Wireless communications: from simple stochastic geometry models to practice
II Connectivity

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Workshop on Probabilistic Methods in Telecommunication
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Outline

- **COVERAGE**
- **CONNECTIVITY and ROUTING**
- **CAPACITY**

CONNECTIVITY

- Multi-hop connecting at least two users (source and destination) distant in space. Existence of routes.

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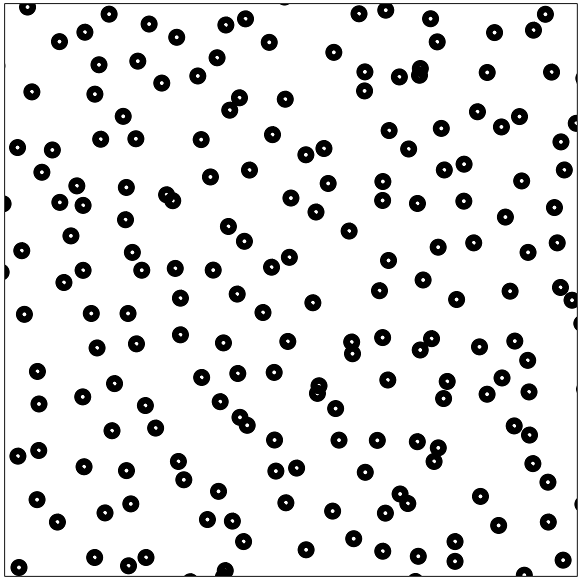
- Multi-hop connecting at least two users (source and destination) distant in space. Existence of routes.
- Percolation theory provides tools to study macroscopic connectivity.
- First passage percolation to study the speed of message propagation on long routes.
- Mostly qualitative results.
- Comparisons methods for non-Poisson models.
- We shall present some results on connectivity and routing on the SINR graph.

Ad-hoc (D2D) Network

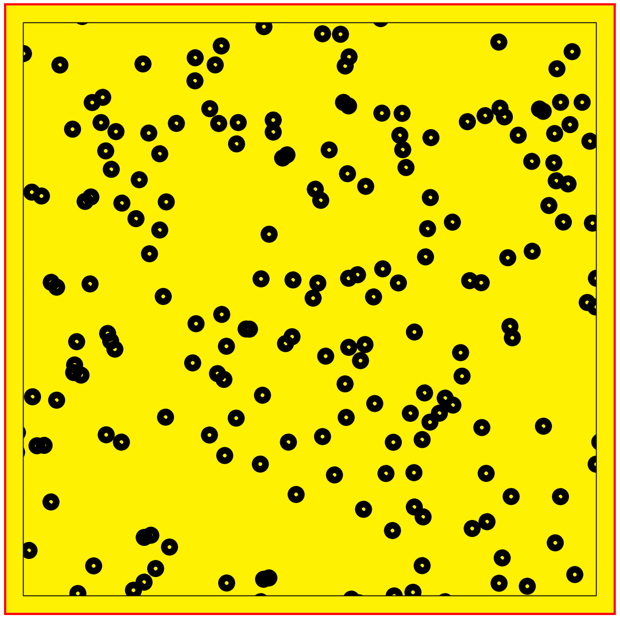
Network made of nodes arbitrarily repartitioned in some region, exchanging packets either transmitting or receiving them on a common frequency, use intermediate retransmissions by nodes lying on the path between the packet source node and its destination nodes.

Ad-hoc ≡ random, usually Poisson

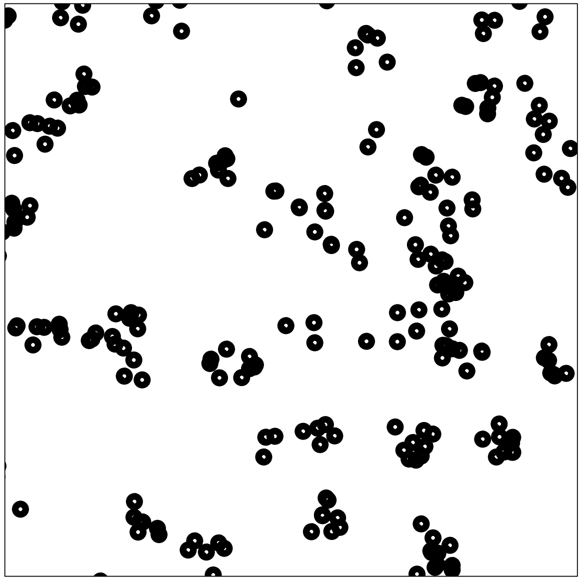
Nodes “arbitrarily” placed ≡ modeled as an instance of a point process. Usually Poisson.



more regular
(sub-Poisson)



Poisson



more clustering
(super-Poisson)

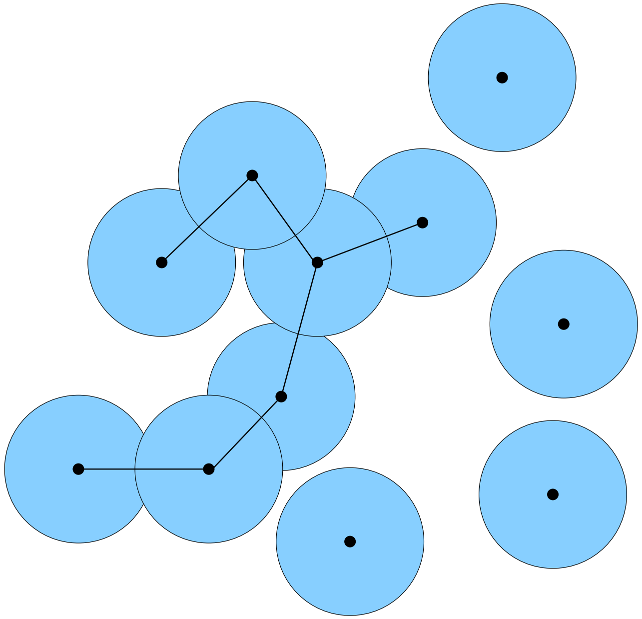
CONNECTIVITY

Macroscopic connectivity via percolation

Gilbert graph – a simplest connectivity model

Gilbert graph $C(\Phi, r)$ on Φ:
edge between any two points which are closer than r from each other.

Interference free model.
Equivalent to connectivity of Boolean model on Φ with spherical grains of radius $r/2$.
Another name: Random Geometric Graph.



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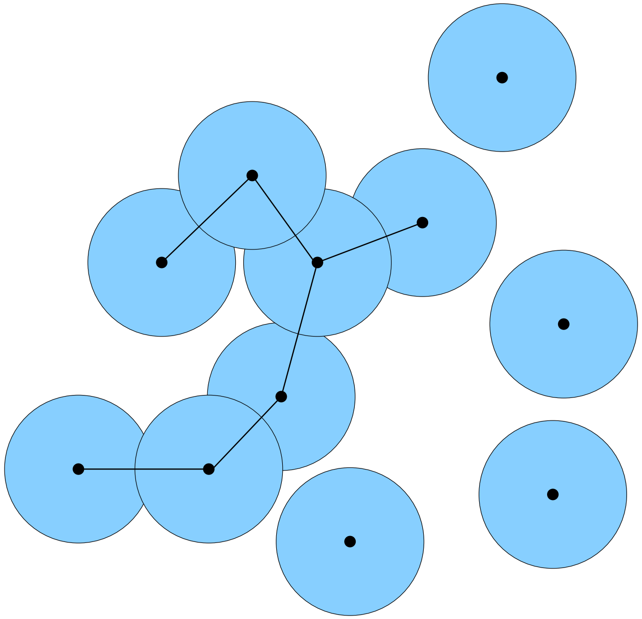

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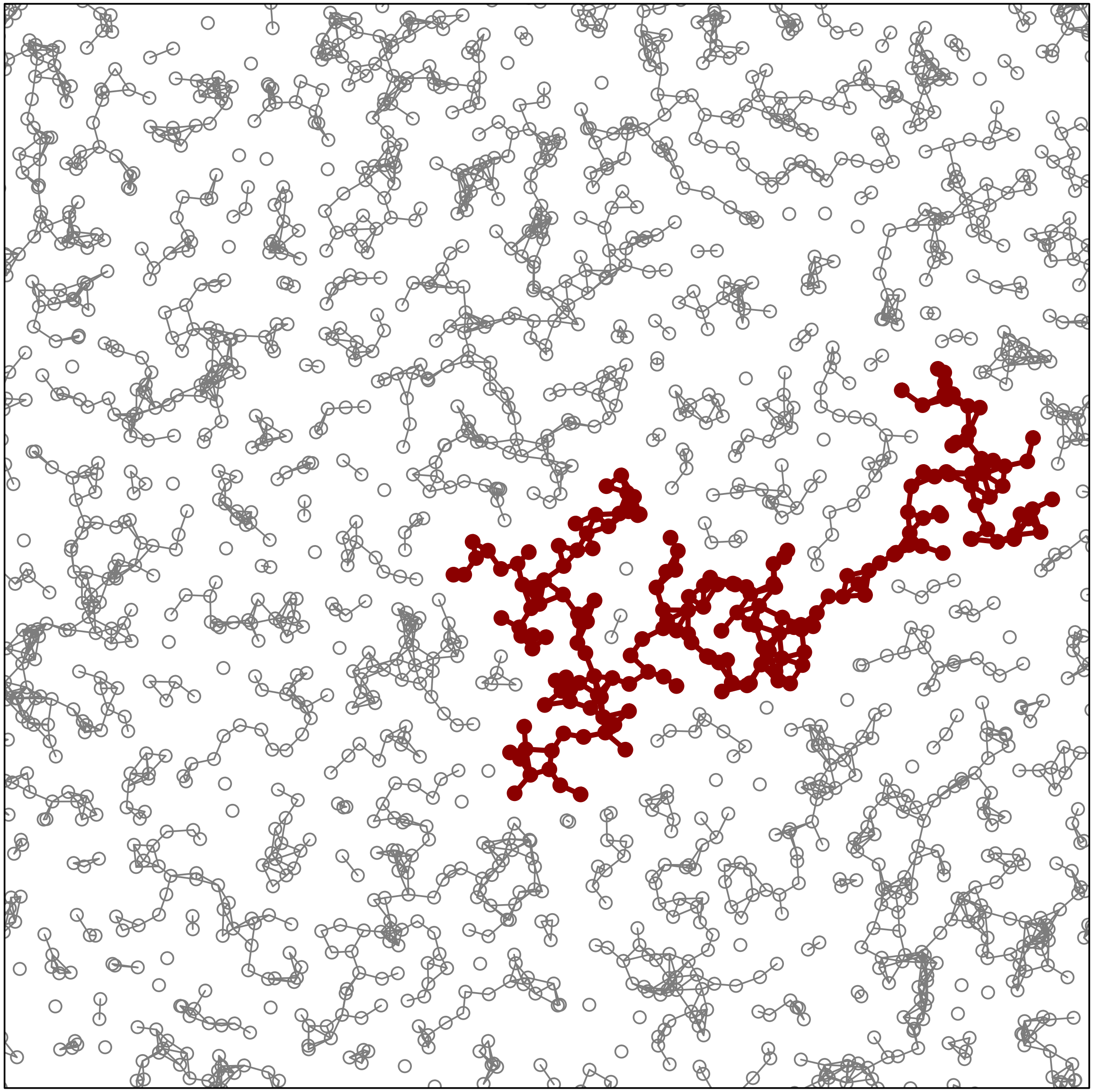
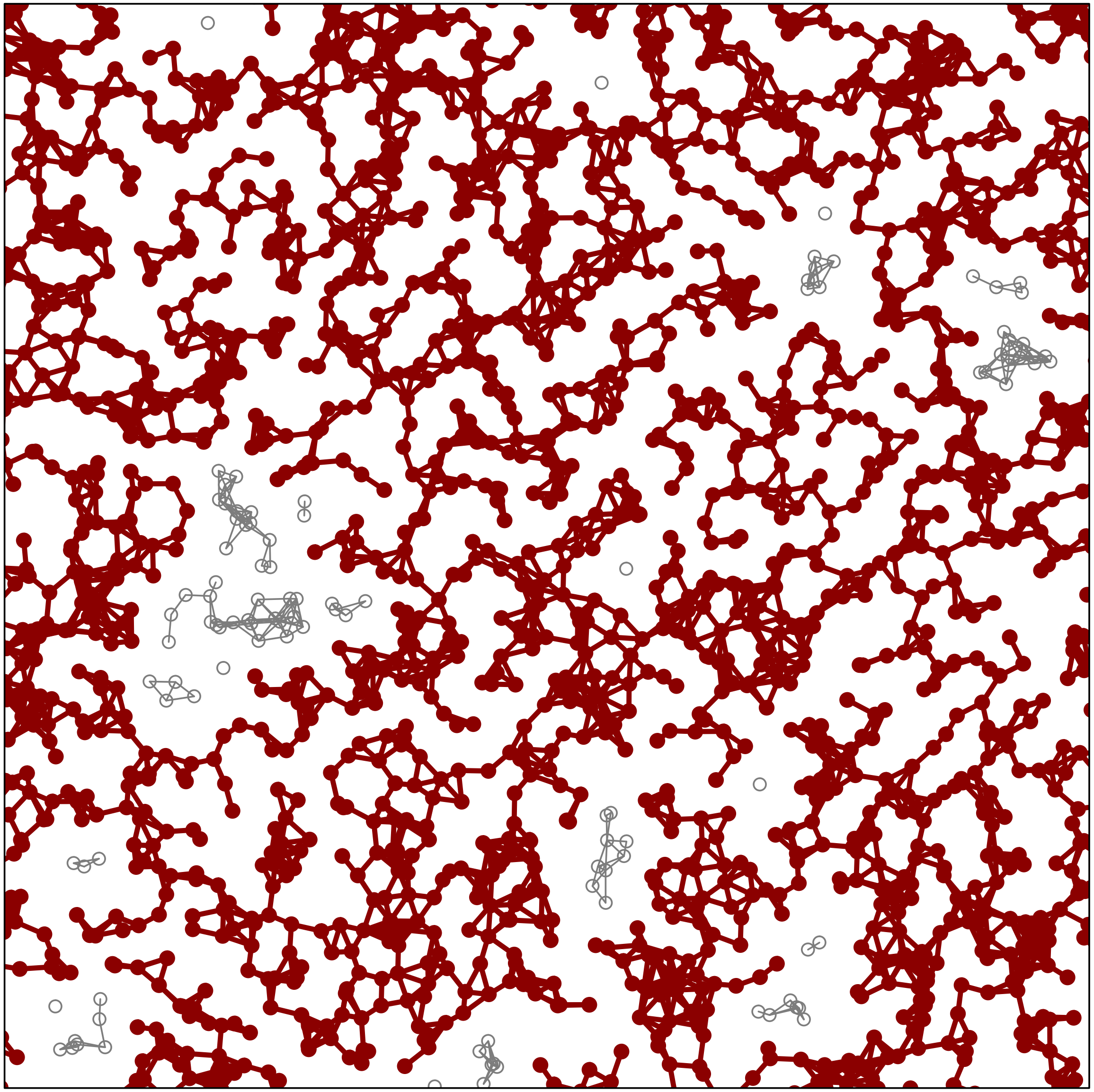
Full connectivity not possible for any finite r in case of homogeneous Poisson Φ on $\mathbb{R}^d$ and many other interesting processes.

