Géométrie aléatoire: un cadre pour la modélisation des réseaux sans fil

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INRIA/ENS TREC

Séminaire DGA/INRIA

INRIA, Rocquencourt, le 19 mars 2009

A comprehensive

STOCHASTIC GEOMETRY (SG) FRAMEWORK

for the modeling of WIRELESS NETWORKS

wireless networks $\Leftarrow$ SG

$\Uparrow$

signal detection models

$\Uparrow$

radio wave propagation models

Developed at INRIA by several researchers (F. Baccelli, BB, Ph. Jacquet P. Mühlethaler, ...) in collaborations with several academic and industrial partners (EPFL, Stanford, SPRINT, Orange Labs, Thomson, ...).

WHAT IS STOCHASTIC GEOMETRY (SG)

an ancient theory and modern contexts

SG is now a reach branch of applied probability, which allows to study random phenomena on the plane or in higher dimension; it is intrinsically related to the theory of point processes.

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Initially its development was stimulated by applications to biology, astronomy and material sciences. Nowadays, it is also used in image analysis and in the context of communication networks.

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One can consider Gilbert's paper of 1961 both as the first paper on continuum percolation (percolation of the Boolean model) and as the first paper on the analysis of the connectivity of large wireless networks by means of stochastic geometry.

The paper of 62 is on Poisson-Voronoi tessellations.

Followers

PHASE I — domination of the cable: The first papers following Gilbert's ideas appeared in the modern engineering literature shortly before year 2000 (before the massive popularization of wireless communications) and were using mainly the classical stochastic geometry models (as Voronoi tessellations or Boolean model) trying to fit them to existing networks.

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PHASE II — wireless revolution: Nowadays, the number of papers using some form of stochastic geometry is increasing very fast in conferences like Infocom or Mobicom, where one of the most important observed trends is **an attempt to better take into account in geometric models specific mechanisms of wireless communications**.

Classical SG models

- Poisson point process,
- Voronoi tessellation,
- Boolean model.

Poisson Point Process

Planar Poisson point process (p.p.) Φ of intensity λ:

- Number of Points $\Phi(B)$ of Φ in subset B of the plane is Poisson random variable with parameter $\lambda|B|$, where $|\cdot|$ is the Lebesgue measure on the plane; i.e.,

$$\mathbf{P}\{\,\Phi(B) = k\,\} = e^{-\lambda|B|}\,\frac{(\lambda|B|)^k}{k!}\,,$$

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Laplace transform of the Poisson p.p.

$$\mathcal{L}_\Phi(h) = \mathbf{E}\big[e^{\int h(x)\,\Phi(\mathrm{d}x)}\big] = e^{-\lambda\int(1-e^{h(x)})\,\mathrm{d}x}\,,$$

where $h(\cdot)$ is a real function on the plane and $\int h(x)\,\Phi(\mathrm{d}x) = \sum_{X_i\in\Phi} h(X_i)$.

Poisson p.p. is a very basic model. Used to represent:

- the repartition of users in all kind of networks,
- locations of nodes in ad hoc, mesh and sensor networks,
- locations of base stations (access points) in irregular cellular network architectures.

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Need for models exhibiting more clustering, attraction, repulsion of points $\Rightarrow$ Cox models (doubly stochastic Poisson p.p's), Gibbs p.p., Hard-core p.p. and others (Determinental/Permimental p.p's ?)

Voronoi Tessellation (VT)

Given a collection of points $\Phi = \{X_i\}$ on the plane and a given point x, we define the Voronoi cell of this point $\mathcal{C}_x = \mathcal{C}_x(\Phi)$ as the subset of the plane of all locations that are closer to x than to any point of Φ; i.e.,

$$\mathcal{C}_x(\Phi) = \{y \in \mathbb{R}^2 : |y - x| \leq |y - X_i| \; \forall X_i \in \Phi\} \, .$$

