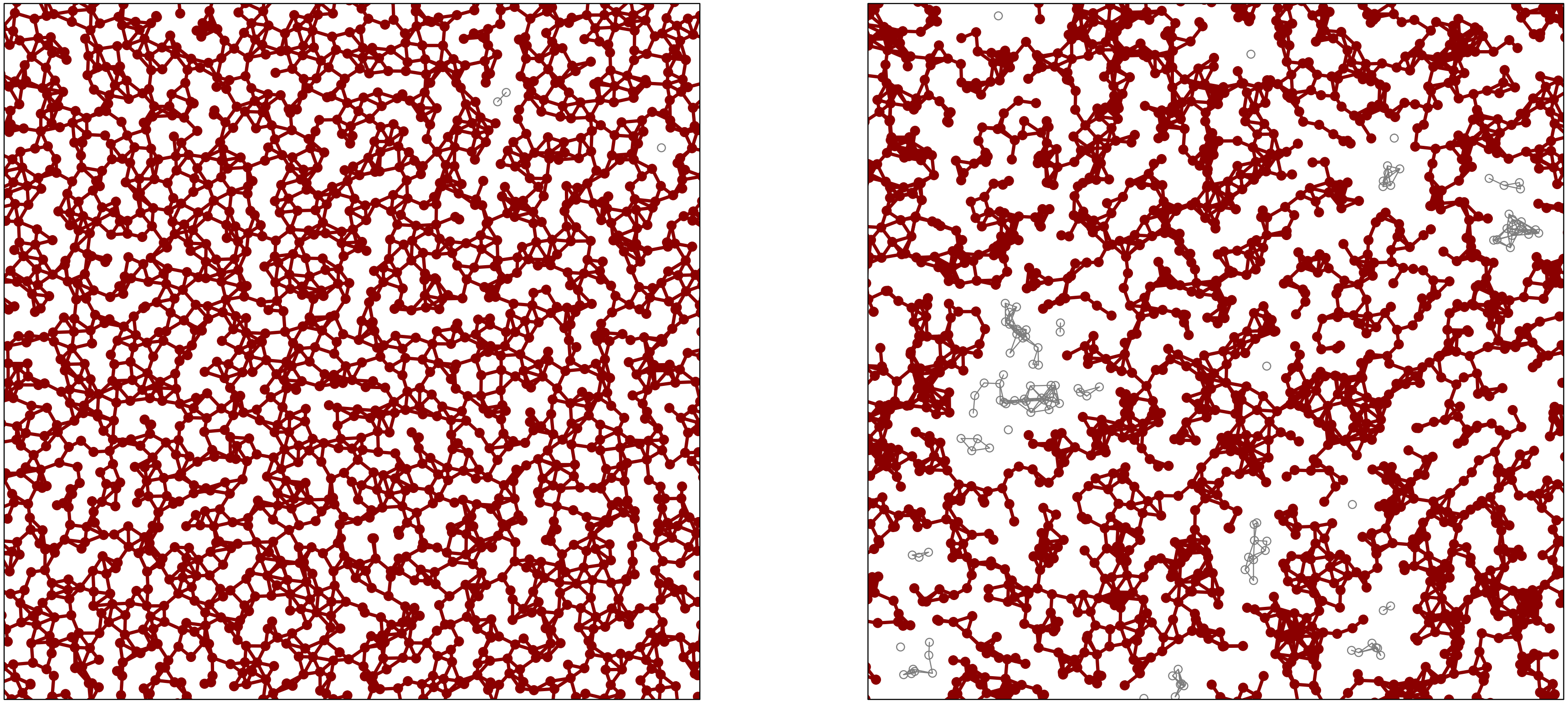


$$r = 112$$

Clustering and percolation



$$r = 120$$

Outline of the remaining part of the lecture

- directionally convex (dcx) order for point processes

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- concluding remarks

dcx ordering of point processes

Stochastic comparison

Integral orders of random vectors:
For two real-valued random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ of the same dimension and a family of test functions $\mathcal{F}$, one says that

$$\boldsymbol{X} \leq_{\mathcal{F}} \boldsymbol{Y} \ \text{ if } \ \mathbf{E}(f(\boldsymbol{X})) \leq \mathbf{E}(f(\boldsymbol{Y})) \quad \forall f \in \mathcal{F},$$

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Various choices of the family of test functions $\mathcal{F}$ allow to compare various aspects of the distributions of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Strong order

Let $\mathcal{F} = st$ be all component-wise increasing functions.
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Strassen's theorem: If $X \leq_{st} Y$ then one can construct both X, Y on a common probability space (couple them) such that $X \leq Y$ almost surely.

Strong order; extension to point processes

One says that $\boldsymbol{\Phi_1 \le_{st} \Phi_2}$ if

$$\boldsymbol{(\Phi_1(B_1),\ldots,\Phi_1(B_n)) \le_{st} (\Phi_2(B_1),\ldots,\Phi_2(B_n))}$$

for every possible finite collection of sets $\boldsymbol{B_1,\ldots,B_n}$.

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Strong order is not suitable for the comparison of point processes with equal mean measures. Indeed, Strassen's theorem implies then equality of the compared processes.

dcx (directionally convex) functions

Function $f : \mathbb{R}^d \to \mathbb{R}$ twice differentiable is dcx if $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0$ for all $x \in \mathbb{R}^d$ and $\forall i, j$.

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Definition can be extended to all functions by saying that f is dcx if all difference operators $\Delta_i^\delta f(x) := f(x + \delta e_i) - f(x)$ are non-negative; $\Delta_i^\epsilon \Delta_j^\delta f(x) \geq 0$, $\forall x \in \mathbb{R}^d$, $i, j \in \{1, \ldots, d\}$, $\delta > 0, \epsilon > 0$.

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Examples:

- $f(x) = e^{-\sum_i a_i x_i}$, $a_i \geq 0$.
- $f(x) = \prod_i \max(x_i, a_i)$, a_i constants,

$\boldsymbol{dcx}$ ordering of random vectors

An integral order generated by $\boldsymbol{\mathcal{F} = dcx}$ functions: For two real-valued random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ of the same dimension

$$\boldsymbol{X} \le_{\boldsymbol{dcx}} \boldsymbol{Y} \quad \text{if} \quad \mathbf{E}(\boldsymbol{f}(\boldsymbol{X})) \le \mathbf{E}(\boldsymbol{f}(\boldsymbol{Y})) \quad \forall \boldsymbol{f} \ \boldsymbol{dcx}\,,$$

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Allows to compare dependence structures and variability of the marginals of random vectors with the same mean $\mathsf{E}(X) = \mathsf{E}(Y)$. (Indeed, both $f(x) = x$ and $f(x) = -x$ are dcx).

