Stochastic geometry and communication networks

Bartek Błaszczyszyn

tutorial lecture

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OUTLINE

I Panorama of some stochastic-geometry models in action,

II Basic geometric models,

III Signal-to-Interference-and-Noise ratio (SINR) coverage model,

IV Modeling ad-hoc networks,

V Power control in CDMA: from static to dynamic modeling — a spatial Erlang formula.

I PANORAMA

- The mathematical principle of the Voronoi tessellation is widely used as a simple idealization of many complex "real" partitions of the plane (cells in cellular communication, Gupta & Kumar protocol model of ad-hoc networks; it takes into account only locations of antennas and ignores all other physical aspects of the communication technology as a.g. additive interference)
- The dual Delaunay graph can be used as a "protocol model" of neighbuorhood in networks ⇒ topology for routing

- Boolean model is a first model of coverage of wireless network (it does not take into account interference). As the underlying model for the study of continuum percolation it can be used to address the questions of connectivity of ad-hoc networks in the absence of interference.
- Mathematical representation of interferences based on (Poisson) shot noise processes ⇒ a variety of results on coverage, connectivity and capacity of large interference-limited networks.

II BASIC GEOMETRIC MODELS

- Poisson point process,
- Voronoi tessellation and Delaunay graph,
- Boolean model,
- Shot-Noise model,
- References.

Poisson Point Process

Planar Poisson point process (p.p.) Φ of intensity λ:

- Number of Points $\Phi(B)$ of Φ in subset B of the plane is Poisson random variable with parameter $\lambda|B|$, where $|\cdot|$ is the Lebesgue measure on the plane; i.e.,

$$\mathbf{P}\{\,\Phi(B)=k\,\} = e^{-\lambda|B|}\,\frac{(\lambda|B|)^k}{k!}\,,$$

- Numbers of points of Φ in disjoint sets are independent.

Laplace transform of the Poisson p.p.

$$\mathcal{L}_\Phi(h) = \mathsf{E}[e^{\int h(x)\,\Phi(\mathrm{d}x)}] = e^{-\lambda\int(1-e^{h(x)}),\mathrm{d}x}\,,$$

where $h(\cdot)$ is a real function on the plane and $\int h(x)\,\Phi(\mathrm{d}x) = \sum_{X_i\in\Phi} h(X_i)$.

Poisson p.p. is the basis of the stochastic-geometry modeling of communication networks.

This modeling consist in treating the given architecture of the network as a snapshot of a (homogeneous) random model, and analyzing it in a statistical way. In this approach the physical meaning of the network elements is preserved and reflected in the model, but their geographical locations are no longer fixed but modeled by random points of, typically, homogeneous planar Poisson point processes.

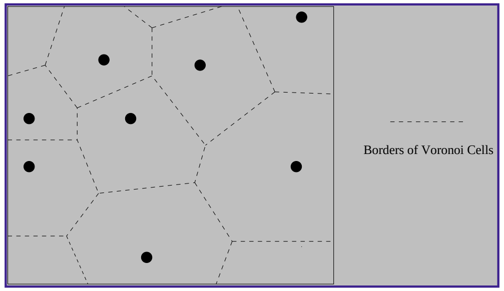
Consequently, any particular detailed pattern of locations is no longer of interest. Instead, the method allows for catching the essential spatial characteristics of the network performance basically through the densities of these point processes (i.e., the densities of the network devices).

Voronoi Tessellation (VT) and Delaunay graph

Given a collection of points $\Phi=\{X_i\}$ on the plane and a given point x, we define the Voronoi cell of this point $\mathcal{C}_x=\mathcal{C}_x(\Phi)$ as the subset of the plane of all locations that are closer to x than to any point of Φ; i.e.,

$$\mathcal{C}_x(\Phi) = \{y\in\mathbb{R}^2 : |y-x| \le |y-X_i|\ \forall X_i\in\Phi\}\,.$$

When $\Phi=\{X_i\}$ is a Poisson p.p. we call the (random) collection of cells $\{\mathcal{C}_{X_i}(\Phi)\}$ the Poisson-Voronoi tessellation (PVT).



Edges of the Delaunay graph connect nuclei of the adjacent cells.

VT is a frequently used generic model of tessellation of the plane.

Points denote locations of various structural elements (devices) of the network (base station antennas and/or network controllers in cellular networks, concentrators in fixed telephony, access nodes in ad hoc networks, etc.).

Cells denote mutually disjoint regions of the plane served in some sense by these devices.

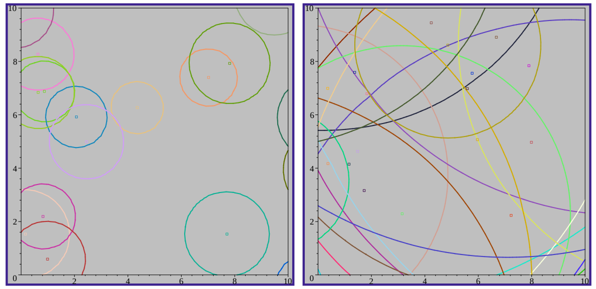
Boolean Model (BM)

Let $\tilde{\Phi} = \{(X_i, G_i)\}$ be a marked Poisson p.p., where $\{X_i\}$ are points and $\{G_i\}$ are iid random closed stets (grains). We define the Boolean Model (BM) as the union

$$\Xi = \bigcup_i X_i \oplus G_i \qquad \text{where } x \oplus G = \{x + y : y \in G\}.$$

Known:

- Poisson distribution of the number of grains intersecting any given set.
- Asymptotic results ($\lambda \to \infty$) for the probability of complete covering of a given set.



BM with spherical grains of random radii

BM is a generic coverage model.

Points denote locations of various structural elements (devices) of the network.

Granis denote independent regions of the plane served these devices .

In wireless networks it is a simplified model (it *does not* take into account interference) for the study of coverage and connectivity.

Shot-Noise (SN) model

Let $\tilde{\Phi} = \{(X_i, S_i)\}$ be a marked p.p., where $\{X_i\}$ are points and $\{S_i\}$ are iid random variables. Given a real response function $L(\cdot)$ of the distance on the plane we define the Shot-Noise field

$$I_{\tilde{\Phi}}(y) = \sum_i S_i L(y - X_i)\,.$$

When $\tilde{\Phi}$ is a marked Poisson p.p. then we call $I_{\tilde{\Phi}}$ the Poisson SN.

For the Poisson SN, the Laplace transform of the vector $(I_{\Phi}(y_1), \ldots, I_{\Phi}(y_n))$ is known for any $y_1, \ldots, y_n \in \mathbb{R}^2$ (via Laplace transform of the Poisson p.p.).

SN is a good model for interference in wireless networks.

Marks S_i correspond to emitted powers.

Response function correspond to attenuation function.

References

1. Stoyan & Kendal & Mekke (1995) *Stochastic Geometry and its Applications.* Wiley, Chichester

2. Okabe & Boots & Sugihara (2001) *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams.* Willey, Chichester.

III SINR COVERAGE MODEL

$\Phi = \{X_i, (S_i, T_i)\}$ marked point process (Poisson)

$\{X_i\}$ points of the p.p. on $\mathbb{R}^2$ — antenna locations,

$(S_i, T_i) \in (\mathbb{R}^+)^2$ possibly random mark of point X_i — (power,threshold)

cell attached to point X_i: $\boxed{C_i(\Phi, W) = \left\{ y : \frac{S_i l(y - X_i)}{W + \kappa I_\Phi(y)} \geq T_i \right\}}$

where $I_\phi(y) = \sum_{i \neq 0} S_i l(y - X_i)$ shot noise process, κ interference factor, $W \geq 0$ external noise, $l(\cdot)$ attenuation (response) function.