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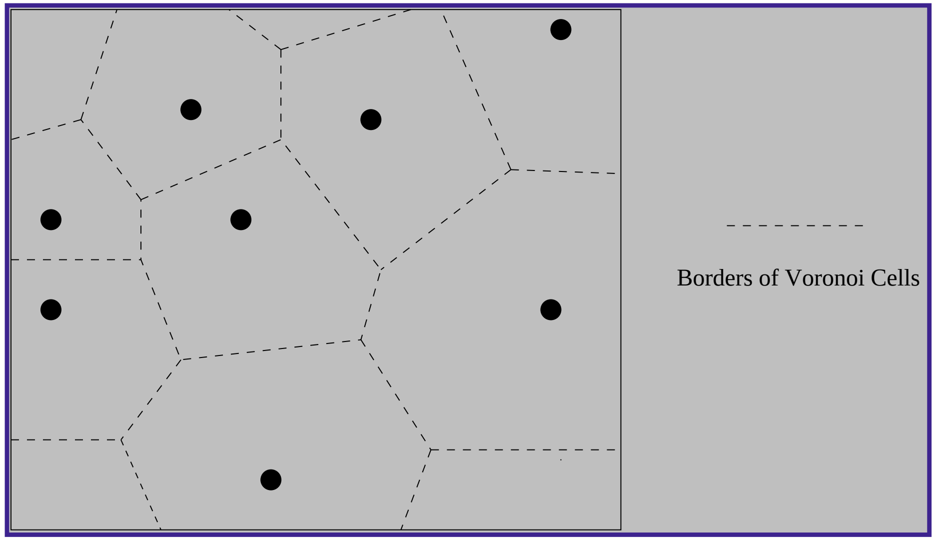
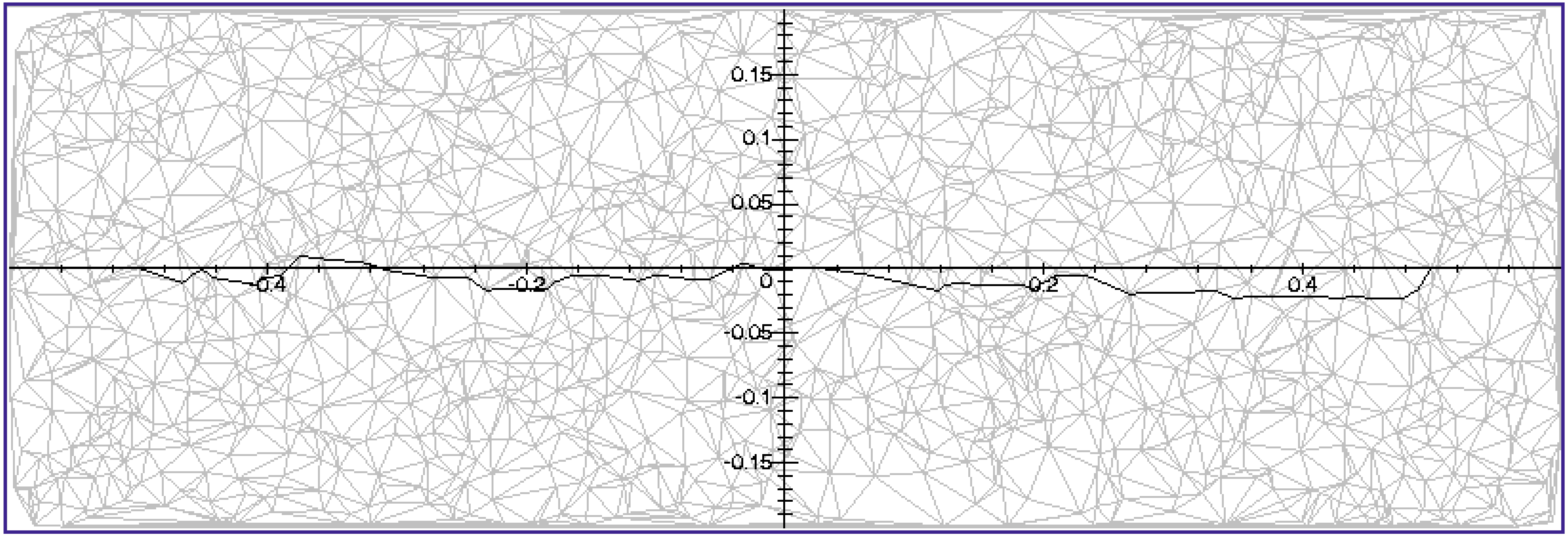
When $\Phi = \{X_i\}$ is a Poisson p.p. we call the (random) collection of cells $\{\mathcal{C}_{X_i}(\Phi)\}$ the Poisson-Voronoi tessellation (PVT).



A dual model to Voronoi tessellation, the Delauney triangulation is used as a (connected) graph of nearest neighbours (e.g. for routing purpose).