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Other "convex-like" orders can be considered; cf. Müller, Stoyan (2002) *Comparison Methods for Stochastic Models and Risk*.

dcx ordering of point processes

Define: $\Phi_1 \le_{dcx} \Phi_2$ if for all bounded Borel subsets $B_1, \ldots, B_n$,

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Sufficient condition: Enough to verify the inequality on disjoint bounded Borel subsets (bBs).

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dcx is a partial order (reflective, antisymmetric and transitive) of point process with locally finite mean measure (to ensure transitivity).

dcx for point processes; properties

- If $\Phi_1 \leq_{dcx} \Phi_2$ then Φ_1 and Φ_2 have equal mean measures; $\mathbf{E}(\Phi_1(\cdot)) = \mathbf{E}(\Phi_2(\cdot))$.

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- If $\Phi_1 \leq_{dcx} \Phi_2$ then Φ_1 and Φ_2 have equal mean measures; $\mathsf{E}(\Phi_1(\cdot)) = \mathsf{E}(\Phi_2(\cdot))$.
- dcx is preserved by independent thinning, marking and superpositioning; i.e.,

$$\text{If } \Phi_1 \leq_{dcx} \Phi_2 \text{ then } \tilde{\Phi}_1 \leq_{dcx} \tilde{\Phi}_2 \,,$$

where $\tilde{\Phi}_i$ is a version of Φ_i independently thinned (or marked, or superposed with a given point process).

dcx and shot-noise fields

Given point process Φ and a non-negative function $h(x, y)$ on $(\mathbb{R}^d, S)$, measurable in x, where S is some set, define shot noise field: for $y \in S$

$$V_\Phi(y) := \sum_{X \in \Phi} h(X, y) = \int_{\mathbb{R}^d} h(x, y)\Phi(dx) \,.$$

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Proposition 1.1 *If* $\Phi_1 \leq_{dcx} \Phi_2$ *then*

$$\big(V_{\Phi_1}(y_1), \ldots, V_{\Phi_1}(y_n)\big) \leq_{dcx} \big(V_{\Phi_2}(y_1), \ldots, V_{\Phi_2}(y_n)\big)$$

for any finite subset $\{y_1, \ldots, y_n\} \subset S$*, provided the RHS has finite mean. In other words,* dcx *is preserved by the shot-noise field construction.*

dcx and shot-noise fields; cont'd

Proof.

- Approximate the integral by simple functions as usual in integration theory: $a.s.$ and in L_1 $\sum_{i=1}^{k_n} a_{in}\Phi(B_{in}^j) \to \int_{\mathbb{R}^d} h(x,y)\Phi(dx) = V_\Phi(y_j)$, $a_{in} \geq 0$.

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- Increasing linear operations preserve dcx hence approximating simple functions are dcx ordered.
- dcx order is preserved by *joint* weak and L_1 convergence. Hence limiting shot-noise fields are dcx ordered.

dcx and extremal shot-noise fields

In the setting as before define for $y \in S$

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$$\mathsf{P}\big(U_{\Phi_1}(y_i) \leq a_i, 1 \leq i \leq m\big) \leq \mathsf{P}\big(U_{\Phi_2}(y_i) \leq a_i, 1 \leq i \leq m\big);$$

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Corollary 1.1 *One-dimensional distributions of the extremal shot-noise fields are strongly ordered with reversed inequality* $U_{\Phi_2}(y) \leq_{st} U_{\Phi_1}(y)$*,* $\forall y \in S$*.*

dcx and extremal shot-noise fields; cont’d

Proof.

- Reduction to an (additive) shot noise:

$$
\begin{aligned}
&\mathsf{P}\left(U_\Phi(y_i) \le a_i, 1 \le i \le n\right) \\
&= \mathsf{E}\left(e^{-\sum_{i=1}^{n}\sum_{X\in\Phi} -\log \mathbf{1}[h(X,y_i)\le a_i]}\right) .
\end{aligned}
$$

dcx and extremal shot-noise fields; cont'd

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- $e^{-\sum x_i}$ is dcx function.

dcx and clustering

dcx and statistical spatial homogeneity

Ripley's K function of a stationary point process on $\mathbb{R}^d$ with finite intensity λ:

$$K(r) := \frac{1}{\lambda\|B\|}\mathsf{E}\Big(\sum_{X_i \in \Phi\cap B}(\Phi(B_{X_i}(r)) - 1)\Big)\ ,$$

where $\|B\|$ denotes the Lebesgue measure of a bBs B.

Pair correlation function (probability of finding a particle at a given position with respect to another particle):

$$g(x,y) = g(x-y) := \frac{\rho^{(2)}(x,y)}{\lambda^2}\,,$$

where $\rho^{(2)}$ is the 2 *nd joint intensity*.

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Proposition 2.1 *Consider Φ_1,Φ_2 with the same finite intensity. Denote by K_i and g_i ($i=1,2$) the respective Ripley's and pair correlation functions. If $\Phi_1\le_{dcx}\Phi_2$ then $K_1(\cdot)\le K_2(\cdot)$ and $g_1(\cdot)\le g_2(\cdot)$ almost everywhere.*

dcx and statistics; cont'd

Proof.

- Express Riplay's function using Palm probability $\mathsf{P}^{\mathbf{0}}$ $K(r) = \mathsf{E}^{\mathbf{0}}\big(\Phi(B_0(r))\big)$. Use the fact that dcx ordering of point processes implies $idcx$ ordering of their Palm versions (test functions are increasing and dcx).

dcx and statistics; cont'd

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- Express Riplay's function using Palm probability P^0 $K(r) = \mathsf{E}^0\big(\Phi(B_0(r))\big)$. Use the fact that dcx ordering of point processes implies $idcx$ ordering of their Palm versions (test functions are increasing and dcx).
- For pair correlation function, the result follows from the comparison of moments (to be explained).

dcx and void probabilities

$\nu(B) = \mathbf{P}\left(\Phi(B) = 0\right)$ for bBs B.