Poisson-Gilbert graph — macroscopic view



The largest component in the window is highlighted.

Poisson-Gilbert graph — larger r



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Percolation — large scale connectivity

Percolation $\equiv$ existence of an infinite connected subset (component).

- Well accepted notion of "minimal connectivity property" of a large network, which might not be fully connected.
- Indicates existence of a non-negligible "connected core" within the network.

Critical radius for percolation

Critical radius for the percolation of the Gilbert graph on Φ:
$r_c(\Phi) = \inf\{r > 0 : \mathsf{P}(C(\Phi, r) \text{percolates}) > 0\}$

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Critical radius for the percolation of the Gilbert graph on $\mathbf{\Phi}$:

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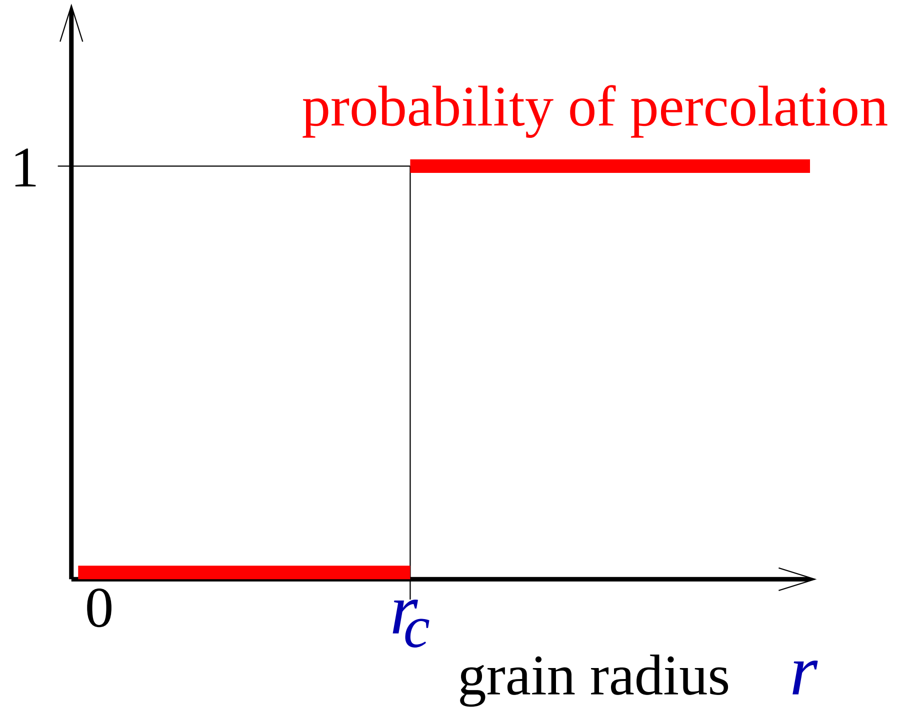
In the case when $\mathbf{\Phi}$ is stationary and ergodic



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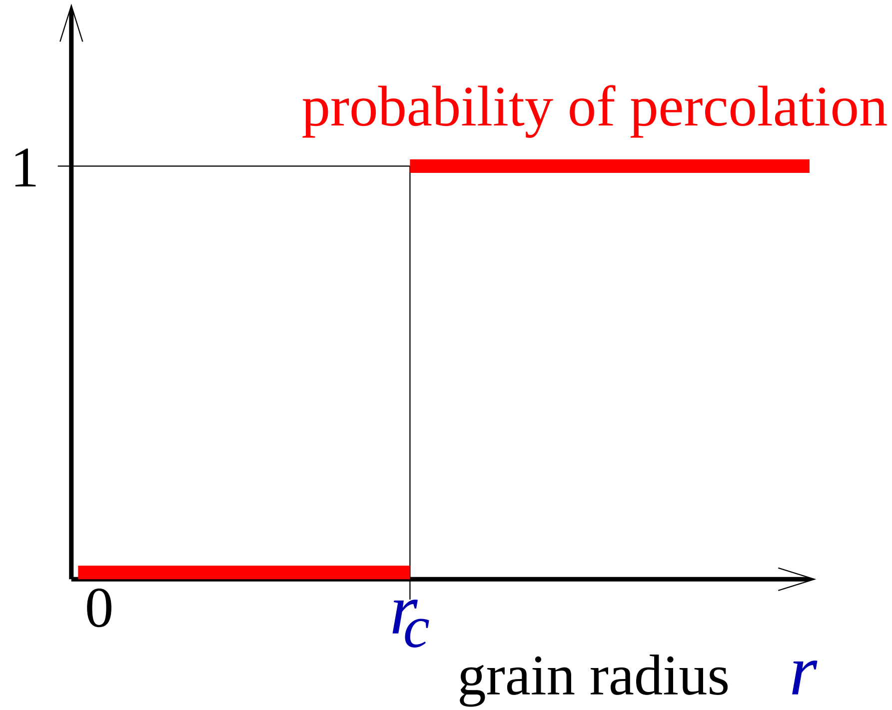

Usually, the no closed form expression for r_c.
If $0 < r_c < \infty$ we say the phase transition is non-trivial.

Non-trivial phase transition for Gilbert graph

THM: For dimension $d \geq 2$ percolation of the Gilbert graph on Poisson point process exhibits non-trivial phase transition

$$0 < r_c(\lambda) < \infty$$

for all values of intensity λ of Poisson process.

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for all values of intensity λ of Poisson process.
In fact,

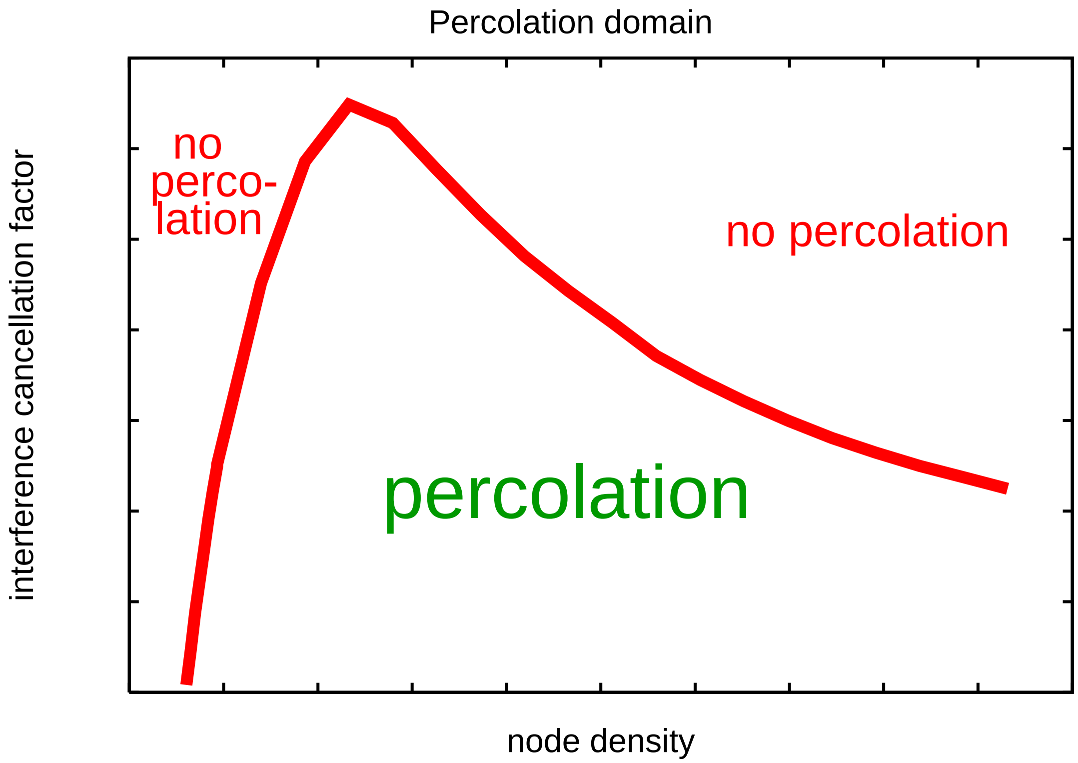
$$r_c(\lambda) = \frac{r_c}{\lambda^{1/d}}.$$

The exact value of r_c is not known. For $d = 2$, $r_c \approx 0.636$.

Percolation in SINR coverage model

Dousse, F. Baccelli, and P Thiran (2003),

Dousse, Franceschetti, Macris, Meester, Thiran (2006)

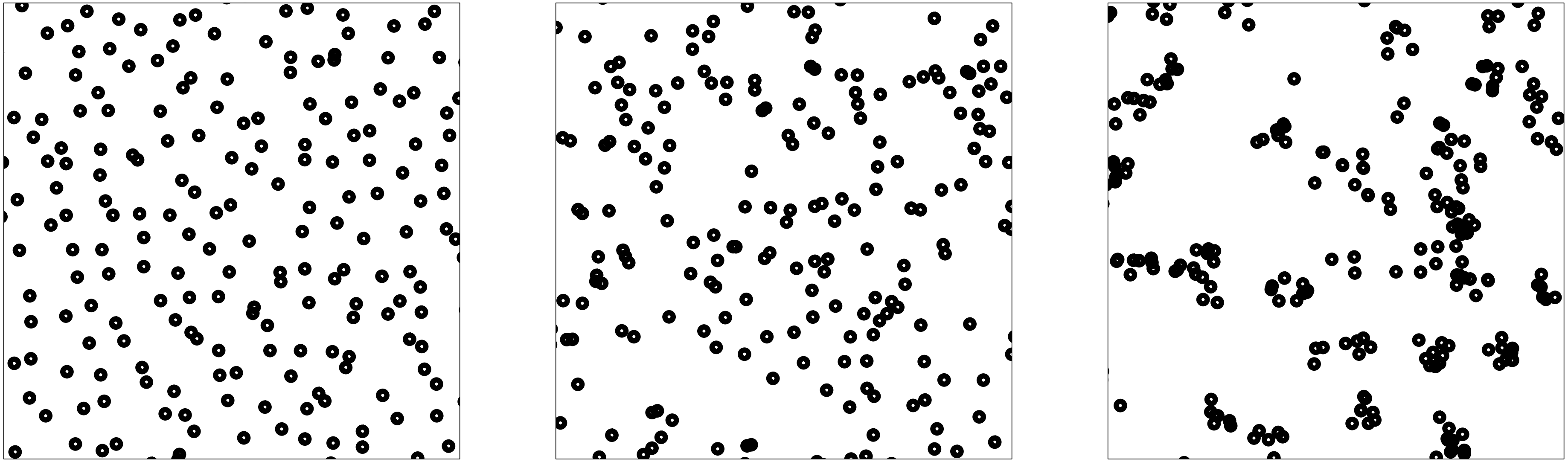


Increasing node density may destroy infinite component(s)!

Percolation for non-Poisson processes by clustering comparison

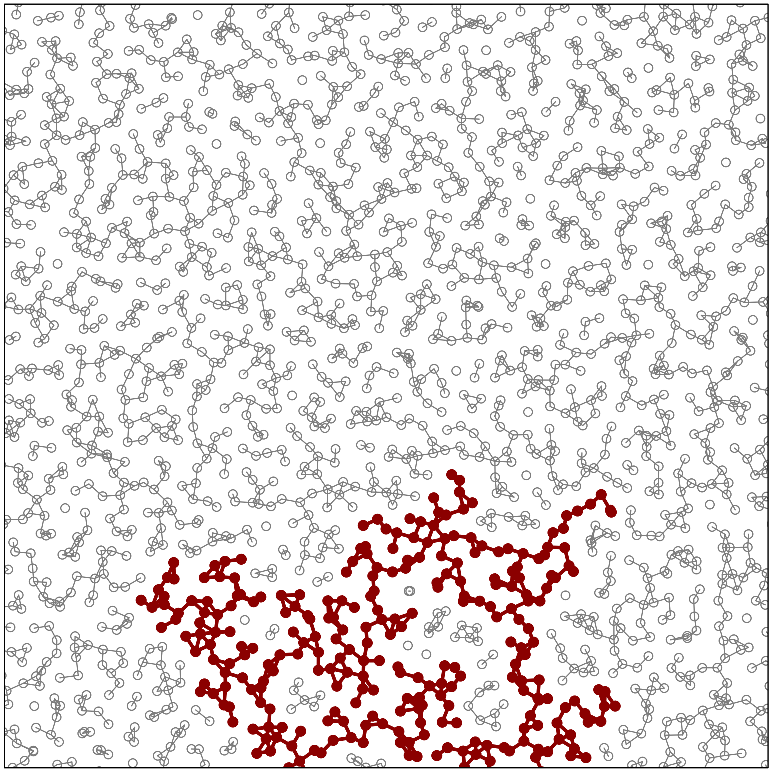
Clustering of points

Clustering in a point pattern roughly means that the points lie in clusters (groups) with the clusters being spaced out.