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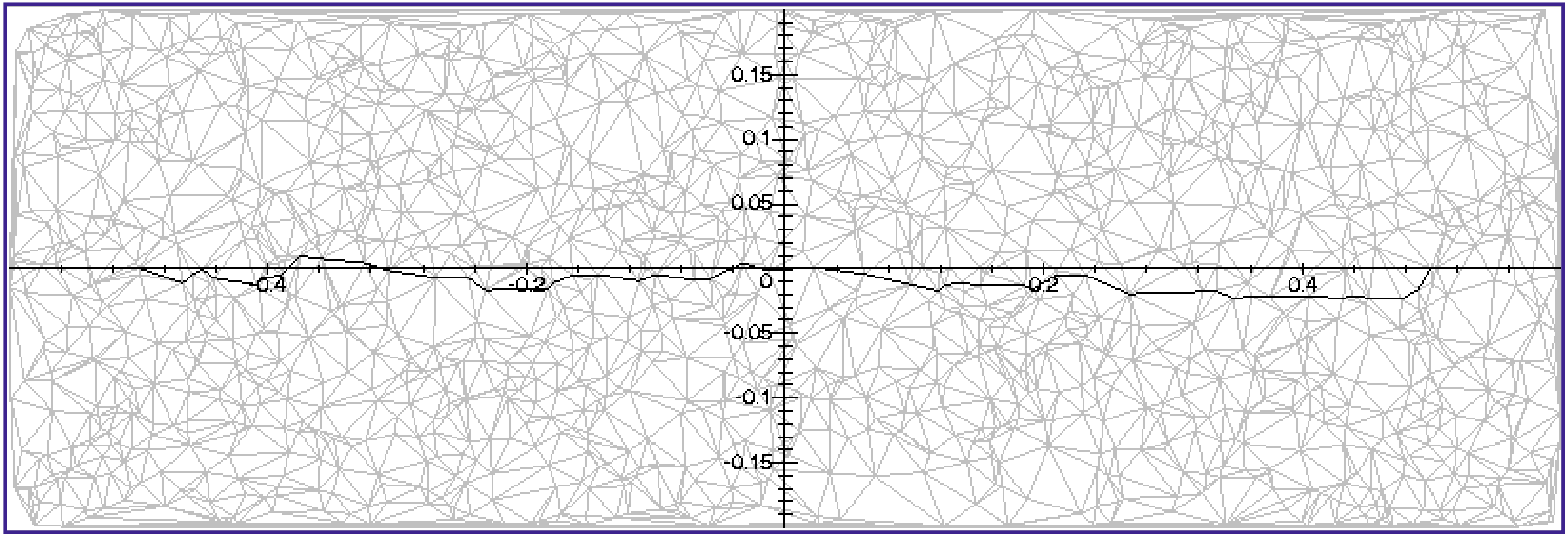

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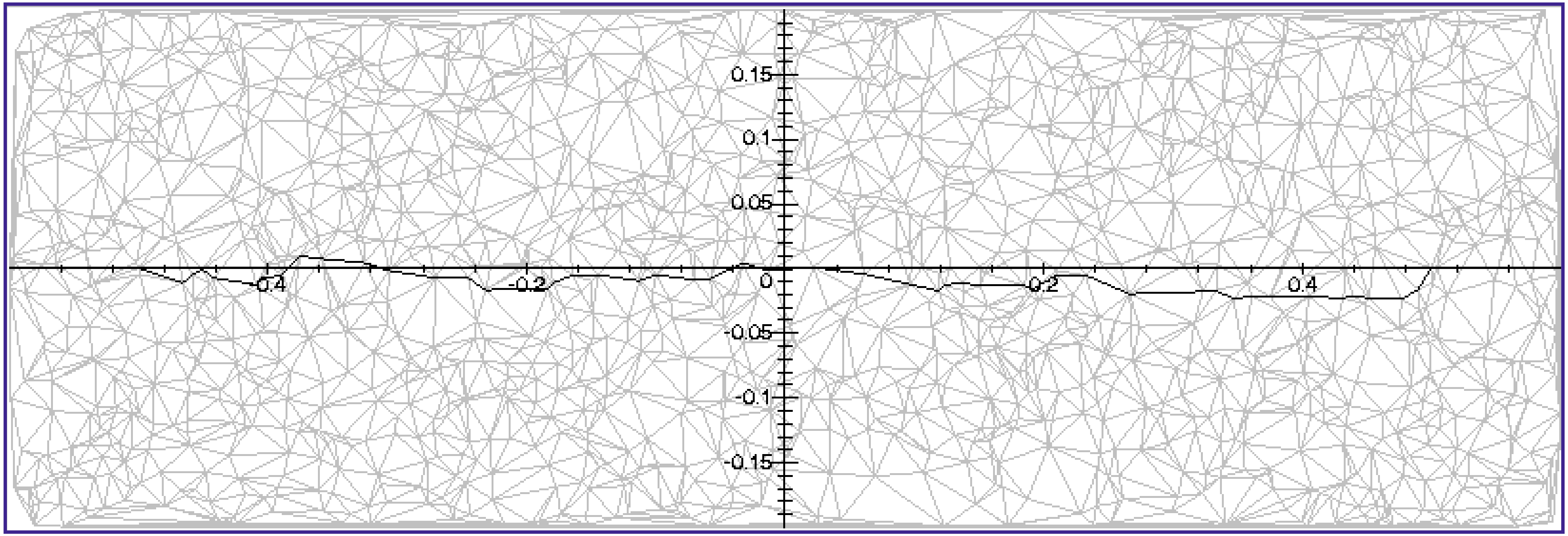


