Clustering comparison of point processes with applications to percolation

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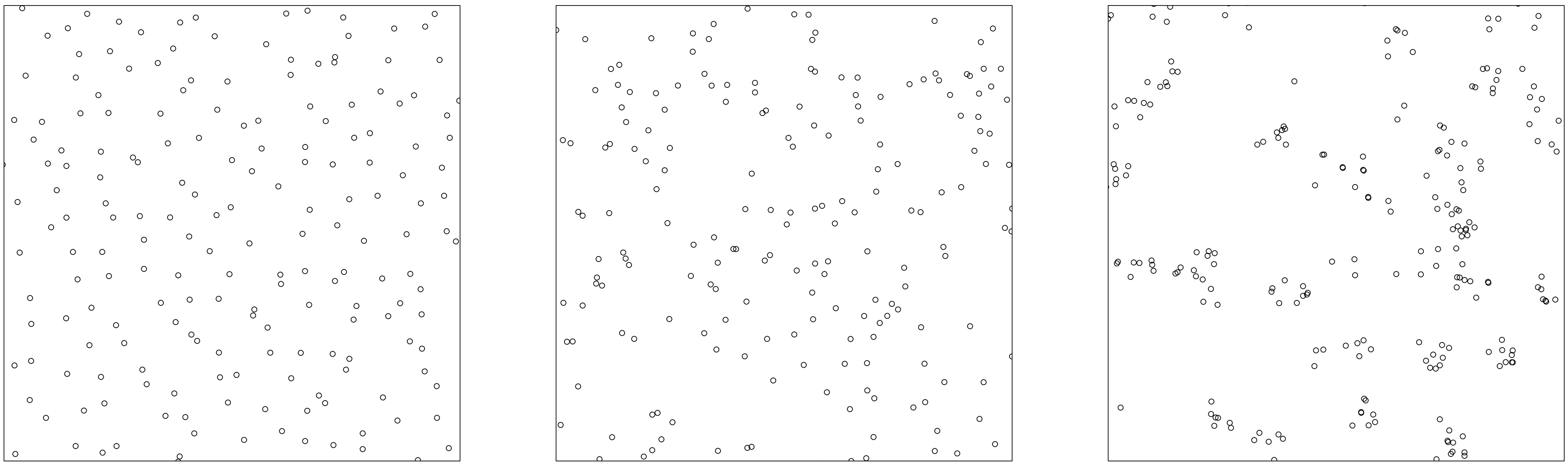
Clustering of points

Clustering in a point pattern roughly means that the points lie in clusters (groups) with the clusters being spaced out.



How to compare clustering of two point processes (pp), say having "on average" the same number of points per unit of space? (More precisely, having the same mean measure.) For simplicity, we consider pp on $\mathbb{R}^d$.

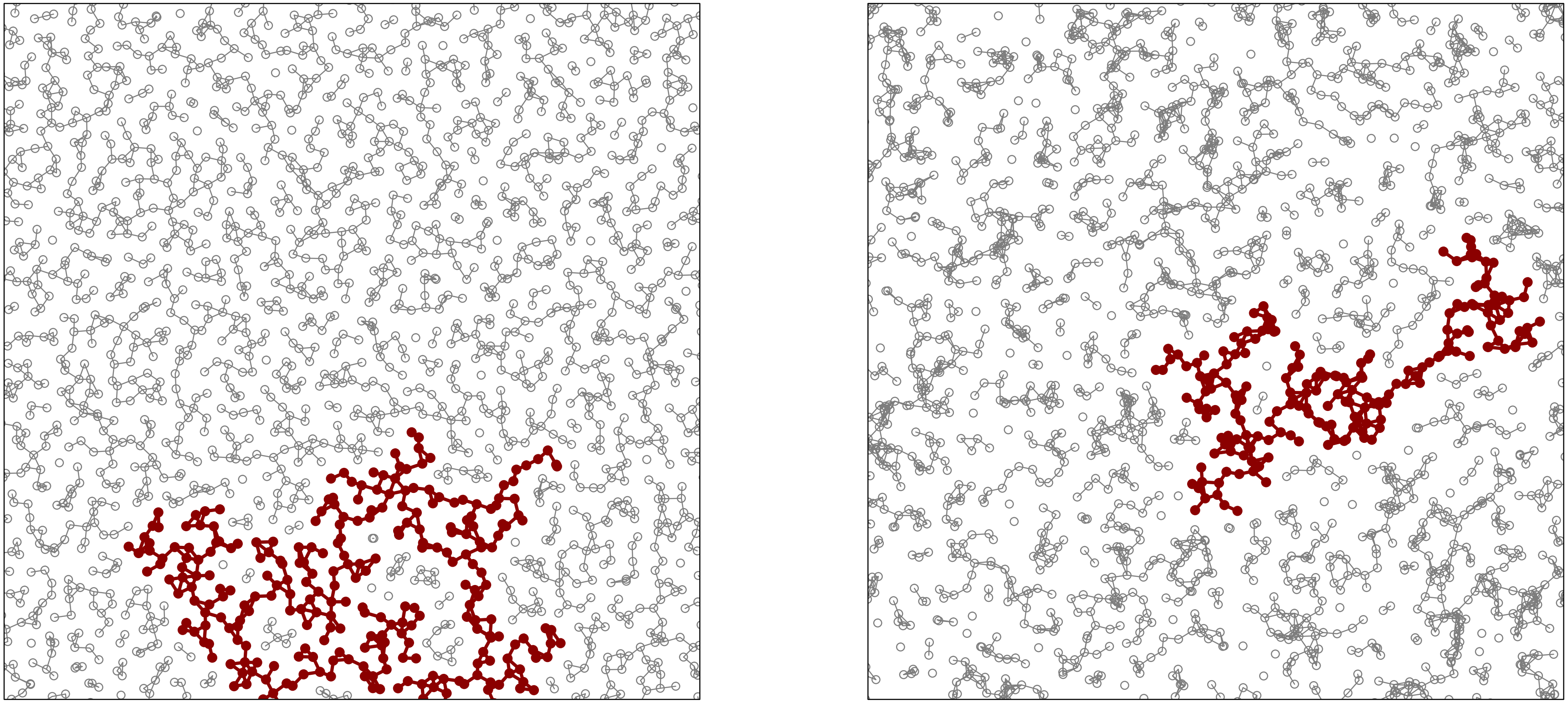
Motivation

- Interesting methods have been developed for studying local and global functionals of geometric structures over Poisson or Bernoulli pp; experts in the audience !
- Try to carry over some results to other point processes by their “cluster-comparison” to Poisson or Bernoulli pp. In this talk we concentrate on percolation-type results.
- The “clustering-comparison” is not the usual strong (coupling) comparison as we compare pp of the same mean measure. Analog of convex comparison of random variables.

Motivation, cont'd

- Program can be reminiscent of Ross-type conjectures in queuing theory (replacing Poisson arrival process in a single-server queue by a Cox PP with the same intensity should increase the average customer delay).
- Actually, more interesting results are on the side of pp "more regular" pp (we call them sub-Poisson) with determinantal pp as prominent examples.
- The notion of sub- and super-Poisson distributions is used e.g. in quantum physics and denotes distributions for which the variance is smaller (respectively larger) than the mean.

