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Complete list of publications — 2007

Contributing to the habilitation dissertation

[1] Risk and duality in multidimensions. *Stoch. Proc. Appl.* **83** (1999), 331–356; with Sigman, K.

[2] Bounds for clump size characteristics in the Boolean model. *Adv. in Appl. Probab. (SGSA)* **31** (1999), 910–928; with Rau, C. and Schmidt, V.

[3] On a coverage process ranging from the Boolean model to the Poisson Voronoi tessellation, with applications to wireless communications. *Adv. in Appl. Probab. (SGSA)* **33** (2001), 293–323; with Baccelli, F.

[4] Approximate decomposition of some modulated-Poisson Voronoi tessellations, *Adv. Appl. Probab. (SGSA)* **35** (2003), 847-862; with Schott, R.

[5] Performance characteristics of multicast flows on random trees, *Stochastic Models* **20**, 341-361 (2004); with Tchoumatchenko, K.

[6] Approximations of functionals of some modulated-Poisson Voronoi tessellations with applications to modeling of communication networks, *Japan Journal of Industrial and Applied Mathematics* (Special Issue on Voronoi diagrams in Science and Engineering), **22**(2) 179–204 (2005); with Schott, R.

[7] An Aloha protocol for multihop mobile wireless networks, *IEEE Transactions on Information Theory* **52**(2), pp. 421–436, (2006); with Baccelli, F. and Muhlethaler, P.

Before PhD

[8] Queues in series in light traffic. *Ann. Appl. Probab.* **3** (1993), 881–896; with Rolski, T.

[9] Factorial-moment expansion for stochastic systems. *Stoch. Proc. Appl.* **56** (1995) 321–335.

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Other original journal papers published after PhD

[13] A note on expansions for functional of spatial marked processes. *Statist. and Probab. Lett.* **36** (1997), 299–306; with Merzbach, E. and Schmidt, V.

[14] Spatial averages of coverage characteristics in large CDMA networks. *ACM Wireless Networks* **8** (2002), 569-586; with Baccelli, F. and Tournois F.

[15] Up and downlink admission / congestion control and maximal load in large homogeneous CDMA networks. *Mobile Networks and Applications (MONET), Special Issue on Optimization of Wireless and Mobile Networks* **9**(6), 605–617 (2004); with Baccelli, F. and Karray, M.

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[16] Spatial averages of downlink coverage characteristics in CDMA network. In *Proc. of IEEE INFOCOM* (2002), 381-390, with Baccelli, F. and Tournois, F.

[17] Downlink admission / congestion control and maximal load in CDMA networks. In *Proc. of IEEE INFOCOM* (2003); with Baccelli F. and Tournois F.,

[18] A spatial reuse Aloha MAC protocol for multihop wireless mobile networks. In *40 th Annual Allerton Conference on Communication, Control, and Computing* (2003); with Baccelli F. and Muhlethaler P.

[19] An Aloha protocol for multihop mobile wireless networks. In *Proc. of 16th ITC Specialist Seminar on Performance Evaluation of Wireless and Mobile Systems* , Antwerp, Belgium (2004); with Baccelli F. and Muhlethaler P.

[20] Blocking rates in large CDMA networks via a spatial Erlang formula. In *Proc. of IEEE INFOCOM* (2005); with Baccelli, F. and Karray, M.

[21] Performance evaluation of scalable congestion control schemes for elastic traffic in cellular networks with power control, in *Proc. of IEEE INFOCOM* (2007); with Karray, M.

[22] M/D/1/1 loss system with interference and applications to transmit-only sensor networks. In *Proc. of SPASWIN 2007* (assciated to *WIOPT*), with Radunovic, B.

[23] Using transmit-only sensors to reduce deployment cost of wireless sensor networks. To appear in *Proc. of IEEE INFOCOM* (2008); with Radunovic, B.

Book

[24] *Podstawy matematyki ubezpieczeń na życie*, WNT, (2004) 391 pages; with Rolski, T.

In preparation

[25] Blocking and cut probabilities of streaming traffic in large cellular networks — approximations accounting for the mean user mobility speed. Submitted; with Karray, M.

[26] On the performance of time-space opportunistic routing in multihop mobile ad hoc networks. In preparation; with Baccelli F. and Muhlethaler P.

[27] Directionally convex ordering of random measures, shot-noise fields and some applications to wireless networks. In preparation; with Yogeshwaran D.

[28] *Spatial Modeling of Wireless Communications, a Stochastic Geometry Approach*, monograph for *Foundations and Trends in Networking*, NOW Publishers; ca 200 pages. In preparation; with Baccelli, F.

Patents

[29] 3 patents concerning admission/congestion control in large CDMA networks (pending or issued in Europe, extended to USA).

Presentation of the habilitation dissertation (autoreferat)

Stochastic Geometry Methods and their Applications in Queueing and Telecommunications

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1 Introduction

This dissertation shows some *examples of the development of stochastic geometry tools in relation with queueing theory and modeling of communication networks.*

1.1 General context

Stochastic geometry is now a reach branch of applied probability, which allows to study random phenomena on the plane or in higher dimension. It is intrinsically related to the theory of point processes. Initially its development was stimulated by applications to biology, astronomy and material sciences. Nowadays, it is also used in image analysis and in the context of communication networks. It this latter case its role is similar to this played by the theory of point processes on the real line in the classical queueing theory.

At first glance, the usage of stochastic geometry for modeling of communication networks is a relatively new idea. In fact, we would like to stress the pioneering role of E. Gilbert in this domain. One can consider Gilbert's paper of 1961 both as the first paper on continuum percolation (percolation of the Boolean model) and as the first paper on the analysis of the connectivity of large wireless networks by means of stochastic geometry. Similar observations can be made on Gilbert's seminal paper of 1962 on Poisson-Voronoi tessellations.

The first papers following Gilbert's ideas appeared in the modern engineering literature shortly before year 2000; i.e., before the massive popularization of wireless communications. They were using mainly the classical stochastic geometry models (as Voronoi tessellations or Boolean model) in this new context; see e.g. (Baccelli et al. 1997; Baccelli and Zuyev 1997; Baccelli and Zuyev 1999). Nowadays, the number of papers using some form of stochastic geometry is increasing very fast in engineering journals and conferences, where one of the most important observed trends is an attempt to better take into account in geometric models specific mechanisms of *wireless* communications.

Wireless networks consist of nodes distributed on the plane (or in 3D space) and communicating by sharing a common radio medium. Key elements of the analysis of these networks are: *power of the signal received* at different locations on the plane (space) from some particular emitting node (e.g. the closest one to this location, with the strongest received signal, the most remote within some range) and the *total power received* from the whole collection of nodes. It is so because, as both information theory and signal-detection theory teach us, the *ratio of these received powers*, called *signal to interference ratio (SIR)* characterizes the throughput of the radio channel form a given emitter to a given location. Since the power of an emitted signal is attenuated by the distance between the emitter and the receiver, the geometry of the location of nodes plays a key role in determining the SIR's. Stochastic geometry, besides other useful models, offers excellent tools to handle various received powers from random configuration of nodes. These tools are called *shot noise (SN)* field and *extremal SN* field.

More generally, stochastic geometry provides a natural way of defining and computing macroscopic properties of communication networks, by some averaging over all potential geometrical patterns of the nodes, in the same way as queuing theory provides averaged response times or congestion over all potential arrival patterns within a given parametric class. When the underlying random model is spatially ergodic, this probabilistic analysis also provides a way of estimating spatial averages which often capture the key dependencies of the network performance characteristics (connectivity, stability, capacity, etc.) in function of a relatively small number of parameters. stochastic geometry modeling of communication networks seems particularly relevant for large scale network performance analysis. This is a very natural approach e.g. for ad hoc networks, or more generally to describe user positions, when these are best described by random processes. But it can also be applied to represent both irregular and regular network architectures as observed in cellular wireless networks.

1.2 Objectives

An objective of this dissertation is to show some examples of works, which develop and apply stochastic geometry tools in queueing and network theory. We begin with an application to queueing theory which allows to express the stationary distribution of a given multidimensional content process by the ruin probabilities of some corresponding dual risk process. Then, we continue with the study of some problems related to the Boolean model and the Voronoi tessellations in the context of their applications to, respectively, broadcast and access networks. Finally, we present two articles, which give foundations for a stochastic geometry framework for the modeling of wireless communication networks.

1.3 Composition of the dissertation

The dissertation is composed of seven articles, which can be classified in the following four groups.

1. The first one consist of only one article, [1], that utilizes the Choquet's capacity functional — a basic tool of the theory of random closes sets — to construct a Markov process on the space of closed sets, which is in some particular relation to a given Markov process. The relation, called in the queueing theory *content-risk duality*, allows to express the stationary distribution of the content process by the ruin probabilities of the risk process.

The six remaining articles deal with two classical stochastic geometry models: the *Boolean* one ([2], [5]), the *Voronoi tessellation* ([4], [6]) and propose a *new coverage model* that is inspired by wireless communications ([3],[7]). Each subject is treated by two articles. The first of them presents results constituting a direct contribution to the stochastic geometry, and is followed by a more application oriented one, where these results or developed techniques are exploited in some communication network context. More precisely:

2. Article [2] gives bounds for the size of some aggregates of sets (called clumps) in the *Boolean model*. A coupling of the Boolean clump with some Galton-Watson branching process, which is the main technique used in this paper, is also applied in [5] to evaluate performance of some *multicast flows* on random trees spanning more general aggregates of points in Poisson point process.

3. The *Voronoi tessellation* generated by a non-homogeneous Poisson or stationary double-stochastic Poisson (Cox) point process is considered in [4]. It is shown how the distribution of the cell generated by a given point and the typical cell (in the stationary Cox case) can be approximated by means of the distribution of the typical cell of some stationary Poisson point processes. These results are extended in [6] to (possibly) unbounded characteristics of the cell and discussed in the context of their applications to modeling of non-homogeneous communication networks.

4. Finally, in [3], a new stochastic geometry *coverage model* (or *germ-grain model*) with *dependent grains* was proposed and studied. Motivated by the interference effect, observed in wireless communications, we let the grains (cells) of this model depend on the shot-noise field created by the

point process of germs (denoting locations of antennas) and some additional marks (playing the role of emitted powers), via a response function (interpreted as the power attenuation function). A variant of this model is latter used in [7] to address some important question of *ad hoc* networks.

Each of the above four subjects was studied by the author with different collaborators.

1.4 Main results

Besides of the theoretical results presented in [1], extending the stationary recursive and Siegmund's dualities to higher dimension, the main *results contributing to the stochastic geometry* are:

- The necessary and sufficient conditions on the existence of higher moments and exponential moments of the clump size in the Boolean model with grains having unbounded support, given in [2]. This provides some complements to the result on the finiteness of the first moment presented in (Hall 1985).
- Development of an approximation technique which allows to treat distributions of cells of some non-homogeneous Poisson Voronoi tessellations, done in [4]. It is, to the best of our knowledge, the first study of this tessellation in non-stationary Poisson case.
- The stochastic geometry model proposed in [3] is, to the best of our knowledge, novel. Important theoretical results concern its well-definiteness and coverage properties.

The main *contributions to modeling and analysis of communication networks* are:

- A stochastic geometry model of flows on multicast trees and a study of its characteristics, done in [5].
- A class of analytically tractable non-homogeneous Poisson Voronoi tessellations considered in [4] and [6] might be is used in macroscopic models of access networks.
- The stochastic geometry model of [3] can be straightforwardly used e.g. to study the coverage of cellular networks (e.g. CDMA networks, as shown in [14]). Moreover, in [7], we develop its variant catching important aspects of ad hoc networks. These two articles give foundations for a comprehensive stochastic geometry framework for the modeling of wireless communication networks. A special feature of this framework is the usage of the shot noise in conjunction with other classical stochastic geometry models to study the geometry of SIR's.

The remaining part of this document is organized as follows. In Sections 2–5 we briefly review the articles contributing the the subjects 1–4 described in Section 1.3 above. In Section 6 we briefly comment on the impact that they have on other works of the author and more generally on the domain. The list of all other then author's own publications, cited in this document by "(Author(s) Year)", is given at the end.

2 Content–risk duality in multidimensions

One-dimensional case, the state of art. In one-dimensional Euclidean space a nice duality theory has been developed between "content" like processes (such as storage or queues) and insurance "risk" like processes. The main result is that the probability that the steady-state content exceeds level x equals the probability that a dual risk process, starting off at level $x \ge 0$ (units of money), is eventually ruined (hits level 0). Specifically, given a real-valued stochastic process $\{V_n : n \ge 0\}$ with steady-state given by $\mathbf{P}(V \ge x) = \lim_{n\to\infty} \mathbf{P}(V_n \ge x)$, there exists a real-valued process $\{R_n : n \ge 0\}$ with "ruin" time

$$\tau(x) = \begin{cases} \min\{n : R_n = 0 | R_0 = x\}, & \\ \infty & \text{if } R_n > 0 \text{ for all } n \ge 0, \end{cases} \tag{2.1}$$

such that

$$\mathbf{P}(V \geq x) = \mathbf{P}(\tau(x) < \infty) \tag{2.2}$$

provided a certain stochastic monotonicity condition holds. In effect, a steady-state probability can be replaced by a first-passage time probability.

The two main general approaches to duality have been the classic Markovian approach of Siegmund (Siegmund 1976) ("Siegmund's duality") and the more recent stationary recursive approach of Asmussen and Sigman (Asmussen and Sigman 1996). The application of such duality is severely limited due to the one-dimensional framework. For example, it can not be applied to c-dimensional queueing processes such as the classic Kiefer-Wolfowitz (see (Kiefer and Wolfowitz 1955)) workload vector for $G/G/c$ queues.