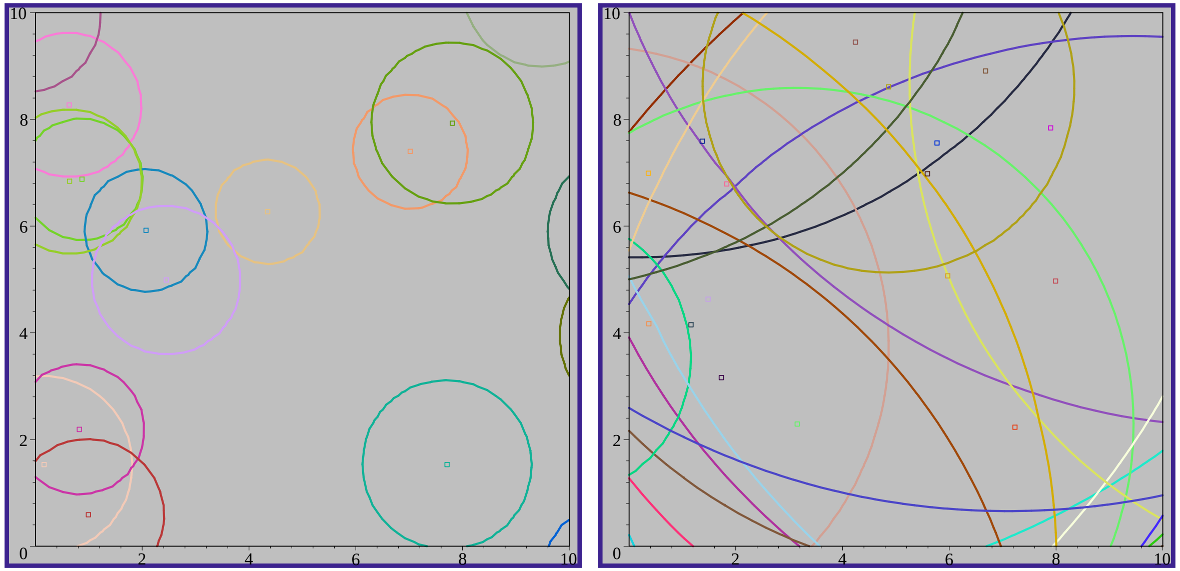
:-) VT takes into account distance to nearest BS's (neighbourhood model)

:-(ignores other physical aspects of the communication as path loss, interference.

Boolean Model (BM)

Let $\tilde{\Phi} = \{(X_i, G_i)\}$ be a marked Poisson p.p., where $\{X_i\}$ are points and $\{G_i\}$ are iid random closed stets (grains). We define the Boolean Model (BM) as the union

$$\Xi = \bigcup_i X_i \oplus G_i \qquad \text{where } x \oplus G = \{x + y : y \in G\}.$$



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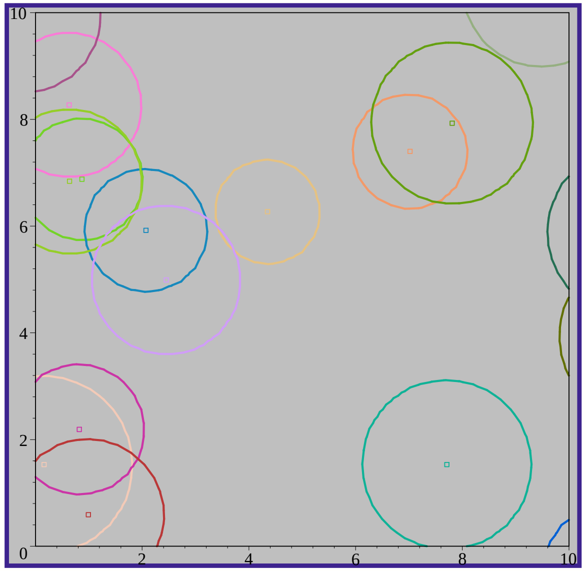