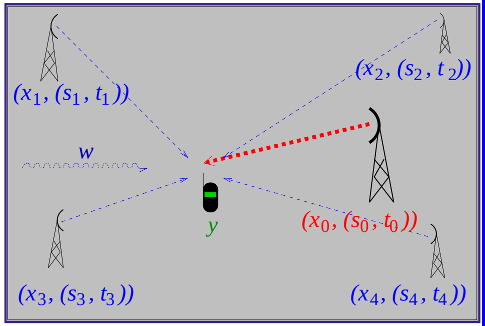
C_i is the region where the SINR from X_i is bigger than the threshold T_i.

Coverage PROCESS: $\boxed{\Xi(\Phi; W) = \bigcup_{i \in \mathbb{N}} C_i(\Phi, W).}$

- Motivations,
- Snapshots and qualitative results,
- Typical cell study
  (coverage probability for the point is some distance to the antenna, simultaneous coverage of several points, mean area of the cell),
- Handoff study
  (overlapping of cells, coverage probability for a typical point, distance to different handoff states),
- Macroeconomic optimization example,
- References.

Motivation I: CDMA handoff cells

x_0 — a point in $\mathbb{R}^2$ (location of an antenna),
$s_0 \geq 0$ and $t_0 \geq 0$ — (pilot signal power of the antenna and SINR threshold (bit energy-to-noise spectral power density $E_b/\mathcal{N}_O$) for the pilot signal),
$\phi = \{x_i, (s_i, t_i)\}$ — pattern of antennas,
$w \geq 0$ — external noise,
$0 \geq \kappa \leq 1$ — orthogonality factor,
$l(\cdot)$ — attenuation function



$$\frac{s_0 l(y - x_0)}{w + I_\phi(y)} \geq t_0$$

Parameter values

Intensity of Poisson process of base stations $\lambda_{BS} \sim 0.2$ BS/km^2.

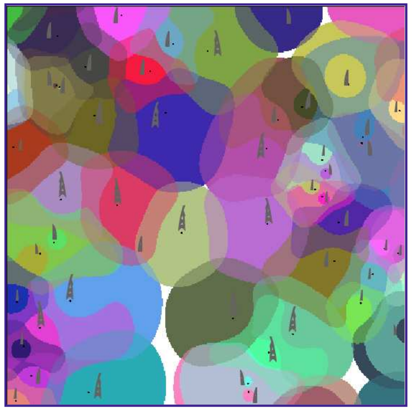
Pilot signal power $s_0 \sim 30$ mW

SINR threshold (bit energy-to-noise spectral power density $E_b/\mathcal{N}_O$) for the pilot $t_0 \sim -14$ dB

External noise $w \sim -105$ dB

Interference factor for pilots from different BS's $\kappa = 1$

Attenuation function
$l(x) = A\max(|x|, r_0)^{-\alpha}$ or
$l(x) = (1 + A|x|)^{-\alpha}$ with $\alpha \sim 3 - 6$.



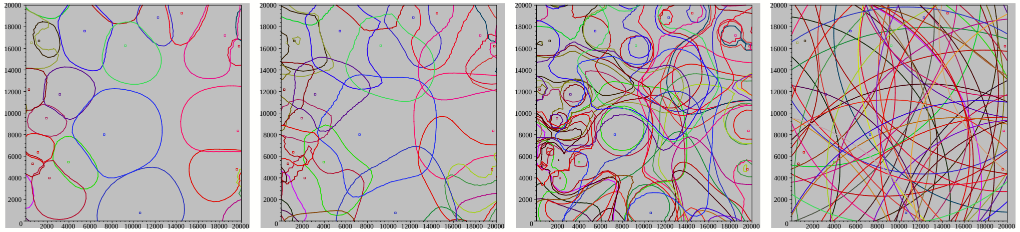
This is a relatively simple model, which takes into account only locations of the Base Stations, their pilot signal powers and SIR's for the pilots.

In particular there is no any pattern of mobiles assumed yet and it does not take into account power control issues.

Gupta & Kummar physical model for ad-hoc networks

(see Modeling of ad-hoc networks)

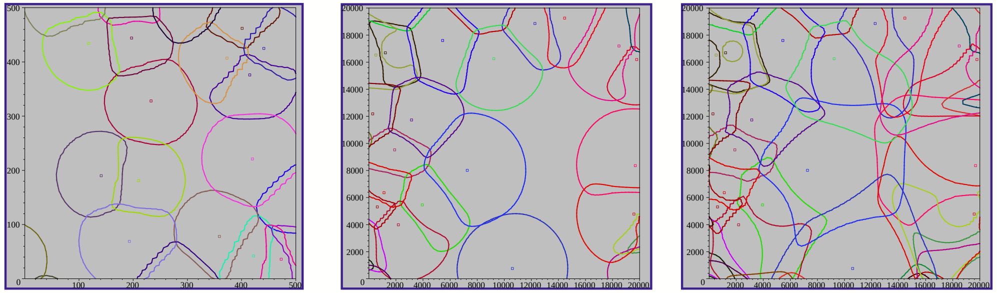
Snapshots and qualitative results



$\kappa = 0.5$ $\kappa = 0.1$ $\kappa = 0.01$ $\kappa = 0$

Constant emitted powers S_i, $e_i \equiv 1$, $T = 0.4$ and interference factor $\kappa \to 0$.

Small interference factor allows one to approximate SINR cells by a Boolean model (via perturbation methods)



Constant emitted powers S_i, $e_i \equiv 1$, $T = 0.4$, $W = 0$, $l(r) = (Ar)^{-\beta}$ and attenuation exponent $\beta \to \infty$.

SIR cells tend to Voronoi cells whenever attenuation is stronger, e.g. in urban areas.

Probability for a typical cell to cover a point

Given: Φ — marked Poisson point process representing antennas in $\mathbb{R}^2$, $(0,(S,T))$ — additional antenna located at fixed point 0 with random (S,T) distributed as any mark of Φ, independent of it (thus $\Phi \cup \{(0,(S,T)\}$ has Poisson Palm distribution), y — location (of a mobile) in $\mathbb{R}^2$.

Probability for C_0 to cover a given point y located at the distance R to the origin:

$$p_R = \mathsf{P}\Big(y \in C_0 \Big)$$
$$= \mathsf{P}\Big(S(1/T-1)l(R) - W - I_\Phi(y) > 0 \Big).$$

Res. For M/G case (general distribution of (S,T)) the coverage probability p_R can be given via Laplace transforms of $S(1/T-1)$, W and the Laplace transform of $I_\Phi(y)$ that is

$$\mathsf{E}[\exp(-\xi I_\Phi(y))] = \exp\left[-\int_{\mathbb{R}^d}\Big(1-\mathcal{L}_S(\xi l(y-z))\Big)\,\mu(\mathrm{d}z)\right],$$

where $\mathcal{L}_S(\xi) = \mathsf{E}[e^{-\xi S}]$ is the Laplace transform of S.