Our results. In [1] we generalize, in discrete time, both stationary recursive and Siegmund's approach to the case when the content process has values in a general state space.

Approach. The main idea is to allow a risk process to be set-valued, and to define ruin as the first time that the risk process becomes the whole space. The risk process can also become infinitely rich, which means that it eventually takes on the empty set as its value. [1]

In what follows we will briefly review the results of Section 3 of [1] that deals with Siegmund's duality, because in this case the used tools of stochastic geometry are more involved. In particular the Choquet's capacity is used to define an appropriate probability measure on the space of *random closed sets*. The Choquet's capacity is a fundamental concept of the probability theory on this space and plays a similar role as the (cumulative) probability distribution function for random variables on $\mathbb{R}$.

2.1 Siegmund's duality on a general state space via Choquet's capacity

The crucial observation of this approach, in the case of a non-negative real-valued Markov process $\{V_n : n \geq 0\}$ with $P_x(V_n \in \cdot)$ denoting $\mathbf{P}(V_n \in \cdot | V_0 = x)$, is that

$$P_y(R_1 \leq x) \stackrel{\text{def}}{=} P_x(V_1 \geq y) \tag{2.3}$$

defines a "dual" Markov process $\{R_n : n \geq 0\}$ on $[0, \infty]$, with $P_y(R_n \in \cdot)$ denoting $\mathbf{P}(R_n \in \cdot | R_0 = y)$, such that

$$P_y(R_n \leq x) = P_x(V_n \geq y), \quad \text{for } all \ \ n \geq 1$$

if and only if $P_x(V_1 \geq y)$ is a right-continuous, nondecreasing function of x for each fixed $y \in \mathbb{R}$. The point here is that only under such restrictions does $P_{(\cdot)}(V_1 \geq \cdot)$ define a Markov transition kernel (M-t-k) on $\mathbb{R}$. The dual process has both 0 and ∞ as absorbing states.

A naive way of trying to extend this approach to $\mathbb{R}^d$-valued Markov processes, is to try again to use (2.3) as the starting point. But the rather mild monotonicity and right-continuity conditions, sufficient in one dimension, are no longer sufficient in higher dimensions to ensure that this indeed defines a M-t-k. Recall that the necessary condition for a function to be a multidimensional distribution function is having *positive increments* and it seems unreasonable to assume such conditions here (for this would seriously limit the applicability of our theory).

Instead, inspired by stochastic geometry, under suitable mild conditions, by specifying an appropriate M-t-k we can define a set-valued "dual" Markov process $\{\overline{V}_n : n \geq 0\}$ that satisfies

$$P_D(\overline{V}_n \ni x) = P_x(V_n \in D), \ n \geq 1, \tag{2.4}$$

[1] One might find it a bit awkward that in our set-valued risk processes, "ruin" corresponds to the event that the risk process eventually takes its value as the whole space and "infinitely rich" to the event that the empty set $\emptyset$ is hit. But one can simply re-define the risk process to be its set complement, in which case ruin will correspond to hitting the empty set and infinitely rich to the whole space. We chose things as we did for mathematical convenience, specifically to work on the space of closed sets.

where $P_D(\overline{V}_n \in \cdot) = \mathbf{P}(\overline{V}_n \in \cdot | \overline{V}_0 = D)$.

Assume for the rest of this section that $\{V_n : n \geq 0\}$ is a given Markov process, with M-t-k $K(x, D) = P_x(V_1 \in D)$, on a locally compact, Hausdorff and separable space $\mathbb{E}$ with Borel σ-algebra $\mathcal{B}(\mathbb{E})$. Our objective is to find a measurable space, being a family of subsets of $\mathbb{E}$, and a M-t-k $\overline{K}$ on it so that the "dual" Markov process $\{\overline{V}_n : n \geq 0\}$ corresponding to $\overline{K}$ satisfies (2.4) for all subsets D of this family and $x \in \mathbb{E}$.

There are (at least) two main families of subsets of $\mathbb{E}$ with their standard σ-algebras considered in stochastic geometry: the family $\mathcal{P}(\mathbb{E})$ of *all* subsets of $\mathbb{E}$ and the family $\mathcal{F}$ of *closed* subset of $\mathbb{E}$, with their appropriate σ-algebras denoted respectively by $\mathbb{M}$ and σ_f (for technical details we refer to Matheron (Matheron 1975).)

Note that (2.4) itself is not enough to define uniquely a M-t-k: for example it does not tell us how to compute $P_D(\{\overline{V}_1 \ni x_1\} \cup \{\overline{V}_1 \ni x_2\})$ for distinct elements x_1, x_2.

Note however, that any such M-t-k would have to satisfy the inequalities

$$\max_{x \in I} P_x(V_1 \in D) \leq P_D\Big(\bigcup_{x \in I} \{\overline{V}_1 \ni x\}\Big) \leq \sum_{x \in I} P_x(V_1 \in D) \tag{2.5}$$

for any finite set I of distinct elements $x \in \mathbb{E}$. The upper bound above might exceed 1, so it can't in general define a probability measure, but the lower *max* bound can be used to define for each fixed $D \in \mathcal{B}(\mathbb{E})$ the so called *space law* and consequently the unique probability measure $\overline{K}(D, \cdot)$ on the space $(\mathcal{P}(\mathbb{E}), \mathbf{M})$ of all subset of $\mathbb{E}$. This is proved in Proposition 3.1 in [1].

Unfortunately, $\overline{K}(D, \cdot)$ is well defined for $D \in \mathcal{B}(\mathbb{E})$ only (not all $D \subseteq \mathbb{E}$), and thus one cannot construct from it a bonafide M-t-k on $\mathcal{P}(\mathbb{E})$ using Chapman-Kolmogorov equations. What we have is, for fixed $D \in \mathcal{B}(\mathbb{E})$, a random set $\overline{V}_1$ satisfying (2.4) for $n = 1$. But this random set might take its value not in $\mathcal{B}(\mathbb{E})$ ruling out our construction to a next transition $\overline{V}_2$. Moreover, in general $\mathcal{B}(\mathbb{E}) \notin \mathbf{M}$.

Thus we have to refine the our analysis taking another space that serves our purpose and involves the topology of $\mathbb{E}$. It happens that the space $\mathcal{F}$ of closed subsets of $\mathbb{E}$ is a right choice.

More precisely, under some additional condition that we call *upper semi-continuity* of the M-t-k K ((USC) in [1]) the same lower bound in (2.5) can be used to define for each closed $D \in \mathcal{F}$ a *Choquet's capacity* and consequently the unique probability measure $\overline{K}(D, \cdot)$ on the space $(\mathcal{F}, \sigma_f)$ of closed subsets of $\mathbb{E}$. This is proved in Proposition 3.3 in [1].

The main objective of Section 3 of paper [1]; i.e., a Siegmund's duality for Markov process in a general state space is stated in Proposition 3.4. Namely, the existence of the dual Markov process $\{\overline{V}_n : n \geq 0\}$ with M-t-k $\overline{K}$ satisfying (2.4) for all $x \in \mathbb{E}$, $D \in \mathcal{F}$ and $n \geq 1$. This result follows from the Chapman-Kolmogorov equations.

The dual Markov process $\{\overline{V}_n : n \geq 0\}$ has interesting monotonicity properties. Namely, with no further conditions, it is stochastically monotone with inclusion being the partial order on $\mathcal{F}$ (Proposition 3.7 in [1]. Moreover, if some stochastic monotonicity is assumed for K (conditions (D1)–(D2) in [1]) then the steady-state probabilities of the Markov process given by M-t-k K can be replaced by first-passage time probabilities for the dual set-valued Markov process given by $\overline{K}$, as in the classical one-dimensional relation (2.2). This is the subject of Proposition 3.8 in [1].

3 Bounds for characteristics of trees generated by aggregates of points in Poisson point process, with applications to broadcast networks

General model. Trees that we consider in [5] have vertex sets embedded in a homogeneous Poisson point process $\Phi = \{X_i\}_i$ in $\mathbb{R}^d$ with intensity λ. Assume that the points of Φ have some independent identically distributed (i.i.d.) marks $R_i = R(X_i)$, which take values in $\mathbb{E} = \{1, 2, \ldots, \ell\}$. The space $\mathbb{E}$

may be finite (as in [5]) or infinite ($\ell = \infty$, as in [2]). The distribution of the generic mark R is given by the set of probabilities

$$q_k = \mathbf{P}(R = k), \quad k \in \mathbb{E}\,. \tag{3.1}$$

We will denoted the above independently marked pint process by $\tilde{\Phi} = \{(X_i, R_i)\}$.

Let us define bonds between points of different marks using a collection of closed bounded sets $\{G_{kl} \subset \mathbb{R}^d :\ k, l \in \mathbb{E}\}$. Denote by $X_i \rightsquigarrow X_j$ the relation $X_j \in X_i + G_{R(X_i)R(X_j)}$. Put $A_0(X_i) = \{X_i\}$ and define by induction the set of n-accessible points $A_n(X_i)$ as all $X_j \in \Phi$, such that, for some $X_m \in A_{n-1}(X_i)$, $X_m \rightsquigarrow X_j$ and $X_j \notin A^{(n-1)}(X_i) \equiv \bigcup_{k=0}^{n-1} A_k(X_i)$. The set of all points accessible from X_i is then given by

$$A(X_i) = \bigcup_n A_n(X_i)$$

and can be seen as a mark of X_i. We will call the set $A(X_i)$ the *aggregate* associated with X_i. In [2] and [5] we are interested in the properties of a *typical* aggregate; i.e., having the Palm distribution (as the mark of the underlying Poisson point process $\tilde{\Phi}$). By Slivnyak's theorem (see, e.g., (Stoyan, Kendall, and Mecke 1995), § II.4.4, pages 121–123), its distribution coincides with the distribution of the aggregate $A(0)$ constructed with respect to the process $\Phi \cup \{0\}$ with an independent $R(0)$ having the common mark distribution. Considering the Palm distribution, we will write simply A and A_n to refer, respectively, to the typical aggregate and to the set of n-accessible points with respect to the origin.

With the typical aggregate A we associate an oriented tree $\mathcal{T}_C = (A, E_C)$ (the subscript stands for "clump") rooted at the origin and having the edge set

$$E_C = \{(v(X_i), X_i),\ X_i \in A \setminus \{0\}\},$$

where for every X_i such that $X_i \in A_n$, $n \geq 1$, the ancestor $v(X_i)$ is chosen by independent sampling from $\{X_j \in A_{n-1} :\ X_j \rightsquigarrow X_i\}$ assuming equal probability for all elements. Such construction yields a tree connecting the origin to every point of A in the least possible number of hops.

Classical example. A classical special case of the above construction is the *tree spanning clump in Boolean model with spherical grains.* Recall that a classical Boolean model in $\mathbb{R}^d$ is a random set $\Xi = \bigcup_i (X_i + B_i)$ generated by a Poisson point process $\Phi = \{X_i\}_i$ and a sequence of i.i.d. compact subsets $\{B_i\}_i$ of $\mathbb{R}^d$, independent of Φ, where $x + B = \{y + x \in \mathbb{R}^d : y \in B\}$. Maximal connected subsets of Ξ are called clumps. Consider a Boolean model with all $B_i = B(R_i)$ being balls centered at the origin of random radii $R_i \in \mathbb{E}$. Then $(X_i + B_i) \cap (X_j + B_j)$ is non-empty if and only if $|X_j - X_i| \leq R_i + R_j$ that is equivalent to $X_j \in X_i + G_{R_i R_j}$, with $G_{r_1 r_2} = B(r_1 + r_2)$. Hence, the aggregate $A(X_i)$ consists of all points of Φ covered by the Boolean clump containing X_i; see Figure 1.

Our goals.

- The principal goal of [2] is to answer the question of existence of moments of typical Boolean clump size (defined as the number of grains) allowing for unbounded random grains; i.e, under the assumption that q has unbounded support on $\mathbb{E} = \{1, 2, \ldots\}$. We also investigate other distributional properties of the clump size and structure, For example, we derive bounds for tail probabilities of the vector of numbers of grains of any particular size present in the clump as well as estimates for the radius of the maximal ball present in it. We briefly review these results in Section 3.1 below.

- In [5] we are interested in some flows on the trees generated by more general aggregates in a Poisson point process. These aggregates are supposed to represent connected subsets of nodes of an ad-hoc network distributed in the space. Our definition of flows is motivated by the broadcast of data packets from one source to many destinations in these networks. We provide some bounds on characteristics of these flows, which can be used to approximate load induced on a network by a particular broadcast session. We briefly review these results in Section 3.2 below.

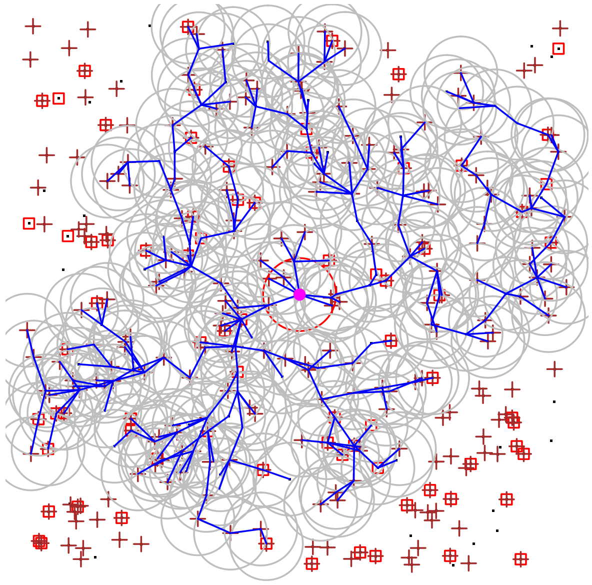


Figure 1: Tree spanning the clump of a given ball in the simulation of a Boolean Model with spherical grains.