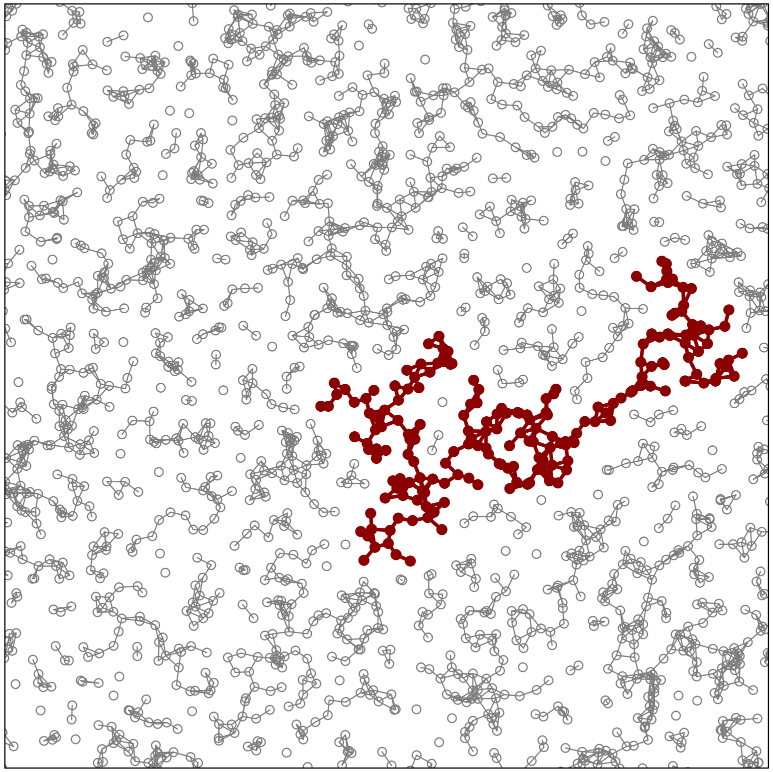


Point processes having the same intensity (on average the same number of points per unit of space).

Clustering and macroscopic connectivity



less clustering



more clustering

Gilbert graph with highlighted the largest component in the window.

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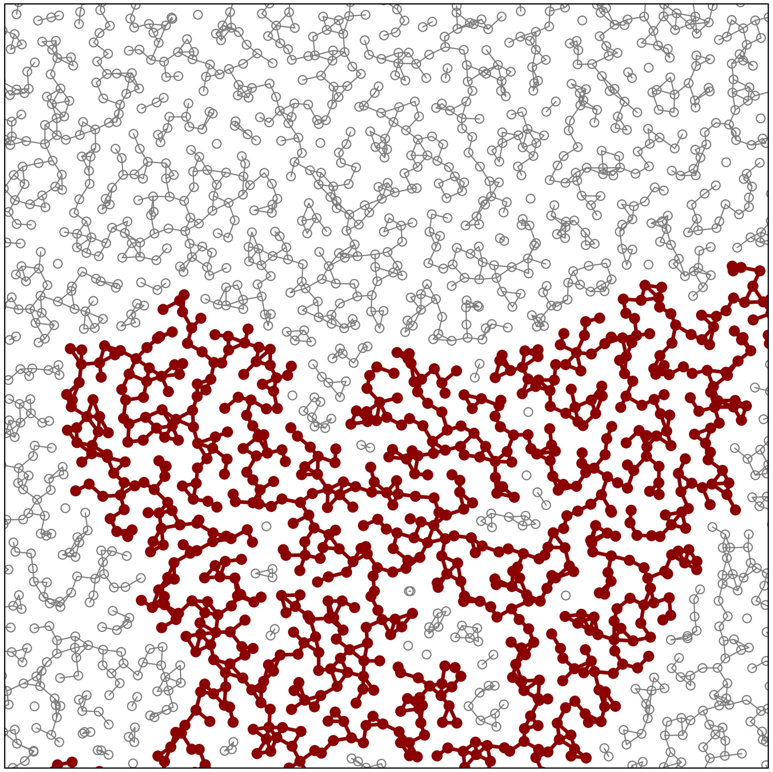
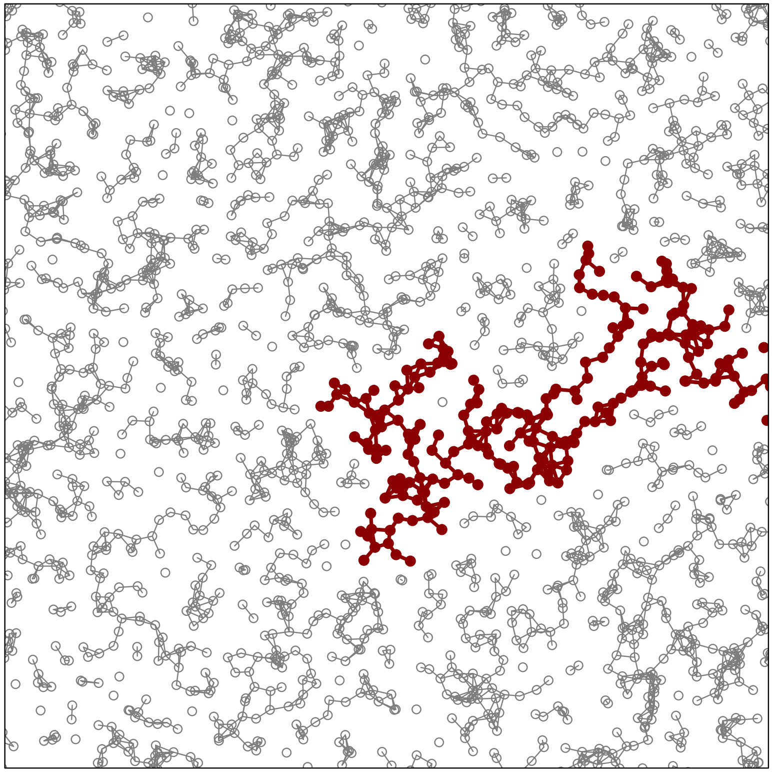


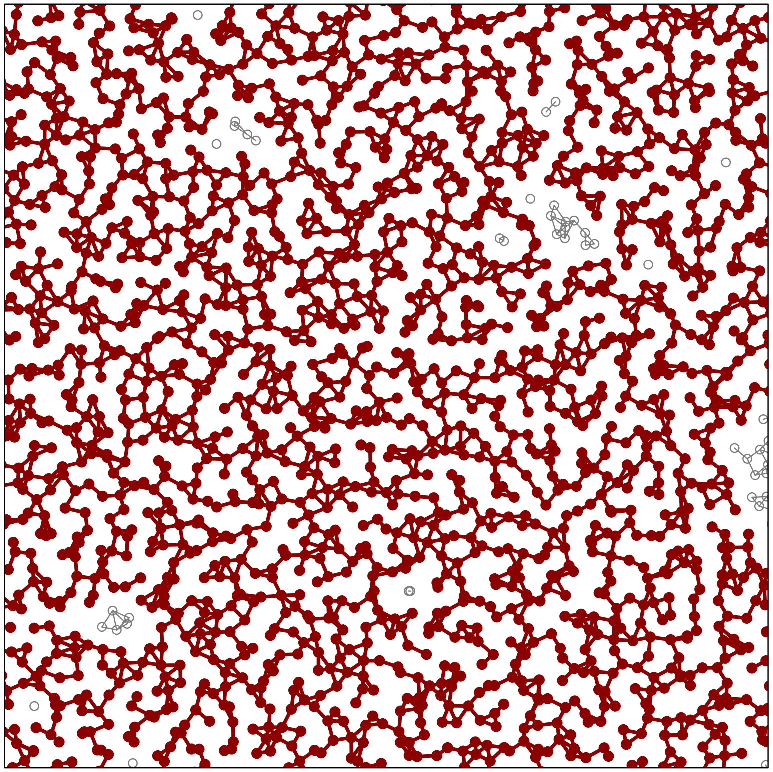


less clustering more clustering

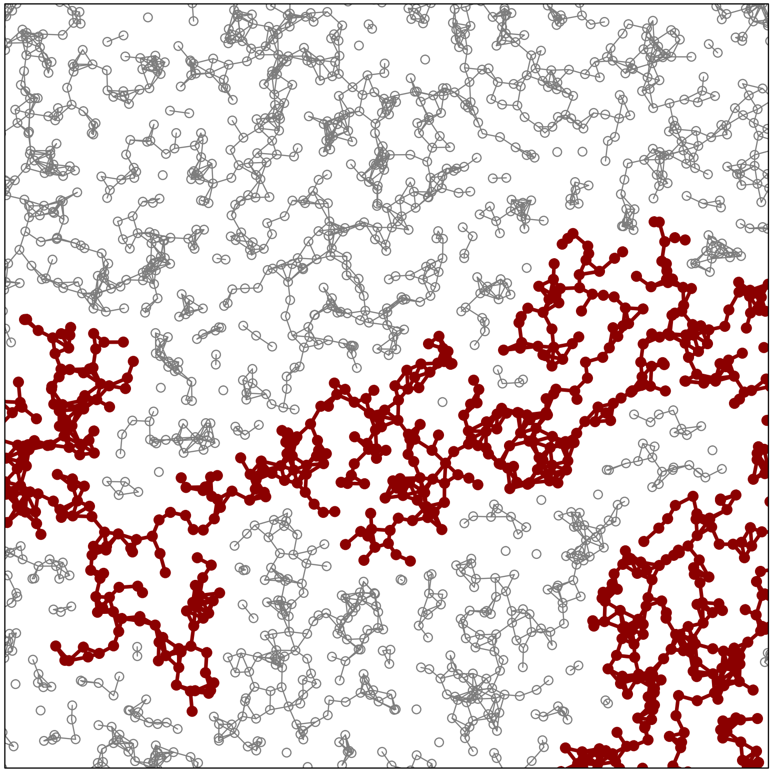
Gilbert graph with highlighted the largest component in the window.

Increasing r.

Clustering and macroscopic connectivity



less clustering



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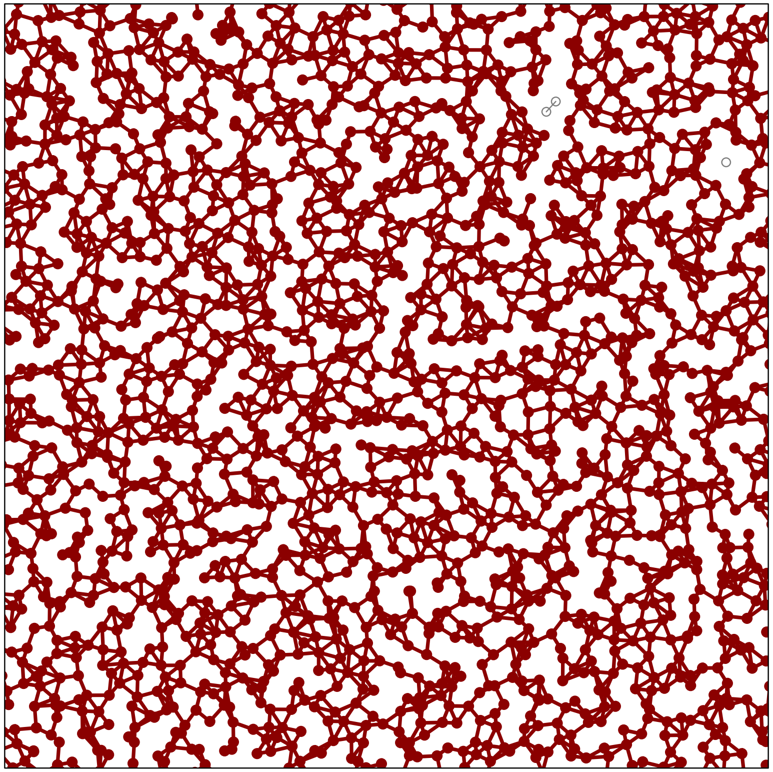
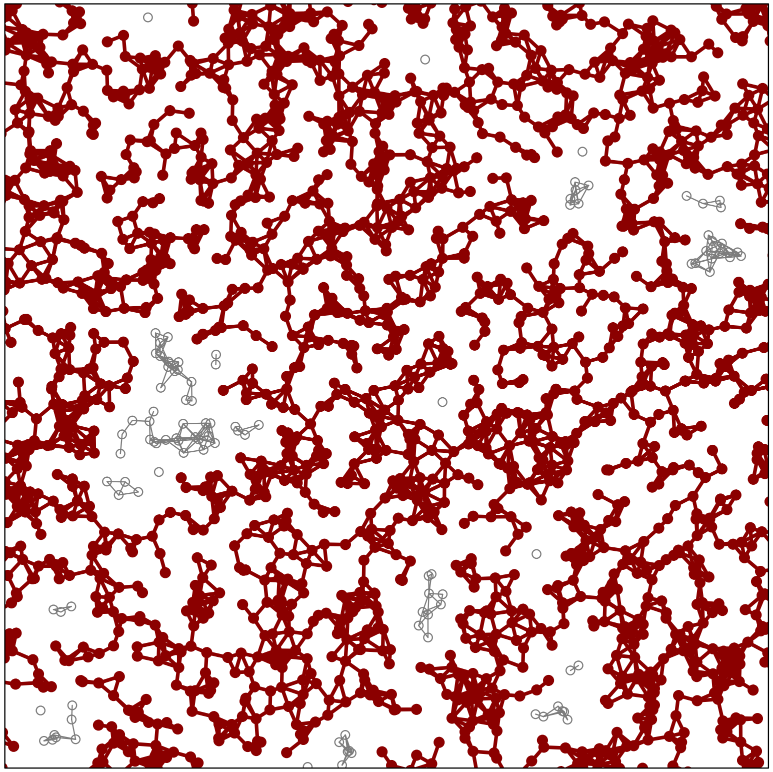


less clustering



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Increasing r.

Conjecture: Clustering worsens percolation

Point processes exhibiting more clustering should have larger critical radius r_c for the percolation of their continuum percolation models.

$$\Phi_1 \text{ "clusters less than" } \Phi_2 \quad \Rightarrow \quad r_c(\Phi_1) \leq r_c(\Phi_2),$$

where $r_c(\Phi) = \inf\{r > 0 : \mathsf{P}(C(\Phi, r)\text{percolates}) > 0\}$

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Heuristic: Interconnecting well spaced-out clusters (necessary to obtain an infinite connected component) requires large r. Spreading points from clusters "more homogeneously" should result in a decrease r for which the percolation takes place.

Comparison tools

- dcx ordering of pp. Natural extension of dcx ordering of random vectors (recall Ross's conjecture), a generalization of convex ordering of random variables. Larger in dcx pp represent more variability (in probability and in state space — clustering).

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- Positive and negative association of pp. Way of comparing dependence of points to the complete independence property of Poisson pp.
- Statistical tools. Ripley function, correlation function, ... (local hence relatively weak tools).

Voids and moments

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- Factorial moment measures: $\alpha^{(k)}(\cdot)$ for simple pp, truncation of the measure $\alpha^k(\cdot)$ to "off the diagonals" $\{(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k : x_i \neq x_j$ for $i \neq j\}$

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- In a general (not necessarily simple pp) $\{\alpha^{(k)}(\cdot) : k\}$ can be expressed in terms of $\{\alpha^k(\cdot) : k\}$ and vice versa. Each of the three families of three functionals (voids, moments and factorial moments) determine the distribution of pp.