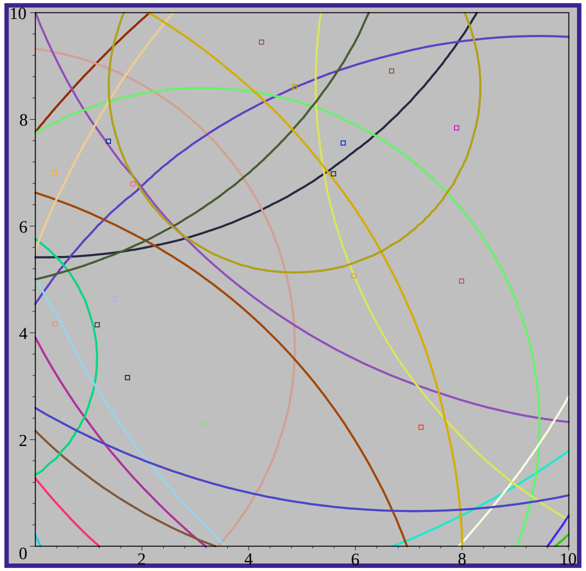

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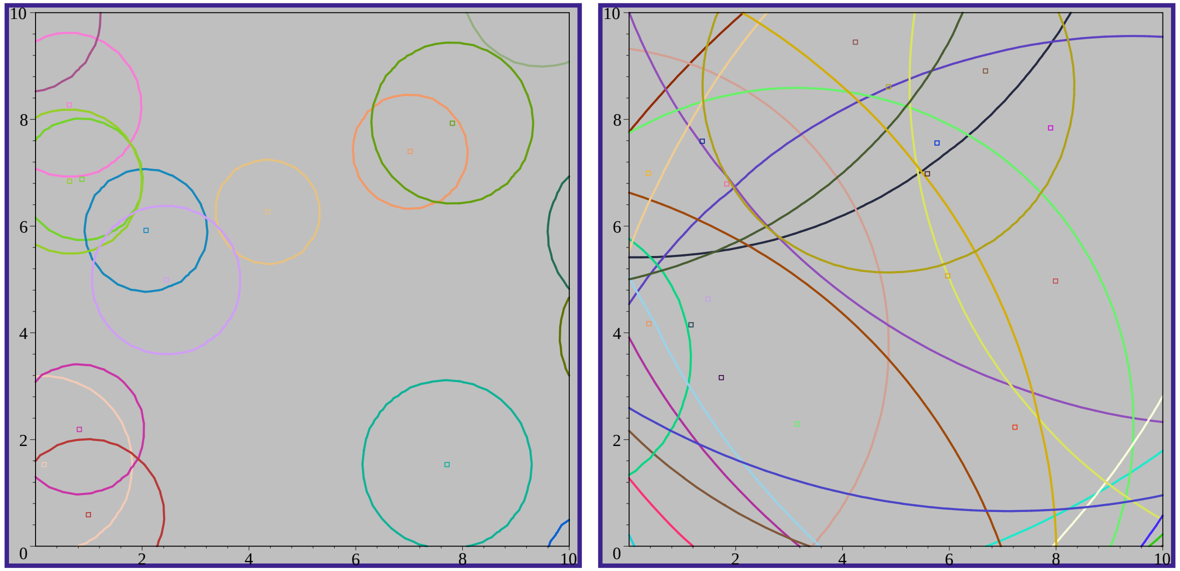
Known:

- Poisson distribution of the number of grains intersecting any given set.
- Asymptotic results ($\lambda \to \infty$) for the probability of complete covering of a given set.



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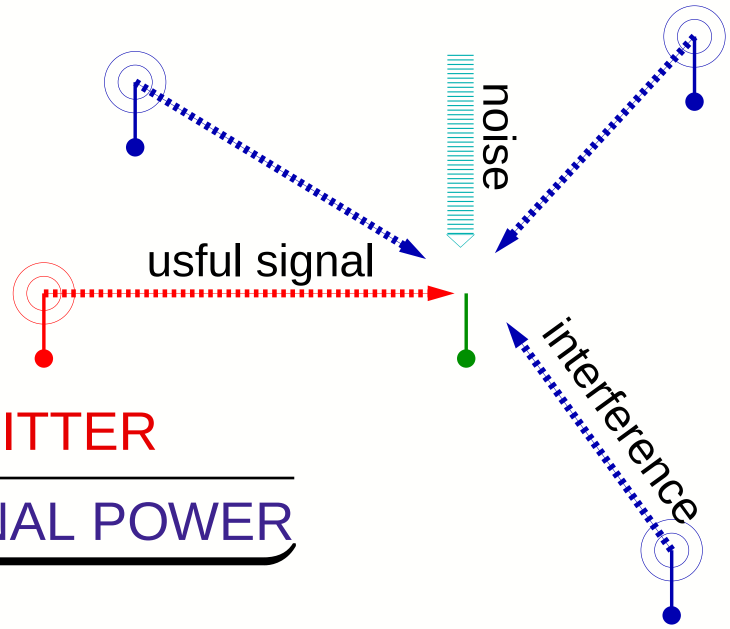
:-(Ignores interference effect (as coverage regions are independent).