Approach. Trees generated by Poisson aggregates have dependent branches, and this makes the analysis of their distributions difficult. We obtain our results in [2] and [5] using the technique of stochastic domination by Galton–Watson trees. This idea is due to Hall (see (Hall 1988)), who used it to give sufficient conditions for finiteness of Boolean clumps and their first moments.

3.1 Bounds for clump size characteristics in the Boolean model

Consider an independently marked Poisson point process $\Phi = \{(X_i, R_i)\}$ as described in the paragraph **Classical example** above and the Boolean model $\Xi = \bigcup X_i + B(R_i)$ it generates. Consider a ball G centered at the origin, with integer radius. Let M_k ($k \geq 1$) denote the number of those balls $B(R_i)$ with radius $R_i = k$ in the Boolean model which contribute to the clump C_G; i.e., to the maximal connected subset of $\Xi \cup G$ containing G. The clumps size $M = \sum_{k=1}^{\infty} M_k$ is the is the total number of balls in C_G. Our principal goal in [2] is to investigate conditions on the intensity λ of the underlying Poisson p.p. and the distribution of R, which guarantee the existence of higher moments, and exponential moments, of the clump size M. (We speak of existence of an exponential moment of M if the probability generation function (p.g.f.) $\mathbf{E}[s^M]$ is finite for some $s > 1$.) The radius of the initial ball G, to which the definition of the clump size M refers, is not essential for the moment conditions and thus may be taken to be equal to 1.

The state of art. Problems of clumping in the Boolean model are treated extensively in Chapter 4 of Hall (Hall 1988). Theorem 4.11 in Section 4.7 there, whose content appeared first in the paper (Hall 1985) of the same author, gives sufficient conditions for the size of each clump to be finite and conditions which characterize the finiteness of the expected size of the clump containing an "arbitrary sphere"; i.e. when a Palm version of the clump size is considered. The idea presented in the proof of this theorem (see pp. 278–279 there) is to study a multitype branching process $(Z_1^{(n)}, Z_2^{(n)}, \dots)$ $(n \geq 0)$ — the set of types $k = 1, 2, \ldots$ being countably infinite — whose total number of individuals $Z_k^{(\infty)} = \sum_{n=0}^{\infty} Z_k^{(n)}$ of any given type k stochastically dominates the number of balls M_k with radius equal to k in the clump C_G.

It is a well-known fact from the theory of branching process, at least in the case of finitely many types, that the expected number of individuals in the n th generation of this process is described by the n th power of its matrix of means. (The proof of the equation given in Harris (Harris 1963, p. 37) may be adapted easily to the infinite-type case.) Therefore, upper bounds on the entries of the matrix powers can be used to ensure finiteness of the first moment of the total number of individuals M_k in the clump

$$\mathbf{E}[M_k] \leq \mathbf{E}[Z_k^{(\infty)}] = \sum_{n=0}^{\infty} \mathbf{E}[Z_k^{(n)}] \, .$$

This branching process $(Z_1^{(n)}, Z_2^{(n)}, \ldots)$ $(n \geq 0)$, however, cannot be used straightforwardly for higher moment approximations because the dependency on the matrices of higher moments is much more complicated (cf. *loc cit* for the case of the second moments, i.e. the covariance matrix), and because of technical difficulties in advanced theory of branching processes whose types belong to an infinite set.

Our contribution. The crucial point in our approach is to construct another multitype branching process $(\bar{Z}_1^{(n)}, \bar{Z}_2^{(n)}, \ldots \bar{Z}_k^{(n)}, \bar{Z}^{(n)})$ $(n \geq 0)$ with only a *finite* number of types. Our process resides on the same probability space and dominates the one constructed by Hall. More precisely, we construct a family of such processes indexed with finite subsets $I = \{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots\}$. Each such auxiliary process almost surely dominates the process $(Z_1^{(n)}, Z_2^{(n)}, \ldots)$ $(n \geq 0)$ coordinate-wise on the set I, and its last coordinate dominates the total number of individuals $\sum_{k \notin I} Z_k^{(n)}$ with types outside this set. We call this device a *coupling* procedure.

According to the standard theory of branching processes (with a finite number of types), the joint probability generating function (p.g.f.) of the total number of individuals of different types $(\bar{Z}_1^{(\infty)}, \bar{Z}_2^{(\infty)}, \ldots \bar{Z}_k^{(\infty)}, \bar{Z}^{(\infty)})$, with $\bar{Z}_k^{(\infty)} = \sum_{n=0}^{\infty} \bar{Z}_k^{(n)}$ is given as the solution of a finite system of functional equations (equations (3.2)–(3.3) in [2]).

Our main tool result, a stochastic bound, relating the clump size vector $(M_1, M_2, \ldots)$ of infinite dimension to the total number of individual $(\bar{Z}_1^{(\infty)}, \bar{Z}_2^{(\infty)}, \ldots \bar{Z}_k^{(\infty)}, \bar{Z}^{(\infty)})$ in our branching process with finite number of types, is formulated in Theorem 3.1 in [2]. The coupling argument appears in its proof.

Using this stochastic bound and studding analytically the solution of equations (3.2)–(3.3) in [2] we derive in Theorems 4.2–4.3 sufficient conditions for higher moments of the clump size to be finite.

These conditions turn out to be also necessary, as stated in Theorems 4.4–4.5 in [2]. Utilizing Theorem 3.1 once more, in Section 5 of [2] we derive other distributional properties of the structure of the Boolean clump.

Our approach allows also to derive in Corollary 4.7 sufficient conditions for the finiteness of the *Lebesgue's measure of the clump.*

3.2 Performance characteristics of multicast flows

Motivation. The motivation for this study comes from telecommunications. A wide range of networking applications, such as conferencing, media streaming, and software distribution, require simultaneous delivery of data from a single source to multiple destinations. One-to-many routing protocols usually construct a distribution tree composed of paths connecting the source to all receivers. *Unicast* protocols treat packet delivery over different paths separately, which results in redundant transmissions of packets over edges belonging to several paths. *Multicast* provides a more efficient alternative (at the cost of an additional intelligence of network nodes): packets can be duplicated at the vertices where routing paths diverge making the transmission of a single packet copy per tree edge sufficient (see (Almeroth 2000) for a survey on multicast routing techniques). Our aim in [5] is to evaluate the performance of a network running a multicast session and, in particular, to quantify the load reduction with respect to the unicast session.

Our model. We model the support of a multicast session by a marked oriented tree $T = (V, E, M)$, with the vertices constituting a countable set V of an arbitrary nature, the edges $E \subset V \times V$ directed from the root vertex $\mathbf{i}_0$, and the vertex marks $M = (m(\mathbf{i}))_{\mathbf{i} \in V}$.

The marks $m(\mathbf{i}) = (r(\mathbf{i}), \sigma(\mathbf{i}), \tau(\mathbf{i}))$ represent vertex characteristics. Here $\sigma(\mathbf{i}) \in \{0, 1\}$ is the indicator of the multicast ability; i.e., the ability to replicate a received packet. If $\sigma(\mathbf{i}_1) = 0$ the number of packet copies transported by any edge $(\mathbf{i}_1, \mathbf{i}_2)$ equals the number of vertices in the sub-tree rooted at $\mathbf{i}_2$, whereas if $\sigma(\mathbf{i}_1) = 0$ it is at most 1. Moreover, $\tau(\mathbf{i}) \in \{0, 1, \ldots\}$ is the number of end receivers at the vertex; i.e., the number of identical packet copies requested by this vertex i, and $r(\mathbf{i}) \in \mathbb{E} = \{1, \ldots, \ell\}$ is the vertex type. We assume $\ell < \infty$.

Denote by $D(\mathbf{i})$ the set of immediate descendants of $\mathbf{i}$ in T and by $T(\mathbf{i}) = (V(\mathbf{i}), E(\mathbf{i}), M(\mathbf{i}))$ the sub-tree of T generated by the vertex $\mathbf{i}$.

Characteristics. The principal performance characteristics that we define are *local flow volume* $K(\mathbf{i})$ defined as the total number of packets sent by vertex $\mathbf{i}$ to its immediate descendants, and *total flow volume* $L(\mathbf{i})$ defined as the total number of packet transfers within the whole sub-tree $T(\mathbf{i})$. They are formally defined by formulas (2.1)–(2.3) in [5].

The following simple recurrence equation, being *packet preservation principle*, is our main algebraic tool when studding these principal performance characteristics:

$$L(\mathbf{i}) = \sum_{\mathbf{j} \in D(\mathbf{i})} \left(\overline{K}(\mathbf{j}) + L(\mathbf{j}) \right), \tag{3.2}$$

where $\overline{K}(\mathbf{j})$ is the number of packet copies received by vertex $\mathbf{j}$ in order to satisfy every packet request within $T(\mathbf{j})$ (formally defined in (2.2) in [5].

The analysis in [5] is done also for a general additive and multiplicative edge weights and yields, as special cases, several other characteristics related to transmissions (packet loss rate, maximal delay) and to the underlying tree (total number of vertices, summary edge length).

Probabilistic scenario. Two scenarios are considered in [5].

- First, we study $K(\mathbf{i})$ and $L(\mathbf{i})$ in an *exact way*, in the case when T is generated by a multitype Galton–Watson (G-W) branching processes. In this case the main analytical tools are recurrent equations on the joint p.g.f. of $K(\mathbf{i}), L(\mathbf{i})$. These equations result from the packet preservation relation (3.2) and the standard recurrent equations concerning the number of individual is successive in successive generations of the G-W process. We briefly review these results in Section 3.2.1 below.

- The obtained exact results in the case of G-W process are used in Section 4 of [5] to give upper bounds on the respective characteristics of the flows on the trees generated by aggregates of points of a Poisson process, which is the second stochastic scenario considered in this paper. The connection between the former and the latter scenario is is again the Hall's branching process which stochastically dominates the number of vertices of different types in the tree spanning the Poisson aggregate.

3.2.1 Flow characteristics for Galton–Watson trees

Now, we place ourselves in a probabilistic setting and consider a multitype Galton–Watson branching process with types in $\mathbb{E} = \{1, 2, \ldots, \ell\}$ (for the theory of branching processes see, e.g., (Harris 1963) and (Athreya and Ney 1972)). Such process can be seen as a random tree $\mathcal{T}_B$ whose edges connect every individual to its direct descendants.

Fix the root type $r(\mathbf{i}_0) = k_0$. The progeny of an individual $\mathbf{i}$ is the vector $Z(\mathbf{i}) = \big(Z_m(\mathbf{i})\big)_{m\in\mathbb{E}}$, where $Z_m(\mathbf{i})$ is the number of direct descendants of $\mathbf{i}$ having type m. We make a standard assumption for branching processes that for $\mathbf{i}$'s of the same generation, all $Z(\mathbf{i})$ are mutually independent. Hence, the process is defined by the distribution of $Z(\mathbf{i})$, or equivalently, by the family of conditional p.g.f's $\psi = (\psi_k)_{k\in\mathbb{E}}$ acting on $z = (z_m)_{m\in\mathbb{E}}$, $z_m \geq 0$

$$\psi_k(z) = \mathbf{E}\Big[\prod_{m\in\mathbb{E}} z_m^{Z_m(\mathbf{i})} | r(\mathbf{i}) = k\Big]. \tag{3.3}$$

We denote by $\Lambda = (\lambda_{km})_{k,m\in\mathbb{E}}$ the so called *matrix of the first moments* of the G-W process.

The marks of the non-root vertices $r(\cdot) \in \mathbb{E}$ correspond to the individuals' types. Regarding the multicast ability and the number of end receivers $(\sigma(\cdot), \tau(\cdot))$, we assume that their joint distribution depends only on $r(\cdot)$ and is given by the set of probabilities

$$p^k_{ij} = \mathbf{P}\big(\sigma(\cdot) = i,\ \tau(\cdot) = j \,|\, r(\cdot) = k\big), \quad i \in \{0,1\}, \quad j \in \{0,1,\ldots\}. \tag{3.4}$$

Hence, the distribution of the marked tree $\mathcal{T}_B = (V_B, E_B, M_B)$ is completely defined.

Consider a family $\phi = (\phi_k)_{k\in\mathbb{E}}$ where $\phi_k = \phi_k(z_1, z_2)$ is the joint p.g.f. of the couple $(K(\mathbf{i}), L(\mathbf{i}))$ under condition $r(\mathbf{i}) = k$

$$\phi_k(z_1, z_2) = \mathbf{E}\big[z_1^{K(\mathbf{i})} z_2^{L(\mathbf{i})} \big| r(\mathbf{i}) = k\big]\,.$$

Introduce also $\phi^{(n)} = \big(\phi^{(n)}_k\big)_{k\in\mathbb{E}}$ as the family of the p.g.f.'s of $\big(K^{(n)}(\mathbf{i}), L^{(n)}(\mathbf{i})\big)$ given $r(\mathbf{i}) = k$, where $K^{(n)}(\mathbf{i})$ and $L^{(n)}(\mathbf{i})$ are respectively, local and total flow volumes evaluated with respect to the truncation of the tree $T(\mathbf{i})$ at the depth n; i.e., the sub-tree including only those vertices $\mathbf{j}$ of $T(\mathbf{i})$ that can be reached from $\mathbf{i}$ by a path $\pi(\mathbf{i}, \mathbf{j})$ of at most $n+1$ vertices.