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Extension to Boolean models with typical grain G:

Proposition 2.3 *If* $\Phi_1 \leq_{dcx} \Phi_2$ *then* $\mathbf{P}\left(C(\Phi_1, G) \cap B = \emptyset\right) \leq \mathbf{P}\left(C(\Phi_2, G) \cap B = \emptyset\right)$ *for all bBs* B *provided* G *is fixed (deterministic) compact grain or* Φ_i *are simple and have locally finite moment measures.*

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Proof. Void probabilities can be expressed using the distribution function an extrema shot-noise:

$$\mathbf{P}(\Phi(B) = 0) = \mathbf{P}\left(\max_{X \in \Phi} 1(X \in B) \le 0\right).$$

Comparison of voids; interpretation

smaller in dcx order

$\Downarrow$

equal mean measure and smaller void probabilities

$\Downarrow$

more “spatial homogeneity”

dcx and moment measures

$\alpha^k(B_1 \times \ldots \times B_k) = \mathsf{E}\left(\prod_{i=1}^k \Phi(B_i)\right)$ for $B_1, \ldots, B_k$ bBs.

$\alpha(\cdot) := \alpha^1(\cdot)$ — the mean measure.

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Proposition 2.4 *If* $\Phi_1 \leq_{dcx} \Phi_2$ *then* $\alpha_1(\cdot) = \alpha_2(\cdot)$ *and* $\alpha_1^k(\cdot) \leq \alpha_2^k(\cdot)$ *for* $k \geq 1$ *provided these measures are* σ*-finite.*

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Interpretation: larger in dcx order

$\Downarrow$

equal mean measure but more expected k-tuples

$\Downarrow$

more clustering

A weaker clustering comparison

Inequalities for void probabilities and/or moment measures

$$\Downarrow$$

a weaker (than dcx) comparison of clustering properties.

Still stronger than usual statistical descriptors as K-function, and pair correlation function.

Comparison to Poisson point process

We say that $\mathbf{\Phi}$ is sub(super)-Poisson if it is dcx smaller (larger) than Poisson pp (of the same mean measure).

We say that $\mathbf{\Phi}$ is weakly sub(super)-Poisson if it has void probabilities and moment measures smaller than Poisson pp of the same mean measure.

Conjecture ?

Critical radius for percolation of the Boolean model $r_c(\Phi)$ is monotone with respect to dcx

$$\Phi_1 \ \le_{dcx} \ \Phi_2 \ \ \Rightarrow \ \ r_c(\Phi_1) \le r_c(\Phi_2).$$

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In general not true! We will show a counterexample.

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In general not true! We will show a counterexample.

However, dcx is related to percolation... (to be explained)

dcx — examples

Poisson point process

Given deterministic, locally finite measure $\Lambda(\cdot)$ on $\mathbb{E} = \mathbb{R}^d$.

Definition. $\Phi = \Phi_\Lambda$ is Poisson point process on $\mathbb{E}$ of intensity $\Lambda(\cdot)$ ($Poi(\Lambda)$) if for any $B_1, \ldots, B_n$, bounded, pairwise disjoint subset of $\mathbb{E}$

- $\Phi(B_1), \ldots, \Phi(B_n)$ are independent random variables and
- $\Phi(B_i)$ has Poisson distribution with parameter $\Lambda(B_i)$.

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Homogeneous case: $\Lambda(\mathrm{d}x) = \lambda \mathrm{d}x$ for some $0 < \lambda < \infty$.

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Λ is the mean measure of Φ_Λ.

Poisson point process, cont’d

- Void probabilities:

$$\nu_{\Phi}(B) = \mathsf{P}\left(\Phi(B) = 0\right) = e^{-\Lambda(B)}\,.$$

Poisson point process, cont'd

- Void probabilities:

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- Moment measure of order k:

$$\alpha^{(k)}(B_1 \times \ldots \times B_k) = \mathsf{E}\left(\prod_{i=1}^{k} \Phi(B_i)\right) = \prod_{i=1}^{k} \Lambda(B_i)$$

for mutually disjoint $B_1, \ldots, B_k$

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- In Homogeneous case: Riplay's function $K(r) \equiv \pi r^2$ and pair correlation function $g(x) \equiv 1$.

Cox point process

or doubly stochastic Poisson point process.
Suspected to cluster more than Poisson.

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Given random, locally finite measure $\boldsymbol{\mathcal{L}}(\cdot)$ on $\mathbb{E} = \mathbb{R}^{\boldsymbol{d}}$.

Definition. $\boldsymbol{\Phi}_{\boldsymbol{\mathcal{L}}}$ is Cox point process on $\mathbb{E}$ of intensity $\boldsymbol{\mathcal{L}}(\cdot)$ ($\boldsymbol{Cox}(\boldsymbol{\mathcal{L}})$) if conditionally, given $\boldsymbol{\mathcal{L}}(\cdot) = \boldsymbol{\Lambda}(\cdot)$, $\boldsymbol{\Phi}_{\boldsymbol{\mathcal{L}}}$ is Poisson point process with intensity measure $\boldsymbol{\Lambda}$.

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- More assumptions on $\mathcal{L}$ needed to get inequality for moment measures and dcx order.

Super-Poisson pp (cluster more)

strongly ($\boldsymbol{dcx}$-larger) than Poisson

- Poisson-Poisson cluster pp; $\boldsymbol{\mathcal{L}}(\mathrm{d}\boldsymbol{x}) = \sum_{\boldsymbol{Y}\in\boldsymbol{\Psi}} \boldsymbol{\Lambda}(\mathrm{d}\boldsymbol{x} + \boldsymbol{Y})$, where $\boldsymbol{\Psi}$ is a Poisson (“parent”) process; (we will show an example)

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- Lévy based Cox pp; $\mathcal{L}(B_1), \ldots, \mathcal{L}(B_n)$ are independent variables for pair-wise disjoint $B_i's$ (complete independence property) [Hellmund, Prokěová, Vedel Jensen’08];

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- some perturbed Poisson pp (to be explained)
- some perturbed lattice pp (to be explained)

Super-Poisson pp (cluster more); cont’d

weakly (voids and moments larger than for Poisson of the same mean)

- Cox pp with associated intensity measures; **Cov** $(f(\mathcal{L}(B_1),\ldots,\mathcal{L}(B_k))g(\mathcal{L}(B_1),\ldots,\mathcal{L}(B_k))) \geq 0$ for all $B_1,\ldots,B_k$, $0 \leq f, g \leq$ continuous and increasing functions; [Waymire’85]

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- Permanental processes; density of the k th factorial moment measure is given by $\rho^{(k)}(x_1,\ldots,x_k)=\text{per}(K(x_i,x_j))_{1\leq i,j\leq k}$, where per stands for permanent of a matrix and K is some kernel (assumptions needed). It is also a Cox process!; [Ben Hough'09]

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  (like square, or hexagonal grid on $\mathbb{R}^2$) ?
- processes with some "repulsion mechanism" between points (like some Gibb's point processes)?
- Well..., not immediately. Some (much) extra assumptions and modification are needed.