Voids & moments and clustering

- The “most spatially homogeneous” (“non-clustering”) way of spreading points of $\mathbf{\Phi}$, with a given mean measure $\alpha(\cdot)$, would be to place them according to the (deterministic) measure $\alpha(\cdot)$. But this is not a point process.

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- Voids and moments allow for upped bounds on these probabilities $\longrightarrow$ concentration inequalities.

Concentration inequalities

- Chernoff's bounds:
  $\mathbf{P}\left(\Phi(B) - \alpha(B) \geq a\right) \leq e^{-t(\alpha(B)+a)}\mathbf{E}\left(e^{t\Phi(B)}\right)$
  and
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- and
  $\mathbf{E}\left(e^{-t\Phi(B)}\right) = \sum_{k=0}^{\infty} e^{-tk}\mathbf{P}\left(\Phi(B) = k\right) = \mathbf{P}\left(\Phi'(B) = 0\right)$
  is the void probability of the point process Φ' obtained from Φ by independent thinning with retention probability $1 - e^{-t}$. Ordering of voids is preserved by independent thinning.

Comparison to Poisson pp — Laplace ordering

- Consider pp Φ having voids and moments smaller than Poisson pp (of the same mean). We call them weakly sub-Poisson (a weaker comparison than dcx).

  $\mathsf{P}\left(\Phi(B)=0\right) \le e^{-\mathsf{E}(\Phi(B))}$ for all bBs B (V)

  $\mathsf{E}\left(\prod_{i=1}^{k}\Phi(B_i)\right) \le \prod_{i=1}^{k}\mathsf{E}(\Phi(B_i))$ for all disjoint B_i (M)

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- Prop. For simple pp Φ of mean measure α: Φ has smaller voids than Poisson ((V) holds true) if and only if for all $f \ge 0$

  $\mathbf{E}\left(\exp\left[-\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right) \le \exp\left[\int_{\mathbb{R}^d}(e^{-f(x)}-1)\,\alpha(\mathrm{d}x)\right]$

  (*)

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- Prop. For simple pp Φ of mean measure α: If Φ has smaller moments than Poisson ((M) holds true) than (*) holds for all $f \le 0$.

Voids & percolation — a sufficient condition

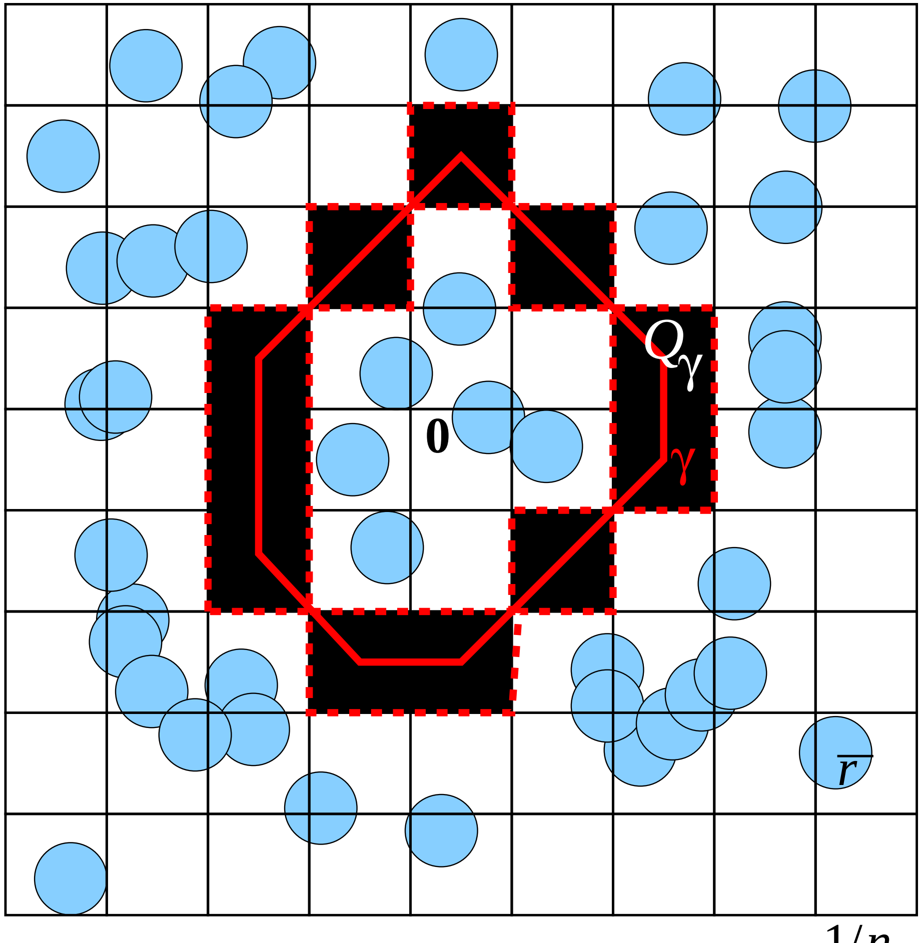
An upper bound on r_c using voids

$$\overline{r}_c = \inf\Big\{ r > 0 : \forall n \geq 1, \sum_{\gamma \in \Gamma_n} \mathbf{P}\left(C(\Phi, r) \cap Q_\gamma = \emptyset\right) < \infty \Big\}.$$

By Peierls argument

$$r_c(\Phi) \leq \overline{r}_c(\Phi).$$

Smaller voids imply smaller $\overline{r}_c(\Phi)$



Moments & percolation — a necessary cond.

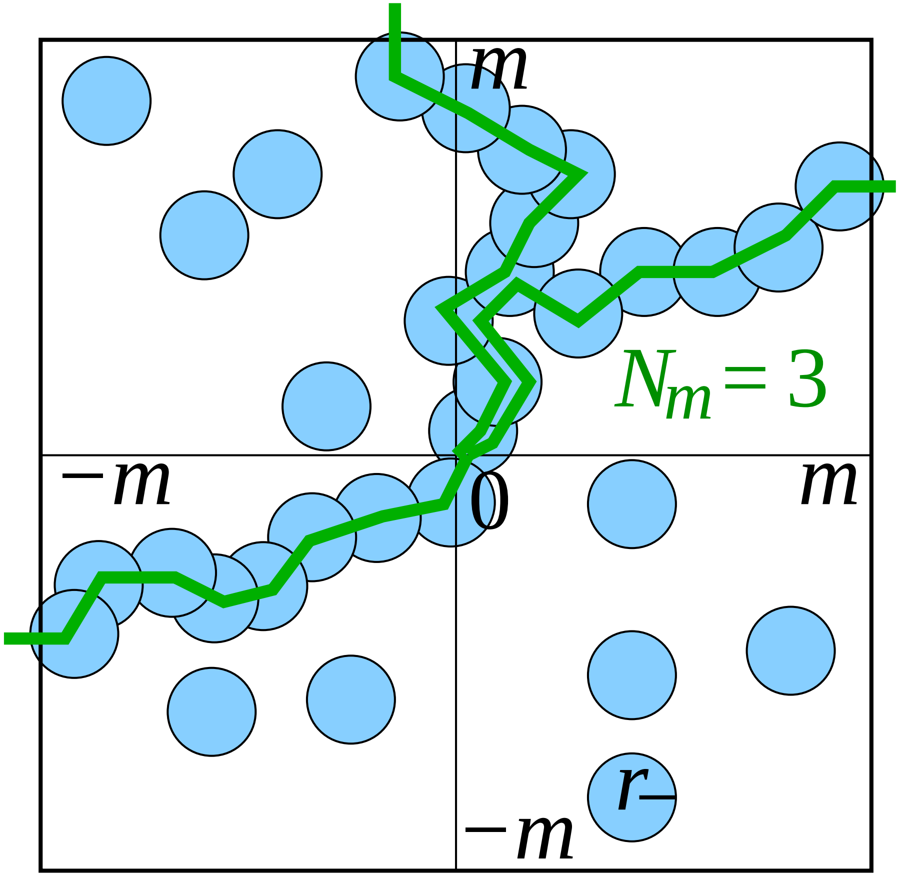
A lower bound on r_c related to moments measures

$$\underline{r}_c(\Phi) := \inf\left\{r > 0 : \liminf_{m\to\infty} \mathbf{E}(N_m(\Phi, r)) > 0\right\}.$$

By Markov inequality

$$\underline{r}_c(\Phi) \le r_c(\Phi).$$

Smaller moments imply larger(!) $\underline{r}_c(\Phi)$



Clustering & percolation phase-transition

THM Let Φ be a stationary sub-Poisson pp on $\mathbb{R}^d$ (void probabilities and moment measures smaller than for the Poisson pp of some intensity λ) and $r_c(\Phi)$ its critical percolation radius on Gilbert graph. Then

$$0 < \frac{1}{(2^d\lambda(3^d-1))^{1/d}} \le r_c(\Phi) \le \frac{\sqrt{d}(\log(3^d-2))^{1/d}}{\lambda^{1/d}} < \infty;$$

[BB-Yogeshwaran (2013)].

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[BB-Yogeshwaran (2013)]. Similar results for

- k-percolation (percolation of k-covered subset) for dcx sub-Poisson.
- word percolation,
- SINR-graph percolation (graph on a shot-noise germ-grain model).

Examples of sub- and super-Poisson processes

sub-Poisson processes

strongly (dcx)

Voronoi perturbed lattices with replication kernel $\mathcal{N} \leq_{cx}$ Pois, in particular binomial, determinantal

negatively associated

binomial, determinantal(*)

weakly (voids and moments)

dcx sub-Poisson, negatively associated, determinantal

super-Poisson processes

strongly (dcx)

Poisson-Poisson cluster, Lévy based Cox, mixed Poisson, Neyman-Scott with mean cluster size 1, Voronoi perturbed lattices with replication kernel $\mathcal{N} \geq_{cx}$ Pois.

associated

Poisson-center cluster, Neyman-Scott, Cox associated with associated intensity measure.

weakly (voids and moments)

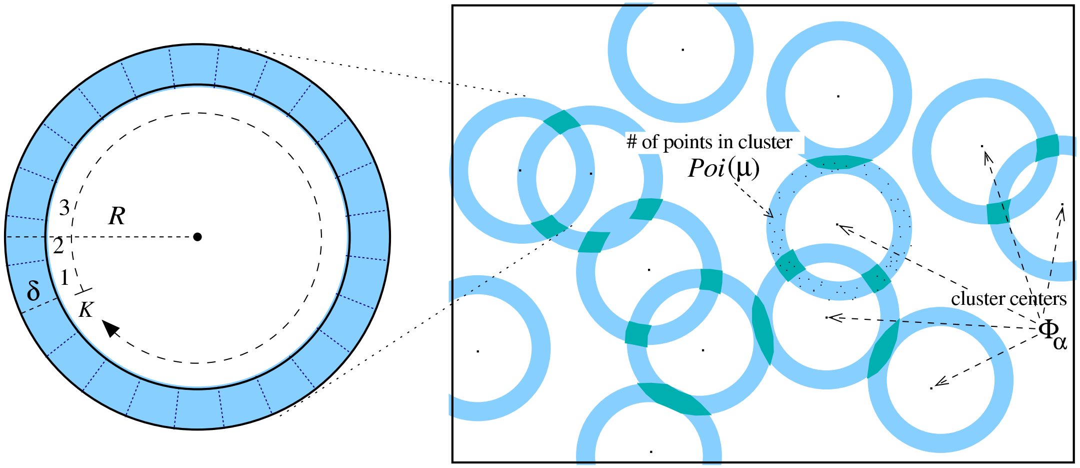
dcx super-Poisson, associated, permanental

(*) Ghosh arXiv:1211.2435

Counterexample: a super-Poisson pp with $r_c = 0$

Poisson-Poisson cluster pp $\Phi_{\alpha}^{R,\delta,\mu}$ with annular clusters

Φ_α — Poisson (parent) pp of intensity α on $\mathbb{R}^2$, Poisson clusters of total intensity μ, supported on annuli of radii $R-\delta, R$.



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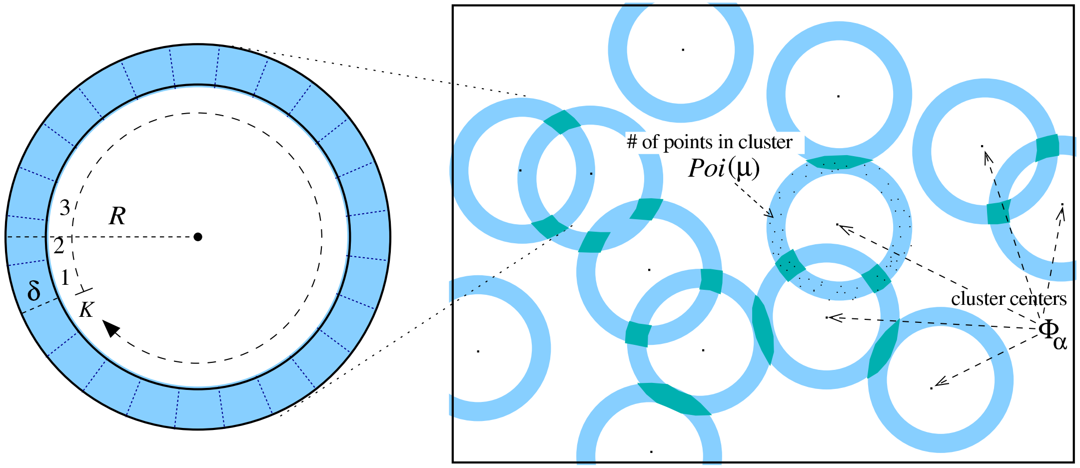

We have $\Phi_{\lambda} \le_{dcx} \Phi_{\alpha}^{R,\delta,\mu}$, where Φ_{λ} is homogeneous Poisson pp of intensity $\lambda = \alpha\mu$.

Prop. Given arbitrarily small $a, r > 0$, there exist constants α, μ, δ, R such that $0 < \alpha, \mu, \delta, R < \infty$, the intensity $\alpha\mu$ of $\Phi_{\alpha}^{R,\delta,\mu}$ is equal to a and the critical radius for percolation $r_c(\Phi_{\alpha}^{R,\delta,\mu}) \le r$. Consequently, one can construct Poisson-Poisson cluster pp of intensity a and $r_c = 0$.

Concluding on clustering & percolation

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- Other clustering comparison tools?