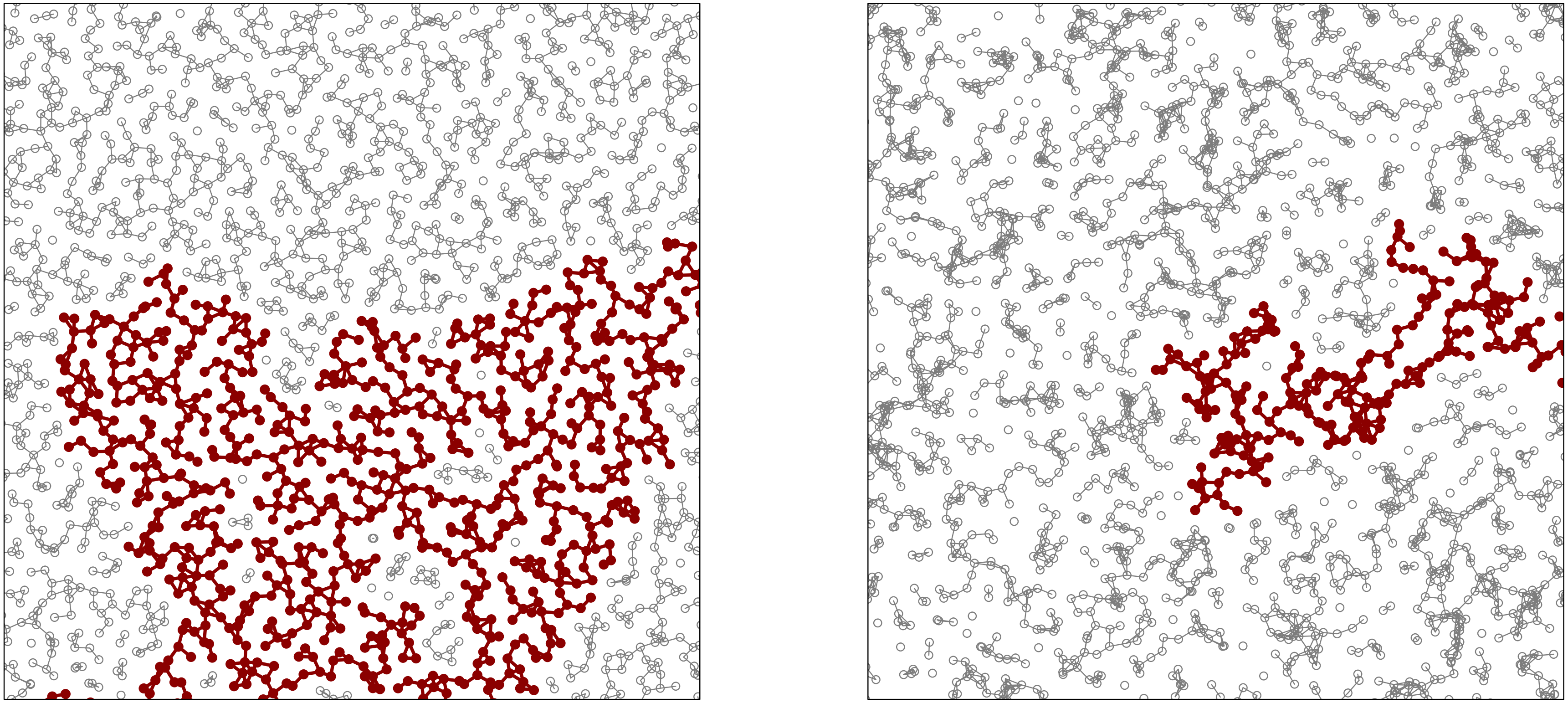
Clustering and percolation



RGG with $r = 98$.

The largest component in the window is highlighted.

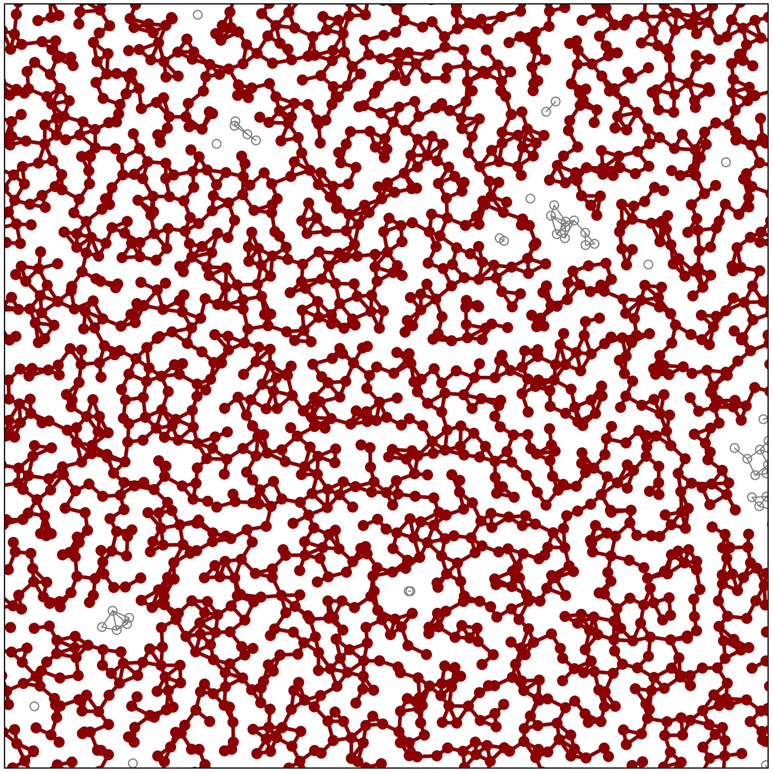
Clustering and percolation

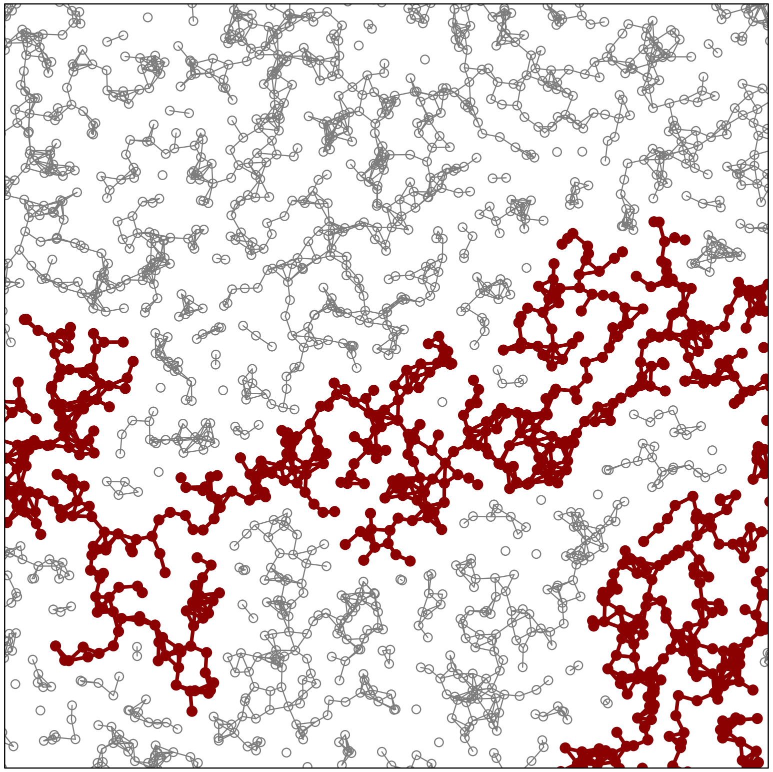


RGG with $r = 100$.

The largest component in the window is highlighted.