Cor. Fourier transform of the Poisson shot-noise variable $I_\phi(y) \to$ $\boxed{\text{Rieman Boundary Problem}}$ $\to$ probability of coverage by the typical cell.

Example

Fourier transform $\mathcal{F}_{I_\Phi}(\xi)$ of the homogeneous Poisson (intensity λ) shot noise with exponential S (parameter m) and attenuation $l(x) = A\max(|x|, r_0)^{-4}$

$$\begin{aligned}\mathcal{F}_{I_\Phi}(\xi) &= \mathsf{E}\big[e^{-i\xi I_\Phi}\big] \\ &= \exp\left[\lambda\pi\sqrt{\frac{iA\xi}{m}}\arctan\left(r_0^2\sqrt{\frac{m}{iA\xi}}\right) - \frac{\lambda}{2}\pi^2\sqrt{\frac{iA\xi}{m}} \right. \\ &\qquad \left. + \lambda\pi r_0^2\frac{r_0^4 - iA\xi - r_0^4 m}{iA\xi + r_0^4 m}\right],\end{aligned}$$

for $\xi \in \mathbb{R}$, where the branch of the complex square root function is chosen with positive real part.

Special M/M case

Res. [Baccelli&BB&Muhlethaler (2004)]: Assume that $\{S_i\}$ are exponential r.vs. with par. μ and $T_i = T$ are constant. Then the probability for C_0 to cover a given point located at the distance R:

is equal
$$p_R = \exp\left\{-2\pi\lambda\int_0^\infty \frac{u}{1 + l(R)/(Tl(u))}\,\mathrm{d}u\right\}.$$

proof: Say the emitter is at the origin and consider the corresp. Palm distribution $\mathbf{P}$;

$$\begin{aligned} p_R &= \mathbf{P}(S \ge TI_{\Phi^1}/l(R)) \\ &= \int_0^\infty e^{-\mu sT/l(R)}\,\mathrm{d}\mathbf{P}(I_\Phi \le s) \\ &= \psi_{I_\Phi}(\mu T/l(R)),\end{aligned}$$

where $\mathcal{L}_{I_\Phi}(\cdot)$ is the Laplace transform of the value of the hom. Poisson SN I_Φ.

Cor. For the attenuation function $l(u) = (Au)^{-\beta}$

$$p_R(\lambda) = e^{-\lambda R^2 T^{2/\beta} C},$$

where $C = C(\beta) = \Big(2\pi\Gamma(2/\beta)\Gamma(1-2/\beta)\Big)/\beta$.

Some optimizations

One can study the following optimization problems for the expected effective transmission range $r \times p_r$:

- given the density of stations λ find the targeted range r that optimizes the expected effective transmission range

$$\rho = \rho(p) = \max_{r\ge 0}\{rp_r(p)\} = \frac{1}{T^{1/\beta}\sqrt{2\lambda pC}}$$

$$r_{\max} = r_{\max}(p) = \mathrm{argmax}_{r\ge 0}\{rp_r(\lambda)\} = \frac{1}{T^{1/\beta}\sqrt{2\lambda C}}$$

- given the targeted range R find the density of emitters λ that optimize the expected effective transmission range $R \times p_R$:

$$\lambda_{\max} = \lambda_{\max}(R) = \text{argmax}_{\lambda\geq 0}\{Rp_R(\lambda)\} = \frac{1}{R^2T^{2/\beta}C}$$

$$\max_{\lambda\geq 0}\{Rp_R(\lambda)\} = \frac{1}{R^2T^{2/\beta}eC}$$

Probability for a typical cell to cover two points

(y_1, y_2) — two point to be covered by a given cell $C_0(\Phi, W)$ under Palm distribution of $\Phi \cup \{(0,(S,T)\}$

We need the joint Laplace transform of $(I_\Phi(y_1), I_\Phi(y_2))$ that is given by

$$\mathsf{E}\Big[\exp\Big(-\xi_1 I_\Phi(y_1) - \xi_2 I_\Phi(y_2)\Big)\Big]$$
$$= \exp\Big[-\int_{\mathbb{R}^d}\Big(1 - L_S(\xi_1 l(y_1 - z) + \xi_2 l(y_2 - z))\Big)\,\mu(\mathrm{d}z)\Big]\,.$$

Coverage probability via perturbation of Boolean model

valid for small interference factor κ

Denote $p_R^{(\kappa)} = \mathsf{P}(x \in C_0^{(\kappa)})$, where $|x| = R$ and $C_0^{(\kappa)} = \Big\{y \in \mathbb{R}^2 : Sl(y) \geq \kappa I_\Phi(y) + W\Big\}$.

Assume $F_*(u) = \mathsf{P}((Sl(x) - W) \leq u)$ admits Taylor approximation at 0:

$$F_*(u) = F_*(0) + \sum_{k=1}^{h}\frac{F_*^{(k)}(0)}{k!}u^k + \mathcal{R}_*(u)$$

and $\mathcal{R}_*(u) = o(u^h) \quad u \searrow 0$.

Res.

$$p_R^{(\kappa)} = \overbrace{\mathsf{P}\Big(Sl(x) \geq W\Big)}^{\text{value for the Boolean model}} \overbrace{-\sum_{k=1}^{h}\kappa^k\frac{F_*^{(k)}(0)}{k!}\mathsf{E}\left[(I_\Phi(y))^k\right]}^{\text{correcting terms}} + \overbrace{o(\kappa^h)}^{\text{error}}\,,$$

provided $\mathsf{E}[(I_\Phi(x))^{2h}] < \infty$.

Mean cell area formula

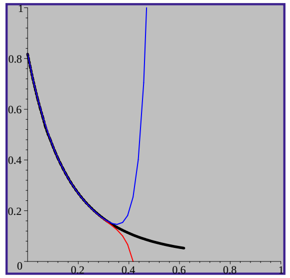
Denote the mean area of the cell of the BS located at 0 by $v_0 = \mathsf{E}[|C_0|]$.

Recall that p_R is the coverage probability for location at distance R.

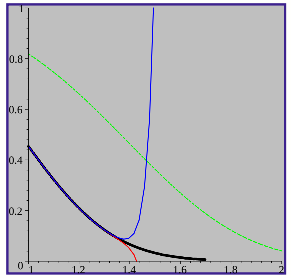
Then

$$v_0 = \int_{\mathbb{R}^d} p_{|y|}\, \mathrm{d}y.$$

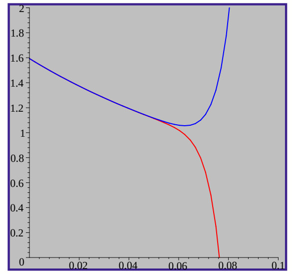
Numerical examples



Probability of coverage as a function of the interference coeffi cient



Probability of coverage as a function of the distance.



Mean cell area as a function of the interference coeffi cient.