Define the operator $F[\cdot] = (F_k[\cdot])_{k\in\mathbb{E}}$ acting on the family $u = (u_k)_{k\in\mathbb{E}}$ of non-negative functions $u_k = u_k(z_1, z_2)$ of $z_1, z_2 \geq 0$ as follows

$$F_k[u] = \psi_k\big(f(u)\big), \tag{3.5}$$

where $f(u) = \big(f_m(u)\big)_{m\in\mathbb{E}}$ is given by

$$f_m(u) = (1 - z_1 z_2)\, u_m(0, z_2) p^m_{10} + z_1 z_2\, u_m(1, z_2) \sum_{j=0}^{\infty} p^m_{1j} + u_m(z_1 z_2, z_2) \sum_{j=0}^{\infty} (z_1 z_2)^j p^m_{0j}. \tag{3.6}$$

Our first result for the multicast flows on the G-W process stated in Proposition 3.1 in [5] says that the family of p.g.f.'s $\phi^{(n)}$, $n = 1, 2, \ldots$, satisfies the recurrence relation $\phi^{(n+1)} = F[\phi^{(n)}]$.

The main fix-point result concerning the p.g.f of the vector $(K(i), L(i))$ is stated in Proposition 3.4 in [5]. It says that if the matrix Λ of the first moments of the G-W process is positively regular, non-singular, with spectral radius $\rho(\Lambda) < 1$ then the family of p.g.f's ϕ is the only solution of $F[\phi] = \phi$, such that $\|\phi\|_\infty \leq 1$.

Several conclusions follow from the above general results, in particular concerning first moments of $K^{(n)}(\mathbf{i}), L^{(n)}(\mathbf{i})$, and $(K(\mathbf{i}), L(\mathbf{i}))$ stated in Section 3.2 in [5].

4 Approximate decomposition of some modulated-Poisson Voronoi tessellations, with applications to modeling of communication networks

General context. *Voronoi tessellation* (VT) is a frequently used model of tessellation of the space (an extensive list of areas of applications can be found in (Stoyan, Kendall, and Mecke 1995; Okabe, Boots, and Sugihara 1995)). For a given locally finite system of points in the Euclidean space, VT is a division of the space into polyhedra (into polygons in the case of the plane) "about" the points of the system. Precisely, the Voronoi polygon (*cell* in common terminology) about a chosen point of the system is the subset of points of the space that lie closer to the chosen point than to any other point of the system. If the underlying system of points is a Poisson point process we call the resulting *random* tessellation the *Poisson Voronoi tessellation* (PVT).

In order to study statistical properties of random VT's one introduces the so called *typical cell* of the tessellation. Very roughly speaking, in stationary case, it can be seen as "randomly chosen" from the set of cells. In non-stationary case its distribution depends on the location and is interpreted as conditional, given the underlying process has its point at this location (formal definitions require Palm theory). Known formulas for distributional properties of the typical cell of PVT's are almost entirely confined to the stationary (homogeneous Poisson) case. Even then, formulas are very complicated and mainly approximations are known (see a review in Section 10.6 of (Stoyan, Kendall, and Mecke 1995), and (Hayen and Quine 2000; Goldman and Calka 2001; Calka 2002; Hayen and Quine 2002) for some new results).

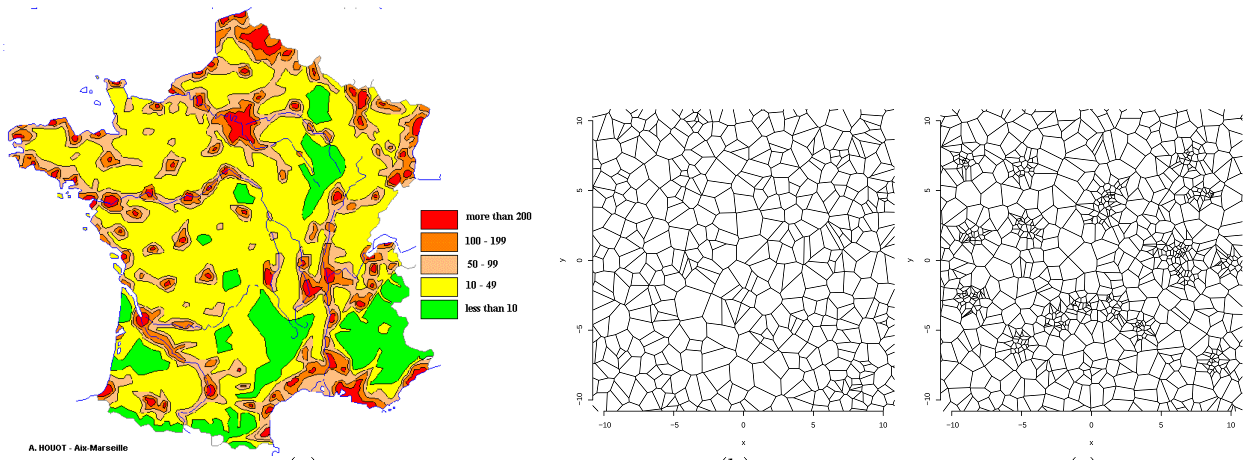
Goal. We want to propose [4] and [6] some class of non-homogeneous Poisson Voronoi tessellations, and develop an approximation technique that allows to approximate the statistical properties of the cells with an explicitly controlled precision.

Motivation. One of the motivations for the this study is modeling of modern communication networks, where application of the PVT has already proved to give some interesting results (see eg. (Baccelli, Klein, Lebourges, and Zuyev 1997; Baccelli and Zuyev 1997; Gupta and Kumar 2000)) and [17]. Generally speaking, within this setting points of the Poisson process represent various communication devices (concentrators, routers, base stations, etc.) and the associated cells represent the regions of the plane or space served by these devices. Adopting Poisson assumption reflects (in a statistical way) various irregularities of a real network architecture. Assuming that the underlying Poisson process is stationary implies that these irregularities are however homogeneous, meaning e.g. that the respective mean densities of the repartition of the network elements are constant on the plane. This scenario is often too simplistic, since it ignores spatial fluctuations of the traffic (large cities versus rural areas etc; cf. Figure 2 (a)). On the other hand, more adequate, non-homogeneous Poisson models rapidly become to difficult to analyze. A possible attitude to take if we want to improve upon this situation is to find a general framework, in which already available results concerning homogeneous cases could be integrated as "local solutions" into a "global" non-homogeneous model.

Our approach. Modeling of inhomogeneity is not an easy task. In order to preserve the postulate formulated above we propose in [4] to use simple parametric models of *modulated-Poisson (Cox) point processes*. For the VT's generated by these models (cf. Figure 2 (c)) we develop in [4] and in [6] an approximation technique for the distribution and mean functionals of the typical cell. The idea is to approximate the unknown distribution in the non-homogeneous case by a mixture of the known distributions for homogeneous Poisson cases. As the main result, we give analytically tractable bounds for the error of the approximation. This approach makes possible the analysis of a wide class of nonhomogeneous PVT's by means of the formulas and estimates already established for homogeneous cases.

4.1 Modulated-Poisson Voronoi tessellations and a quasi-tessellation

Let $\Phi_u = \sum_i \varepsilon_{X_i^u}$, $u = 1, \ldots, \ell$ be independent stationary Poisson point processes on $\mathbb{R}^d$, with intensities, respectively $\lambda_u > 0$; here and throughout ε_z is the atom measure at z. Let a measurable partition



(a) (b) (c)

Figure 2: (a) Density of the population in France in 1990, in habitants/km^2, (b) a simulation of the homogeneous PVT tessellation, (c) a simulation of an inhomogeneous PVT, the inhomogeneity is modeled by taking the density of nuclei in 20 circular, randomly chosen regions of radius 1, to be 10 times bigger than in the remaining part of the plane.

$\chi = \{\chi_u : u = 1, \dots, \ell\}$ of $\mathbb{R}^d$ be given. We call the inhomogeneous Poisson point process

$$\Phi_\chi \equiv \sum_{u=1}^{\ell} \sum_i \mathbb{I}(X_i^u \in \chi_u)\varepsilon_{X_i^u}$$

the *χ-modulated Poisson process (χ-mod PP)*. Obviously χ-mod PP is an inhomogeneous (in general) Poisson point process with intensity measure $\Lambda_\chi(\cdot)$ given by

$$\Lambda_\chi(\mathrm{d}x) \equiv \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbb{I}(y \in \mathrm{d}x)\,\Phi_\chi(\mathrm{d}y)\right] = \sum_{u=1}^{\ell} \mathbb{I}(x \in \chi_u)\lambda_u\,\mathrm{d}x. \tag{4.1}$$

Denote by $V(x, \phi)$ the subset of points of $\mathbb{R}^d$ (called *Voronoi cell of x in ϕ*) of points in $\mathbb{R}^d$ that lie closer to $x \in \mathbb{R}^d$ than to any other point of the point measure ϕ on $\mathbb{R}^d$; i.e.,

$$V(x, \phi) = \{y \in \mathbb{R}^d : |y - x| \le \inf_{\phi \ni z \ne x} |y - z|\},$$

where $|x|$ is the Euclidean norm in $\mathbb{R}^d$. For a given point process $\Phi = \sum_i \varepsilon_{X_i}$, the *Voronoi tessellation (VT)* generated by Φ is the marked point process

$$\tilde{\Phi} = \sum_i \varepsilon_{(X_i, V_i(\Phi) - X_i)}\,,$$

where marks are shifted to the origin random closed sets $V_i(\Phi) = V(X_i, \Phi)$. The Voronoi tessellation generated by Φ_χ will be called the *χ-modulated-Poisson Voronoi tessellation (χ-mod PVT)*. Note that we consider the Voronoi tessellation as a marked point and as such, it has its intensity measure

$$\tilde{\Lambda}^v_\chi(\mathrm{d}x \times L) \equiv \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbb{I}(y \in \mathrm{d}x)\mathbb{I}(V(0, \Phi_\chi - x) \in L)\,\Phi_\chi(\mathrm{d}y)\right],$$

where L is an appropriately measurable subset of the space of closed subsets of $\mathbb{R}^d$ and $\sum_i \varepsilon_{x_i} + x = \sum_i \varepsilon_{x_i + x}$.

Note at this stage, that this intensity does not admit any exact decomposition analogous to (4.1). A reason for this is that the Voronoi cell $V(x,\Phi_\chi)$ of a given point x in the the χ-mod PP Φ_χ depends on its neighboring points, which might be in different sets of the partition χ. However, sometimes such a decomposition might be a good approximation.

A quasi-tessellation. In order gain some intuition lest consider the following *χ-modulated-Poisson Voronoi quasi-tessellation (PVqT).* Let $\mathcal{V}(\Phi_u)=\sum_i \varepsilon_{(X_i^u,V_i^u-X_i^u)}$ be the PVT's generated by independent homogeneous Poisson point processes Φ_u with intensities λ_u, $u=1,\dots,\ell$. We will call the following marked point process

$$\mathcal{V}^q(\Phi_\chi)=\sum_{u=1}^{\ell}\sum_i \mathbb{1}(X_i^u\in\chi_u)\varepsilon_{(X_i^u,V_i^u-X_i^u)} \tag{4.2}$$

the χ-modulated-Poisson Voronoi quasi-tessellation (χ-mod PVqT). Note that each point $X_i^u\in\Phi_u\cap\chi_u$ belongs to $\mathcal{V}^q(\Phi_\chi)$ and has there as the mark its "original" Voronoi cell of Φ_u (and not the Voronoi cell created by Φ_χ). In consequence χ-mod PVqT $\mathcal{V}^q(\Phi_\chi)$ admits the exact decomposition analogous to (4.1). However, the cells of the quasi-tessellation, unlike cells of a "true tessellation", might not be disjoint and their union might not cover the whole space. Obviously the phenomena of "overlapping" cells and "wholes" are more likely to occur close to the boundary of each χ_u where some points of Φ_u have neighbors among the points of Φ_v, $v\neq u$. (Again: it would be a true tessellation, namely the χ-mod PVT, if all the cells were generated by the common pattern Φ_χ of points and not out of the component point processes Φ_u.) However, we might expect, that at least in some cases, the existence of the cells intersecting boundaries of χ_u is negligible and the distribution of the "true" χ-mod PVT can be approximated by χ-mod PVqT.

Boundary cell identification. We sketch here our basic idea how to identify a cell $V(x,\Phi_\chi)$, with $x\in\chi_u$ for some $u\in 1,\dots,\ell$, such that $V(x,\Phi_\chi)\neq V(x,\Phi_u)$. For simplicity we assume dimension $d=2$. The identification of such cells (in any dimension $d\geq 2$ is the basis of all error approximations developed in [4], and [6].