Clustering and percolation

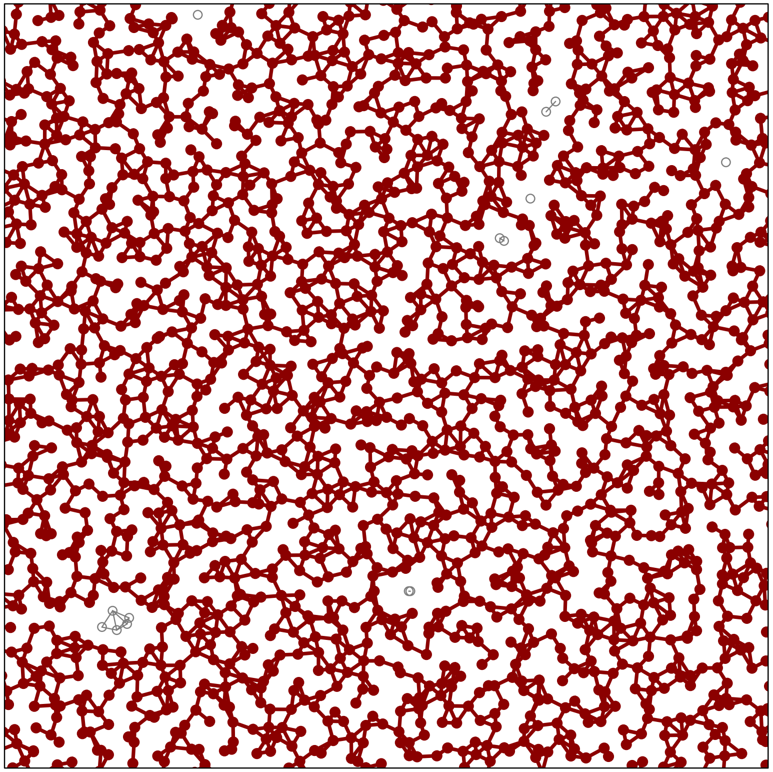


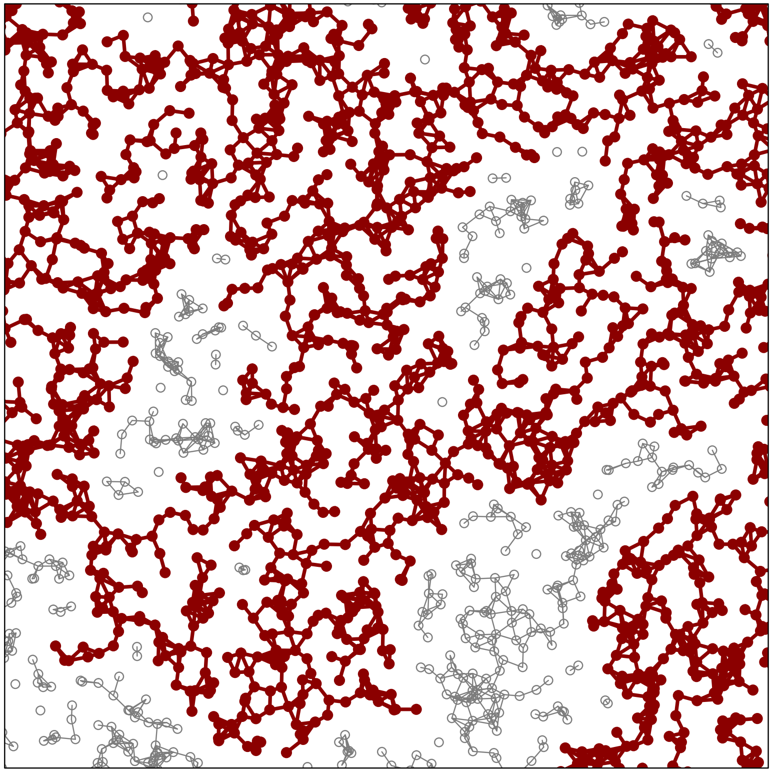


RGG with $r = 108$.

The largest component in the window is highlighted.

Clustering and percolation

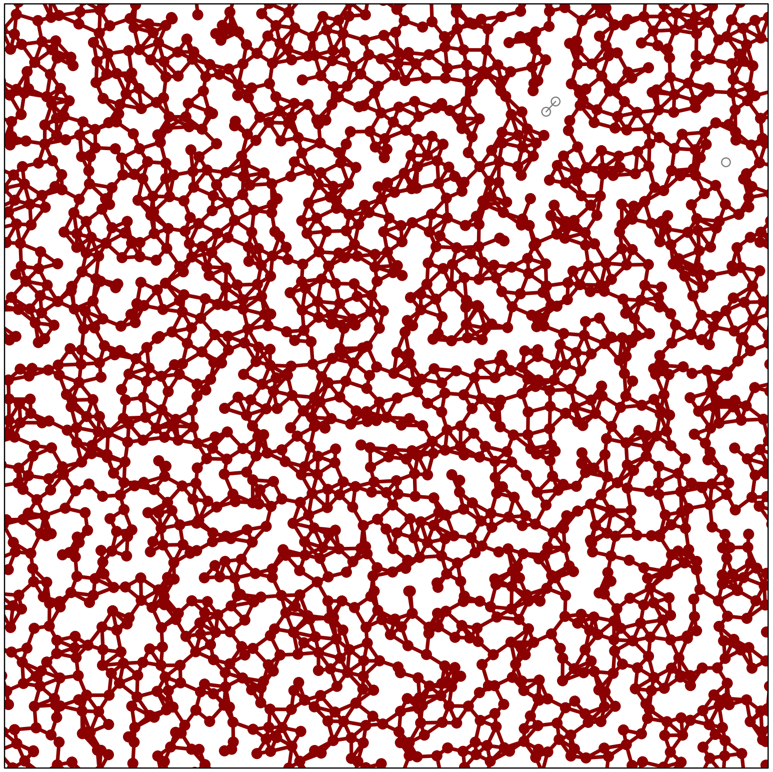


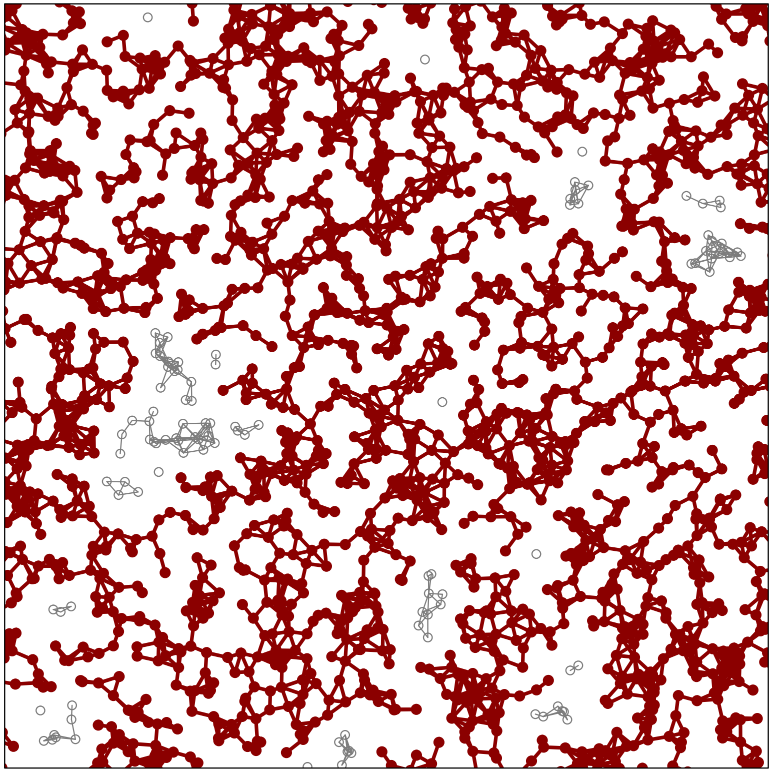


RGG with $r = 112$.

The largest component in the window is highlighted.

Clustering and percolation





RGG with $r = 120$.

The largest component in the window is highlighted.

Conjecture: Clustering worsens percolation

Point processes exhibiting more clustering should have larger critical radius r_c for the percolation of their continuum percolation models.

$$\Phi_1 \text{ "clusters less than" } \Phi_2 \quad \Rightarrow \quad r_c(\Phi_1) \le r_c(\Phi_2),$$

where $r_c(\Phi) = \inf\{r > 0 : \mathsf{P}(C(\Phi, r)\text{percolates}) > 0\}$

Heuristic: Interconnecting well spaced-out clusters (necessary to obtain an infinite connected component) requires large r. Spreading points from clusters "more homogeneously" should result in a decrease r for which the percolation takes place.

Ways of comparing clustering — outline of the talk

Smaller in one of the following ways indicates less clustering:

- Second-order statistics (Ripley's K, L, pair correlation function) $\Rightarrow$ variance comparisons
- Comparisons of void probabilities and all higher-order factorial moment measures.
  $\Rightarrow$ concentration inequalities and percolation results
- Positive and negative association of pp.
  $\Rightarrow$ comparison to Poisson pp
- dcx ordering of pp
  $\Rightarrow$ the strongest (on this list) comparison tool
- examples, counterexamples and conclusions

Second-order statistics

Ripley's K and L function

- Ripley's K function: for a stationary isotropic pp Φ of intensity λ on $\mathbb{R}^d$

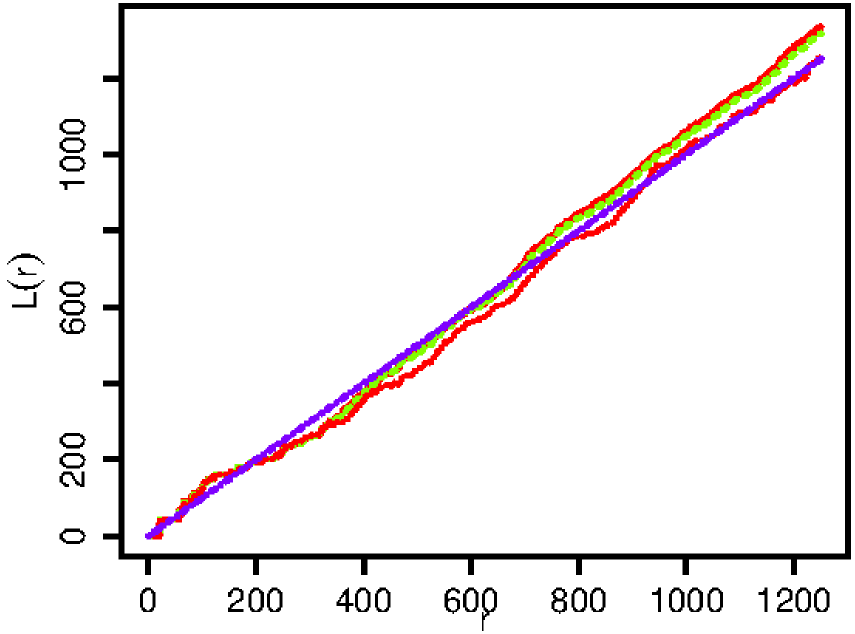
$$K(r) = \frac{1}{\lambda} \mathsf{E}^0[\Phi(\{x : |x| \le r\}) - 1]$$