Overlapping of cells

Deterministic scenario: given n cells $C(x_i, s_i, t_i; \phi, w)$, $i = 1, \ldots, n$

Res. The inequality $\sum_{i=1}^{n} t_i/(1+t_i) < 1$ is a necessary condition for $\bigcap_{i=1}^{n} C(x_i, s_i, t_i; \phi, w) \neq \emptyset$

Random scenario:

Cor. If the distribution of the ratio T is such that $T \geq \tau$ for some $\tau > 0$, then the number K_y of cells of the coverage process Ξ covering any given point y is a.s. bounded

$$K_y < \frac{1+\tau}{\tau}.$$

(Given point cannot be covered by $(1+\tau)/\tau$ or more cells, no matter how close they are located and how their signal is strong — "pole handoff number".)

Example: For the maximal pilot's bit energy-to-noise spectral power density $\tau = E_b/N_O = -14$ dB the pole handoff number (theoretical maximal handoff number) $K \leq 26$.

Moment expansion of the number of cells K_y covering y

Res. The factorial moment of K_y is given by

$$\mathsf{E}[K_y^{(n)}] = \mathsf{E}[K_y(K_y-1)\dots(K_y-n+1)_+]$$
$$= \int_{(\mathbb{R}^d)^n} \mathsf{P}\Big(y \in \bigcap_{k=1}^{n} C\Big(x_k, S_k, T_k; \Phi + \sum_{\substack{i=1\\ i\neq k}}^{n} \varepsilon_{(x_i,(S_i,T_i))}, W\Big)\Big)$$
$$\times \mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n).$$

Little law

In particular, for a homogeneous Poisson point process with intensity λ

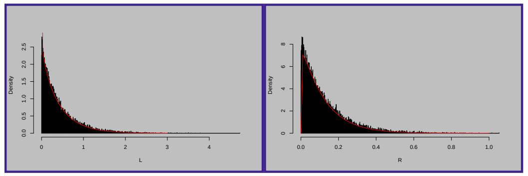
$$\mathsf{E}[K_0] = \lambda \mathsf{E}[|C_0|]\,,$$

where $|C_0|$ is the area of the typical cell. Moreover, in this case the volume fraction p (fraction of the space covered by Ξ) is given by

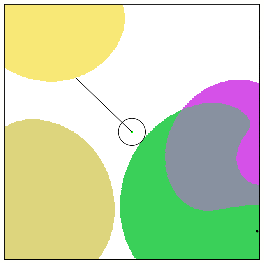
$$p = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \mathsf{E}[(K_0)^{(k)}]\,.$$

Contact distribution functions

Example of contact d.f.'s estimation



Histograms of linear L and spherical R contact d.f. given the point is not covered.



$\mathsf{E}L$	$varL$	$\mathsf{E}R$	$varR$
0.423 km	0.191 km^2	0.121 km	0.013 km^2

Conditional distribution of the model

Two finite sets of points: $z_1,\dots,z_n$ and $z'_1,\dots,z'_p$.

Condition:

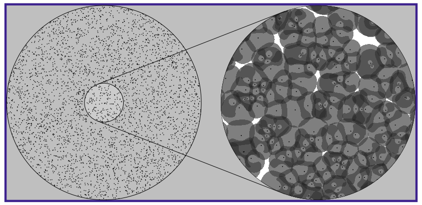
points z_i are covered by at least n_i cells
and
points z'_i are covered by at most n'_i cells,

for some given numbers $n_1,\dots,n_n$ and $n'_1,\dots,n'_p$.

This type of conditions allows one to consider cases where the exact number of cells covering a point is specified.

Almost exact simulation of the shot-noise

For a given size of observation window (radius R) one selects a larger influence window (radius R') in order to get good estimate of the shot-noise term I_ϕ in the smaller observation window.



Th. If the attenuation functions is of the form $l(x,y) < C/|x-y|^\beta$ for some constants $C > 0, \beta > 0$ and if the distribution of S has finite moment $E[S^{1/(\beta/2-\delta)}] < \infty$ for some $\delta \in [1, \beta/2]$, then one can show that for any $R, \varepsilon, \alpha > 0$, there exists $R' > 0$ such that

$$P\left(\sup_{|y|<R} \sum_{|X_i|>R'} S_i l(y, X_i) < \varepsilon\right) > 1 - \alpha\,.$$

Perfect simulation in the observation window

One constructs a Markov process $(\tilde{Z}_t)$ of patterns of points that has for its stationary distribution the conditional distribution.

> Points are generated at exponential periods and located in the window but only if their presence does not violate conditions of maximal coverage of the points z_i'. Points located in the window stay there for exponential times and are removed, but only if their absence does not violate the conditions of maximal coverage of the points z_i'. If a particular removal would lead to the violation, then the point are exponentially perpetuated.

The exact stationary distribution of the Markov process $(\tilde{Z}_t)$ is obtained using backward simulation (coupling from the past) similar to that proposed by Kendall.

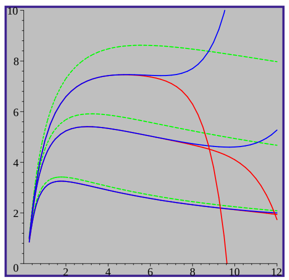
Macroeconomic optimization example: densification / magnification

Increase the mean power m of existing antennas or
increase the density λ of antennas?

C total budget of an operator per km^2,
C_λ cost of one antenna,
C_m cost of increasing the power of one antenna by 1W.

constraint: $\boxed{\lambda C_\lambda + C_m \lambda / m = C.}$

mean handoff level $\mathrm{E}N_0$



mean antenna power $1/m$

Plots of mean handoff as a functions of mean antenna power $1/m$ under budget constraint with $C = 1000, C_\lambda = 500$ and from the top: $C_m = 1, 2, 5$.

Solution: Plot maximum = Optimal confi guration.

References

1. Baccelli & Bł aszczyszyn (2001), On a coverage process ranging from the Boolean model to the Poisson Voronoi tessellation, with applications to wireless communications *Adv. Appl. Probab.* **33**

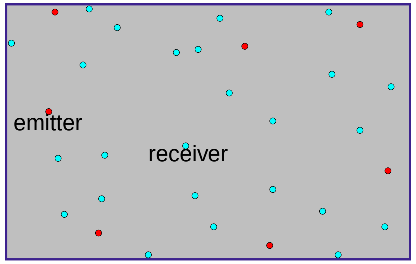
2. Tournois (2002), Perfect Simulation of a Stochastic Model for CDMA coverage *INRIA report* **4348**,

3. Baccelli, Bł aszczyszyn & Tournois (2002), Spatial averages of downlink coverage characteristics in CDMA networks (*INFOCOM*).

IV MODELING AD-HOC NETWORKS

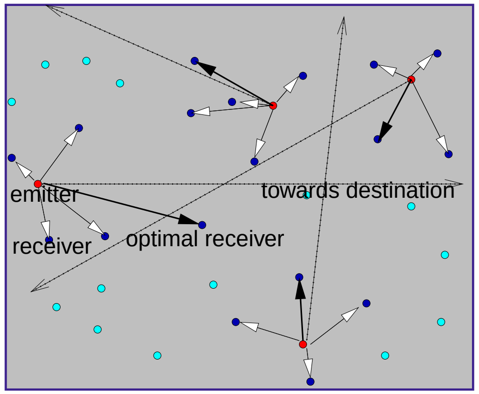
- Ad-hoc networks,
- A few sg "interference aware" models for ad-hoc networks,
- Some optimization problem in capacity and medium access control,
- References.