Sub-Poisson pp (cluster less)

strongly (in dcx)

- some perturbed lattice pp (to be explained)

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weakly (voids and moments)

- Negatively associated point processes; $\mathbf{P}\left(\Phi(B_i)=0, i=1,\ldots,n\right) \le \prod_{i=1}^{n} \mathbf{P}\left(\Phi(B_i)=0\right)$, for mutually disjoint $B_i's$; [Pemantle '00]

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- Determinantal point processes density of the k th factorial moment measure is given by $\rho^{(k)}(x_1,\ldots,x_k) = \det(K(x_i,x_j))_{1\le i,j\le k}$, where det stands for determinant of a matrix and K is some kernel (assumptions needed). It is a Gibbs process!; [Ben Hough'09]

More for determinantal and permanental

dcx comparison to Poisson pp is possible on mutually disjoint, simultaneously observable sets.

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dcx comparison to Poisson pp is possible on mutually disjoint, simultaneously observable sets.

It follows for example that, the pp of radii of the Ginibre$(*)$ pp is (dcx) sub-Poisson.

$(*)$ The determinantal pp with kernel $K((x_1, x_2),(y_1, y_2)) = \exp[(x_1y_1 + x_2y_2) + i(x_2y_1 - x_1y_2)]$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2$, with respect to the measure $\mu(d(x_1, x_2)) = \pi^{-1}\exp[-x_1^2 - x_2^2]\, dx_1dx_2$.

Perturbation of a point processes

Φ a pp on $\mathbb{R}^d$, $\mathcal{N}(\cdot,\cdot)$, $\mathcal{X}(\cdot,\cdot)$ be two probability kernels from $\mathbb{R}^d$ to non-negative integers $\mathbb{Z}^+$ and $\mathbb{R}^d$, respectively. Define a new pp on $\mathbb{R}^d$

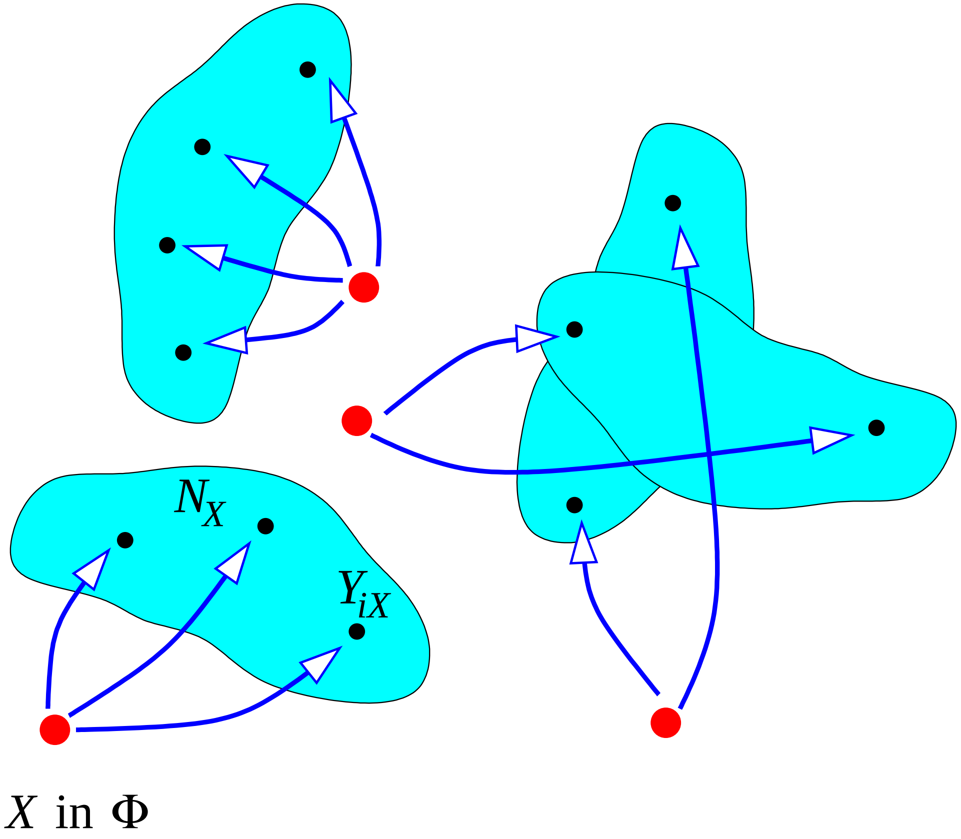
$$\Phi^{pert} := \bigcup_{X\in\Phi} \bigcup_{i=1}^{N_X} \{X + Y_{iX}\},$$

where

- N_X, $X \in \Phi$ are independent, non-negative integer-valued random variables with distribution $\mathbf{P}\left(N_X \in \cdot \,|\, \Phi\right) = \mathcal{N}(X,\cdot)$,
- $\mathrm{Y}_X = (Y_{iX} : i = 1, 2, \ldots)$, $X \in \Phi$ are independent vectors of i.i.d. elements of $\mathbb{R}^d$, with Y_{iX}'s having the conditional distribution $\mathbf{P}\left(Y_{iX} \in \cdot \,|\, \Phi\right) = \mathcal{X}(X,\cdot)$,
- the random elements N_X, Y_X are independent given Φ, for all $X \in \Phi$.

Perturbation of a point processes; cont’d

Φ^{pert} can be seen as independently replicating and translating points from the pp Φ, with replication kernel $\mathcal{N}$ and the translation kernel $\mathcal{X}$.