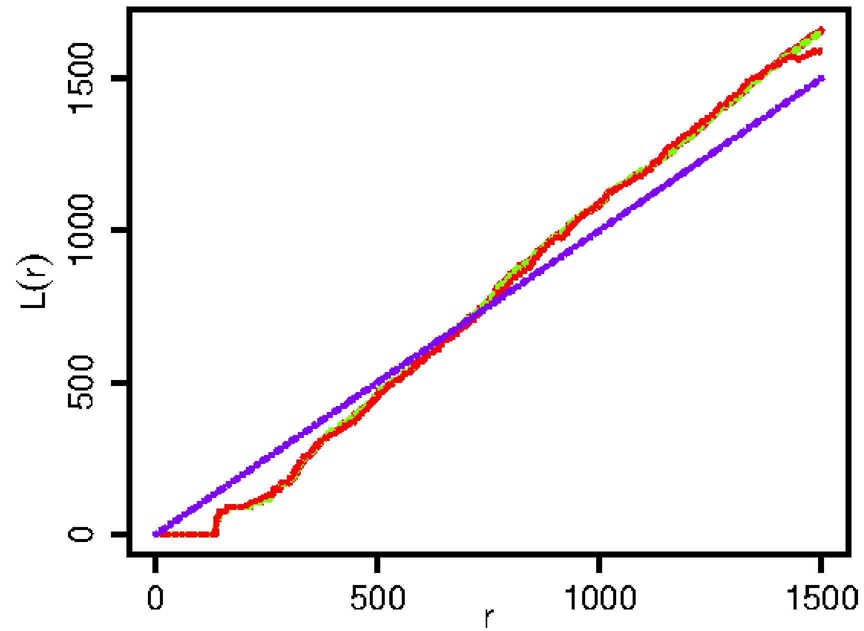
(expected number of points of Φ within the distance r of its typical point)

- Ripley's L function: $L(r) = (K(r)/\kappa_d)^{1/d}$, where κ_d volume of the unit ball in $\mathbb{R}^d$.

Ripley's K and L function; cont'd

Fact: For Poisson pp $K(r) = \kappa_d r^d$, $L(r) = r$ (Slivnyak-Mecke).





"Poisson-like network"　　　　"not so Poisson network"

Empirical Ripley's L function for real positioning of BS in some big European city (Jovanovic&Karray [Orange Labs]).

Allow for <u>local</u> clustering comparison at different scales r.

Pair correlation function

Probability of finding a point at a given position with respect to another point

$$g(x,y) = g(x-y) := \frac{\rho^{(2)}(x,y)}{\lambda^2},$$

where $\rho^{(2)}$ is the density of the 2'nd order moment measure.

Also a local comparison. To weak to capture global (percolation-like) properties.

Ripley's K function and variance comparison

A forerunner in this theory

Fact (Stoyan'83): Consider two stationary isotropic pp Φ_1 and Φ_1 of the same intensity, with the Ripley's functions K_1 and K_2, respectively. If $K_1 \leq_{dc} K_2$ i.e.,

$$\int_0^\infty f(r) K_1(\mathrm{d}r) \leq \int_0^\infty f(r) K_2(\mathrm{d}r)$$

for all decreasing convex f then

$$\mathsf{Var}\left(\Phi_1(B)\right) \leq \mathsf{Var}\left(\Phi_2(B)\right)$$

for all compact convex B.

Stoyan'83 considers applications to some renewal, Cox, Neyman-Scott and fibre processes.

Voids and moments & concentration inequalities via Chernoff bounds

Voids and moments

- probabilities: $\nu(B) = \mathbf{P}(\Phi(B) = 0)$, bounded Borel sets (bBs) B.
- Moment measures: $\alpha^k(B_1 \times \ldots \times B_k) = \mathbf{E}\Big(\prod_{i=1}^k \Phi(B_i)\Big)$ for all (not necessarily disjoint) bBs B_i.
- Factorial moment measures: $\alpha^{(k)}(\cdot)$ for simple pp, truncation of the measure $\alpha^k(\cdot)$ to "off the diagonals" $\{(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k : x_i \neq x_j$ for $i \neq j\}$
- In a general (not necessarily simple pp) $\{\alpha^{(k)}(\cdot) : k\}$ can be expressed in terms of $\{\alpha^k(\cdot) : k\}$ and vice versa. Each of the three families of three functionals (voids, moments and factorial moments) determine the distribution of pp.

Clustering & concentration

- The “most spatially homogeneous” (“non-clustering”) way of spreading points of Φ, with a given mean measure $\alpha(\cdot)$, would be to place them according to the (deterministic) measure $\alpha(\cdot)$. But this is not a point process.
- Consider the probability that Φ deviates from $\alpha(\cdot)$ on B by more than a: $\mathbf{P}\left(|\Phi(B) - \alpha(B)| \geq a\right)$.
- Smaller these probabilities indicate less clustering (more homogeneity).
- Voids and moments allow for upped bounds on these probabilities $\rightarrow$ concentration inequalities.

Concentration inequalities

- Chernoff's bounds:
  $\mathbf{P}\left(\Phi(B) - \alpha(B) \geq a\right) \leq e^{-t(\alpha(B)+a)} \mathbf{E}\big(e^{t\Phi(B)}\big)$
  and
  $\mathbf{P}\left(\alpha(B) - \Phi(B) \geq a\right) \leq e^{t(\alpha(B)-a)} \mathbf{E}\big(e^{-t\Phi(B)}\big)$
- $\mathbf{E}\big(e^{t\Phi(B)}\big)$ and $\mathbf{E}\big(e^{-t\Phi(B)}\big)$ can be expressed in terms of moments and voids of Φ, respectively.
- Indeed: $\mathbf{E}\big(e^{t\Phi(B)}\big) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \alpha^k(B)$
- and
  $\mathbf{E}\big(e^{-t\Phi(B)}\big) = \sum_{k=0}^{\infty} e^{-tk} \mathbf{P}\left(\Phi(B) = k\right) = \mathbf{P}\left(\Phi'(B) = 0\right)$
  is the void probability of the point process Φ' obtained from Φ by independent thinning with retention probability $1 - e^{-t}$. Ordering of voids is preserved by independent thinning.

Comparison to Poisson pp — Laplace ordering

- Consider pp Φ having voids and moments smaller than Poisson pp (of the same mean). We call them weakly sub-Poisson (a weaker comparison than dcx).
  $\mathbf{P}\left(\Phi(B)=0\right) \le e^{-\mathbf{E}(\Phi(B))}$ for all bBs B (V)
  $\mathbf{E}\left(\prod_{i=1}^{k}\Phi(B_i)\right) \le \prod_{i=1}^{k}\mathbf{E}(\Phi(B_i))$ for all disjoint B_i (M)
- Prop. For simple pp Φ of mean measure α: Φ has smaller voids than Poisson ((V) holds true) if and only if for all $f \le 0$
  $\mathbf{E}\left(\exp\left[\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right) \le \exp\left[\int_{\mathbb{R}^d}(e^{f(x)}-1)\,\alpha(\mathrm{d}x)\right]$ (*)
- Prop. For simple pp Φ of mean measure α: If Φ has smaller moments than Poisson ((M) holds true) than (*) holds for all $f \ge 0$.