SG FOR WIRELESS NETWORKS



$$\text{SINR} = \frac{\text{POWER RECEIVED FROM GIVEN EMITTER}}{\text{NOISE POWER} + \underbrace{\text{OTHER RECEIVED SIGNAL POWER}}_{\text{interference}}}$$

- SINR characterizes the throughput of the radio channel to a given location.
- SINR depends very much on the geometry of the location of nodes!
- Nodes' location $\Rightarrow$ realization of some random (point) process (mobile users, mesh networks, etc.)
- Many nodes' and channels' characteristics (MAC states, fading, etc) $\Rightarrow$ random marks of points.

SG for wireless...

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We believe this methodology will play the same role as queuing theory in wireline systems, where it was instrumental in designing the first multiprogramming computers and the basic protocols used in computer networks and in particular in the Internet.

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- SINR coverage model: foundations of the GS framework for wireless networks,

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- MAC in MANETS: an optimal tuning of MAC, MAC/Routing interplay.
- Opportunistic routing in MANETS: Opportunistic scheduling had a major impact on cellular networks with e.g. HSDPA. In mobile ad hoc networks, opportunistic routing strategies, which take advantage of both time and space diversity, seem to significantly outperform classical routing strategies, where packets are routed on a pre-defined path usually obtained by a shortest path routing protocol and then implemented by the MAC.

SG ALLOWS FOR TWO TYPES OF RESULTS

- Scaling laws: Discover main tendencies of the network behaviour when number of nodes increases using mathematical formalism (percolation theory, limiting theory for random graphs, ...). Difficult/impossible via pure, crude simulation.

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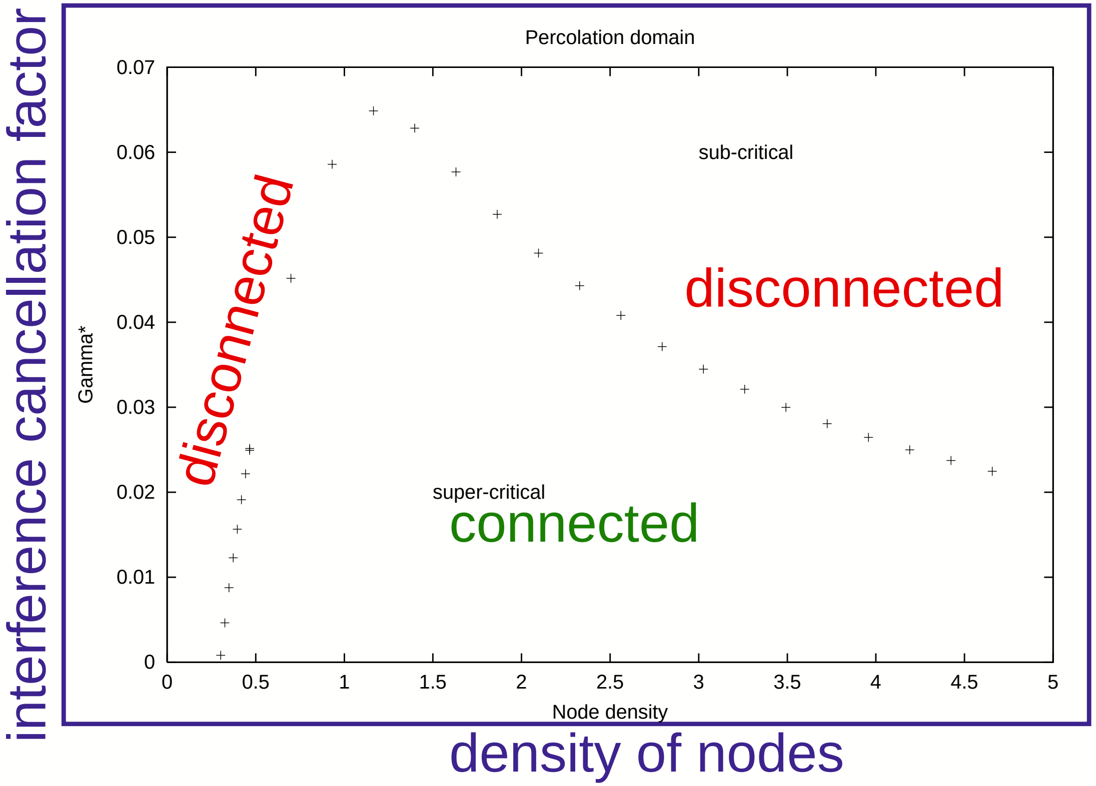
- Scaling laws: Discover main tendencies of the network behaviour when number of nodes increases using mathematical formalism (percolation theory, limiting theory for random graphs, ...). Difficult/impossible via pure, crude simulation.
- Optimal tuning of “real” network parameters: rapid when using (semi)explicit analytical evaluation of the performance metrics. Difficult via pure, crude simulation.