Perturbation of a point processes; cont'd

Perturbation of Φ is dcx monotone with respect to the replication kernel.

Proposition 3.1 *Consider a pp* Φ *with locally finite mean measure* $\alpha(\cdot)$ *and its two perturbations* Φ_j^{pert} $j = 1, 2$ *with the same translation kernel* $\mathcal{X}$ *and replication kernels* $\mathcal{N}_j$*,* $j = 1, 2$*, respectively. If* $\mathcal{N}(x, \cdot) \leq_{cx} \mathcal{N}(x, \cdot)$ *(convex ordering of the number of replicas; test functions* $\mathcal{F}$ *are all convex functions on* $\mathbb{R}$*) for* α*-almost all* $x \in \mathbb{R}^d$*, then* $\Phi_1^{pert} \leq_{dcx} \Phi_2^{pert}$*.*

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Proof. Using dcx comparison of some shot-noise fields; Th. 1.1.

Perturbed Poisson pp

Assume:

$\boldsymbol{\Phi}$ — (possibly inhomogeneous) Poisson pp,

arbitrary translation kernel,

$\boldsymbol{\mathcal{N}_1(x,\cdot)}$ Dirac measure on $\mathbb{Z}^+$ concentrated at 1,

$\boldsymbol{\mathcal{N}_2(x,\cdot)}$ arbitrary with mean number 1 of replications.

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Then

$$\underset{\text{Poisson pp}\ \nearrow}{\mathbf{\Phi}_1^{pert}} \le_{dcx} \underset{\nwarrow\ \text{perturbed Poisson pp}}{\mathbf{\Phi}_2^{pert}}$$

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Then

$$\Phi_1^{pert} \le_{dcx} \Phi_2^{pert}$$

Poisson pp ↗ ↖ perturbed Poisson pp

Indeed, by Jensen's inequality $\mathcal{N}_1 \le_{cx} \mathcal{N}_2$.

Perturbed lattices

Assume:

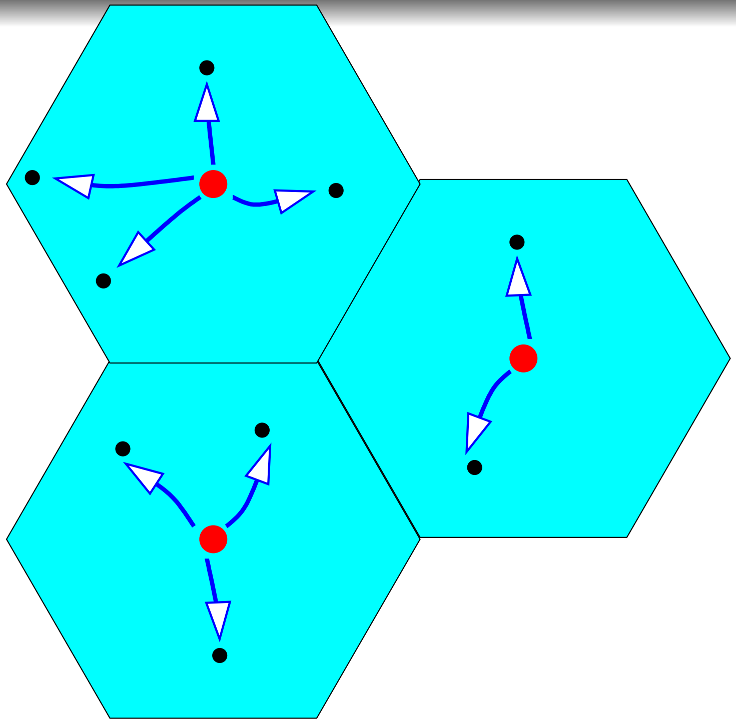
Φ — deterministic lattice,

(say uniform) translation kernel inside lattice cell,

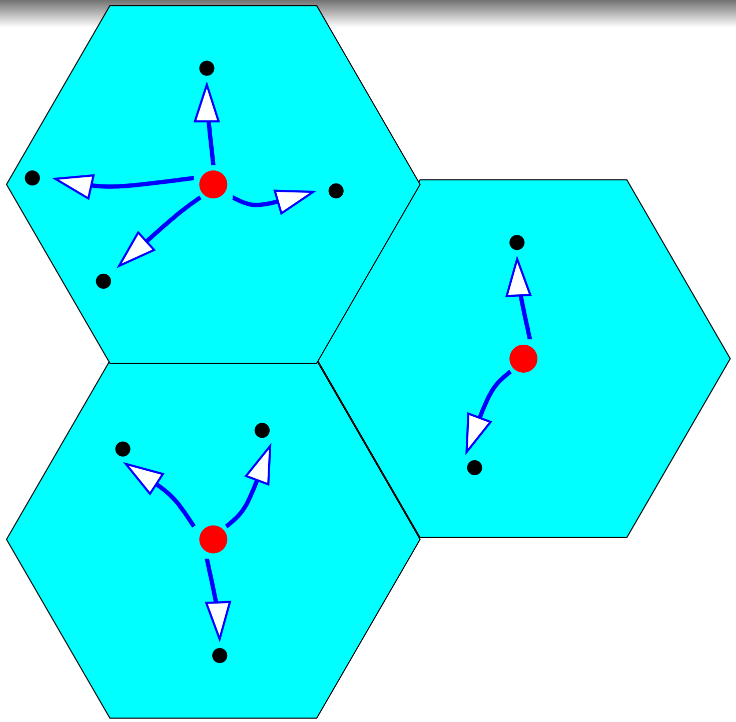
$\mathcal{N}_0(x,\cdot) = Poi(1)$,

$\mathcal{N}_1(x,\cdot) \leq_c Poi(1)$,

$\mathcal{N}_2(x,\cdot) \geq_c Poi(1)$.