Concentration inequality for sub-Poisson

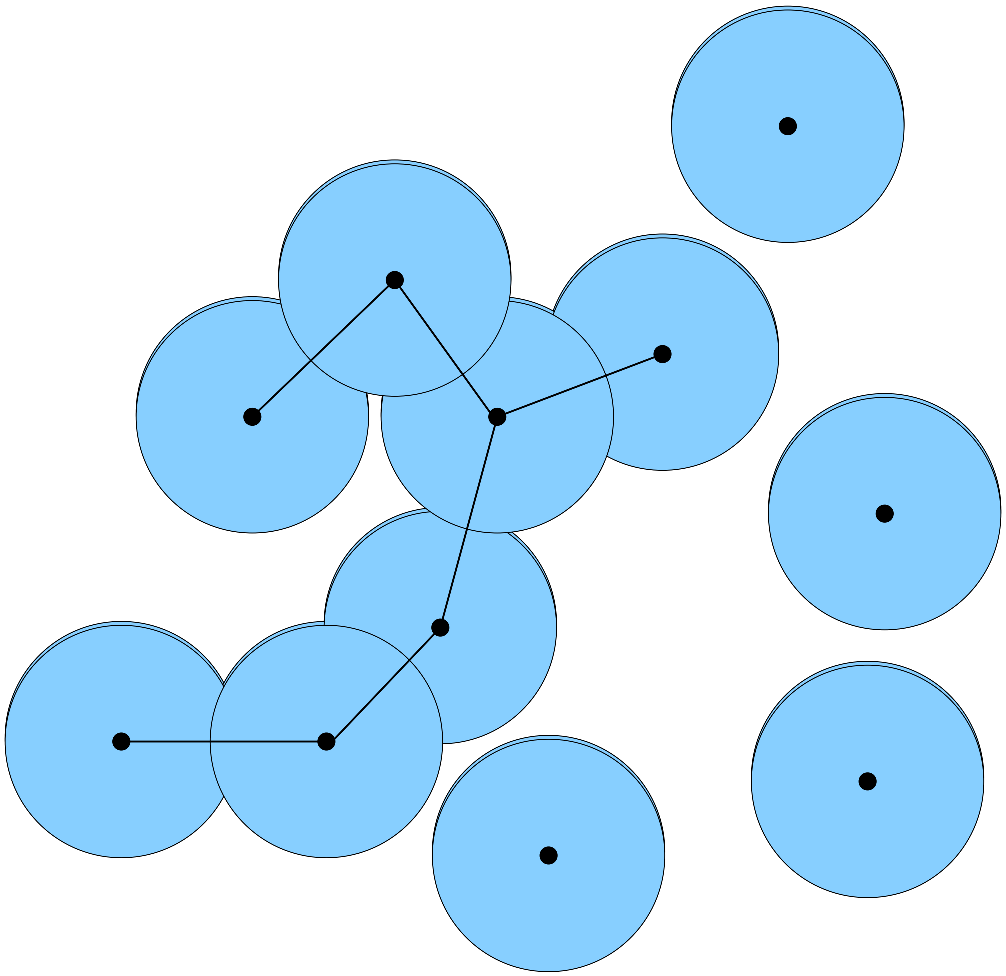
Extension of a result for Poisson pp (cf Penrose (2003)):

- Cor. Let Φ be an unit intensity, simple, stationary, weakly sub-Poisson point process and B_n be a set of Lebesgue measure n. Then, for any $1/2 < a < 1$ there exist $n(a)$ such that for $n \geq n(a)$
  $\mathsf{P}(|\Phi(B_n) - n| \geq n^a) \leq 2\exp\left[-n^{2a-1}/9\right]$.

Voids and moments & percolation

Continuum percolation

Boolean model $C(\Phi, 2r)$:
germs in Φ,
spherical grains of given radius r.

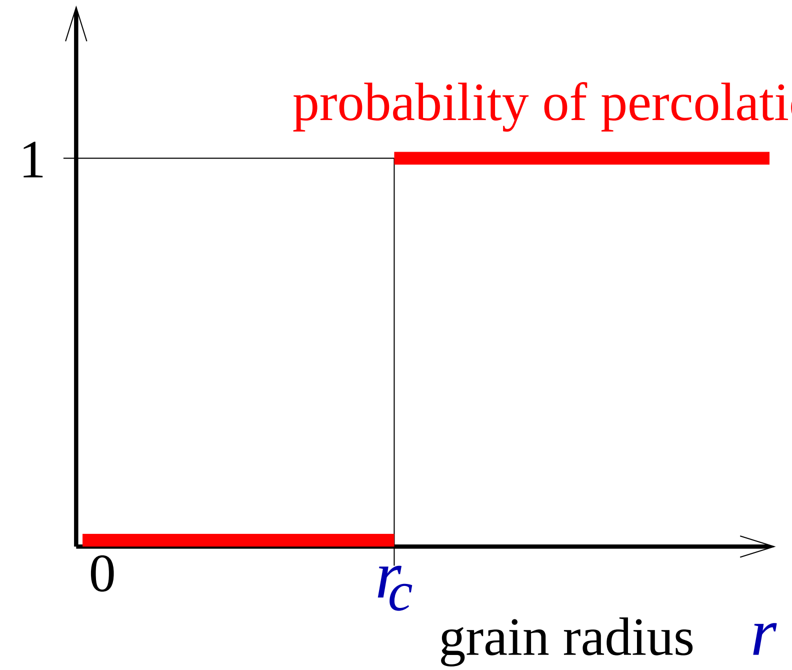


Joining germs whose grains intersect one gets Random Geometric Graph (RGG).

percolation $\equiv$ existence of an infinite connected subset (component).

Critical radius for percolation

- Critical radius for the percolation in the Boolean Model with germs in Φ:
  $r_c(\Phi) = \inf\{r > 0 : \mathsf{P}(C(\Phi, r)\text{percolates}) > 0\}$
- In the case when Φ is stationary and ergodic



- If $0 < r_c < \infty$ the phase transition is non-trivial.

Voids & percolation — a sufficient condition

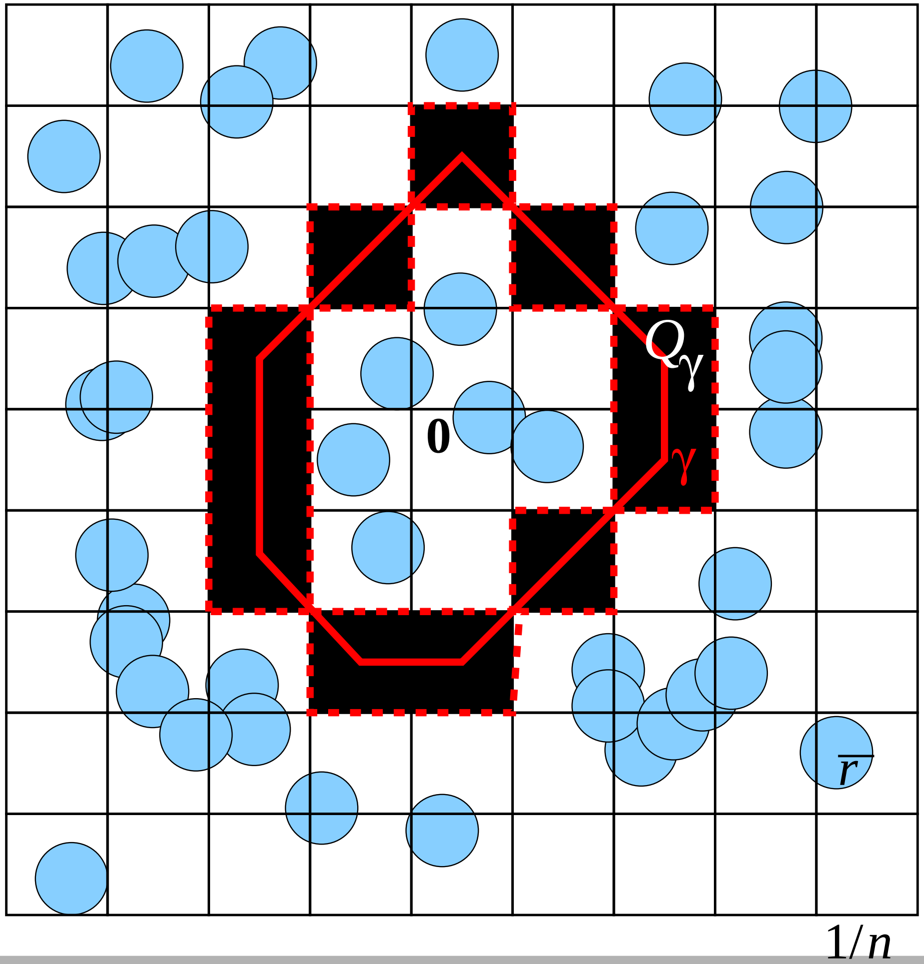
An upper bound on r_c using voids

$$\overline{r}_c = \inf\{r > 0 : \forall n \geq 1, \sum_{\gamma \in \Gamma_n} \mathbf{P}\left(C(\Phi, r) \cap Q_\gamma = \emptyset\right) < \infty\}.$$

By Peierls argument

$$r_c(\Phi) \leq \overline{r}_c(\Phi).$$

Smaller voids imply smaller $\overline{r}_c(\Phi)$



Moments & percolation — a necessary cond.