- A random set of users distributed in space and sharing a common Hertzian medium.
- Users constitute ad-hoc network that is in charge of transmitting information far away via several hops.
- Users switch between emitter and receiver modes.



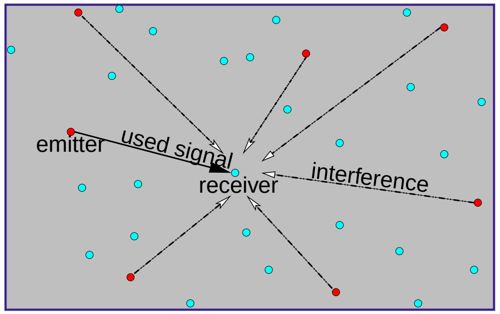
Multi-hop transmissions

- Emitter sends a packet in some given direction far away via several hops.
- The packet is received by some number (possibly 0) of neigbouring receivers.
- An optimal receiver among them is in charge of forwarding this packet in (one of) his next emission time-slots.
- In the case of no reception, emitter re-emits the packet next authorized time.



Interference and successful reception

- Emitter sends a packet emitting some power.
- Transmission distance attenuates the emitted power.
- Emitted power causes interference at all receivers.
- Reception is successful if the Signal to Interference Ratio (SIR) at the receiver is large enough.



A few sg interference aware models

General settings

- Nodes distributed in (a subset of) the plane according to a Poisson p.p. $\Phi = \{X_i\}$,
- Nodes X_i are marked in some way (not necessarily independently) by $e_i = 1$ when node is emitting or 0 when not (it is a potential receiver);
- Call
  - $\Phi^1 = \{X_i \in \Phi : e_i = 1\}$ — emitters and
  - $\Phi^0 = \{X_i \in \Phi : e_i = 0\}$ — potential receivers.

A way of choosing marks e_i for nodes is called medium access control. Two possibilities can be considered:

- Some "central authority" assigns marks e_i (in some optimal way) for each given configuration of nodes
  $\Rightarrow$ $\{e_i\}$ are dependent in some way.
- Each node independently switches between modes $e_i = 1$ and $e_i = 0$
  $\Rightarrow$ $\{e_i\}$ are independent, given configuration of nodes.

Voronoi tessellation principle: Protocol Model [Gupta & Kumar (2000)]

- When the node $X_i \in \Phi^1$ transmits it can be successfully received by node $X_j \in \Phi^0$ if

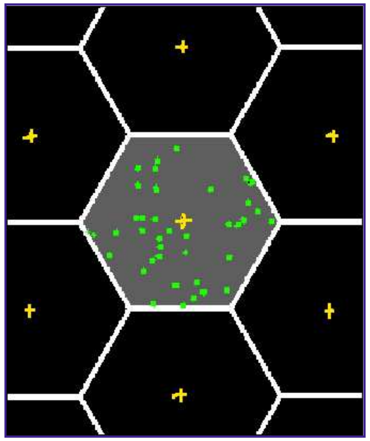
$$|X_j - X_i| \le (1+\Delta)|X_j - X_k| \quad \forall X_k \in \Phi^1\,,$$

where $\Delta \ge 0$ is some constant.

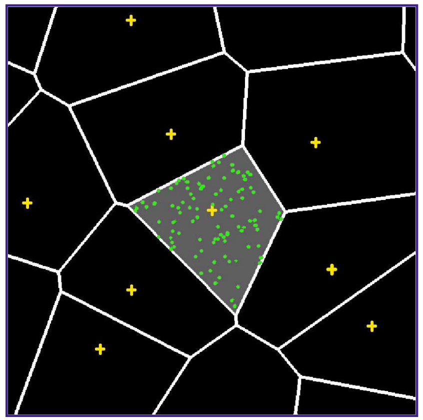
$$\Leftrightarrow \quad X_j \in \text{modifi ed Voronoi Cell } \mathcal{C}_{X_i}(\Phi^1)$$

Examples of medium access

centralized



independent



The Protocol Model with centralized medium access was studied asymptotically (when number of nodes goes to ∞) by G & K (2000); to be commented.

Hard-core principle: Carrier-Sense Model

Close to some currently used protocols implemented e.g. in IEEE 802.11

- When the node $X_i \in \Phi^1$ transmits it can be successfully received by node $X_j \in \Phi^0$ if

$$|X_j - X_i| \le R \quad \text{and} \quad |X_k - X_i| > R_{cs} \;\; \forall X_k \in \Phi^1 ,$$

where $R_{cs} > R \ge 0$ are some constants.

$\Leftrightarrow$ Each communication within the transmission range R is protected by the exclusion disc centered at the transmitter with radius $R_{cs} > R$. (One can also think of the models with exclusion discs centered at the receiver and at both transmitter and receiver.)

Examples of medium access

- Matérn hard-core model:

  $\Rightarrow$ Nodes are first independently marked by some auxiliary marks in $[0, 1]$. A node X_i is selected as emitter ($e_i = 1$) if its auxiliary mark is larger than all auxiliary marks within its neighbourhood of range R_{cs}. Otherwise the node is marked receiver.

- Gibbs hard-core model:

  $\Rightarrow$ The steady state distribution of a spatial birth-and-death process of nodes. When a node is born it is marked as emitter if in its neighbourhood of range R_{cs} there is no any other emitter. Otherwise it is marked receiver.

The Carrier Sense Model, has not been studied yet by means of stochastic geometry tools (to the best of our knowledge).

SINR principle: Physical model

- Points X_i are independently identically marked by some $S_i \ge 0$ representing powers used when emitting signals.
- $l(r) = (Ar)^{-\beta}$ signal attenuation function,
- T signal-to-interference ratio threshold,
- $W > 0$ constant external noise, $\kappa > 0$ interference factor.
- When the node $X_i \in \Phi^1$ transmits it can be successfully received by node $X_j \in \Phi^0$ if

$$\frac{S_i l(|X_j - X_i|)}{W + \kappa I_{\Phi^1 \setminus X_i}(X_j)} \ge T ,$$

where I_{Φ^1} is the shot-noise process of Φ^1 $I_{\Phi^1}(y) = \sum_{X_k \in \Phi^1} S_k l(|y - X_k|)$.

$\Leftrightarrow$ Communication is successful if the signal-to-interference ratio at the receiver is bigger than the threshold T.

Connectivity for the physical model

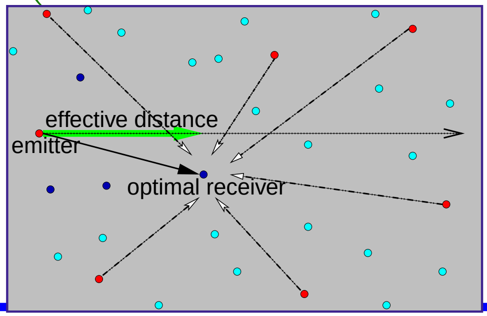
See *Performance'05* tutorial by P. Thiran.

Some optimization problem in capacity and medium access for the physical model

Suppose emitter $X_0 = 0$ has to send information to infinity along the real axis.