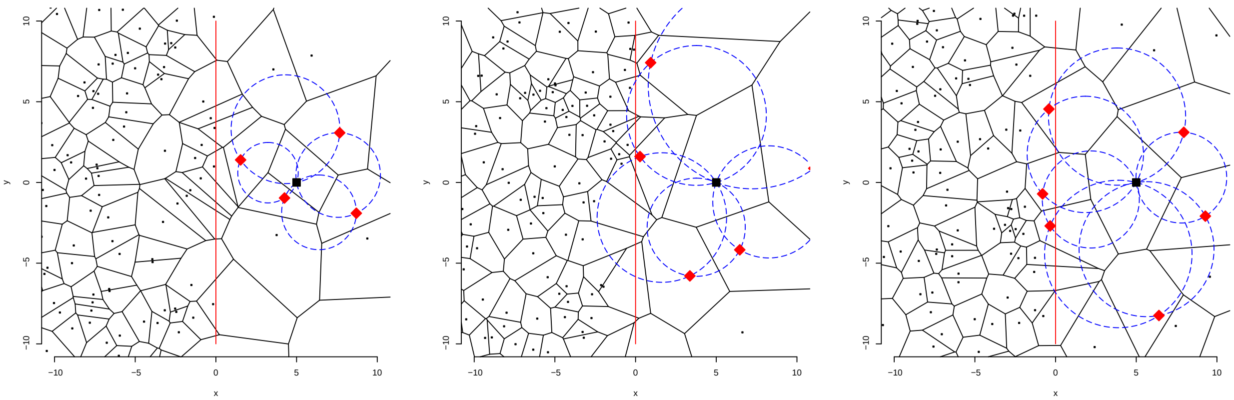
For a given point $x\in\mathbb{R}^2$ and a realization ϕ of a point process on $\mathbb{R}^2$, let $\mathcal{N}(x,\phi)$ denote the subset of points of ϕ, which in the Voronoi tessellation $\mathcal{V}(\phi)$ have cells sharing an edge with the cell $V(x,\phi)$. Formally

$$\mathcal{N}(x,\phi)=\Big\{y\in\phi:\phi\Big(B(x,y,z)\Big)=0 \text{ for some } z\in\phi,\ x,y,z \text{ distinct}\Big\}. \tag{4.3}$$

where $B(x,y,z)$ is the ball circumscribed on the points x,y,z. We have used above the well known principle, saying that *three given elements from the set of the nuclei generating a VT share a common vertex if and only if the ball circumscribed on them does not contain in its interior any other nucleus.* The union of empty balls $B(x,y,z)$ appearing in the definition of $\mathcal{N}(x,\phi)$ is called the *fundamental region* (or the *Voronoi flower*) of the cell $V(x,\phi)$ (see Figure 3). Note that the cell $V(x,\phi)$ preserves its shape when the pattern of nuclei ϕ is subject to changes only outside the fundamental region of $V(x,\phi)$. Thus, a cell $V(x,\Phi_\chi)$ with its nucleus $x\in\chi_u$ for some $u=1,\dots,\ell$, can be different from $V(x,\Phi_u)$ if the fundamental region of $V(x,\Phi_\chi)$ *is not totally contained* in χ_u.

Verifying the inclusion property for the whole fundamental region is not an easy task. In is much easier to verify simply, for a nucleus $x\in\chi_u$, whether at least one of its neigbours in $\mathcal{N}(x,\phi)$ is outside χ_u. Note however that it might be the case that all the neighbours $\mathcal{N}(x,\phi)$ are in χ_u, and that the fundamental region of $V(x,\phi)$ intersects χ_u^c. Then a modification of the pattern ϕ in the intersection of χ_u^c with the fundamental region of $V(x,\phi)$ might modify the shape of the cell $V(x,\phi)$. However any thinning (removing of points) of ϕ in χ_u^c will not modify the fundamental region of the cell $V(x,\phi)$ and consequently will not modify $V(x,\phi)$ itself.

The above observation let us treat cell $V(x,\Phi_\chi)$, with $x\in\chi_u$ for some $u=1,\dots,\ell$, as *possibly* being different than $V(x,\Phi_u)$ only if at least one of its neigbours in $\mathcal{N}(x,\Phi_u^{\max})$ is outside χ_u, where $\Phi_u^{\max}$



(a) (b) (c)

Figure 3: Fundamental region of the cell with the nucleus located at $(5, 0)$; (a) *Any modification* of the pattern of the nuclei left to the vertical line $x = 0$ *cannot* modify this cell. (b) Any *thinning* of the pattern of the nuclei left to the vertical line $x = 0$ *cannot* modify this cell. (c) A *thinning or adding points* to the pattern of the nuclei left to the vertical line $x = 0$ *can* modify this cell.

is some auxiliary pattern, defined in Section 4 in [4], that has more points than Φ_χ and Φ_u, and from whom both can be retrieved by thinning of points in χ_u^c.

Approximation method. In view of the fact that $\mathcal{N}(x, \phi)$ depends on x and some pairs (in planar case; in general d-tuples) of other points of ϕ satisfying some "void" condition (cf (4.3)), counting the expected number of points $x \in \Phi_u \cap \chi_u$ in some fixed set B such that $\mathcal{N}(x, \Phi_u^{\max}) \neq \emptyset$ for Poisson $\Phi_u^{\max}$ involves some integral of the exponential void probabilities with respect to the factorial moment measure of order $d + 1$ of $\Phi_u^{\max}$. This, in turn, gives bounds on the expected number of the cells of $\mathcal{V}^q(\Phi_\chi)$ in B, which are not identical to their counterparts in $\mathcal{V}(\Phi_\chi)$ and, in consequence, the error of the approximation of the intensity measure of the marked p.p. $\mathcal{V}(\Phi_\chi)$ of the "true" tessellation by this of the quasi-tessellation $\mathcal{V}^q(\Phi_\chi)$. This basic approximation is formulated in Lemma 4.1 in [4] with an explicit error evaluation in Proposition 4.1 therein.

4.2 Approximation results, fixed modulation case

Distribution of a given cell. Using Campbell's formula, and the approximation of the intensity measure of $\mathcal{V}(\Phi_\chi)$ (Lemma 4.1 and Proposition 4.1 in [4]) one can obtain an approximation in total variation of the distribution of the cell $V(x, \Phi_\chi + \varepsilon_x)$ for $x \in \chi_u$ by the distribution of the typical cell $\mathcal{M}_u^v$ of the homogeneous PVT $\mathcal{V}(\Phi_u)$ (cf Corollary 4.1 in [4]), and the expectations of *bounded* functionals of this cell (cf Corollary 4.2 in [4]).

The errors of the above approximation depend on the distance of the nucleus x to the boundary of the element of the partition χ_u it belongs to. They are bounded explicitly in [4] for the following examples of simple modulations (which are presented here in the contexts of their possible applications).

- *Cell located at some distance to a "hot spot".* Assume $\chi_1 = B_{(-r,0)}(r)$, where $B_x(r)$ is the disc in $\mathbb{R}^2$

centered at x, with radius r, assume $\chi_2 = \mathbb{R}^2 \setminus \chi_1$ and consider a nucleus $x \in \chi_2$. Let $\lambda_1 > \lambda_2$. This can be a simple model of the following scenario: consider a city (modeled by the disc) with much larger density λ_1 of some kind of communication devices, and consider a particular device located at x, outside the city, where the respective density is considerably smaller. Obviously, if the distance from this particular device to the city ($|x|$ in our example) is large, then the distribution, and hence all the mean functionals, of the cell served by the device located at x, is approximately the same as the distribution of the typical cell in homogeneous Poisson scenario with density λ_2. How good such an approximation is for a given distance $|x|$ is a natural question in this context. One can consider also a reverse situation, with $\lambda_1 < \lambda_2$ (a "cold spot").

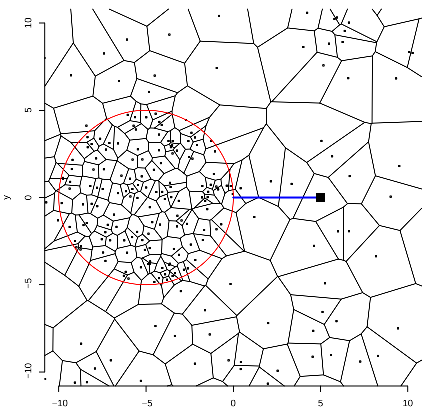


Figure 4: Simulation of a PVT with a hot spot, where the density of nuclei is 10 times bigger than outside it.

- *Cell located at some distance to a "hot wall".* Let χ_1 be a half-plane, $\chi_2 = \mathbb{R}^2 \setminus \chi_1$, and suppose $x \in \chi_2$, with $|x|$ being the distance of x to χ_1. Let $\lambda_1 > \lambda_2$. This model is supposed to describe a similar scenario as previously, but with the region of the larger intensity of devices being so vast (comparing to the distance $|x|$) that it is "seen" from x as a half-plane. It seems to be relevant as well to a network deployed in a coastal region, where the population density is relatively large along the cost.

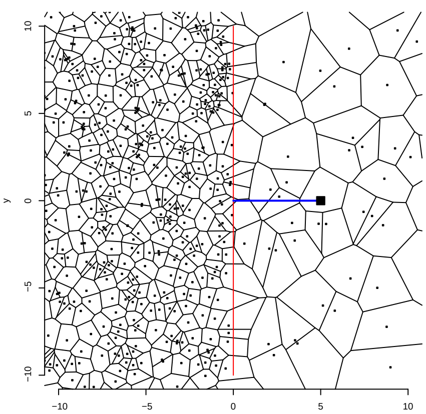


Figure 5: Simulation of a PVT with a hot wall, where the density of nuclei is 10 times bigger than elsewhere.

Unbounded functionals of a given cell in planar case. In [6] we refine the approximations obtained in [4]) and in planar case ($d = 2$) consider *nonnegative*, possibly unbounded, functionals $\Psi(V)$ of the Voronoi cell $V = V(x, \phi)$, which are *translation invariant*; i.e., $\Psi(V) = \Psi(V - y)$ for all $y \in \mathbb{R}^2$ and which satisfies the following property

$$\Psi(V) \leq A\big(\mathcal{R}(V)\big)^{\alpha}, \tag{4.4}$$

where A, α are some positive constants and $\mathcal{R}(V)$ is the minimal radius of the disc centered at the nucleus x of V, that covers V; i.e.,

$$\mathcal{R}(V) = \inf\{r : V \subset B_x(r)\}\,.$$

Examples of functionals Ψ, which satisfy the above conditions are

- $\Psi(V) = |V|$, the area of V,
- $\Psi(V) = \int_V f(y - x)\,\mathrm{d}y$, the total cost or load of connecting of all points in cell V to its nucleus at x, where f is some non-negative cost function,

- $\Psi(V) = \int_V f(y-x)\,\mathrm{d}y \times \int_V g(y-x)\,\mathrm{d}y$ for some non-negative cost functions f, g,
- $\Psi(V) = |\partial V|$, the length of the boundary of V.

These functionals appear naturally in modeling of communication networks as explained in Section 3.1.3 in [6] and their expected values of the typical PV cell in homogeneous Poisson case are known explicitly (see Section 3.1.4 therein).

In order to be able to give bounds on the difference between the expected value of $\Psi\big(V(x,\Phi_\chi+\varepsilon_x)\big)$ for $x \in \chi_u$ and $\int \Psi(V)\mathcal{M}_u^v(\mathrm{d}V)$, where $\mathcal{M}_u^v$ denotes the distribution of the typical cell in homogeneous Poisson p.p. Φ_u, one has, not only to bound the probability that $V(x,\Phi_\chi+\varepsilon_x)$ is a boundary cell (cf **Boundary cell identification** above), but also bound the expectation of $\Psi\big(V(x,\Phi_\chi+\varepsilon_x)\big)$ given this event. This can be done using our assumption (4.4) on Ψ and a bound on $\mathbf{E}\big[\big(\mathcal{R}(V(0,\Phi))\big)^\alpha \big| \Phi\big(B(0,y,z)\big)=0\big]$ for a homogeneous Poisson p.p. Φ, that is developed in Lemma A.1 in [6].

Working out the above program, we are able to give in Proposition 4.3 of [6] some approximations of the mean functionals $\Psi\big(V(x,\Phi_\chi+\varepsilon_x)\big)$ by (presumably known) expected values of these functionals of the typical PV cell in homogeneous case, with the approximation errors explicitly bounded in Proposition 4.4 therein.

4.3 Approximation results, random modulation

Up to now we have considered a fixed modulation χ, a non-homogeneous modulated Poisson p.p. Φ_χ and and a Voronoi cell $V(x,\Phi_\chi+\varepsilon_x)$ of a fixed point x.

Now we assume a stationary decomposing random partition $\Xi = \{\Xi_u : u = 1,\ldots,\ell\}$; that is, that for any vector $x \in \mathbb{R}^d$ the distribution of $\Xi + x = \{\Xi_u + x : u = 1,\ldots,\ell\}$ is the same as Ξ. Moreover, let Ξ be independent of Poisson processes Φ_u, $u = 1,\ldots,\ell$. This makes the Ξ-mod PP Φ_Ξ the stationary *double-stochastic-Poisson* (Cox) point process and the stationary-Cox Voronoi tessellation (CoxVT) $\mathcal{V}(\Phi_\Xi)$ admits the distribution of the typical cell $\mathcal{M}^v_{(\Xi)}$.

Conditioning on $\Xi = \chi$ and using our previous results for a fixed modulation we can approximate the distribution and functionals of the cell of the nucleus located at a given point, say at the origin $x = 0$. Recall that these approximations depend on the law of the typical cell of the element χ_u the nucleus 0 belongs to. Moreover, approximation errors depend on the distance of the nucleus 0 to the boundary of this partition element. Consequently, after de-conditioning with respect to χ, this element of the partition becomes random (as so the distance of 0 to its boundary). We see thus that the typical cell of the CoxVT should have a distribution close to some linear combination (mixture) of the distributions in homogeneous cases, with the weights being the mean fractions of the space covered by the respective elements Ξ_u of the random partition Ξ (usually in stochastic geometry one calls them *volume fractions*). Moreover, approximation errors should depend on *linear contact distribution functions* of the boundaries of the partition elements.