Perturbed lattices



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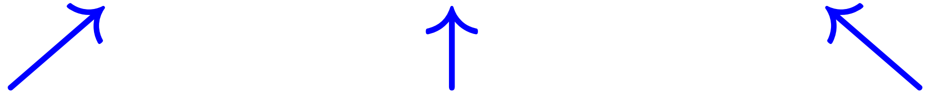

sub-Poisson
perturbed lattice

Poisson pp

super-Poisson
perturbed lattice

Perturbed lattices; cont'd

cx ordered families of (discrete) random variables from smaller to larger:

- deterministic (constant);

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Assuming parameters making equal means, we have $const \leq_{cx} HGeo \leq_{cx} Bin \leq_{cx} Poi \leq_{cx} NBin \leq_{cx} Geo$

Conjecture for perturbed lattices

$$\Phi_1 \quad \leq_{dcx} \quad \Phi_2$$

$$\Downarrow$$

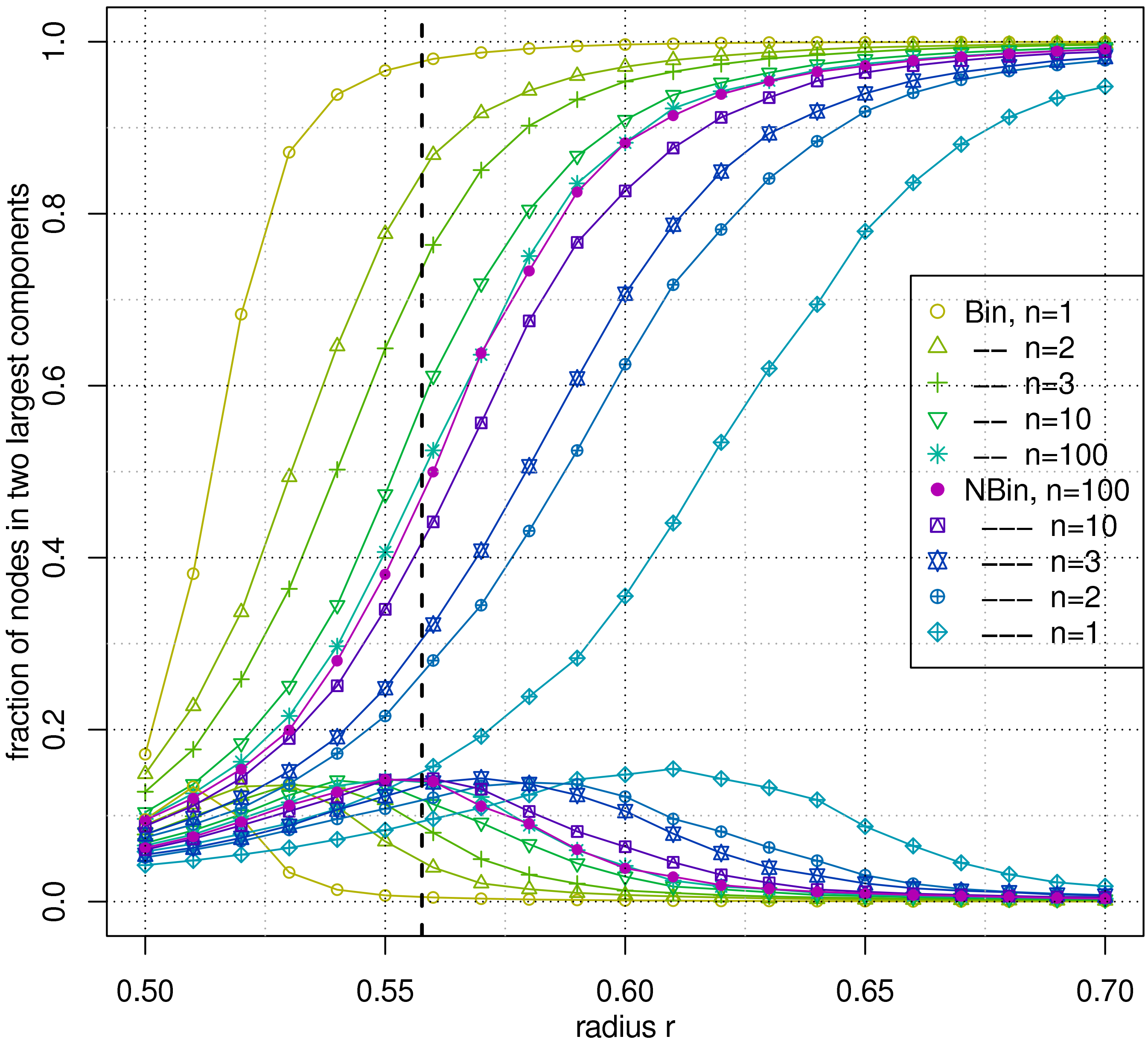
$$r_c(\Phi_1) \quad \leq \quad r_c(\Phi_2)$$

$Bin(1,1) = const$

$Bin(1,1/n) \nearrow_{cx} Poi(1)$

$NBin(n,1/(1+n)) \searrow_{cx} Poi(1)$

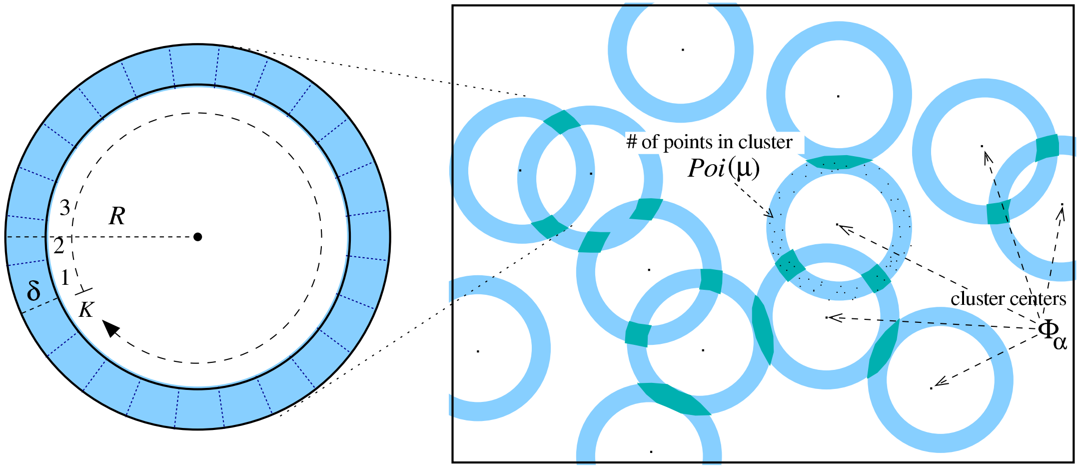
$NBin(1,1/2) = Geo(1/2)$



Counterexample

Poisson-Poisson cluster pp $\Phi_{\alpha}^{R,\delta,\mu}$ with annular clusters

Φ_α — Poisson (parent) pp of intensity α on $\mathbb{R}^2$, Poisson clusters of total intensity μ, supported on annuli of radii $R-\delta, R$.