Define effective distance traversed in one hop, called also progress

$$D = \max_{X_j \in \Phi^0} \Big(\overbrace{\delta(X_j, 0, \Phi^1)}^{\text{successful rec. ind.}} \overbrace{|X_j| \Big(\cos(\arg(X_j))\Big)^+}^{\text{effective distance}} \Big) ,$$



Define also a modified progress

$$\tilde D = \max_{X_j \in \Phi^0} \Big(\overbrace{p_{|X_j|}(\lambda p)}^{\text{probab. of successful rec.}} \overbrace{|X_j| \Big(\cos(\arg(X_j))\Big)^+}^{\text{effective distance}} \Big) .$$

Denote $d(\lambda, p) = \mathsf{E}[D]$, $\tilde d(\lambda, p) = \mathsf{E}[\tilde D]$.

Search for receivers that realize the progresses, D and $\tilde D$ can be implemented.

Fact The mean total distance traversed in one hop by all transmissions initialized in some unit area (called density of (modified) progress) is equal to $\lambda p\, d(\lambda, p)$ (resp. $\lambda p\, \tilde d(\lambda, p)$).

proof: For $B \subset \mathbb{R}^2$ of unit area, by the Campbell's formula

$$\begin{aligned} \mathsf{E}\Big[\sum_{X_i \in \Phi^1 \cap B} D_i \Big] &= \lambda p \int_{\mathbb{R}^2} 1(x \in B) \mathsf{E}[D_0]\, \mathrm{d}x \\ &= \lambda p\, d(\lambda, p) . \end{aligned}$$

Fact For all $\lambda, p > 0$ we have $d(\lambda, p) \ge \tilde d(\lambda, p)$.

proof follows from Jensen's inequality.

For $z \in [0, 1]$, define an auxiliary function

$$G(z) = 2 \int_{\{t: e^t/\sqrt{2et} \le 1/z\}} \arccos\Big(\frac{z e^t}{\sqrt{2et}} \Big)\, \mathrm{d}t .$$

Res. For the M/M model with $W = 0$ The distribution function of the modified progress is given by

$$F_{\tilde D}(z) = \mathbf{P}(\tilde D \le z) = e^{-\lambda(1-p)(r_{\max}(\lambda p))^2 G(z/\rho(\lambda p))} .$$

proof: $\tilde{D}$ is an extremal shot-noise $\max_{X_i\in\Phi^0} g(X_i)$ with the response function $g(x) = |x| p_{|x|}(\cos(\arg(x)))^+$. Its distribution function can be expressed by the Laplace transform of the (additive) shot noise

$$\mathsf{P}\big(\max_{X_i\in\Phi^0} g(X_i) \le z\big) = \mathsf{E}\Big[\exp\Big[\sum_{X_i\in\Phi^0} \ln(1(g(X_i) \le z))\Big]\Big]$$

and thus, for Poisson p.p. Φ^0 with intensity $\lambda(1-p)$

$$\mathsf{P}(\tilde{D} \le z) = \exp\Big[-\lambda(1-p)\int_{\mathbb{R}^2} 1(g(x) > z)\,\mathrm{d}x\Big]\,.$$

Passing to polar coordinates in the integral $\int_{\mathbb{R}^2}\ldots\,\mathrm{d}x$ completes the proof.

Cor. For the M/M model with $W=0$ the expectation of $\tilde{D}$ is equal to

$$\tilde{d}(\lambda,p) = \mathsf{E}[\tilde{D}] = \frac{1}{T^{1/\beta}\sqrt{2\lambda p e C}}\int_0^1 1-\exp\Big[\Big(1-\frac{1}{p}\Big)\frac{G(z)}{2T^{2/\beta}C}\Big]\,\mathrm{d}z\,,$$

The optimization

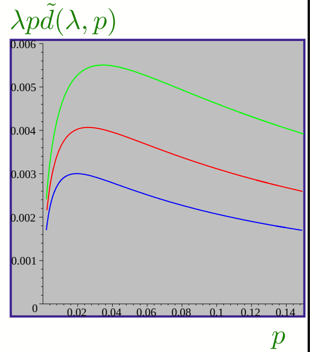
Goal: optimize the density of progress $\lambda p\, d(\lambda,p)$ in MAP p, for fixed λ.

Note that replacing d by $\tilde{d}$ in the optimization problem will result in conservative bound on the density of progress.

Res. The maximal density of modified progress $\lambda p\, \tilde{d}(\lambda,p)$ is attained for p satisfying

$$\int_0^1 \Big(1+\frac{G(z)}{pT^{2/\beta}C}\Big)\exp\Big[\Big(1-\frac{1}{p}\Big)\frac{G(z)}{2T^{2/\beta}C}\Big]\,\mathrm{d}z = 1\,.$$

Example: Modified transport capacities for the special case with $T=\{13,15,17\}$dB (curves from top to bottom).



Relation to Gupta & Kumar's (2000) result

- The transport capacity of a bounded network with Protocol and SIR (Physical) Model with centralized medium access is proportional to $\sqrt{\lambda}$ (as $\lambda \longrightarrow \infty$). This law holds true even if the network architecture and operation is optimally organized in a centralized manner.
- Our density of progress can be seen as an instantaneous transport capacity.
- We show that the independent medium access gives an instantaneous transport capacity of $K(p)\sqrt{\lambda}$.
- Closed form of $K(p)$ allows for optimization of $K(p)$ in p.
- Question: Is this performance *implementable and stable* in the strong sense (existence of positive recurrent Markov process describing the system)?

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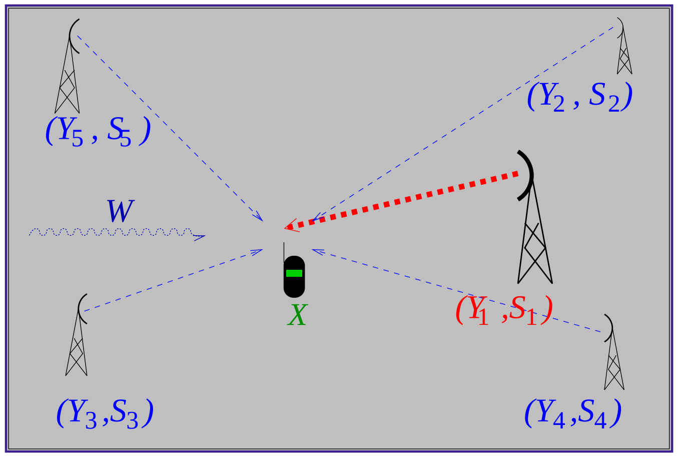
- Dousse & Baccelli & Thiran (2003) Impact of Interferences on the Connectivity of Ad Hoc Networks, *IEEE INFOCOM*,
- Baccelli & Blaszczyszyn & Muhlethaler (2004), An Aloha protocol for multihop mobile ad-hoc wireless networks, *Proc of 16th ITC Specialist Seminar* to appear in *IEEE Inf. Theory*

V POWER CONTROL IN CDMA: FROM STATIC TO DYNAMIC MODELING

- CDMA Power allocation algebra,
- Decentralized Power Allocation Principle DPAP,
- Network architecture models,
- Maximal load estimates,
- Feasibility probabilites,
- Blocking rates via a spatial Erlang formula,
- References.

CDMA: Interference limited radio channel



$$\frac{S_1 l(X - Y_1)}{W + I(X)} \geq C$$

CDMA Power allocation algebra

$\mathcal{N}_{BS} = \{Y_j\}_j$: locations of base-stations (BS's) in $\mathbb{R}^2$,
$\mathcal{N}_M = \{X_i^j\}_i$: locations of mobiles served by BS No. j,
C_i^j: SINR required for mobile X_i^j,
W_i^j: total non-traffic noise at X_i^j (from common overhead channels, thermal noise),
κ_j, γ: orthogonality factors,
$l(x, y)$: path-loss from y to x.