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- We have seen examples regarding concentration inequalities and phase transition in percolation.
- Other clustering comparison tools?
- Conjecture "clustering worsens percolation" is not true in full generality, perhaps restricted to sub-Poisson pp.?

For more details on clustering and percolation

- BB, Yogeshwaran Directionally convex ordering of random measures, shot-noise fields ... *Adv. Appl. Probab. (2009)*
- BB, Yogeshwaran Clustering and percolation of point processes *EJP 2013*.
- BB, Yogeshwaran On comparison of clustering properties of point processes *Adv. Appl. Probab.* (2014).
- BB, Yogeshwaran Clustering comparison of point processes with applications to random geometric models *Stochastic Geometry, Spatial Statistics and Random Fields ...* (V. Schmidt, ed.) Lecture Notes in Mathematics Springer (2014).

ROUTING

Routing in wireless networks, time added

Medium Access Control (MAC)

Everybody can transmit, but not at the same time!

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The Medium Access Control (MAC) layer is a part of the data communication protocol organizing simultaneous transmissions in the network.

Aloha MAC

No central authority in ad-hoc networks. One needs a "decentralized" MAC.

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In our talk we will consider a very simple MAC scheme called Aloha.

Consider slotted time $n = 1, 2, \ldots$.

At each time slot n:
each node independently tosses a coin with some bias p.
if the outcome is heads it transmits at time n,
otherwise it does not transmit at time n and tries at time $n + 1$ (independently tossing a coin again).

Aloha = independent thinning

Slotted Aloha $\equiv$ independent thinning of the pattern of nodes (at a given time slot).

Aloha $=$ independent thinning

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Thinning is a nice operation on a p.p.
Thinning of Poisson p.p. Φ of intensity $\Lambda(\cdot)$ leads to

- Poisson p.p. Φ^1 of intensity $p\Lambda(\cdot)$ of the nodes allowed for transmissions (at a given time slot),
- Poisson p.p. Φ^0 of intensity $(1-p)\Lambda(\cdot)$ of nodes not allowed for transmission, they can serve as receivers at this time slot,
- with Φ^1 and Φ^2 being independent.

SINR and successful transmissions

A given transmission is successful if the SINR is large enough

$$\text{SINR} = \frac{\text{USEFUL SIGNAL RECEIVED POWER}}{\text{ALL OTHER SIGNALS RECEIVED POWER (and/or) NOISE}}$$

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SINR: Signal-to-Interference-and-Noise Ratio

Interference: sum of the powers of signals received from all concurrent transmissions.

Tuning Aloha Parameter p

It is important to tune the value of the Medium Access Probability (MAP) p of Aloha so as to realize a compromise between two extremal scenarios:

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Tuning Aloha Parameter p

It is important to tune the value of the Medium Access Probability (MAP) p of Aloha so as to realize a compromise between two extremal scenarios:

- $p \approx 1$: to many concurrent (interfering) transmissions highly likely failing, retransmissions needed, time is wasted.
- $p \approx 0$: too sparse authorized transmissions (which are very likely successful), but most of the nodes wast time.

Shot-noise (SN)

Given a point process $\Phi = \{X_i\}$ on $\mathbb{R}^d$ and an i.i.d. sequence $\{L_i(\cdot)\}$ of random fields on $\mathbb{R}^d$, the filed

$$I(y) := \sum_i L_i(y - X_i) \quad y \in \mathbb{R}^d$$

is called a Shot-Noise of Φ.

SN is a natural model for the interference (sum of received powers from transmitting nodes).

Laplace transform of the shot-noise

The Laplace transform $\mathcal{L}_{I(y)}(\xi) = \mathsf{E}[e^{-\xi I(y)}]$ of I is related to the Laplace functional $\mathcal{L}_{\Phi}(f) := \mathsf{E}[e^{-\int f(x)\Phi(\mathrm{d}x)}]$ of Φ.

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In particular,
Fact: If Φ is Poisson p.p. of intensity Λ on $\mathbb{R}^d$ then

$$\mathcal{L}_{I(y)}(\xi) = \exp\Big[-\int_{\mathbb{R}^d} (1 - \mathcal{L}_{L(y-x)}(\xi))\, \Lambda(\mathrm{d}x)\Big]\,,$$

where $\mathcal{L}_{L(z)}(\xi)$ is the Laplace transform of the (marginal) law of $L_i(z)$ at z.

Can be extended to joint Laplace transform of vectors $(I(y_1), \ldots, I(y_2))$.

Transmission delay calculus

Successful transmission from x to y

Consider a node located at $x \in \mathbb{R}^2$ seeking to transmit to $y \in \mathbb{R}^2$ in the presence of interfering nodes $\psi = \{y_i \in \mathbb{R}^2\}$. All nodes obey Aloha with MAP p. Assume independent exponential (Rayleigh) fading F in all channels and power law path-loss function $l(r) = (Ar)^\beta$, $\beta > 2$.

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Node x can successfully transmit message to y provided x and y are, respectively, selected and not selected by Aloha, and $\mathrm{SINR}_{(x,y)} \geq T$, where

$$\mathrm{SINR}_{(x,y)} = \frac{F_{(x,y)}/l(|x-y|)}{W + I_{\psi^0 \setminus \{x\}}(y)},$$

with $I_{\psi^0 \setminus \{x\}}(y) = \sum_{y_i \in \psi^0 \setminus \{x\}} F_{(y,X_i)}/l(|y - X_i|)$, ψ^0 nodes in ψ not selected by Aloha, $F_{(y,z)}$ fading from z to y.

Probability of successful transmission

Fact: Probability of successful transmission from x to y in one time slot is equal to

$$\Pi(x, y, \psi) = p(1-p)w(|x-y|)\prod_{z\in\psi} h(|z-y|, |x-y|)\,,$$

where

$$h(s, r) = 1 - \frac{p}{\frac{1}{T}(s/r)^{\beta} + 1} \quad s, r \geq 0,$$

$$w(s) = \exp(-TW(As)^{\beta}) \quad s \geq 0.$$

Successful transmission probability, cont'd

Proof: $\Pi(x, y, \psi) =$

$$p(1-p)\mathsf{P}\Big\{F_{(x,y)}/l(|x-y|) \geq T(W + \sum_{z\in\psi} e_z F_{(z,y)}/l(|y-z|))\Big\}$$

$$= p(1-p)e^{-TWl(|x-y|)} \prod_{z\in\psi} \mathsf{E}\big[e^{-Te_zF_{(z,y)}l(|x-y|)/l(|z-y|)}\big]$$

Successful transmission probability, cont'd

Proof: $\Pi(x, y, \psi) =$

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and

$$\mathsf{E}\Big[e^{-Te_xF_{(z,y)}l(|x-y|)/l(|z-y|)}\Big] \quad (1)$$

$$= (1-p) + p\mathsf{E}\Big[e^{-TF_{(z,y)}l(|x-y|)/l(|z-y|)}\Big] \quad (2)$$

$$= 1 - \frac{p}{\frac{1}{T}\frac{|z-y|^\beta}{|x-y|^\beta}+1}, \quad (3)$$

where we use the assumption that F is an exponential random variable.

(Local) successful transmission delay

After an unsuccessful transmission x tries to retransmit the packet to y (respecting Aloha) possibly several times, until the successful reception.

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Fact: The number of transmissions required to successfully delivered the message from x to y is a geometric random variable with mean

$$\begin{aligned} L(x,y,\psi) &= \frac{1}{\Pi(x,y,\psi)} \\ &= \frac{1}{p(1-p)} w^{-1}(|x-y|) \prod_{z\in\psi} h^{-1}(|z-y|,|x-y|)\,. \end{aligned}$$

Route delay

Consider a route $\mathcal{R} = \{x_0, x_1, \ldots, x_n\}$ of the packet sent from x_0 to x_n via successive transmissions x_k to x_{k+1}.

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Assume a pattern of external (to the route) interfering nodes $\psi = \{z_i\}$; ($\mathcal{R} \cap \psi = \emptyset$).
The route delay is the sum of the (independent, geometric) local delays on successive transmissions from x_0 to x_n .
The average route delay is

$$L(\mathcal{R}, \psi) := \sum_{k=0}^{n-1} L(x_k, x_{k+1}, \psi \cup \mathcal{R} \setminus \{x_k, x_{k+1}\}) \,.$$

The mean packet speed on the route

Denote the mean speed of packet progression on $\mathcal{R}$

$$V(\mathcal{R}, \psi) := \frac{|x_n - x_0|}{L(\mathcal{R}, \psi)}$$

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In what follows, we are mainly interested in the evaluation of V, on long routes, i.e., when $n \to \infty$ and $|x_n - x_0| \to \infty$, under various probabilistic assumptions regarding random $\mathcal{R}$ and ψ.

Fixed route, random external interferers

Probability of successful transmission

Consider $\boldsymbol{x}$, $\boldsymbol{y}$ as before and assume $\boldsymbol{\psi}$ (external interferers) is a realization of a point process $\boldsymbol{\Psi}$ on $\mathbb{R}^{\mathbf{2}}$, random but constant in time.

Probability of successful transmission

Consider x, y as before and assume ψ (external interferers) is a realization of a point process Ψ on $\mathbb{R}^2$, random but constant in time.

Fact: The probability of successful transmission is equal to

$$\begin{aligned}
&\mathsf{E}_\Psi[\Pi(x, y, \Psi)] \\
=&\mathsf{E}_\Psi\left[p(1-p)w(|x-y|)\prod_{z\in\Psi} h(|z-y|, |x-y|)\right] \\
=&\frac{1}{p(1-p)}w(|x-y|)\mathcal{L}_\Psi(-H_{x,y})
\end{aligned}$$

where $\mathcal{L}_\Psi(\cdot)$ is the Laplace transform of Ψ taken of the non-negative function $-H(z) = -H_{x,y}(z) := -\log(h(|z-y|, |x-y|))$, $z \in \mathbb{R}^2$.

Mean local delay

Similarly

Fact: The mean local delay is equal to

$$
\begin{aligned}
&\mathsf{E}_{\Psi}[L(x,y,\Psi)]\\
=&\mathsf{E}_{\Psi}\left[\frac{1}{p(1-p)}w^{-1}(|x-y|)\prod_{z\in\Psi}h^{-1}(|z-y|,|x-y|)\right]\\
=&\frac{1}{p(1-p)}w^{-1}(|x-y|)\mathcal{L}_{\Psi}(H_{x,y})
\end{aligned}
$$

where $\mathcal{L}_{\Psi}(\cdot)$ is the Laplace transform of Ψ taken of the non-positive function $H(z)=H_{x,y}(z):=\log(h(|z-y|,|x-y|))$, $z\in\mathbb{R}^2$.

Mean route delays

Similarly, for a deterministic route $\mathcal{R} = \{x_0, \ldots, x_n\}$ as before and point process of external interferers Ψ, the mean route delay is equal to

$$\mathsf{E}_\Psi[L(\mathcal{R}, \Psi)] = \frac{1}{p(1-p)} \sum_{k=0}^{n-1} w^{-1}(|x_k - x_{k+1}|) \mathcal{L}_\Psi(H_{x_k, x_{k+1}}) \times \prod_{z \in \mathcal{R} \setminus \{x_k, x_{k+1}\}} h^{-1}(|z - x_{k+1}|, |x_k - x_{k+1}|) \,.$$

with $H_{x_k, x_{k+1}} \leq 0$.

Interferers' clustering paradox

Consider two (distributions of) interferers Ψ_1, Ψ_2, which are Laplace transform ordered $\Psi_1 \leq_{LT} \Psi_2$; i.e., $\mathcal{L}_{\Psi_1}(f) \leq \mathcal{L}_{\Psi_2}(f)$ for all functions f of constant sign.

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$$\mathsf{E}[\Pi(x, y, \Psi_1)] \leq \mathsf{E}[\Pi(x, y, \Psi_2)]$$

but also

$$\mathsf{E}[L(x, y, \Psi_1)] \leq \mathsf{E}[\Pi(x, y, \Psi_2)]$$

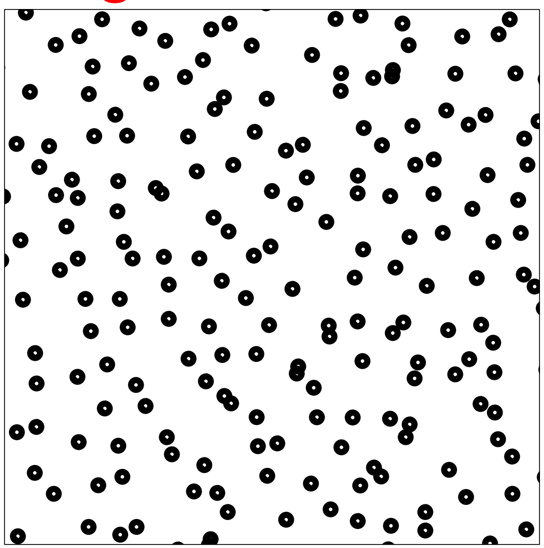
and

$$\mathsf{E}[L(\mathcal{R}, \Psi_1)] \leq \mathsf{E}[L(\mathcal{R}, \Psi_2)]$$

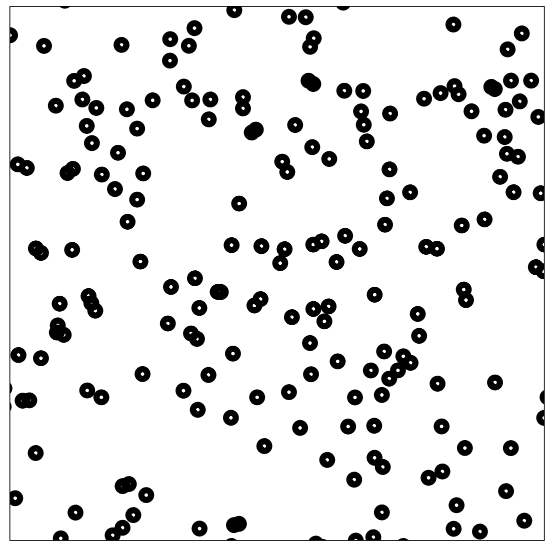
For $\boldsymbol{\Psi_1}$, successful transmission is less likely but delays are smaller also!