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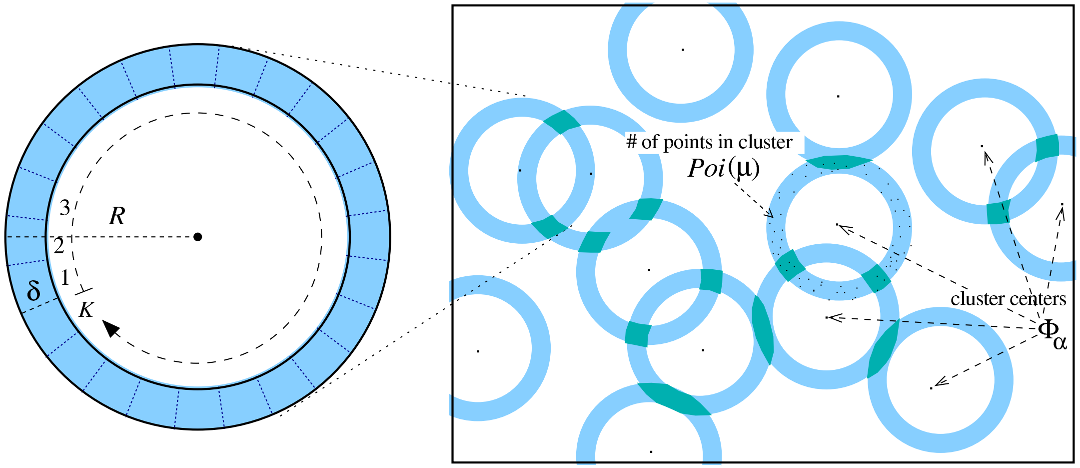

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Proposition 3.2 *Given arbitrarily small $a, r > 0$, there exist constants α, μ, δ, R such that $0 < \alpha, \mu, \delta, R < \infty$, the intensity $\alpha\mu$ of $\Phi_\alpha^{R,\delta,\mu}$ is equal to a and the critical radius for percolation $r_c(\Phi_\alpha^{R,\delta,\mu}) \le r$. Consequently, one can construct Poisson-Poisson cluster pp of intensity a and $r_c = 0$.*

dcx and continuum percolation

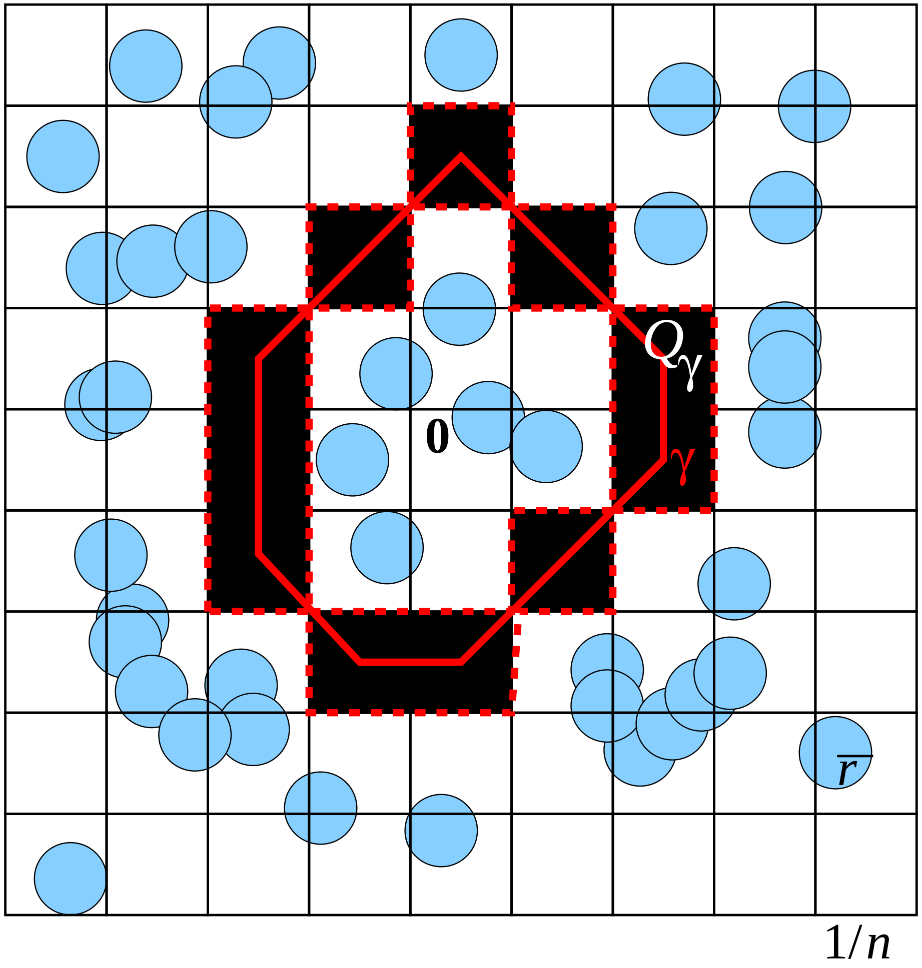
An “upper” critical radius

Define a new critical radius

$$\overline{r}_c = \inf\left\{r > 0 : \forall n \geq 1, \sum_{\gamma\in\Gamma_n} \mathbf{P}\left(C(\Phi, r) \cap Q_\gamma = \emptyset\right) < \infty\right\}.$$

By Peierls argument

$$r_c(\Phi) \leq \overline{r}_c(\Phi).$$



Peierls argument

- A sufficient condition for percolation: the maximal number of closed (not tuching the Boolean Model), disjoint contours around the origin is finite.

Peierls argument

- A sufficient condition for percolation: the maximal number of closed (not tuching the Boolean Model), disjoint contours around the origin is finite.
- Even stronger condition: expected number of such closed contours is finite.

$$
\begin{aligned}
&\mathbf{E}(\text{number of closed contours})\\
&\quad= \mathbf{E}\left(\sum_{\gamma\in\Gamma_n} 1(\text{contour } \gamma \text{ is closed})\right)\\
&\quad= \sum_{\gamma\in\Gamma_n} \mathbf{P}\left(\text{contour } \gamma \text{ is closed}\right)\\
&\quad= \sum_{\gamma\in\Gamma_n} \mathbf{P}\left(C(\Phi, r) \cap Q_\gamma = \emptyset\right) < \infty\,.
\end{aligned}
$$

“Upper” critical radius; cont’d

Proposition 4.1 *If* $\Phi_1 \le_{dcx} \Phi_2$ *then* $\overline{r}_c(\Phi_1) \le \overline{r}_c(\Phi_2)$.