Power allocation feasible if exist antenna powers $0 \le S_i^j < \infty$ such that

$$\boxed{\frac{S_i^j l(Y_j - X_i^j)}{W_i^j + \underbrace{\kappa_j l(Y_j, X_i^j) \sum_{i' \ne i} S_{i'}^j}_{\text{own-cell interference}} + \underbrace{\gamma \sum_{k \ne j} l(Y_k, X_i^j) \sum_{i'} S_{i'}^k}_{\text{other-cell interference}}} \ge C_i^j \text{ all } i, j.}$$

Local and global problem

Power allocation feasible

$\Updownarrow$

Local power allocation feasible for each BS and Global power allocation feasible

Local problem

Fix BS j.

Fix total powers emitted on traffic channels by other BS's: $S_k = \sum_{i'} S_{i'}^k$ ($k \ne j$).

Power allocation is locally feasible in cell j if exist powers $0 \le S_i^j < \infty$ s.t.

$$\boxed{\frac{S_i^j l(Y_j - X_i^j)}{W_i^j + \kappa_j l(Y_j, X_i^j) \sum_{i' \ne i} S_{i'}^j + \underbrace{\gamma \sum_{k \ne j} l(Y_k, X_i^j) \sum_{i'} S_{i'}^k}_{\text{fixed}=\gamma \sum_{k \ne j} l(Y_k, X_i^j) S_k}} \ge C_i^j \text{ all } i.}$$

Res. Local power allocation is feasible iff $\boxed{\kappa_j \sum_i \frac{C_i^j}{1 + \kappa_j C_i^j} < 1.}$

(local) pole capacity condition

Global problem

Suppose for each BS local power allocation is feasible.

Define:

$$a_{jk} = \gamma \sum_i \frac{H_i^j l(Y_k, X_i^j)}{l(Y_j, X_i^j)} \text{ for } j \ne k \quad \text{and} \quad a_{jj} = \sum_i \kappa_j H_i^j,$$

$$b_j = \sum_i \frac{H_i^j W_i^j}{l(Y_j, X_i^j)}, \qquad \text{where } H_i^j = \frac{C_i^j}{1 + \kappa_j C_i^j}.$$

Denote the matrix $(a_{jk}) = \mathbf{A}$, the vector $(b_j) = \mathbf{b}$ and $(S_j) = \mathbf{S}$.

Global power allocation is feasible if exist antenna powers $0 \le S_j < \infty$ (total powers emitted on traffic channels) such that

$$\boxed{\mathbf{S} \ge \mathbf{b} + \mathbf{A}\,\mathbf{S}}$$

Res.

- Global power allocation feasible iff the spectral radius of the matrix $\mathbf{A}$ is less than 1.
- The minimal solution $\mathbf{S}$ is equal to $\sum_n \mathbf{A}^n\,\mathbf{b}$.
- The minimal solution can be obtained as the limit of the iteration $\mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \ldots$
- A sufficient condition for the spectral radius to be less than one is that $\mathbf{A}$ is substochastic (has row-sums less then 1). $\Longrightarrow$ Decentralized Power Allocation Principle

Decentralized Power Allocation Principle (DPAP)

Each BS j verifies for the pattern $\mathcal{N}_M^j$ of the mobiles it controls if

$$\boxed{\underbrace{\kappa_j \sum_{X_i^j \in \mathcal{N}_M^j} H_i^j}_{\text{pole capacity term}} + \underbrace{\gamma \sum_{k \neq j} \sum_{X_i^j \in \mathcal{N}_M^j} H_i^j \frac{l(Y_k, X_i^j)}{l(Y_j, X_i^j)}}_{\text{other-cell-interference correcting term}} < 1\,,}$$

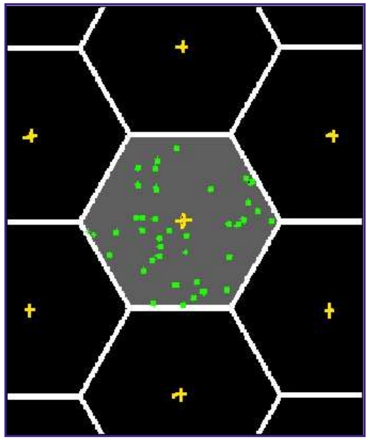
equivalently

$$\boxed{\sum_{X_i^j \in \mathcal{N}_M^j} \underbrace{H_i^j \frac{\text{total path loss of user } i}{\text{own-BS path loss of user } i}}_{\text{user's } i \text{ weight}} < 1\,.}$$

Network architecture models

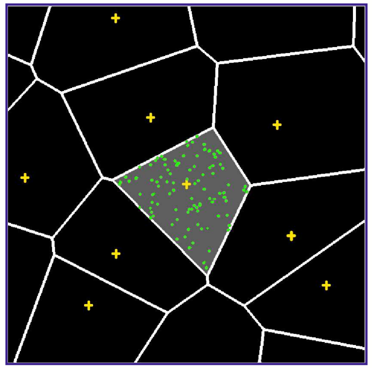
Hexagonal (Hex) model ("too regular")

- BS's $\{Y_j\}$ located according to hexagonal grid, p.p, with spatial density λ_{BS}.
- All antenna parameters are i.i.d. marks.
- All mobiles form independent Poisson p.p. $\mathcal{N}_M$ with spatial density λ_M.
- Each mobile is served by the nearest BS.



Poisson-Voronoi (P-V) model ("too random")

- BS's $\{Y_j\}$ located according to Poisson p.p, with intensity λ_{BS}.
- All antenna parameters are i.i.d. marks.
- All mobiles form independent Poisson p.p. $\mathcal{N}_M$ with intensity λ_M.
- Each mobile is served by the nearest BS. (Equivalently: Each BS j serves mobiles $\mathcal{N}_M^j$ in its Voronoi cell.)



Maximal load estimations of P-V and Hex model

(Wrong) Idea: Given density of BS's λ_{BS} find maximal density of mobiles λ_M, such that power allocation is feasible with probability 1.

Res. Given density of BS's λ_{BS}, for any $\lambda_M > 0$ in both P-V and Hex model, the spectral radius of $\mathbf{A}$ is equal ∞ with probability 1, and thus power allocation in not feasible!

Conclusion: A reduction of mobiles (admission control) is necessary for any $\lambda_M > 0$. Calculate blocking probabilities.