A SCALING LAW FOR MANETS

- Douse et al. (2000,2003):



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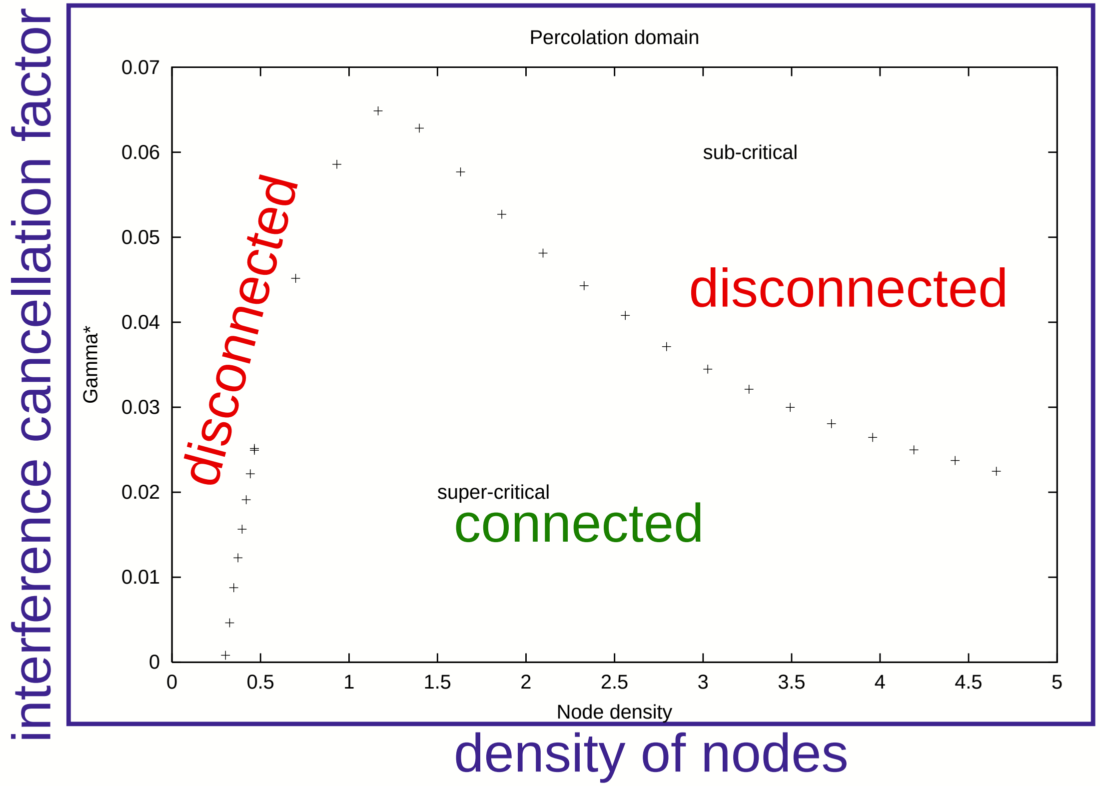

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See also: Gupta & Kumar's (2000): transport capacity per node in MANET is proportional to $1/\sqrt{\text{node density}}$ when node density $\to \infty$.