More precisely, denote by $T^\Sigma_{\partial\Xi}(\cdot)$ the half of the sum of the capacity functionals of the boundaries of the elements Ξ; i.e., for each compact set $K \subset \mathbb{R}^d$

$$T^\Sigma_{\partial\Xi}(K) = \frac{1}{2}\sum_{u=1}^{\ell}\mathbf{P}(\partial\Xi_u \cap K \neq \emptyset) = \frac{1}{2}\sum_{u=1}^{\ell} T_{\partial\Xi_u}(K)\,,$$

where $\partial\Xi_u$ is the boundary of Ξ_u. Let $p_u = \mathbf{P}(\,0 \in \Xi_u\,)$ denote the volume fraction of Ξ_u. For any $x, y \in \mathbb{R}^d$ let $\langle x, y\rangle = \{z \in \mathbb{R}^d : z = \xi x + (1-\xi)y, \xi \in [0,1]\}$ be the segment in $\mathbb{R}^d$.

One proves in Proposition 5.1 in [4] that the distribution $\mathcal{M}^v_{(\Xi)}$ of the typical cell of the stationary-Cox Voronoi tessellation $\mathcal{V}(\Phi_\Xi)$ admits the following decomposition

$$\mathcal{M}^v_{(\Xi)}(L) = \frac{1}{\lambda_{(\Xi)}}\sum_{u=1}^{\ell}\lambda_u p_u \mathcal{M}^v_u(L) + R \tag{4.5}$$

where $\lambda_{(\Xi)} = \sum_{u=1}^{\ell} \lambda_u p_u$ and the remainder term R is explicitly bounded in terms of $T^{\Sigma}_{\partial\Xi}$ (see inequality (5.2) in [4]).

Propositions 4.7 and 4.8 of [6] extend the above result to the case expectations of nonnegative, translation invariant functionals Ψ satisfying (4.4).

The following example of stationary CoxVT is considered in [4] and [6].

• ***Typical cell of the PVT modulated by a Boolean model.*** Let the partition $\Xi = \{\mathbb{X}, \mathbb{X}^c\}$ be given, where $\mathbb{X}$ is a stationary *Boolean model (BM)*; i.e.,

$$\mathbb{X} = \bigcup_i (C_i + Y_i) \tag{4.6}$$

where $\{Y_i\}$ is a Poisson point process (of the so-called *germs*) on $\mathbb{R}^2$ and $\{C_i\}$ is a sequence of (possibly random, independent, identically distributed) subsets of $\mathbb{R}^2$ (called *grains*). Let for example $\lambda_1 > \lambda_2$. In order to demonstrate a possible application context of this model consider a country in which regions of a higher density of communication devices are scattered irregularly. Suppose we are not interested in a given particular location but in some "average device" typical for the whole country (such notion is useful e.g. for a global economical planning). Then, instead of analyzing the given "real" configuration of regions of higher density, it is customary to consider it as a snapshot of a (random) BM Ξ, where germs $\{Y_i\}$ model (e.g.) geographical centers of this regions and $\{C_i\}$ model regions themselves, centered to 0. Now, provided the partitioning of the plane by the BM is not very "fine" with respect to the densities of the devices, the typical cell for the whole country should have a distribution close to the linear combination of the homogeneous cases with densities λ_1 and λ_2, with the coefficients given by the fractions of the plane covered by the BM Ξ and its complement, respectively. The error of such approximation, which we quantify in Example 5.1 in [4] and Example 5.3 in [4], comes from existence of cells whose fundamental regions cross the boundaries between the partitioning sets.

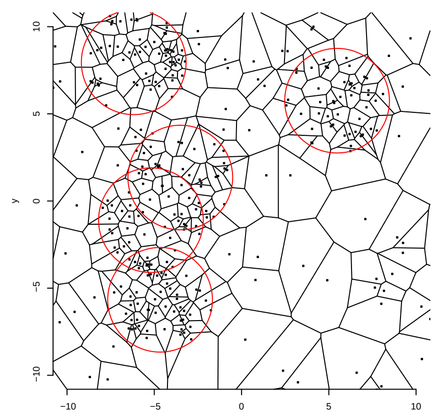


Figure 6: PVT modulated by a Boolean model: the inhomogeneity is modeled by taking the density of nuclei in 5 circular, randomly chosen regions of radius 3, to be 10 times bigger than in the remaining part of the plane.

5 On a coverage process ranging from the Boolean model to the Poisson Voronoi tessellation with applications to wireless communications

The model. Let $\Phi = \{(X_i, Z_i)\}$ be a marked point process on the d-dimensional Euclidean space $\mathbb{R}^d$, where $\{X_i\}$ denotes the locations of points, and where the *marks* $Z_i = (S_i, A_i)$ are such that S_i belong to some metric space $\mathbb{D}$ and $A_i = (a_i, b_i, c_i) \in (\mathbb{R})^3$.

In addition to this marked point process, the model is based on a function $L : \mathbb{D} \times \mathbb{R}^d \to \mathbb{R}^+$, which is continuous w.r.t. its second argument, and such that $L(s, x) \to 0$ when $|x| \to \infty$ (where $|x|$ is the Euclidean norm of x in $\mathbb{R}^d$).

We define the *cell* C_0 attached to the point X_0 as the following subset of $\mathbb{R}^d$

$$C_0 = C_0(\Phi) = \left\{ y : \; a_0 L(S_0, y - X_0) \geq b_0 I_\Phi(y) + c_0 \right\}. \tag{5.1}$$

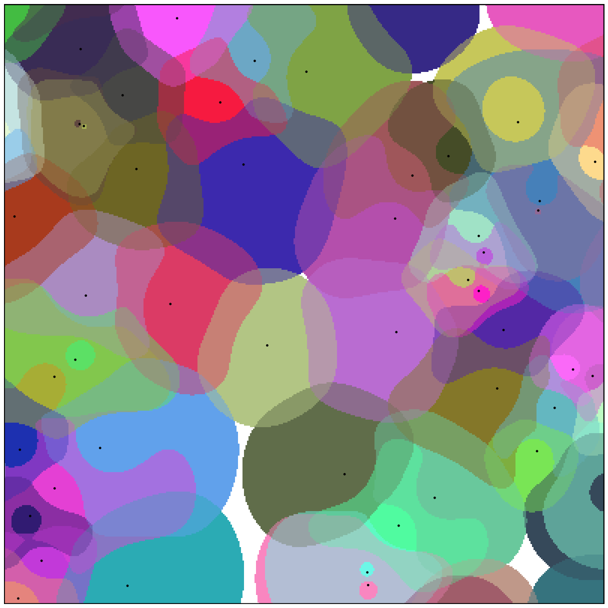


Figure 7: Coverage model Ξ with grains (cells) given by (5.1).

where $I_\Phi(y)$ denotes the value of the *shot noise (SN)* process (see e.g. (Westcott 1976; Rice 1977; Heinrich and Schmidt 1985; Heinrich and Molchanov 1994)) of $\{X_i, S_i\}$ at point y for the response function L, namely :

$$I_\Phi(y) = \sum_i L(S_i, y - X_i) = \int_{\mathbb{R}^d \times \mathbb{D}} L(s, y - x)\, \Phi(d(x, s))\,.$$

The second formula is obtained when considering Φ as the random point measure $\Phi = \sum_i \delta_{(X_i, Z_i)}$ and when using the simplified notation $\Phi(d(x, s)) = \Phi\left(d(x, s) \times (\mathbb{R})^3\right)$.

The union of the cells

$$\Xi = \Xi(\Phi) = \bigcup_i C_i(\Phi)$$

is the associated *coverage* process; see Figure 7 for some snapshot of the simulated coverage model. To the best of our knowledge, this model is new in the stochastic geometry setting and yields several well-known models, as Voronoi tessellation, the Boolean model and the Johnson-Mehl model, as particular limiting cases. It can also be seen as a *germ-grain model with dependent marks.*

Motivations. Consider a collection of transmitters $\{X_i\}$ distributed in the space and emitting, respectively, powers $\{S_i\}$ in some common radio medium. In one of the most simple scenario (which can be easily enriched) the *total power received from this collection of transmitters* at a given location can be modeled by the value of the SN field $I_{\tilde{\Phi}}$ at this location, where the response function is $L(s, z) = s/l(|z|)$ with $l(r)$ being the so called *omnidirectional path-loss* function. This observation explains why SN fields appear naturally in modeling of wireless communications.

The total power received from a set of transmitters scattered in the plane is often be considered as *interference* (or noise) with respect to the signal received from one (or more) transmitter(s) not belonging to this set. Within this setting, this total power plays a key role in signal detection theory (as we explain in Chapter 2 of the monograph [28]; see also e.g. (Tse and Viswanath 2005)). More precisely, the bit-reception error-probability depends on the ratio of the value of received power of the useful signal(s) to the value of the interference and noise power. This ratio, called Signal-to-Interference-and-Noise ratio (SINR) is a key parameter in analysis of wireless communications.

Note that C_0 in (5.1) with the response function L paying the role of the attenuation function, is a subset of the space where the power of the signal received form the emitter located at X_0 exceeds an affine function of the interference. Thus, it can be interpreted as the subset of locations of potential receives, which would be able to sustains a radio channel from X_0 with some given quality (related to the bit-reception error-probability).

What is the typical shape of the cell C_0? What interactions exist between adjacent or remote cells? Under natural stationarity and ergodicity assumptions, what is the proportion of the space which belongs to exactly k cells? Is it possible to have coverage of x by arbitrarily many cells? Answers to the above and other questions typically asked in stochastic geometry can of course be interpreted in terms of certain performance characteristics of the wireless communication networks.

5.1 First results: sufficient conditions for the model to be a random closed set

Section 3 in [3] addresses questions concerning Ξ as a random closed set. The default probabilistic assumptions are the following: Φ is an independently marked Poisson point process (Poisson p.p. for short) where the marks $\{Z_i\}$ constitute a sequence of independent identically distributed random vectors characterized by the distribution of Z_0. We sometimes use Z for a generic random mark. The default option is that when the underlying (non marked) Poisson process is non homogeneous with $\mu(\cdot)$ denoting its intensity measure. We assume that μ is non-atomic; thus Φ is a simple p.p. This assumption allows to derive several computational results.

Some existence results can be extended via Palm calculus to a more general setting. For this, we mainly consider the case when the marked point process Φ is stationary and ergodic, with (constant) intensity λ. In this more general setting the generic mark Z has the same distribution as Z_0 under Palm distribution $\mathbf{P}^0$.

In what follows, in order to avoid degeneracy and/or special cases, we make the following general assumptions: $a, b, c \in \mathbb{R}^+$ a.s. and $\mathbf{P}^0(a_0 = c_0 = 0) = 0$.

Finiteness and continuity of the SN field. Since $L(\cdot)$ is positive, $I_\Phi(y)$ is well defined but can be infinite. We require this random function to be a.s. finite and even more, to have finite expectation $\mathbf{E}[I_\Phi(y)] = \int_{\mathbb{R}^d \times \mathbb{D}} L(s, y-x)\, \mu(dx) H(ds) < \infty$ where H denotes the law of $S_0 \in \mathbb{D}$ (see e.g. (Schmidt 1985)).

Moreover we will require that the cell C_0 under Palm distribution (and in consequence all cells C_i under Palm and stationary distribution) to be a *random closed set*. Since L is a continuous function of its last argument, C_0 is a.s. a closed set provided $I_\Phi(y)$ is also a.s. continuous in y (lower semi-continuity is sufficient). Some sufficient conditions for these properties are given in Proposition 3.1 in [3] under Poisson assumption.

Coverage process. We will also require Ξ to be a random closed set (note that the countable union of closed sets need not be closed). In fact we will require the stronger property that for any given bounded set in $\mathbb{R}^d$ (with compact closure) the number of cells that have non-empty intersection with it is almost surely finite. An equivalent statement is that the *collection of cells* is a.s. a Radon point measure on the space of closed sets, so that it can be treated as a point process $\sum_i \delta_{C_i(\Phi)}$ on the space of closed sets. This is a typical assumption for coverage processes (in particular for the Boolean model, see e.g. (Stoyan, Kendall, and Mecke 1995), eq. (3.1.1), p. 59.).

Some sufficient condition for this latter property are given in Proposition 3.2 and Corollary 3.3 in [3] for the Poisson case. Similar conditions can be derived for a general stationary ergodic case, as explained in the concluding remark of Section 3 therein.

5.2 Typical cell and the coverage process characteristics

Consider the cell $C(x;\Phi)$ attached to a point located at x of the marked Poisson p.p. Φ under the *Palm* distribution $\mathbf{P}_x$. Due to Slivnyak's theorem, the law of this set under $\mathbf{P}_x$ is the same as that of the random closed set

$$C(x;\Phi+\delta_{(x,Z)}) = \{y : aL(S,y-x) \geq b(I_\Phi(y)+L(S,y-x))+c\} \tag{5.2}$$

under $\mathbf{P}$, where Φ is the original Poisson p.p. and $Z=(S,A)=(S,(a,b,c))$ is an "additional mark" distributed like the other marks and independent of Φ. We will refer to (5.2) as the *typical cell located at* x, although the name "typical cell" is natural only in stationary case. Indeed, if the Poisson point process is homogeneous its characteristics are the same for all points x and we will speak of *the typical cell*.