“Upper” critical radius; cont’d

Proposition 4.1 *If* $\boldsymbol{\Phi_1 \le_{dcx} \Phi_2}$ *then* $\boldsymbol{\overline{r}_c(\Phi_1) \le \overline{r}_c(\Phi_2)}$*.*

Ordering of void probabilities of $\boldsymbol{\Phi_i}$ is enough for RGG.
$\boldsymbol{dcx}$ needed for Boolean models with arbitrary grain.

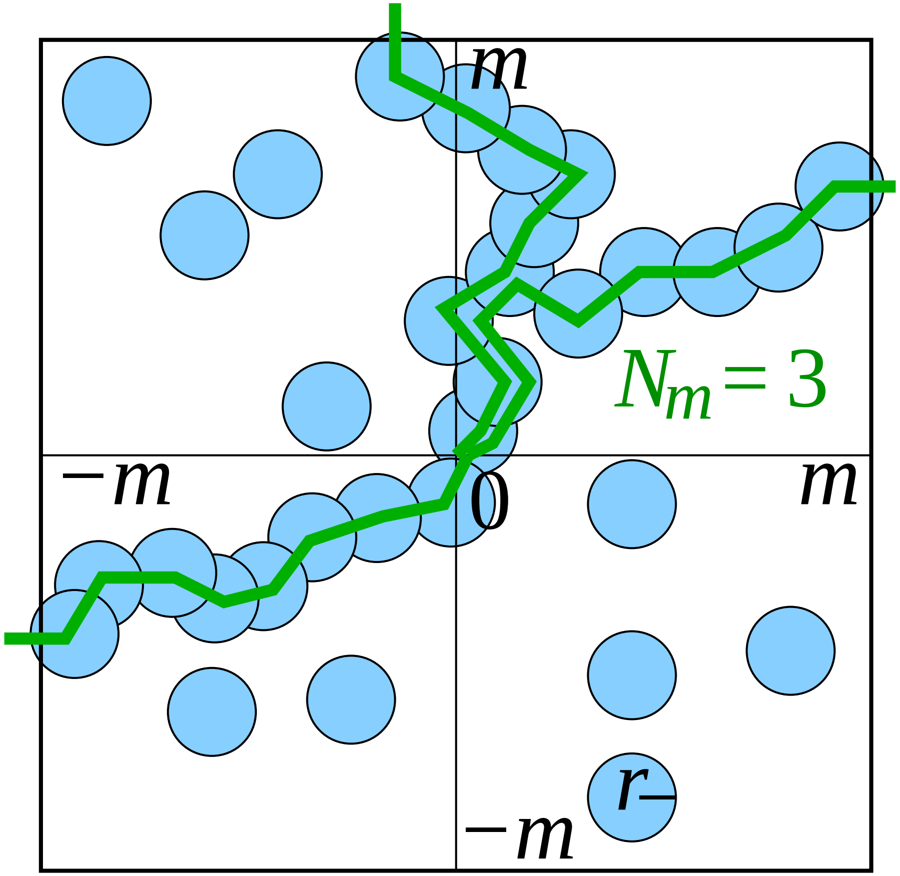
A “lower” critical radius

Define a new critical radius

$$\underline{r}_c(\Phi) := \inf\left\{r > 0 : \liminf_{m\to\infty} \mathbf{E}(N_m(\Phi, r)) > 0\right\}.$$

By Markov inequality

$$\underline{r}_c(\Phi) \le r_c(\Phi).$$



“Lower” critical radius; cont’d

Proposition 4.2 *If* $\boldsymbol{\Phi_1 \le_{dcx} \Phi_2}$ *then* $\boldsymbol{\underline{r}_c(\Phi_1) \ge \underline{r}_c(\Phi_2)}$*.*

"Lower" critical radius; cont'd

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Inequality reversed! In clustering pp, whenever there is at least one path of some given length, there might be actually so many such paths, that the inequality for the expected numbers of paths are reversed.

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Ordering of moment measures of Φ_i is enough for RGG.

Sandwich inequality for the critical radii

Corollary 4.1 *If* $\Phi_1 \leq_{dcx} \Phi_2$ *then*

$$\underline{r}_c(\Phi_2) \leq \underline{r}_c(\Phi_1) \leq r_c(\Phi_1) \leq \overline{r}_c(\Phi_1) \leq \overline{r}_c(\Phi_2).$$

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Double phase transition for Φ_2

$$0 < \underline{r}_c(\Phi_2) \leq \overline{r}_c(\Phi_2) < \infty$$

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usual phase transition for all $\Phi_1 \leq_{dcx} \Phi_2$

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Application: prove the double phase transition for Poisson pp to ensure usual phase transition of all sub-Poisson pp.

Phase transitions for sub-Poisson pp

Proposition 4.3 *Let* Φ *be a stationary pp on* $\mathbb{R}^d$*, weakly sub-Poisson (void probabilities and moment measures smaller than for the Poisson pp of some intensity* λ*). Then*

$$0 < \frac{1}{(2^d\lambda(3^d-1))^{1/d}} \leq r_c(\Phi) \leq \frac{\sqrt{d}(\log(3^d-2))^{1/d}}{\lambda^{1/d}} < \infty.$$

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Similar results for

- k-percolation (percolation of k-covered subset) for dcx sub-Poisson.
- word percolation,
- SINR-graph percolation (graph on a shot-noise germ-grain model).

concluding remarks

- Clustering in a point process should increase the critical radius for the percolation of the corresponding continuum percolation model. However, even a relatively strong tool such as the dcx order falls short, when it comes to making a formal general statement of this heuristic.

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- A rephrased conjecture: any homogeneous sub-Poisson pp has a smaller critical radius for percolation than the Poisson pp of the same intensity
- Phenomena of clustering in random objects (data, graphs, point processes) are currently receiving a lot of attention. Follow the recent literature!

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thank you