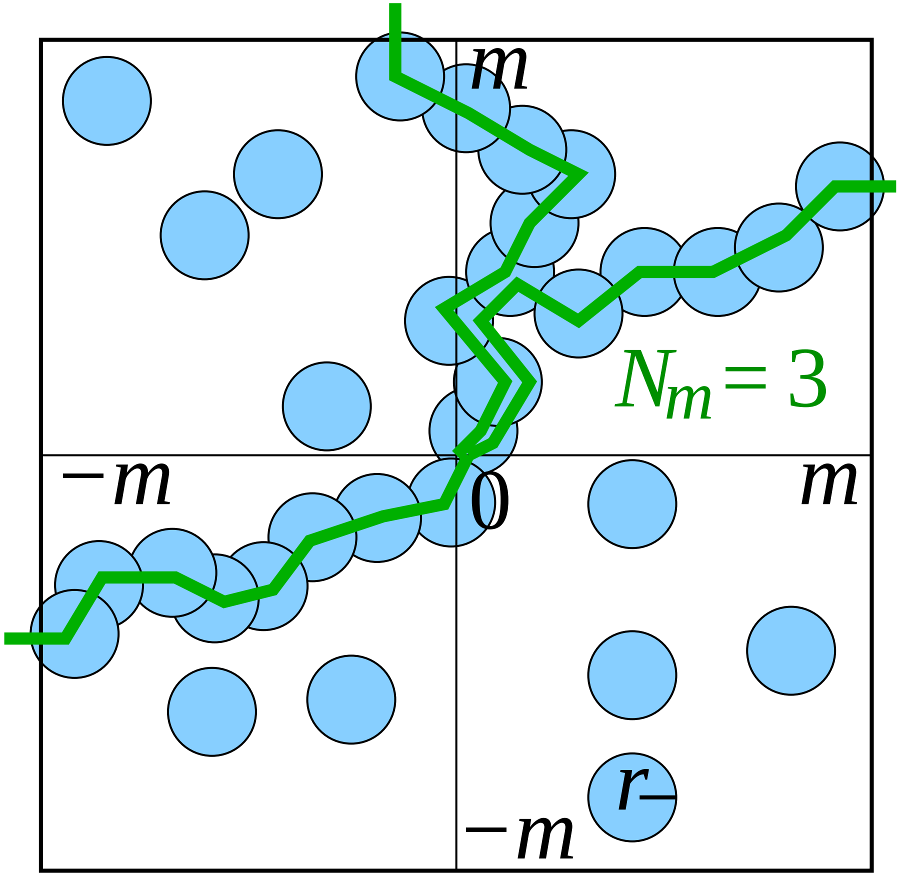
A lower bound on r_c related to moments measures

$$\underline{r}_c(\Phi) := \inf\left\{r > 0 : \liminf_{m\to\infty} \mathbf{E}(N_m(\Phi, r)) > 0\right\}.$$

By Markov inequality

$$\underline{r}_c(\Phi) \le r_c(\Phi).$$

Smaller moments imply larger(!) $\underline{r}_c(\Phi)$



Non-trivial phase transition for sub-Poisson

Extension of the well known result for Poisson pp:

- Prop. Let Φ be a stationary, weakly sub-Poisson pp with intensity λ. Then

  $$0 < \frac{1}{(\kappa_d \lambda)^{1/d}} \le r_c(\Phi) \le \sqrt{d}\left(\frac{\log(3^d - 2)}{\lambda}\right)^{1/d} < \infty.$$

  All weakly sub-Poisson point processes exhibit a non-trivial phase transition in the percolation of their Boolean models. Bounds are uniform over all processes of a given intensity!

- Similar results for k-coverage in Boolean model (clique percolation) and SINR percolation and some other percolation models.

Association of point processes as comparison to Poisson pp

Association of pp

- Φ is called associated if
  Cov $(f(\Phi(B_1),\ldots,\Phi(B_k)),g(\Phi(B_1),\ldots,\Phi(B_k)))\geq 0$
  for bBs $B_1,\ldots,B_k$ and f,g continuous and increasing functions taking values in $[0,1]$ (Burton&Waymire (1985)).
- Φ is called negatively associated if
  Cov $(f(\Phi(B_1),\ldots,\Phi(B_k)),g(\Phi(B_{k+1}),\ldots,\Phi(B_l)))\leq 0$
  for bBs $B_1,\ldots,B_l$ such that
  $(B_1\cup\ldots\cup B_k)\cap(B_{k+1}\cup\ldots\cup B_l)=\emptyset$ and f,g increasing functions (Pemantale (2000)).

Weak sub-poissonianity and association

- Prop. A negatively associated, simple point process with a Radon mean measure is weakly sub-Poisson.

  A (positively) associated point process with a Radon, diffuse mean measure is weakly super-Poisson (voids and moments larger than for Poisson).

- Cor. Assume that Φ is a simple point process of Radon mean measure α. If Φ is negatively associated then for all f of a fixed sign

  $\mathsf{E}\Big(\exp\Big[\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\Big]\Big) \le \exp\Big[\int_{\mathbb{R}^d}(e^{f(x)}-1)\,\alpha(\mathrm{d}x)\Big]$

  provided the integrals are well defined.

directionally-convex ordering of point processes

dcx ordering of point processes

- $\Phi_1 \leq_{dcx} \Phi_2$ if for all bounded Borel subsets $B_1, \ldots, B_n$,

  $$\mathbf{E}\Big(f\big(\Phi_1(B_1), \ldots, \Phi_1(B_n)\big)\Big) \leq \mathbf{E}\Big(f\big(\Phi_2(B_1), \ldots, \Phi_2(B_n)\big)\Big).$$

  for all dcx f. Function $f : \mathbb{R}^d \to \mathbb{R}$ twice differentiable is dcx if $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0$ for all $x \in \mathbb{R}^d$ and $\forall i, j$; extended to all functions by considering difference operators.
- dcx is a partial order (reflective, antisymmetric and transitive) of point process with locally finite mean measure (to ensure transitivity).
- If $\Phi_1 \leq_{dcx} \Phi_2$ then $\mathbf{E}(\Phi_1(\cdot)) = \mathbf{E}(\Phi_2(\cdot))$ (equal mean measures).
- dcx is preserved by independent thinning, marking and superpositioning of pp., creating of Cox pp.

dcx and shot-noise fields

Given point process Φ and a non-negative function $h(x, y)$ on $(\mathbb{R}^d, S)$, measurable in x, where S is some set, define shot noise field: for $y \in S$

$$V_\Phi(y) := \sum_{X \in \Phi} h(X, y) = \int_{\mathbb{R}^d} h(x, y)\Phi(dx)\,.$$

Prop. If $\Phi_1 \leq_{dcx} \Phi_2$ then

$$\big(V_{\Phi_1}(y_1), \ldots, V_{\Phi_1}(y_n)\big) \leq_{dcx} \big(V_{\Phi_2}(y_1), \ldots, V_{\Phi_2}(y_n)\big)$$

for any finite subset $\{y_1, \ldots, y_n\} \subset S$, provided the RHS has finite mean. In other words, dcx is preserved by the shot-noise field construction.

dcx and shot-noise fields; cont'd

Proof.

- Approximate the integral by simple functions as usual in integration theory: $a.s.$ and in L_1
  $\sum_{i=1}^{k_n} a_{in}\Phi(B_{in}^j) \to \int_{\mathbb{R}^d} h(x,y)\Phi(dx) = V_\Phi(y_j)$, $a_{in} \geq 0$.
- Increasing linear operations preserve dcx hence approximating simple functions are dcx ordered.
- dcx order is preserved by *joint* weak and L_1 convergence. Hence limiting shot-noise fields are dcx ordered.

dcx and extremal shot-noise fields

In the setting as before define for $y \in S$

$$U_\Phi(y) := \sup_{X \in \Phi} h(X, y)\,.$$

Prop. If $\Phi_1 \leq_{dcx} \Phi_2$ then for all
$y_1, \ldots, y_n \in S$; $a_1, \ldots, a_n \in \mathbb{R}$,
$\mathsf{P}\big(U_{\Phi_1}(y_i) \leq a_i, 1 \leq i \leq m\big) \leq \mathsf{P}\big(U_{\Phi_2}(y_i) \leq a_i, 1 \leq i \leq m\big)$;
i.e, the (joint) finite-dimensional distribution functions of the extremal shot-noise fields are ordered (lower orthant order).

Cor. One-dimensional distributions of the extremal shot-noise fields are strongly ordered with reversed inequality
$U_{\Phi_2}(y) \leq_{st} U_{\Phi_1}(y)$, $\forall y \in S$.

dcx and extremal shot-noise fields; cont'd

Proof.

- Reduction to an (additive) shot noise:

$$\mathbf{P}\left(U_\Phi(y_i) \le a_i, 1 \le i \le n\right) = \mathbf{E}\left(e^{-\sum_{i=1}^n \sum_{X\in\Phi} -\log \mathbf{1}[h(X,y_i)\le a_i]}\right).$$

- $e^{-\sum x_i}$ is dcx function.

dcx and voids & moments

Prop. If $\Phi_1 \leq_{dcx} \Phi_2$ then $\nu_1(B) \leq \nu_2(B)$.

Prop. If $\Phi_1 \leq_{dcx} \Phi_2$ then $\alpha_1(\cdot) = \alpha_2(\cdot)$ and $\alpha_1^k(\cdot) \leq \alpha_2^k(\cdot)$ for $k \geq 1$ provided these measures are σ-finite.