Clustering paradox, cont'd

Intuition: $\Psi_1 \leq_{LT} \Psi_2$ means Ψ_2 "clusters" more its points.
Clustered interfering nodes leave statistically large "void" regions but also dense clusters



$\leq_{LT}$



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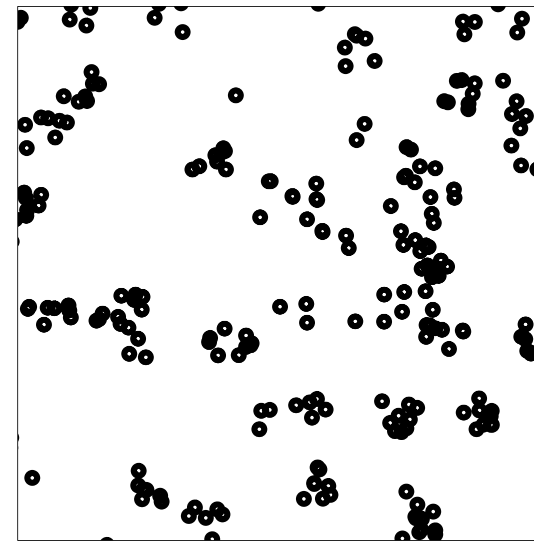

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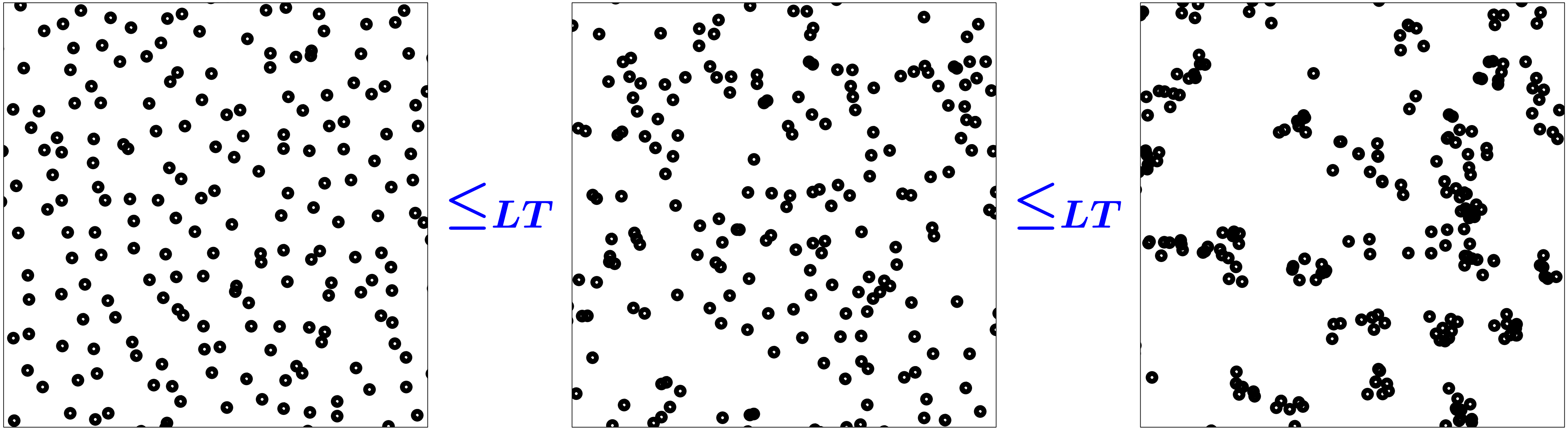

Transmissions in voids (of interferers) are more likely successful.

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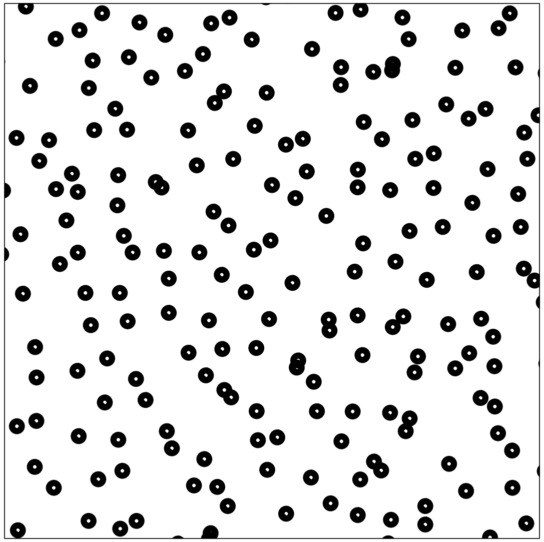

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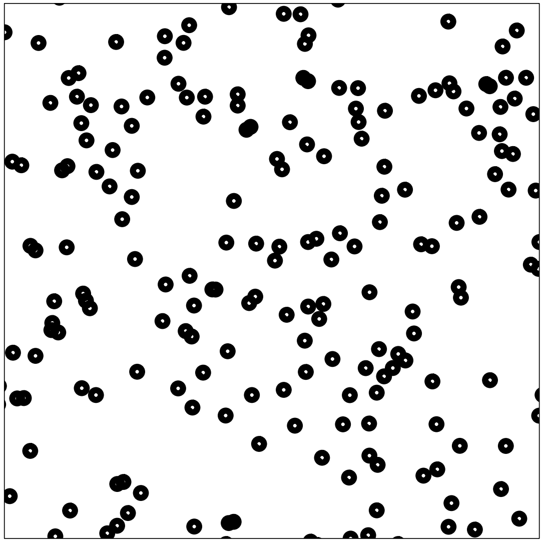

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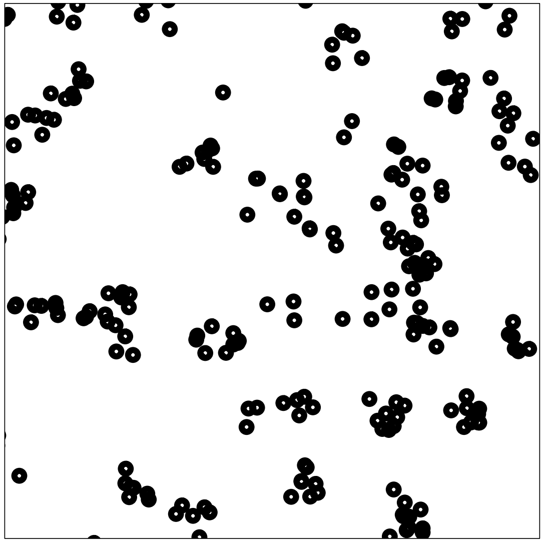
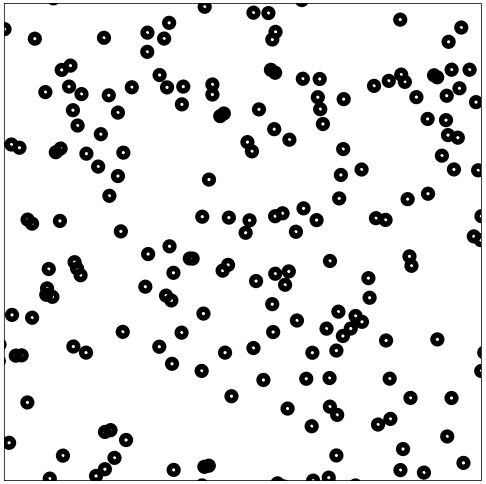


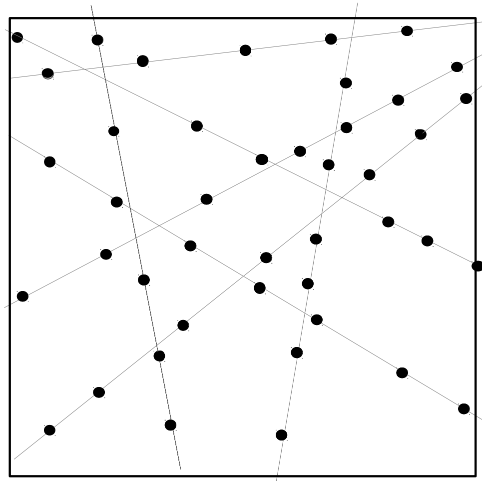
Transmissions in voids (of interferers) are more likely successful.
Transmissions in clusters of high density (of interferers) experience many failures, making the expected delay larger even if the probability of falling within a cluster is small.

Poisson $\leq_{LT}$ Poisson-line Cox

Consider Ψ_P homogeneous Poisson point process on $\mathbb{R}^2$, intensity μ and
Ψ_{PL} Cox point process with 1D Poisson processes distributed on Poisson-line process of the same mean intensity.



Poisson



Poisson-line Cox

$$\Psi_P \leq_{LT} \Psi_{PL} .$$

LT of Poisson and Poisson-line Cox

$$\mathcal{L}_P(f) = \exp\left(-2\pi\mu\int_0^\infty 1-\exp(-f(s))s\,\mathrm{d}s\right),$$

$$\mathcal{L}_{PL}(f) = \exp\left(-2\nu\int_0^\infty 1-e^{-2\lambda'\int_0^\infty 1-\exp(-f(\sqrt{s^2+t^2}))\mathrm{d}t}\mathrm{d}s\right),$$

where in for PL, ν the mean line-length per unit of surface λ' mean number of nodes per unit of line length.

If $\mu = \lambda'\nu$ then $\mathcal{L}_P(f) \leq \mathcal{L}_{PL}(f)$, all f.

Poisson-line route

Poisson line

Suppose that $\mathcal{R} = \Phi = \{X_i\}$ forms a Poisson point process of intensity λ, on the line $\mathbb{R}$. The notational convention is such that $X_i < X_{i+1}$. Note that the Poisson assumption means that the 1-hop distances $X_{i+1} - X_i$ are independent (across i) exponential random variables with some given mean $1/\lambda$.

Local Poisson-line delay

Denote

$$\mathcal{D}_1(p) = \mathcal{D}_1(p; T, \beta)$$

$$= T^{1/\beta}\Big(\int_{T^{-1/\beta}}^{\infty} \frac{1}{u^\beta + 1 - p}\,\mathrm{d}u + \int_0^{\infty} \frac{1}{u^\beta + 1 - p}\,\mathrm{d}u\Big)\,.$$

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Prop.: The mean local delay on Poisson route is equal to

$$\mathsf{E}^0[L_0] := \mathsf{E}^0[L(0, X_1, \Phi \setminus \{0, X_1\})] = \frac{1}{p(1-p)(1 - p\mathcal{D}_1(p))}$$

provided

$$p\mathcal{D}_1(p) < 1$$

and $\mathsf{E}^0[L_0] = \infty$ otherwise.

Local Poisson-line delay

Prof: From the previous expression for $L(x, y, \psi)$

$$
\begin{aligned}
&\mathsf{E}^0[L(0, X_1, \Phi \setminus \{0, X_1\})] \\
=&\frac{1}{p(1-p)}\mathsf{E}^0\Big[\prod_{i, i \neq 0,1} h^{-1}(|X_i - X_1|, |X_i - X_1|)\Big] .
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\end{aligned}
$$

The proof follows by conditioning of the Palm probability P^0 on X_1 (exponential r.v.) and using the fact that given $X_1 = r$, the point process $\Phi \setminus \{0, X_1\}$, is Poisson with intensity λ on $\cap((-\infty, 0) \cup (r, \infty))$ and 0 elsewhere and finally using the known expression for Poisson Laplace transform.

Long-distance Poisson-line speed

Denote by

$$v = \lim_{k\to\infty} \frac{|X_k - X_0|}{\sum_{i=0}^{k} L_i},$$

where $L_i := L(X_i, X_{i+1}, \Psi \setminus \{X_i, X_{i+1}\})$, the long-distance speed of the packet progression on the Poisson-line route.

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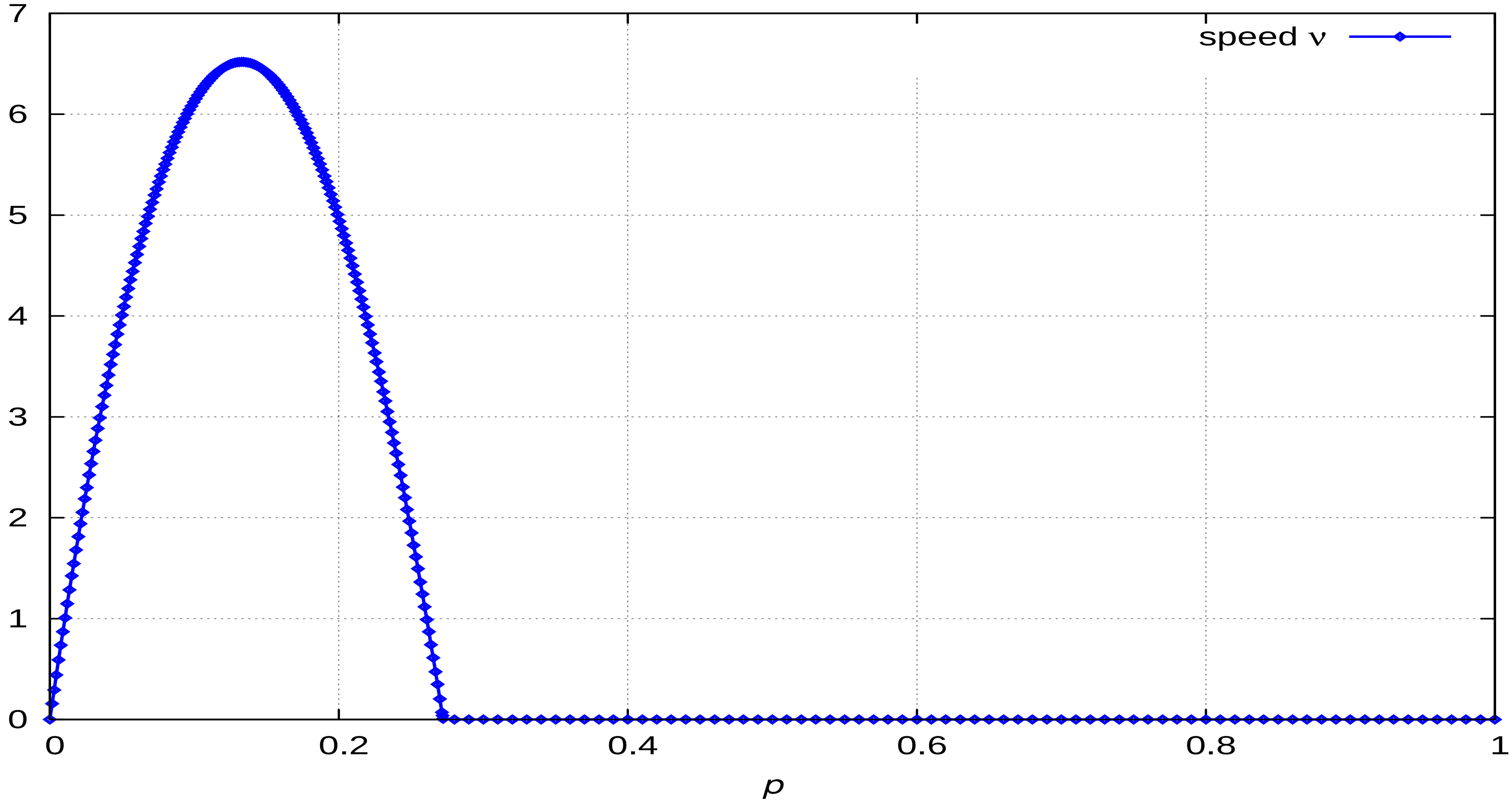
Prop.: We have

$$v = \frac{1}{\lambda \mathsf{E}^0[L_0]} = \begin{cases} \frac{p(1-p)(1-p\mathcal{D}_1(p))}{\lambda} & \text{provided } p\mathcal{D}_1(p) < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Poisson-line speed, cont'd



Poisson-line speed, cont'd

Proof: The mean empirical speed during the first k hops is:

$$\frac{|X_0 - X_k|}{\sum_{i=0}^{k-1} L_i} = \frac{(\sum_{i=0}^{k-1}(X_{i+1} - X_i))/k}{(\sum_{i=0}^{k-1} L_i)/k} .$$

Poisson-line speed, cont'd

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By the ergodic theorem for (marked ergodic) point processes, When $k \to \infty$
the numerator tends almost surely to $1/\lambda$,
the denominator tends almost surely to $\mathsf{E}^0[L_0]$.