Cell coverage probability. Denote by $p_x(y)$ the probability that point $y \in \mathbb{R}^d$ is covered by $C(x;\Phi+\delta_{(x,Z)})$. We have

$$\begin{aligned} p_x(y) &= \mathbf{P}\Big(y \in C(x;\Phi+\delta_{(x,Z)})\Big) \\ &= \mathbf{P}\Big(aL(S,y-x)-c \geq 0, b=0\Big) \\ &\quad +\mathbf{P}\Big((\frac{a}{b}-1)L(S,y-x)-\frac{c}{b}-I_\Phi(y)\geq 0 \Big| b>0\Big)\mathbf{P}(b>0)\,. \end{aligned} \tag{5.3}$$

The distribution of the mark Z can be considered as given. Note also that the random variables $I_\Phi(y)$ and Z involved in (5.3) are independent. Thus in order to determine the probability $p_x(y)$, we need to know the marginal distribution of the shot-noise process $I_\Phi(\cdot)$ at y. This distribution is usually not known explicitly, but only via its transforms. For example the *characteristic functional of the process* $I_\Phi(\cdot)$ is given by

$$\begin{aligned} \varphi_I(\nu) &= \mathbf{E}\exp\Big[i\int_{\mathbb{R}^d} I_\Phi(y)\,\nu(\mathrm{d}y)\Big] \\ &= \exp\Big[\int_{\mathbb{R}^d\times\mathbb{D}}\Big(\exp\Big[i\int_{\mathbb{R}^d} L(s,y-x)\,\nu(dy)\Big]-1\Big)\,\mu(\mathrm{d}x)H(ds)\,, \end{aligned} \tag{5.4}$$

where ν is any measure on $\mathbb{R}^d$ such that the outer integral in (5.4) is finite (see e.g. (Rice 1977)). The joint characteristic function of the vector $(I_\Phi(y_1),\ldots,I_\Phi(y_n))$ can be obtained from (5.4) by setting $\nu=\sum_{k=1}^n \xi_k\delta_{y_k}$. Knowing the transforms: $\varphi_I(\xi)$ of the variable $I_\Phi(y)$, and $\varphi(\xi)$ of $(a/b-1)L(S,y-x)-c/b$ (recall that we assumed $b>0$ a.s.) and assuming that at least one of these two variables has a density with respect to Lebesgue's measure then we can express the coverage probability in terms of the following contour integral

$$p_x(y) = \frac{1}{2} - \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{\nu(\xi)\nu_I(-\xi)}{\xi}\,\mathrm{d}\xi\,, \tag{5.5}$$

where the singular contour integral in the right-hand side, which has a pole at $\xi=0$, is understood in the principal value sense. This result follows from the solution of some *Riemann boundary problem* on the real line explained in the proof of Proposition A.1 in [3].

The above expression, albeit numerically tractable (cf Figure 8) in not very explicit. However, there are some special (and interesting) cases, when the coverage can be evaluated more explicitly. Assume for example that S is exponential[2] (i.e., $H(s)=1-e^{-\mu s}$, where $1/\mu$ is the mean of S). Assume moreover

[2]This assumption is not only for mathematical convenience; a constant emitted power combined with the so called *Rayleigh fading model* of a radio channel leads to a random exponential received power; as we explain in Chapter 1 of the monograph [28]; also e.g. (Tse and Viswanath 2005)).

(as in paragraph **Motivations** above) that $L(s,z) = s/l(|z|)$ for some nonnegative, nondecreasing (this is not essential) function, and for simplicity take $a/b - 1 = 1$ and $c/b = 1$. Then by the independence and the exponential assumption

$$p_x(y) = \mathbf{P}\Big(S \geq l(|x-y|)(1 + I_\Phi(y)) \geq 0\Big) = e^{-\mu l(|x-y|)(1+I_\Phi(y))} = e^{-\mu l(|x-y|)} \mathcal{L}_{I_\Phi}(\mu l(|x-y|)) \,, \quad (5.6)$$

where $\mathcal{L}_{I_\Phi}$ is the Laplace transform of $I_\Phi(y)$, that is explicit in view of (5.4). The above observation made in [7] is the basis for all explicit optimization study of some ad-hoc network model considered in [7], which we review in Section 5.4 below.

Integrating $p_x(y)$ with respect to the Lebesgue's measure over $\mathbb{R}^d$ one obtains the mean d dimensional volume v_x of the cell of the point x.

The coverage process characteristics. The inequality $\sum_{i=1}^n b_i/a_i < 1$ is a necessary condition for the set of cells C_i, $i = 1, \ldots, n$, to have a common nonempty intersection (cf Lemma 5.1 in [3]). This condition gives a stochastic bound on the number of overlapping cells in terms of the distribution of marks a_0, b_0 (see Corollary 5.2). In particular, if $b_0/a_0 \geq \rho$ a.s for some fixed $\rho > 0$, then the number of cells N_x covering any given point x is never larger that $1/\rho$ (no matter how large the intensity of the underlying point process is. This is in strong contras to the situation observed in the Boolean model, where N_x has unbounded support.

In proposition 5.3 of [3], if Poisson p.p. case, we express the factorial moments of N_x in terms of the factorial moment measures of the underlying point process. Proposition 5.4 and Corollary 5.5 therein gives some sufficient conditions on the finiteness of these moments.

Other characteristics of the coverage process Ξ, as e.g. its volume fraction $\mathbf{P}(\,0 \in \Xi)$, can be obtained via the *factorial moments expansions* — an approximation technique proposed in [9] and extended to the spatial case in [13]. The first order expansion formula for the volume fraction p is presented in Section 5.3 in [3].

5.3 From the Boolean model to the Poisson-Voronoi tessellation

Our coverage model yields several well-known models, as Voronoi tessellation, the Boolean model and the Johnson-Mehl model, as particular limiting cases.

Towards the Boolean model. Note that the cells of Ξ given by (5.1) are not mutually independent because of the presence of the shot-noise variable I_Φ. However, if we assume $b = 0$ a.s. the cells are independent, and Ξ is a Boolean model. In Section 6.1 of [3] we study the following continuity problem: assume that $b \to 0$ in some sense. In what sense and under what conditions does the typical cell $C(x, \Phi + \delta(x, Z))$ and the whole process $\Xi(\Phi)$ tend to their counterparts in the Boolean model obtained by assuming $b \equiv 0$? In Proposition 6.1 we answer this question considering Painlevé-Kuratowski convergence on the space of closed sets. Then, in Propositions 6.4–6.7 we study convergence of some mean functionals, as $\mathbf{E}[N_x]$, capacity functional of the typical cell and the whole process, mean volume of the typical cell.

The above continuity is only a first step in the direction of the following more interesting differentiability question: assume the above continuity holds, and take b small in some sense. What first (and higher) order perturbation should one apply to the characteristics of the Boolean cells (which are explicitly known) to get the characteristics of the dependent cells? Propositions 6.8–6.10 in [3] give such *perturbation results* for the cell coverage probability and the mean cell volume; cf also Figure 8.

Towards the Voronoi tessellation. Recall that the form of the Voronoi cell $V(x, \Phi)$ attached to point x is determined by some "neighboring" points of x in $\in \Phi$ only. It is quite reasonable to expect that if we let the response function $L(s, z)$ decrease fast in z, we will get the same effect. We formalize

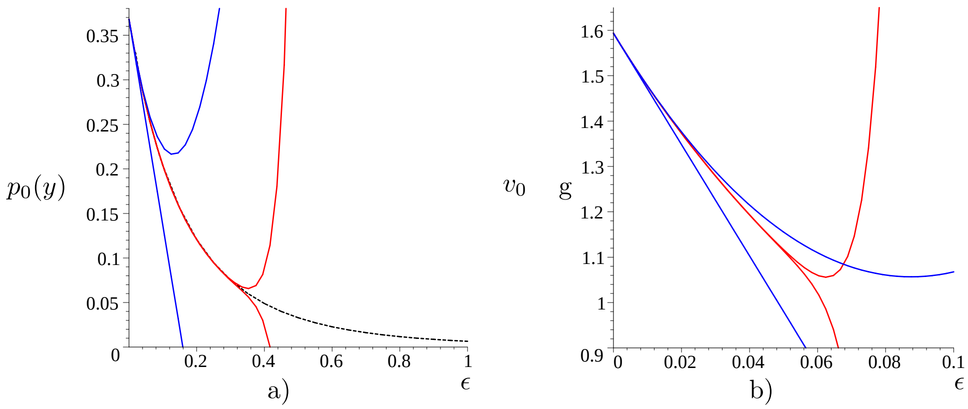


Figure 8: a) Exact values of the cell (located at 0) coverage probability $p_0(y)$ of some given point y. The dashed line is obtained via the singular integral representation (5.5). Other curves are first, second, 14 th and 15 th order Taylor expansions of $p_0(y)$ as a function of the factor ϵ with which we multiply the shot-nose I_Φ in (5.1); $\epsilon = 0$ gives a Boolean model. b) Similar approximations for the mean area of the typical cell.

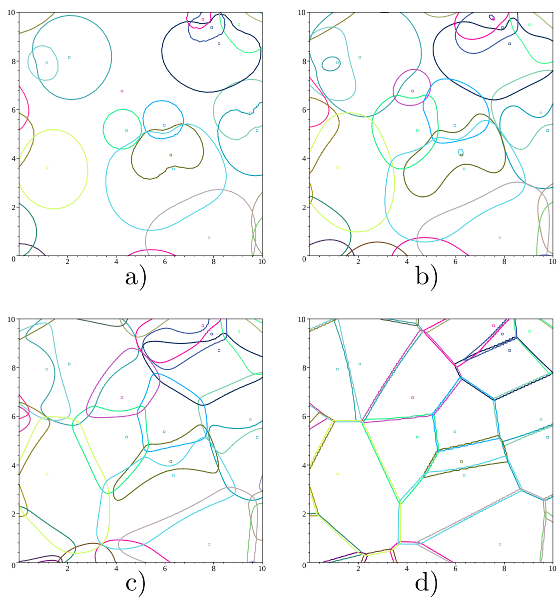


Figure 9: The coverage process Ξ tending to the Voronoi tessellation of the plane. We have: a) $\alpha = 3$, b) $\alpha = 5$, c) $\alpha = 12$, d) $\alpha = 100$.

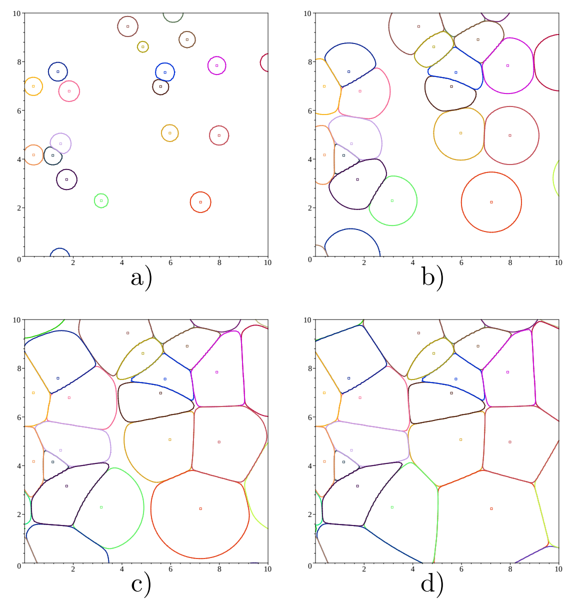


Figure 10: The coverage process Ξ growing as in the Johnson-Mehl model to the Voronoi tessellation of the plane.

this observation in Section 6.2 of [3], assuming $L(s, z) = sl(|z|)$ and taking appropriate families of l-functions.

Figure 9 illustrates Proposition 6.13 in [3] showing some patterns of our coverage process Ξ with $l(y) = (1 + |y|)^{-\alpha}$ at various α tending (Painlevé-Kuratowski convergence) to the Voronoi tessellation of the plane.

Under some parametrization our model behaves also as some Johnson-Mehl model; see Section 6.2 in [3] and cf. Figure 10.

5.4 Modeling wireless ad-hoc networks

The stochastic geometry model described in [3] can be straightforwardly used e.g. to study the *cellular networks* (see e.g. applications to CDMA networks shown in [14]). However some results, as e.g. the formulas for the cell coverage probability, are universal for wireless communications, and can be used in the context to other network architectures. In [7] we use a variant of our coverage model to analyze the so called *ad hoc networks*.

Challenges. *Ad-hoc network* is a generic term denoting a communication network architecture, in which there is no fixed infrastructure of antennas (base stations) serving mobile users, but these users (called also nodes) constitute an auto-organized network of nodes communicating to each other typically via *mutihops*. This means that each packet from some given source-node is transmitted to its its destination-node possibly using other nodes as relays. In such a scenario, the subset of nodes emitting at a given time generates the corresponding shot-noise dependent cells of the coverage process, and other nodes, which do not emit at this moment, are considered as potential receivers (see Figure 11).

The mechanism that decides which nodes are emitting and which are potential receivers at a given time is called *medium access (MAC)* protocol. It is supposed to prevent simultaneous *neighboring* transmissions from occurring, since such transmissions create interference to each other and hence are more likely unsuccessful. On the other hand, to enable an efficient use of the network, the same MAC protocol should allow as many simultaneous and successful transmissions as possible *over different parts of the network*. This desired ability of wireless networks is known as *spatial reuse*.

A very simple MAC protocol, considered in [7], let at each time slot[3], any potential node independently toss a coin with some bias p (which is be referred to as the medium access probability (MAP)) and it allows this node to emit if the outcome is heads; otherwise this node delays its transmission. This protocol, called *Aloha*, is a widely deployed and studied access protocol (the initial paper presenting Aloha was published by (Abramson 1970)). The aim of this scheme in the spatial context is to create *random exclusion zones* around each emitting node. It is of course far not perfect, in the sense that allowing a node to emit does not guarantee the success of the transmissions. The advantage of this scheme is that it is decentralized, meaning that there is no need for any central authority to implement it.

When tuning the value of the MAP parameter p it is important to find a compromise between the average number of concurring transmission per surface area and the guarantee that a given authorized transmission will be successful. In fact, taking larger p one obtains more concurrent transmissions but (statistically) smaller exclusion zones making these transmissions more vulnerable. On the other hand, smaller p gives fewer transmissions with higher probability of success.