Explicit mean formulas

$\bar{M}$ – mean number of users per cell, $R\ \sim$ – mean distance between BS's;
$\lambda_{BS} = 1/(\pi R^2)$, $L(r) = (Kr)^\alpha$ – path-loss, α – path-loss exponent,
κ – (downlink) orthogonality factor, $\bar{H}\ \sim$ – bit rate (SINR threshold)
P_{max} – power limit

downlink:

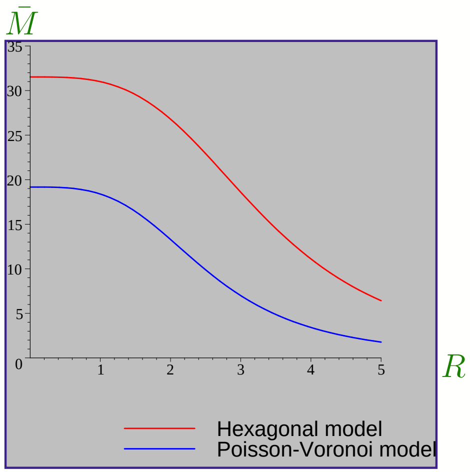
$$\boxed{\bar{M} \le \frac{1 - P_{cch}/P_{max}}{\bar{H}(\kappa + \bar{f} + L(R)g(\bar{M},\alpha)/P_{max})}}$$

$\bar{f}_H = 1/(\alpha - 2)$, $\bar{f}_{PV} = 2/(\alpha - 2)$, $\quad \bar{g}_H() \approx \dots,\ \bar{g}_{PV}() = \dots$

uplink:

$$\boxed{\bar{M} \le \frac{1}{\bar{H}(1 + \bar{f} + L(R)h(\bar{M},\alpha)/P_{max})}}$$

$\bar{h}_H() \approx \dots,\ \bar{h}_{PV}() = \dots$



Mean maximal load for downlink with power constraints;
number of users in cell as a function of the distance between BS's.

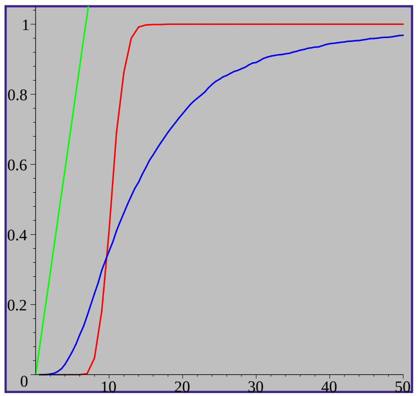
Feasibility probabilities for DPAP

$$\mathbf{P}\Bigg(\underbrace{\sum_{X_i^j \in \mathcal{N}_M^j} H_i^j \frac{\text{total path loss of user } i}{\text{own-BS path loss of user } i} < 1}_{\text{DPAP satisfied}}\Bigg).$$

It says says how often an non-constrained Poisson configuration of users in a given cell cannot be entirely accepted by the admission scheme DPAP.

POWER CONTROL / Feasibility probabilities ...

Pr. of DPAP



λ_M

$\lambda_{BS}=0.18$ BS/km^2
$C=0.011797$
$\gamma=1$
$\kappa=0.2$
$\alpha=3$

Simulated DPAP failure probability for P-V model (more flat curve) and Hex model (more steep) curve. The straight line corresponds to the explicit “mean capacity” of the P-V model.

POWER CONTROL ...

Blocking rates under DPAP — spatial dymamic modeling

- Fix one BS, say $Y^0=0$. Denote its cell, considered as a subset of $\mathbb{R}^2$, by C_0.
- Spatial Birth-and-Death (SBD) process of call arrivals to C_0:
  - for a given subset $A\subset C_0$, call inter-arrival times to A are independent exponential random variables with mean $1/\lambda(A)$, where $\lambda(\cdot)$ is some given intensity measure of arrivals to C_0 unit of time,
  - call holding times are independent exponential random variables with mean τ.
- Call acceptance/rejection: given some configuration of calls in progress $\{X_m\ inC_0\}$, accept a new call at x if $f(x)+\sum_m f(X_m)<1$, where $f(\cdot)$ is the call weight function defined on C_0, and reject otherwise.

POWER CONTROL / Blocking rates ...

Define blocking rate associated with a given location in the cell is the fraction of users arriving according to the SBD process at this location that are rejected.

Res.

- The stationary distribution (time-limit) of the (non-constrained) SBD process of call arrivals is the distribution of a (spatial) Poisson process Π with density $\lambda(\cdot)$.
- The stationary distribution of calls accepted is the distribution of a Poisson process truncated to the state space $\{\sum_m f(X_m)<1\}$.
- Blocking rate b_x at $x\in C_0$ is given by the spatial Erlang formula

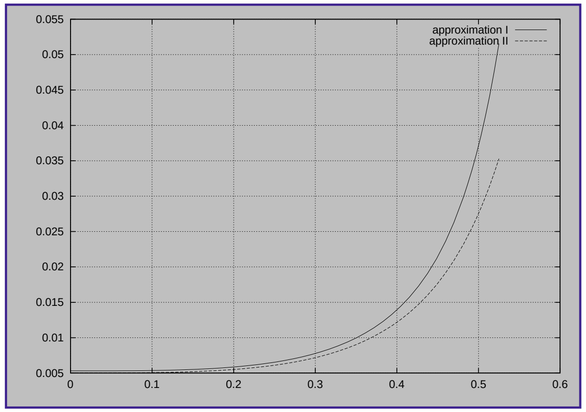
$$b_x=\frac{\Pi\{1-f(x)\le\sum_m f(X_m)<1\}}{\Pi\{\sum_m f(X_m)<1\}}\,.$$

Rem. Note that $\sum_m f(X_m)$ is a compound Poisson r.v., whose distribution can be effectively approximated by Gaussian distribution.

POWER CONTROL / Blocking rates ...

Numerical results (assumptions correspond to UMTS)

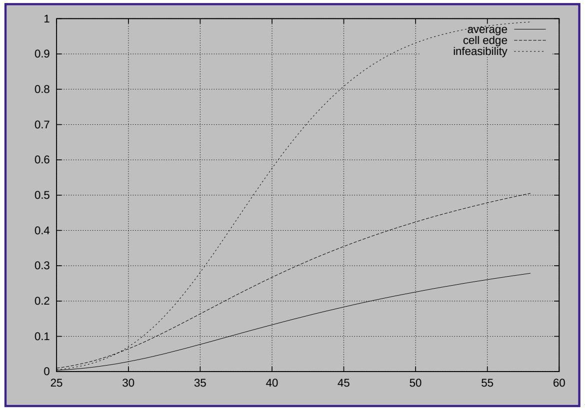
Blocking rate



normalized distance

Approximations of the blocking probability as functions of the distance to BS for the mean number $\bar{M}=27$ of users per cell.

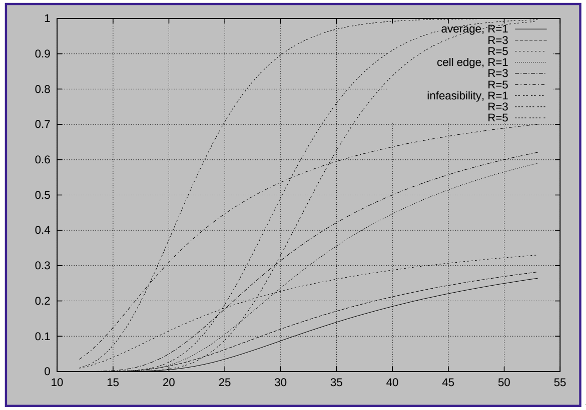
Blocking rate



$\bar{M}$

Blocking probability at the cell edge, average blocking probability, and feasibility probability as functions of the mean number of users in hexagonal cell.

Blocking rate



$\bar{M}$

Blocking probability at the cell edge, average blocking probability, and feasibility probability as functions of the mean number of users in hexagonal cell; comparison for various cell radii R.

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