Poisson-line route with external noise and/or 2D filed of external interferers

Noise and external interferers

Consider a noise random variable W (one realization for all time slots) and a point process Ψ on $\mathbb{R}^2$ of external interferers. We assume that W, Ψ, Φ are independent. All nodes, in Φ and Ψ obey Aloha with MAP p.

Noise and external interferers

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Prop.: The mean local delay is infinite, $\mathsf{E}^0[L_0] = \infty$, provided $\mathsf{P}(W > w) \geq \epsilon$ for some $w, \epsilon > 0$,
or Ψ is super-Poisson in negative Laplace transform order (has Laplace transforms of non-positive functions not smaller than a Poisson point process).
In this case the speed of packet progression on long routes is almost surely null.

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Consequently

$$\text{E}[L] = \int_0^\infty e^{-ax} e^{bx} \, \text{d}x = \int_0^\infty e^{-(a-b)x} \, \text{d}x \,,$$

which is infinite if $a < b$.

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Expected local delay such a distance without noise is $e^{\lambda r\, p\mathcal{D}_1(p)}$. If $p\mathcal{D}_1(p) < 1$ it is OK!
However, with noise $W = w$ this delay increases by the factor $e^{Tw(Ar)^\beta}$. Recall $\beta > 2$.
Consequently

$$\int_0^\infty e^{-\lambda r + \lambda r p\mathcal{D}_1(p) + Tw(Ar)^\beta}\, \mathrm{d}r = \infty\,.$$

when $w > 0$.

Why infinite?

External field Ψ of interferers increases the delay of the transmission on the distance r by the factor $\mathcal{L}_{\Psi}^{H}(r) := \mathcal{L}_{\Psi}(H_r(\cdot))$, with $H_r(x) = \log(h(|x|, r)) = 1 - \frac{p}{\frac{1}{T}(|x|/r)^{\beta}+1} \leq 0$.

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In case of homogeneous Poisson Ψ we have $\mathcal{L}^H_\Psi(r) \sim \exp[r^2 p/(1-p)^{1-2/\beta} \times Const]$ for large r and thus $\int_0^\infty e^{-\lambda r(1-p\mathcal{D}_1(p))} \mathcal{L}^H_\Psi(r)\,\mathrm{d}r = \infty$ for all $p > 0$.

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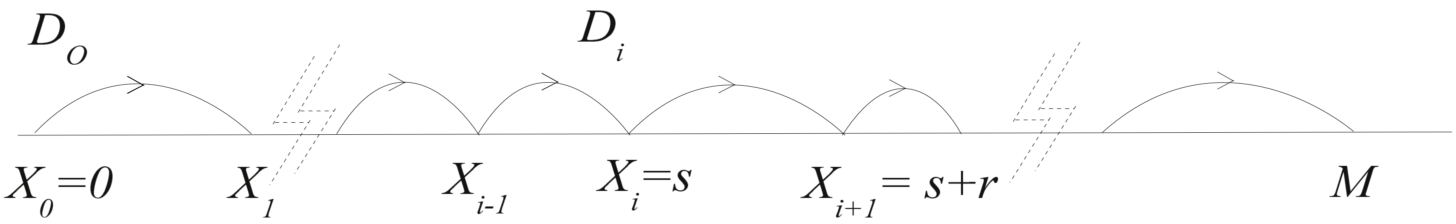
By the comparison to the Poisson case, this integral is divergent for point processes Ψ which have larger Laplace transforms of negative functions (cluster more).

End-to-end transmissions on finite routes

Under two-point palm P^{0M}



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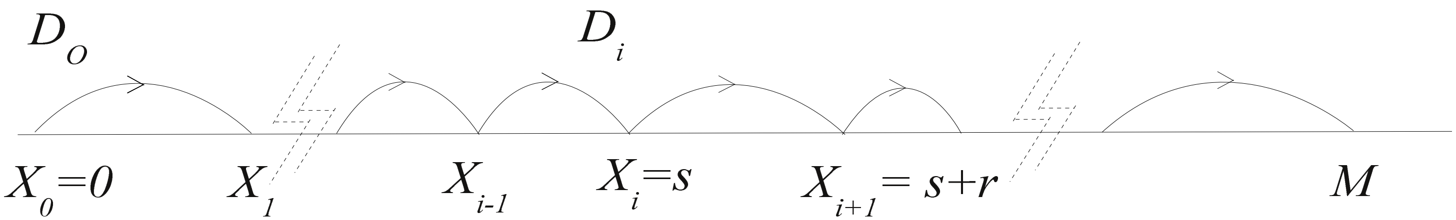


$$\mathsf{E}^{0M}[L_{0M}] = \frac{1}{p(1-p)}\Big(e^{-\lambda M}E(M) \qquad \text{(a)}$$

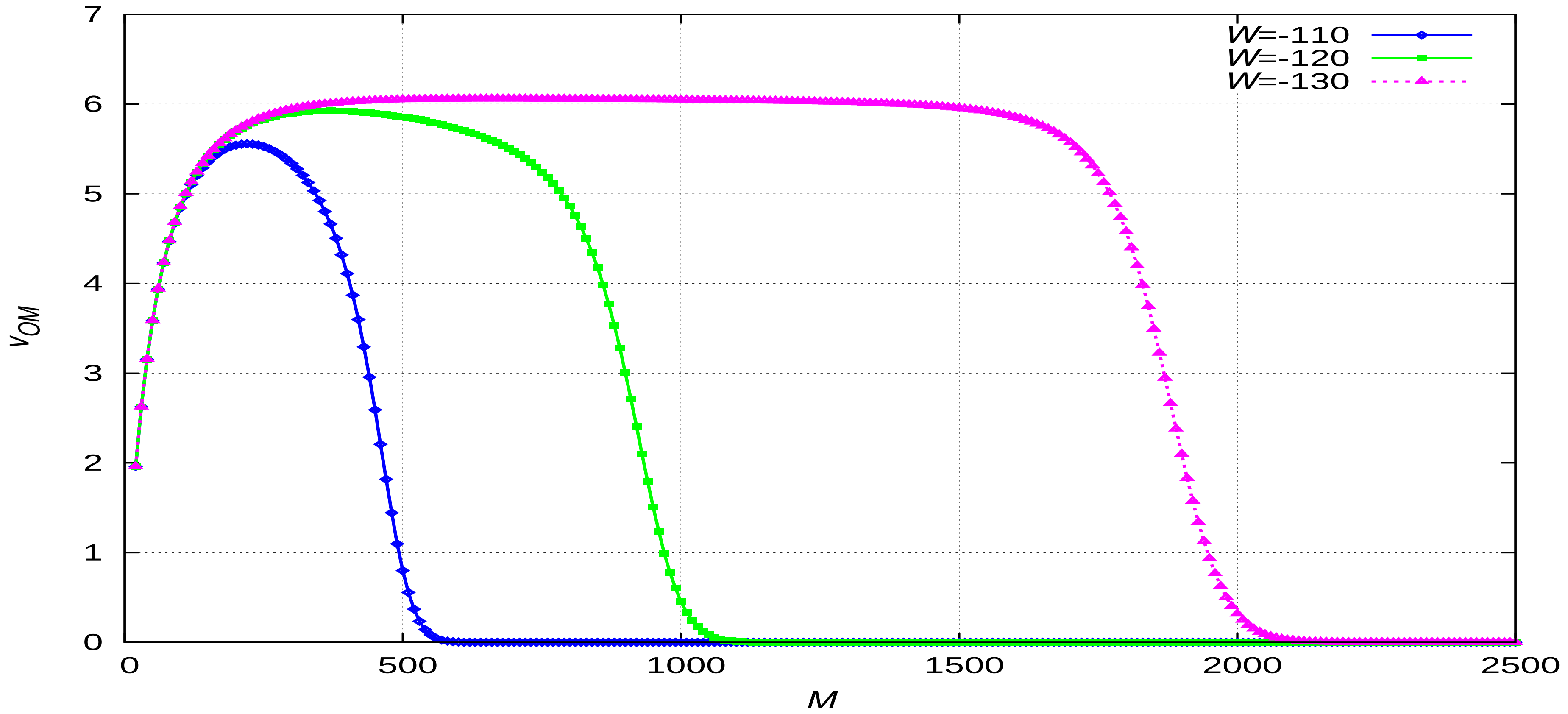
$$+\int_0^M \lambda e^{-\lambda r}E(r)G_M(0,r)\mathrm{d}r \qquad \text{(b)}$$

$$+\lambda\int_0^M\int_0^{M-s} E(r)G_0(s,r)G_M(s,r)\lambda e^{-\lambda r}\mathrm{d}r\mathrm{d}s \qquad \text{(c)}$$

$$+\lambda\int_0^M E(M-s)G_0(s,M-s)e^{-\lambda(M-s)}\mathrm{d}s\Big) \qquad \text{(d)}$$

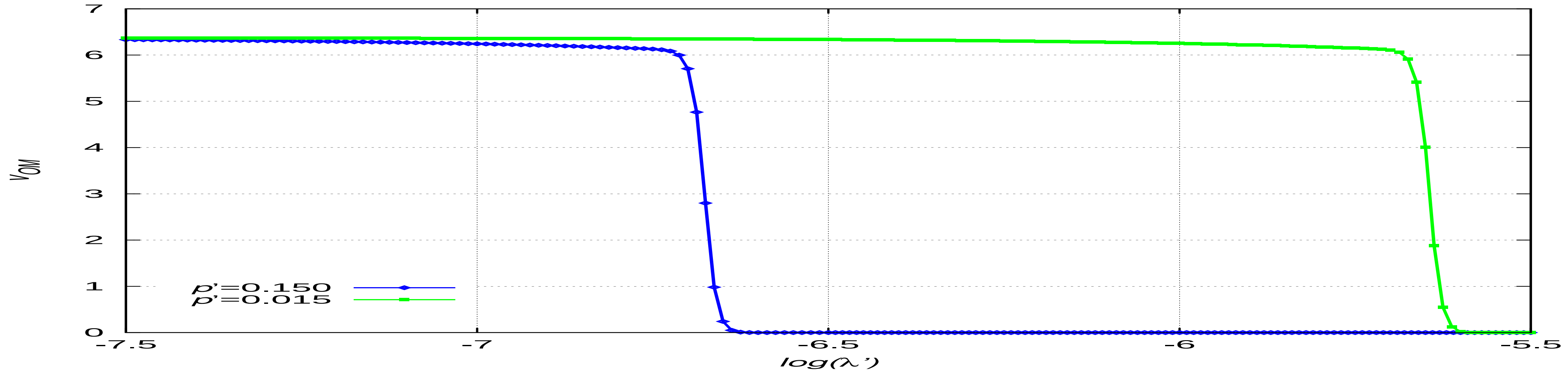
with $E(r) = e^{\lambda p r \mathcal{D}_1(p)}e^{TW(Ar)^\beta}\mathcal{L}^H_\Psi(r)$, $G_0(s,r) = h(s+r,r)^{-1}$ and $G_M(s,r) = h(M-s-r,r)^{-1}$.

Mean speed from 0 to M with noise W

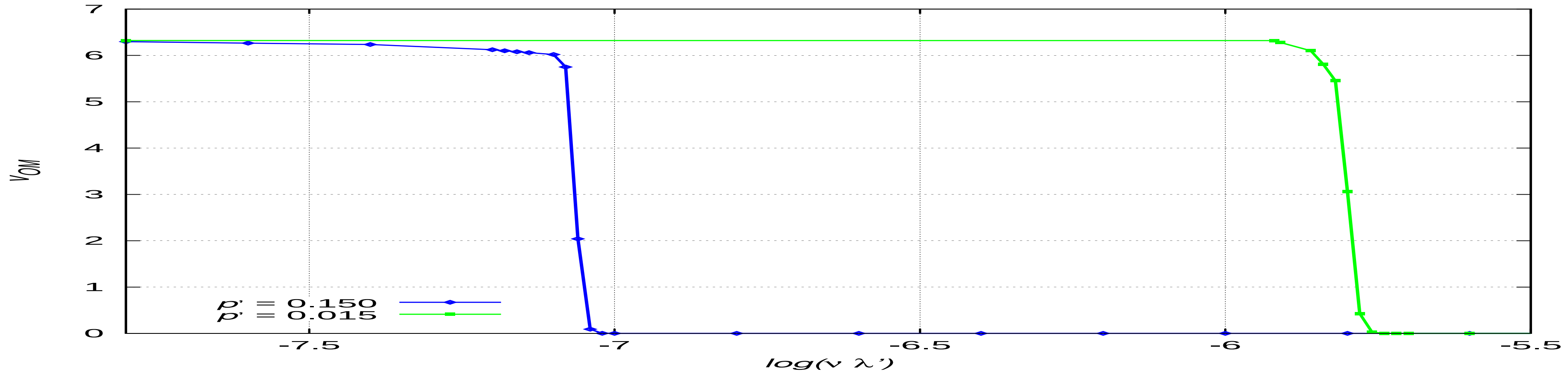


Mean speed from $\mathbf{0}$ to M with interferers Ψ

- Poisson



- Posson-line Cox



Mean speed on a given distance $\mathbf{0}M$ in function of the density of interferers $\mu = \lambda' \nu$.