Another important geometric characteristic is one-hop distance on which the transmissions are effectuated. A smaller such distance makes the transmissions more likely successful but involves more relaying nodes to communicate on some given (large) distance. On the other hand, a larger one-hop distance reduces the number of hops but might increase the number faults and retransmissions on a given hop.

The model we presented in [7] does not yet address more difficult routing issues (as it is attempted in [26]). However, it is enough to consider the above problems. In particular we are able to define in it and study the following performance characteristics related to the spatial repartition of nodes: the probability of successful transmission to an optimal (in one hop) receiver, the mean number of successful transmissions per unit surface area (called *density of successful communications*), the mean *effective progress* made in one transmission, the mean number of communication-meters per surface area (*density of progress*) and the *spatial reuse*.

Ad hoc network model with Aloha MAC. Let $\Phi = \{(X_i, (e_i, S_i, T_i))\}$ be a marked Poisson point process with intensity λ on the plane $\mathbb{R}^2$, where

[3]We assume a slotted Aloha model; a non-slotted model is analyzed [22].

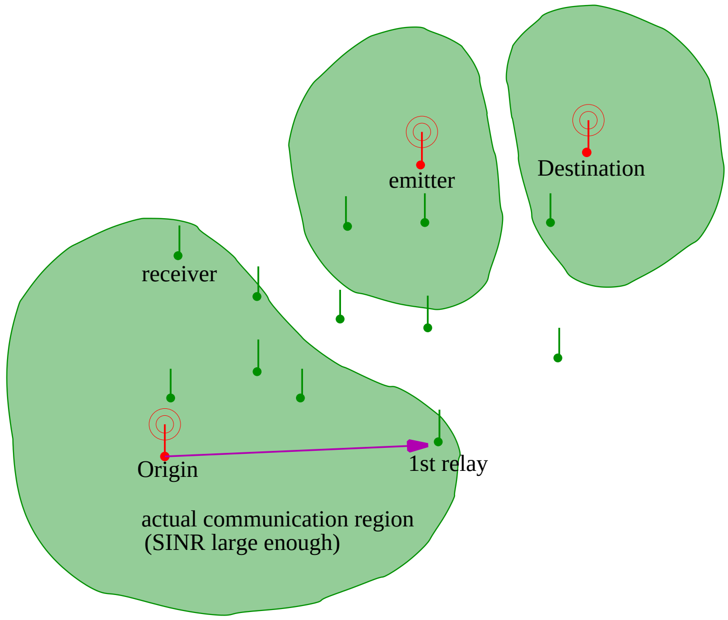


Figure 11: Coverage model for ad hoc network.

- $\{X_i\}$ denote the locations of the nodes,
- $\{e_i\}_i$ the medium access indicator of node i; $e_i = 1$ for the nodes which are allowed to transmit and $e_i = 0$ means that the node is (a potential) receiver. The random variables e_i are independent, with $\mathbf{P}(e_i = 1) = p$.
- $\{S_i\}_i$ denote the powers emitted by the nodes (for which $e_i = 1$); the random variables $\{S_i\}$ are assumed independent and identically distributed with mean $1/\mu$. An important special case considered in [7], which we assume here, is that with exponentially distributed S_i. It allows for a quantitative analysis of the model.
- $\{T_i\}$ are the SINR thresholds corresponding to some channel bit rates or bit error rates; here, for simplicity, we will take $T_i \equiv T$ constant.

In addition to this marked point process, the model is based on a path-loss $l(r)$ where r is the distance between emitter and receiver. We also consider a random variable W, independent of Φ, modeling an external noise.

Note first that Φ can be represented as a pair of independent Poisson p.p. representing transmitters $\Phi^1 = \{X_i : e_i = 1\}$, and receivers $\Phi^0 = \{X_i : e_i = 0\}$, with intensities, respectively, λp and $\lambda(1-p)$.

Let us suppose there is a node located at x that transmits with power S. Suppose there is a node located at $y \in \mathbb{R}^2$. We say that the node at x can communicate to y if and only if

$$\frac{S/l(|x-y|)}{W + I_{\Phi^1}(y)} \geq T\,, \tag{5.7}$$

where I_{Φ^1} is the shot-noise process of Φ^1: $I_{\Phi^1}(y) = \sum_{X_i \in \Phi^1} S_i/l(|y - X_i|)$. Denote by $\delta(x, y, \Phi^1)$ the indicator that (5.7) holds. Note that by stationarity of Φ^1, that the probability $\mathbf{E}[\delta(x, y, \Phi^1)]$ depends only on the distance $x - y$ and *not* on the specific locations of (x, y); so we can use the notation $p_{|x-y|}(\lambda p) = \mathbf{E}[\delta(x, y, \Phi^1)]$, where λp is the intensity of the transmitters Φ^1. Note that this probability

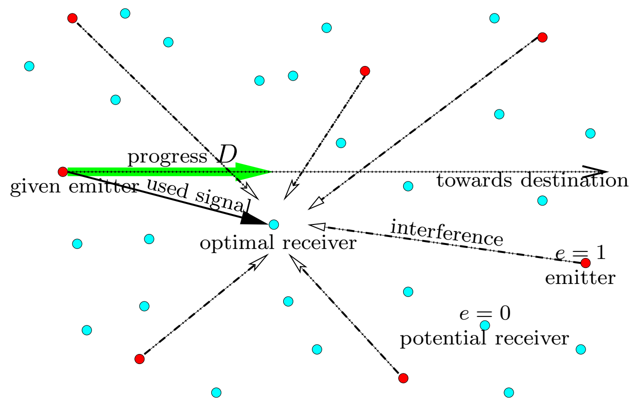


Figure 12: Progress.

is equal to the cell coverage probability of some variant of the coverage model considered in Section 5.2. Moreover, due to exponential assumption on S we have

$$p_R(\lambda) = \exp\left\{ -2\pi\lambda \int_0^\infty \frac{u}{1 + l(R)/(Tl(u))}\,\mathrm{d}u \right\} \mathcal{L}_W(\mu T/l(R))\,, \tag{5.8}$$

where $\mathcal{L}_W(\cdot)$ is the Laplace transform of W (cf (5.6) and see Lemma 3.1 in [7]). Assuming some particular path-loss function (e.g. $l(r) = (Ar)^\beta$) the above expression becomes even more explicit.

Successful transmission versus spatial reuse. The above result is a key ingredient in further analysis of the model. For example, by the Campbell's formula we can state that the mean number of successful transmissions on the distance R per unit surface of our network is equal to $p\lambda p_R(\lambda p)$. This characteristic can be explicitly optimized in p yielding some trade-off between the number of simultaneous transmissions and the probability of the success of a typical transmission; see Section IV in [7].

Optimal relay node. The question of an optimal relay node (already related to routing) can be formulated in our model in the following way. Suppose that a transmitter, say X_0, located under Palm distribution $\mathbf{P}^0$ at the origin $X_0 = 0$ has to send information in some given direction (say along the x axis) to some destination located far from it (say at infinity – see Figure 12). Since the destination is too far from the source to be able to receive the signal in one hop, the source tries to find a non-transmitting node in Φ^0 such that the hop to this node maximizes the distance traversed towards the destination, among these which are able to receive the signal. This node will be later in charge of forwarding the data to the destination or a next intermediary node (this is not taken into account in the model). In this case, the "effective" distance traversed in one hop, which we will call the progress, is equal to

$$D = \max_{X_j \in \Phi^0} \left(\delta(0, X_j, \Phi^1)|X_j| \Big(\cos(\arg(X_j))\Big)^+ \right), \tag{5.9}$$

where $\arg(y)$ is the argument of the vector $y \in \mathbb{R}^2$ ($-\pi < \arg(y) \leq \pi$) and $\delta(x, y, \Phi^1)$ the indicator that (5.7) holds. We are interested in the expectation $d(\lambda, p) = \mathbf{E}^0[D]$ that only depends on λ and on the MAP p, given all other parameters concerning emission and reception. By the Campbell's formula, the mean total distance traversed in one hop by all transmissions initialized in some unit area called *density of progress* is equal to $\lambda p d(\lambda, p)$.

For a given λ, there is again the following trade-off in p between the spatial density of communications and the range of each transmission. For a small p, there are few transmitters per unit area, but each of them can likely reach a very remote receiver as a consequence of the fact that I_{Φ^1} is small. On the other

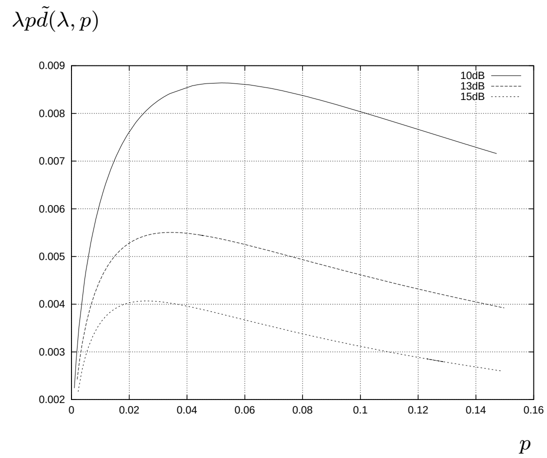


Figure 13: Lower bound of the density of progress for the model with exponential S and path-loss function with function with $\beta = 3$, $\lambda = 1$ and $W = 0$, with $T = \{10, 13, 15\}$dB (curves from top to bottom). The optimal values (argmax, max) are respectively $\{(0.052, 0.0086), (0.034, 0.0055), (0.026, 0.0040)\}$.

hand, a large p means many transmitters per unit area that create interference and thus possibly prevent each other from reaching a remote receiver. Another feature associated with large p is the paucity of receivers, which makes the chances of a jump in the right direction smaller.

It would be thus reasonable to quantify this trade-off and find p that maximizes the density of progress. Unfortunately the mean total distance traversed in one hop d does not admit any explicit expression in our model. We decided thus to analyze the above optimization problem using some lower bound $\tilde{d}(\lambda, p) = \mathbf{E}[\tilde{D}]$ of d, where $\tilde{D}$ is defined as

$$\tilde{D} = \max_{X_j \in \Phi^0} \left(p_{|X_j|}(\lambda p) |X_j| \Big(\cos(\arg(X_j)) \Big)^+ \right), \tag{5.10}$$

(see Proposition 5.2 in [7] for the inequality between d and $\tilde{d}$).

The distribution function of $\tilde{D}$ can be expressed using the known formula for the distribution function of some extremal shot-noise (see Proposition 5.4 in [7]). The formula for its mean $\tilde{D}$ follows easily by integration of its tail distribution function (see Proposition 5.5 therein).

Having expressed the lower bound of the density of progress $\lambda p\tilde{d}$ we can optimize it in p, which is done in Section V.C in [7], and in this way give some conservative (due to the lower bound) answer about the optimal tuning of the MAC parameter p; cf Figure 13.

6 Concluding remarks

We briefly comment here on the impact that some papers contributing to the dissertation have had on other works.

- Results of [4] and [6] contribute to the methodology proposed in (Baccelli et al. 1997) for the macroscopic modeling of communication networks by means of spatial tessellations. This methodology is currently actively developed at the University of Ulm (Germany) in collaboration with France Telecom R&D (France). In particular, the modulated-Poisson Voronoi tessellation model proposed and studied in [4] and [6] is the subject of further analysis in (Fleischer et al. 2007).

- The paper [3] seems to be quite seminal both in the theory and in applications. I has been cited (according to *Science Citation Index*) 7, 2 and, 6 times, respectively, in, mathematical, physical and engineering journals (not counting for auto-citations). It triggered some mathematical research on stochastic geometry models integrating a shot-noise field. For example (Dousse et al. 2006) studies the continuum percolation problem in a variant of our coverage model.

- Paper [7] (or its preliminary version [18]), which proposed a new stochastic geometry approach to ad-hoc networks, was remarked in the literature (cited at least 6 times). In particular, the idea of the explicit calculus of the coverage probability in the case of Poisson repartition of nodes and exponentially distributed powers (cf. (5.6) and the footnote on page 20 for some explanation) was recently extended in (Hunter et al. 2007).

- As far as author's own works are concerned, the articles [3] and [7] give foundations for a *stochastic geometry framework for the modeling of wireless communication networks*. A special feature of this comprehensive framework is the usage of the shot noise in conjunction with other stochastic geometry models to study the geometry of SIR's. This frameworks is being developed through several papers. In particular [14], [7] and [22], [23] developed it in the context of cellular, ad-hoc and sensor networks.

  In a series of more technical papers [17], [15], [20], [21] a dynamic aspect of users served by a large cellular network with power control is considered, engaging elements of queueing theory within the stochastic geometry framework. Using a spatial birth-and-death process to model arrivals, departures and mobility of customers, we obtain (via a spatial Erlang formula) explicit formulas for blocking probabilities as well as the steady state mean throughput and mean delay for elastic (data) traffic in the case of proportional fair service policies. This study was the subject of the PhD thesis (Karray 2007) co-supervised by the author. Three patents [29] were filled on this subject and the results are exploited by Orange, implemented in its UMTS dimensioning tools.

  Other interesting subjects, as routing in mobile ad hoc networks are currently investigated (cf [26]). The whole approach will be presented in the monograph [28].

- Spatial stochastic modeling of wireless networks (using spatial point processes, stochastic geometry, random geometric graphs, etc) is the subject of the SpaSWiN workshop (see `www.spaswing.org`) held each year in conjunction with WiOpt. The author was a co-organizer and a co-chair of the first edition of this workshop in 2005 and is currently organizing its forth edition. The community of researchers who contribute to spatial stochastic modeling of wireless networks is growing and currently the *IEEE Journal on Selected Areas in Communications* (J-SAC) is planning to publish a special issue on this subject.

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