We call pp dcx smaller (larger) than Poisson sub-Poisson super-Poisson is (stronger) dcx sens.

dcx versus association

Prop. A negatively associated point processes with convexly sub-Poisson one-dimensional marginal distributions is dcx sub-Poisson.
An associated point processes with convexly super-Poisson one-dimensional marginal distributions is dcx super-Poisson.

Clustering comparison tools — recap.

dcx ordering

dcx-functions of
$(\Phi(B_1), \ldots, \Phi(B_k))$
and shot-noise fields, in particular
$\mathcal{L}_\Phi(f)$ for $f \le 0$ or $f \ge 0$

with marginals cx ordered to Poisson

negative & positive association
comparison with respect to the Poisson
point process

$\mathcal{L}_\Phi(f)$ for $f \le 0$ or $f \ge 0$

comparison of void probabilities

$\mathcal{L}_\Phi(f)$ for $f \ge 0$

comparison of moment measures

$\mathcal{L}_\Phi(f)$ for $f \le 0$

statistical comparison

pair correlation function,
Ripley's K-function

Implications between methods, and some characteristics their allow to compare; $\mathcal{L}_\Phi(f) = \mathsf{E}[\exp \int -f \mathrm{d}\Phi]$.
Smaller in any comparison means clusters less.

EXAMPLES ???

Comparison to Poisson pp

sub-Poisson processes

strongly (dcx)

Voronoi perturbed lattices with replication kernel $\mathcal{N} \le_{cx}$ Pois, in particular binomial, determinantal(?)

negatively associated

binomial, determinantal(?)

weakly (voids and moments)

dcx sub-Poisson, negatively associated, determinantal

super-Poisson processes

strongly (dcx)

Poisson-Poisson cluster, Lévy based Cox, mixed Poisson, Neyman-Scott with mean cluster size 1, Voronoi perturbed lattices with replication kernel $\mathcal{N} \ge_{cx}$ Pois.

associated

Poisson-center cluster, Neyman-Scott, Cox associated with associated intensity measure.

weakly (voids and moments)

dcx super-Poisson, associated, permanental

Some point processes comparable to Poisson point process according to different methods.

Determinantal pp
— voids, moments and more

Determinantal pp

- Examples of weakly sub-Poisson pp? Theory fits well to determinantal pp Φ^{det} defined as having density of the k th factorial moment measure with respect to $\mu^{\otimes d}$, for some $\mu(\cdot)$, given by $\rho^{(k)}(x_1, \ldots, x_k) = \det(K(x_i, x_j))_{1 \le i,j \le k}$, where det stands for determinant of a matrix and K is some kernel. Assumptions on K needed!
- Assumptions: Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ locally square-integrable kernel with respect to $\mu^{\otimes 2}$, defining Hermitian, positive semi-definite, trace-class operator $\mathcal{K}_B$ on on $L^2(B, \mu)$, for all compact B, with all eigenvalues in $[0, 1]$. (cf. Ben Hough(2009))

Determinantal pp is weakly sub-Poisson

- By Hadamard's inequality, $\det\left(K(x_i,x_j)\right)_{1\le i,j\le k} \le \prod_{i=1}^{k} K(x_i,x_i)$ hence $\Phi^{\det}$ has moments smaller than Poisson pp of mean $K(x,x)\mu(dx)$.
- Distribution of $\Phi^{det}(B)$ is equal to sum of independent Bernoulli variables with parameters given by the eigenvalues of $\mathcal{K}_B$. Hence $\Phi^{\det}(B)$ is convexly smaller than Poisson which implies smaller voids.
- Cor. All determinantal pp exhibit non-trivial phase transition in percolation of their RGG. New result!

Determinantal pp and dcx

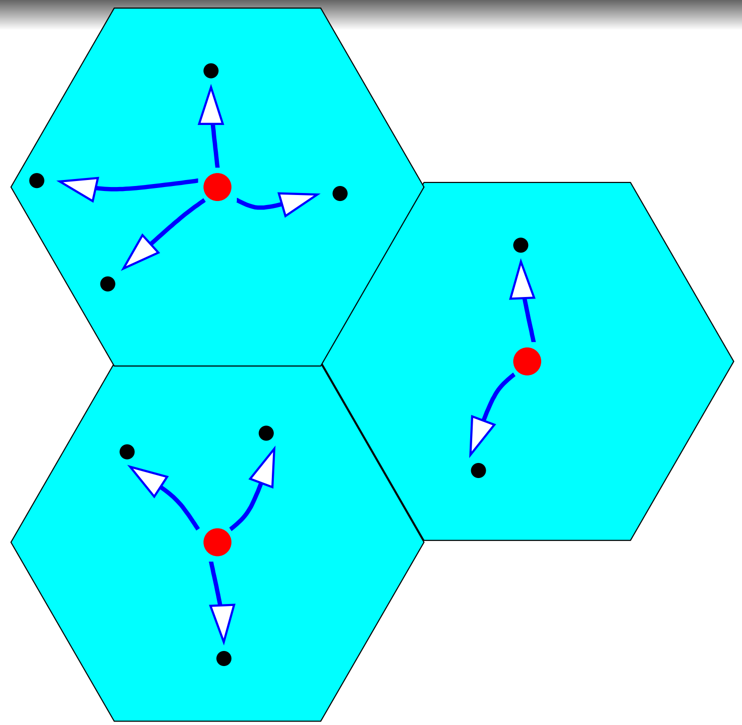
- Prop.
  $$\big(\Phi^{det}(B_1),\ldots,\Phi^{det}(B_n)\big) \le_{dcx} \big(\text{Pois}(B_1),\ldots,\text{Pois}(B_n)\big),$$
  for disjoint, simultaneously observable B_i
  (eigenfunctions of $\mathcal{K}_{\bigcup B_i}$, restricted to B_i are also eigenfunctions of $\mathcal{K}_{B_i}$ for all i).
- A partial proof of the fact that stationary determinantal pp are negatively associated can be found in the current version of Ghosh'12 arXiv:1211.2435.
  If this is true than determinantal pp are not only weakly sub-Poisson, but having convexly smaller marginals are actually dcx sub-Poisson.

Ginibre pp

- Example: Ginibre pp is the the determinantal point process on $\mathbb{R}^2$ with kernel $K((x_1, x_2), (y_1, y_2)) = \exp[(x_1 y_1 + x_2 y_2) + i(x_2 y_1 - x_1 y_2)]$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2$, with respect to the measure $\mu(d(x_1, x_2)) = \pi^{-1} \exp[-x_1^2 - x_2^2]\, dx_1 dx_2$.
- Spherical annuli are its simultaneously observable sets.
- Consequently, pp of the squared radii $\{|X_i|^2\}$ of the Ginibre point process is dcx sub-Poisson. Interestingly $\{|X_i|^2\} =_{distr} \{T_n = \sum_n \sum_{i=1}^n Z_i^n\}$, where Z_i^n are i.i.d. exponential.

Clustering worsens percolation? — examples and ... a counterexample

Perturbed lattices



Assume:

$\mathbf{\Phi}$ — deterministic lattice,

(say uniform) translation kernel inside lattice cell,

$\mathcal{N}_0(x,\cdot) = Poi(1)$,

$\mathcal{N}_1(x,\cdot) \leq_c Poi(1)$,

$\mathcal{N}_2(x,\cdot) \geq_c Poi(1)$.

Then

$$\Phi_1^{pert} \leq_{dcx} \Phi_0^{pert} \leq_{dcx} \Phi_2^{pert}$$

sub-Poisson perturbed lattice

Poisson pp

super-Poisson perturbed lattice

Perturbed lattices; cont'd

cx ordered families of (discrete) random variables from smaller to larger:

- deterministic (constant);
- Hyer-Geometric $p_{HGeo(n,m,k)}(i) = \binom{m}{i}\binom{n-m}{k-i}/\binom{n}{k}$ $(\max(k-n+m,0) \le i \le m)$.
- Binomial $p_{Bin(n,p)}(i) = \binom{n}{i}p^i(1-p)^{n-i}$ $(i = 0, \dots, n)$
- Poisson $p_{Poi(\lambda)}(i) = e^{-\lambda}\lambda^i/i!$ $(i = 0, 1, \dots)$
- Negative Binomial $p_{NBin(r,p)}(i) = \binom{r+i-1}{i}p^i(1-p)^r$.
- Geometric $p_{Geo(p)}(i) = p^i(1-p)$

Assuming parameters making equal means, we have $const \le_{cx} HGeo \le_{cx} Bin \le_{cx} Poi \le_{cx} NBin \le_{cx} Geo$