Adding fixed relays

Consider regularly spaced “fixed” relay nodes $\mathcal{G} = \{n\Delta + U_\Delta, n \in \mathbb{Z}\}$, where $\Delta > 0$ and U_Δ is uniform r.v. on $[0, \Delta]$ (to make $\mathcal{G}$ stationary).

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\begin{aligned}
\mathsf{E}^0[L_0] \quad = \quad & \frac{\lambda}{p(1-p)(\epsilon+\lambda)} \Bigg(\frac{1}{\Delta} \int_0^\Delta E(z) E'(z) e^{-\lambda z} e^{H(0,z)} \mathrm{d}z \\
& + \frac{\lambda}{\Delta} \int_0^\Delta \int_0^z E(r) E'(z) e^{-\lambda r} e^{H(z-r,r) + \log(h(z-r,r))} \mathrm{d}r \mathrm{d}z \Bigg) \\
& + \frac{\epsilon}{p(1-p)(\epsilon+\lambda)} \Bigg(\frac{\lambda}{\Delta} \int_0^\Delta E(z) E'(z) e^{-\lambda z} e^{H(-z,z)} \mathrm{d}z \\
& \qquad + e^{-\lambda\Delta} B(\Delta) E(\Delta) e^{H(0,\Delta) - \log(h(\Delta,\Delta))} \Bigg)
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with $\epsilon = \frac{1}{\Delta}$ and $H(z,r) = \sum_{n \in \mathbb{Z}, n \neq 0} \log(h(n\Delta + z, r))$.

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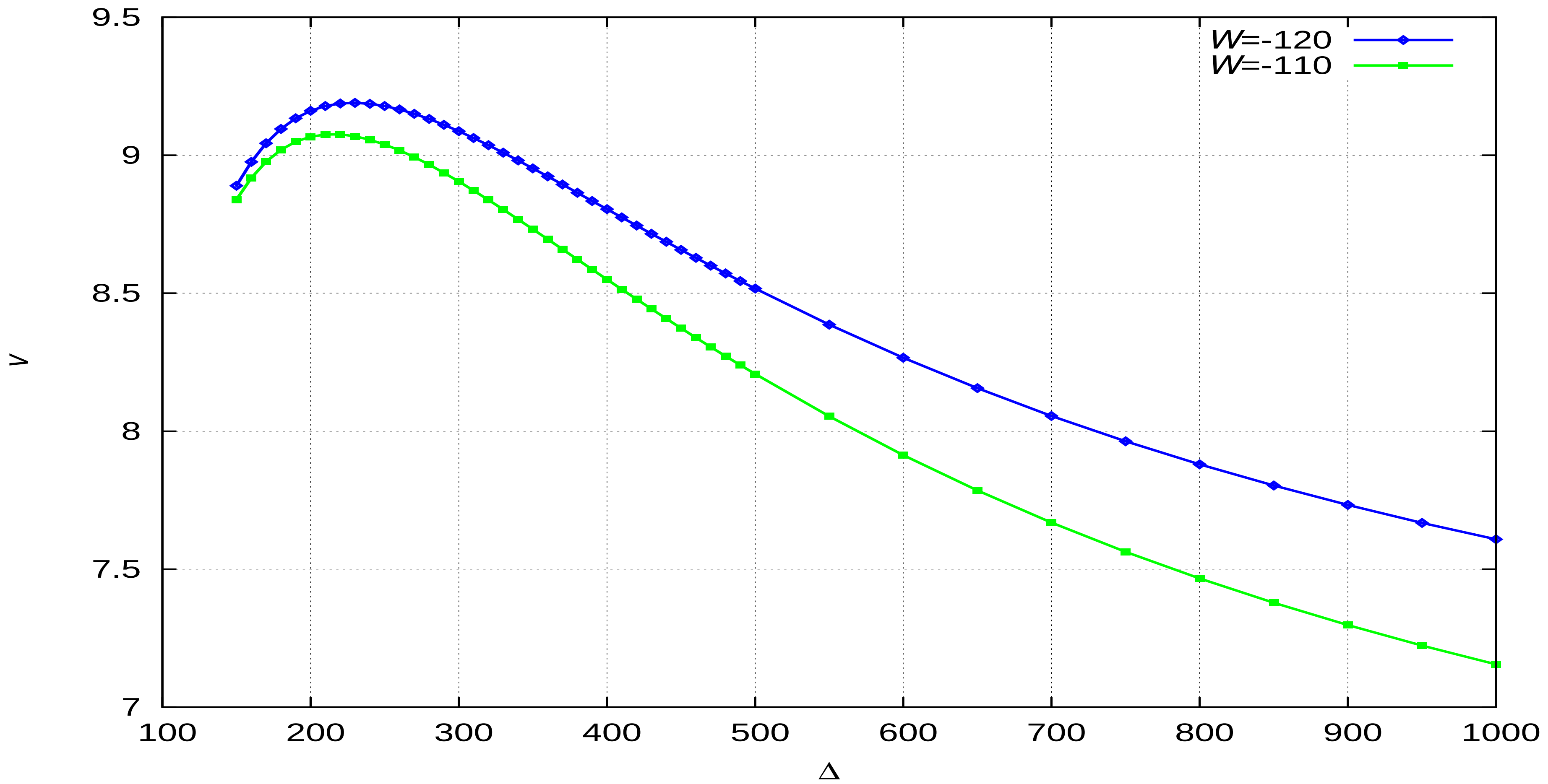
$\mathsf{E}^0[L_0] < \infty$ hence long distance speed $v = 1/\mathsf{E}^0[L_0]$ finite!

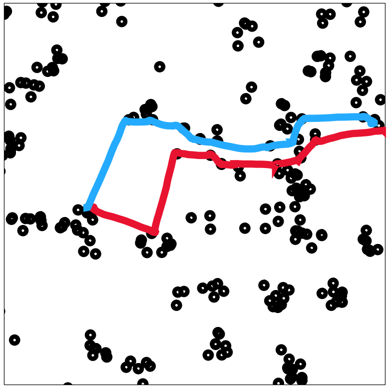
Mean speed on Poisson line with fixed relays



Note the optimal inter-relay distance **Δ**.

Routing on the plane

Problems in 2D



Which route? No natural notion of route $\mathcal{R}$ in a given direction!

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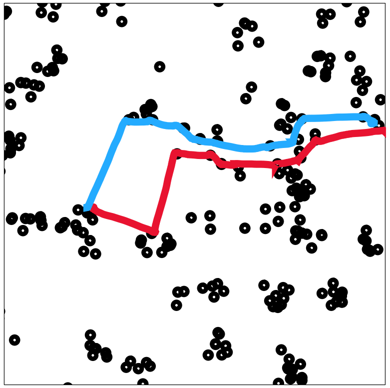

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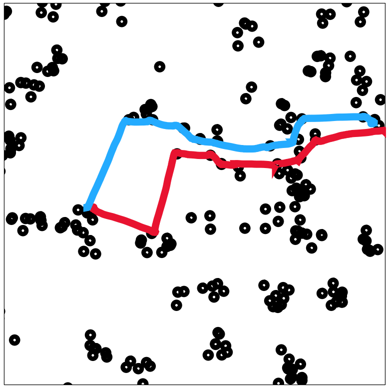

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Unless the routing algorithm is a bijectif point map. Such point maps are known to preserve palm distribution P^0. But no bujectif point map is known to be reasonable routing in a given direction!

Space-time SINR random graph

Assuming Aloha and independent exponential fading $\boldsymbol{F}$ channels as before we define a graph that allows us to "trace" in space and time all possible paths (routs) of packets send in the model on 2D point process $\boldsymbol{\Phi}$.

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Directed edges of this oriented graph connect

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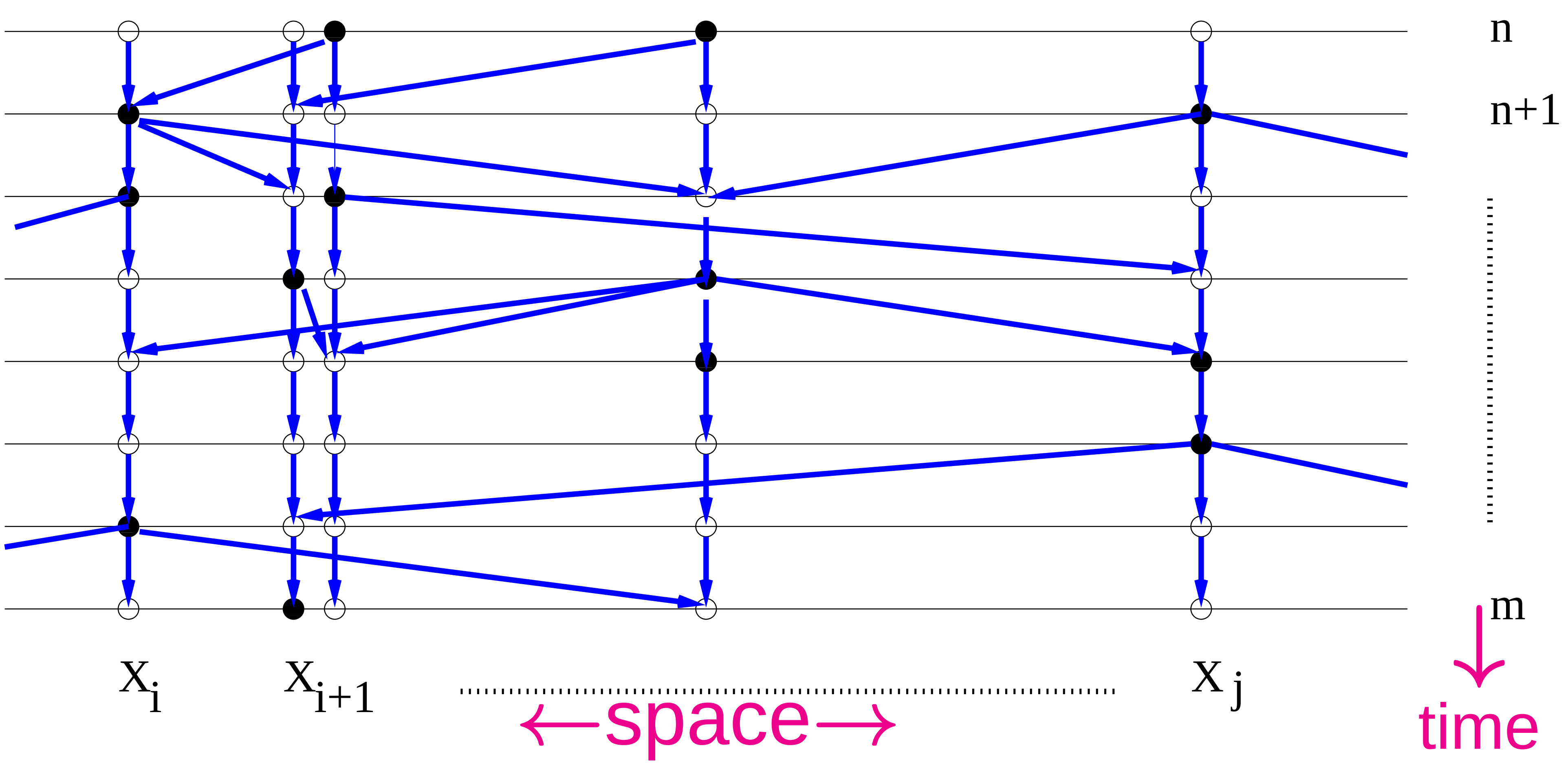
i.e. all possible moves of a tagged packet from X_i at time n.

SINR Graph $\mathcal{G}$



● emitting nodes, ○ non-emitting nodes (receives)

↘ ↙ successful packet transmissions

↓ packet stays at the given node.

First passage percolation problem

Existence and finiteness (or not) of the limit

$$\frac{\text{minimal number of hops on } \mathcal{G} \text{ from node } O \text{ to node } D}{\text{Euclidean distance } |O - D|}$$

when $|O - D| \to \infty$, called time constant.

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The number of hops on $\mathcal{G}$ in the numerator above, corresponds to the end-to-end delay (from O to D); it is the sum of the local delays at all nodes visited on the shortest-time path by some tagged packet, which does not experience any queuing at nodes before being scheduled for transmission.

Three qualitative results in 2D

1. In in Poisson network without noise ($W = 0$) $\mathsf{E}^{X,Y}[L_{X,Y}(0)] < \infty$ where $L_{X,Y}$ is the shortest (in time) possible path from X to Y and $\mathsf{E}^{X,Y}$ is two-point Palm expectation.

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   This is due to statistically too large voids in Poisson process.
3. Adding an arbitrarily sparse, stationary periodic infrastructure of nodes (superposing it with Poisson p.p.) makes end-to-end delay scale linearly with $|O - D|$ (time constant positive and finite).

The reason for the negative result

The mean local exit time from the typical node (expected time to leave the typical node) is infinite.

Proof of the positive result with “extra relays”

Denote by $p(x, y, \Phi)$ the expected minimal time (number of time-slots) to go from $x \in \mathbb{R}^2$ to $y \in \mathbb{R}^2$ using any path on the SINR space-time graph on Φ.

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One has to work (using explicit expressions for the mean one-hop delay in exponential fading case) to prove that this limit is positive and finite.

Yet another positive result

Achieving non-zero information propagation speed on 2D Poisson process.

Individual Aloha and power control: in case of a long hop node X_i makes less transmission attempts (decreases p_i) but increases the transmission power P_i keeping

$$p_i \times P_i = const.$$

[Iyer, Vaze 2015+]

Concluding on routing

- He have studied the performance of simple routing algorithms on long routes in random environment.
- Existence of statistically too large voids in 2D Poisson process causes zero information propagation speed on the time-space Aloha-SINR model with fixed Aloha parameter p.
- Individual choice of p by the nodes and power control allows one to achieve a positive speed.
- How about some less clustering processes (with smaller void probabilities, e.g. determinantal)?

For more details on routing

- Baccelli, F., B.B. and Mirsadeghi, O. (2011). Optimal paths on the space-time SINR random graph. *Adv. Appl. Probab.*
- B.B. and Muhlethaler, P. (2015). Random linear multihop relaying in a general field of interferers using spatial Aloha. *IEEE Trans. Wireless Commun.*

Thank you for today.
Tomorrow: Capacity