Conjecture for perturbed lattices

$$\Phi_1 \quad \leq_{dcx} \quad \Phi_2$$

$$\Downarrow$$

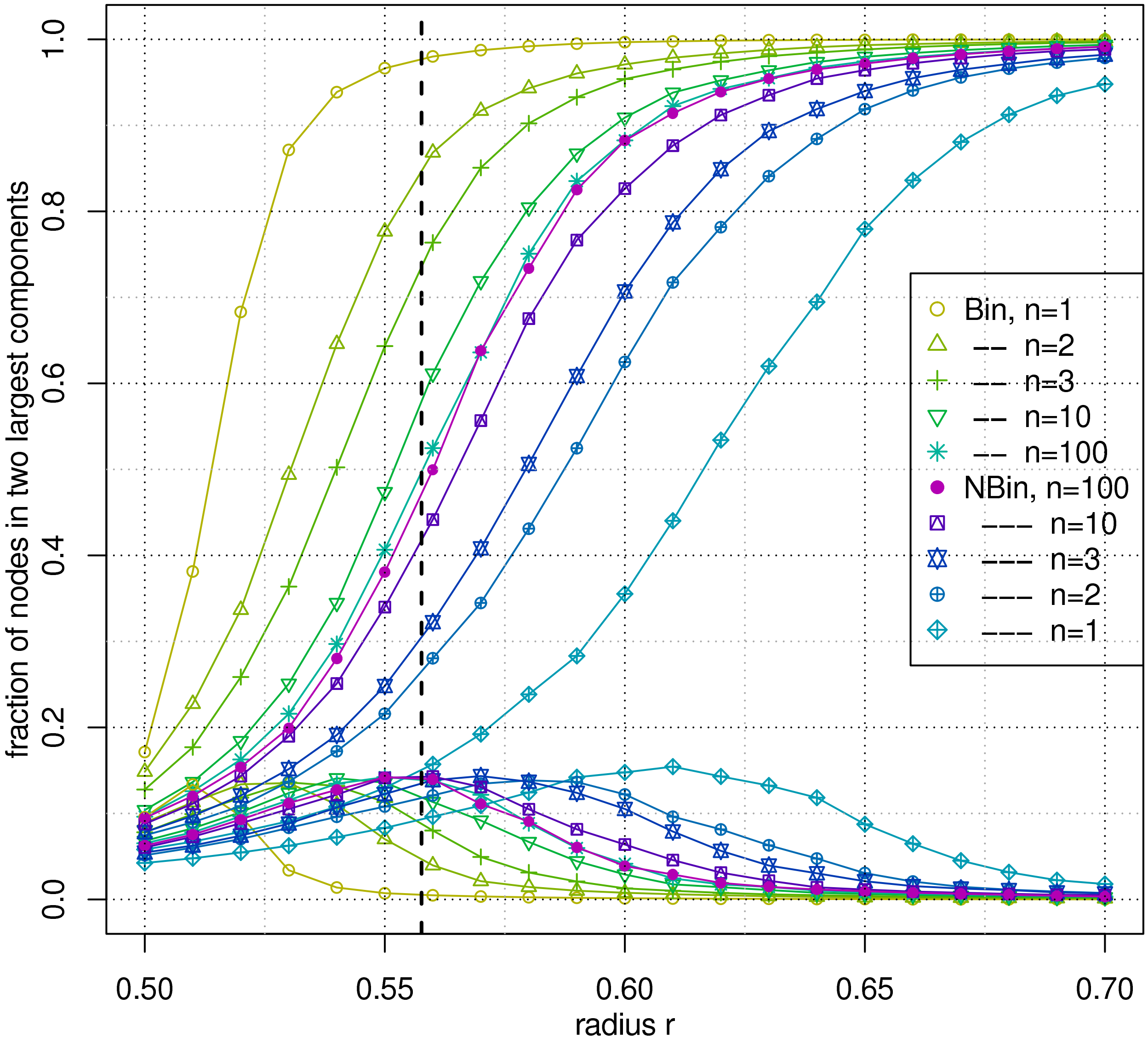
$$r_c(\Phi_1) \quad \leq \quad r_c(\Phi_2)$$

$Bin(1,1) = const$

$Bin(1,1/n) \nearrow_{cx} Poi(1)$

$NBin(n,1/(1+n)) \searrow_{cx} Poi(1)$

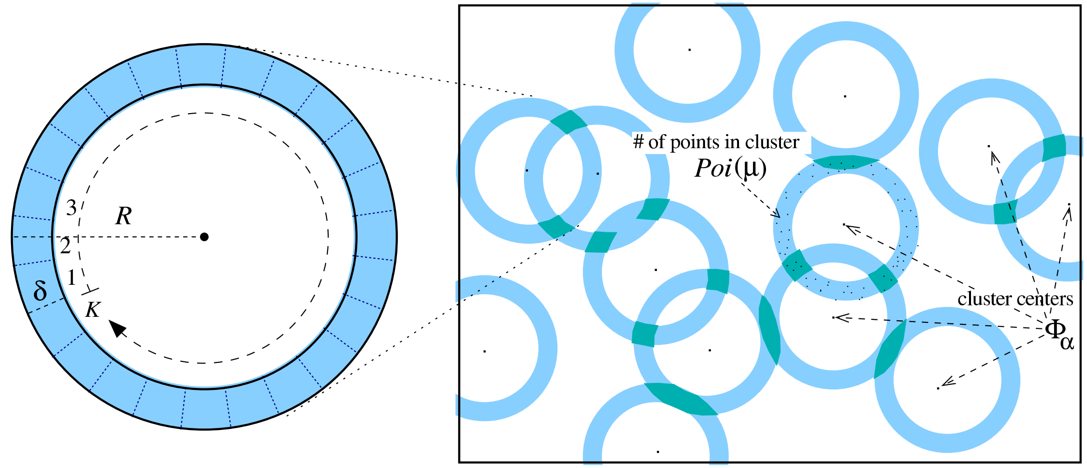
$NBin(1,1/2) = Geo(1/2)$



Counterexample: a super-Poisson pp with $r_c = 0$

Poisson-Poisson cluster pp $\Phi_\alpha^{R,\delta,\mu}$ with annular clusters

Φ_α — Poisson (parent) pp of intensity α on $\mathbb{R}^2$, Poisson clusters of total intensity μ, supported on annuli of radii $R-\delta, R$.



We have $\Phi_\lambda \le_{dcx} \Phi_\alpha^{R,\delta,\mu}$, where Φ_λ is homogeneous Poisson pp of intensity $\lambda = \alpha\mu$.

Prop. Given arbitrarily small $a, r > 0$, there exist constants α, μ, δ, R such that $0 < \alpha, \mu, \delta, R < \infty$, the intensity $\alpha\mu$ of $\Phi_\alpha^{R,\delta,\mu}$ is equal to a and the critical radius for percolation $r_c(\Phi_\alpha^{R,\delta,\mu}) \le r$. Consequently, one can construct Poisson-Poisson cluster pp of intensity a and $r_c = 0$.

Conclusions

- Voids and moment measures allow for a simple comparison of comparison of clustering properties of pp.
- We believe that these tools can be used to generalize some results derived for Poisson to “more homogeneous” (less clustering) — sub-Poisson pp.
- We have seen examples regarding concentration inequalities and phase transition in percolation.
- Other clustering comparison tools?
- Conjecture restricted to sub-Poisson pp.?

Sub-poissonianity used in

- Daley Last Descending chains, the lilypond model, and mutual-nearest-neighbour matching (2005)
- Hirsch, Neuhaeuser, Schmidt Connectivity of random geometric graphs related to minimal spanning forests (2012)
- Yogeshwaran, Adler On the topology of random complexes built over stationary point processes (2012).

Other related works

- Benjamini and Stauffer (2011) Perturbing the hexagonal circle packing: a percolation perspective.
- Franceschetti, Booth, Cook, Meester and Bruck (2005) Continuum percolation with unreliable and spread-out connections. J. Stat. Phy.
- Franceschetti, Penrose, and Rosoma (2010) Strict inequalities of critical probabilities on Gilbert's continuum percolation graph. arXiv
- Jonasson (2001) Optimization of shape in continuum percolation. Ann. Probab.
- Roy and Tanemura (2002) Critical intensities of boolean models with different underlying convex shapes.
- Ghosh, Krishnapur, Peres (2012) Continuum Percolation for Gaussian zeroes and Ginibre eigenvalues.

For mode details ...

- BB, Yogeshwaran Directionally convex ordering of random measures, shot-noise fields ... *Adv. Appl. Probab. (2009)*
- BB, Yogeshwaran Clustering and percolation of point processes *EJP 2013*.
- BB, Yogeshwaran On comparison of clustering properties of point processes *Adv. Appl. Probab.* (2014).
- BB, Yogeshwaran Clustering comparison of point processes with applications to random geometric models arXiv:1112.5285 to appear in *Stochastic Geometry, Spatial Statistics and Random Fields ...* (V. Schmidt, ed.) Lecture Notes in Mathematics Springer.

thank you