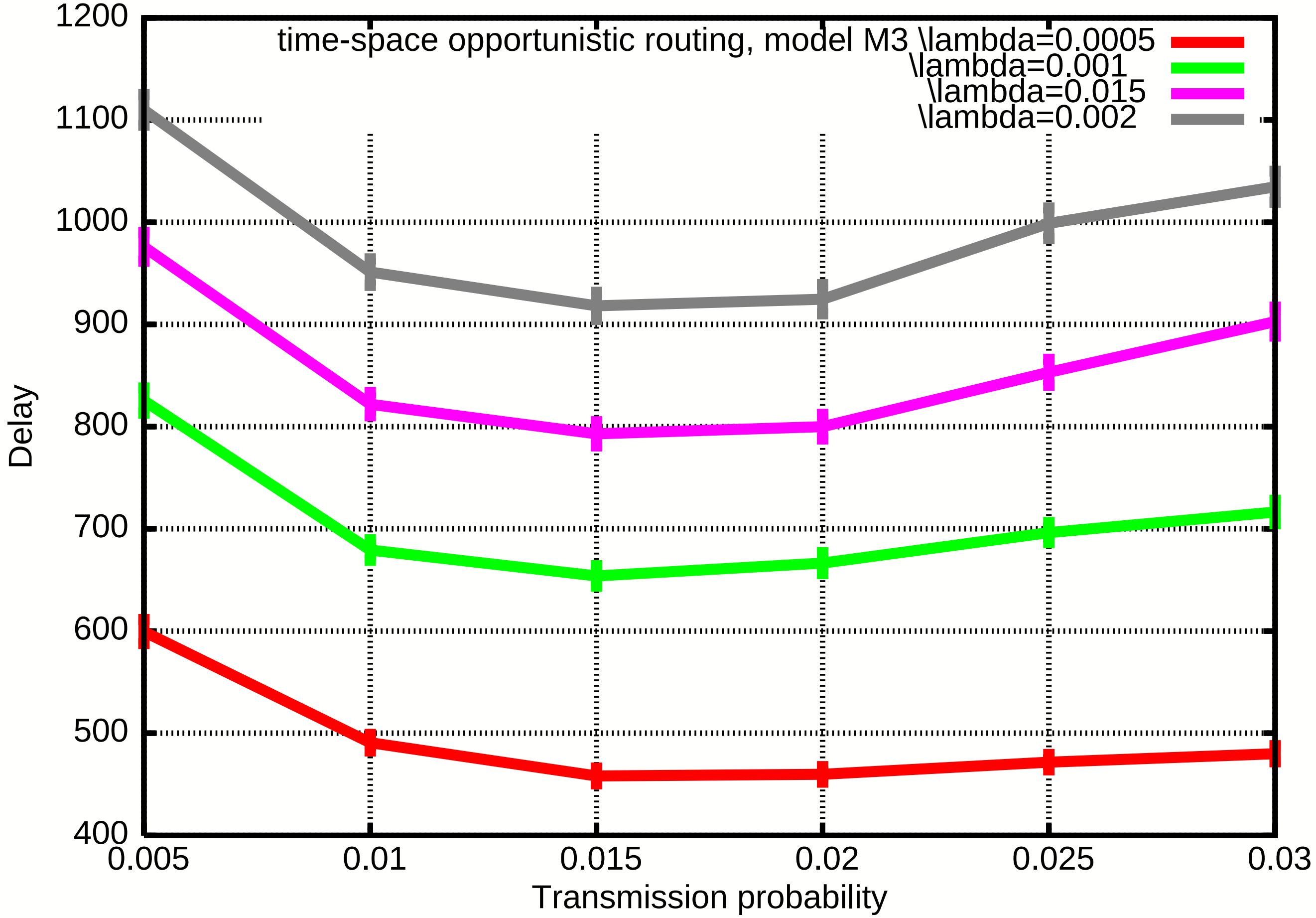
MAC TUNING FOR MANETS

Baccelli, BB, Mühlethaler (2008):

Optimal tuning of MAP p

for mean end-to-end delay in opportunistic routing.

optimal p does not depend on the density of nodes



THANK YOU

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Probability that the given antenna covers y with SINR higher than some threshold T is

$$p_R = \mathsf{P}\Big(SL(R)/T - W - I_{\tilde{\Phi}}(y) > 0 \Big).$$

Res. The coverage probability p_R can be given via Fourier transforms of (independent) r.v.'s S, W and $I_{\tilde{\Phi}}(y)$ that is explicitly known (as Laplace transform) in Poisson SN case

$$p_R = \frac{1}{2} - \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\mathsf{E}[e^{i\xi I_{\tilde{\Phi}}(y)}]\mathsf{E}[e^{i\xi W}]\mathsf{E}[e^{-i\xi SL(R)/T}]}{\xi} \, d\xi \, .$$

Proof via Rieman boundary problem.

Example

Fourier transform $\mathcal{F}_{I_{\tilde{\Phi}}}(\xi)$ of the homogeneous Poisson (intensity λ) SN with exponential S (parameter m) and attenuation $L(r) = A\max(r, r_0)^{-4}$

$$\begin{aligned}\mathcal{F}_{I_\Phi}(\xi) &= \mathsf{E}\left[e^{-i\xi I_\Phi}\right] \\ &= \exp\left[\lambda\pi\sqrt{\frac{iA\xi}{m}}\arctan\left(r_0^2\sqrt{\frac{m}{iA\xi}}\right) - \frac{\lambda}{2}\pi^2\sqrt{\frac{iA\xi}{m}}\right. \\ &\qquad \left. + \lambda\pi r_0^2\frac{r_0^4 - iA\xi - r_0^4 m}{iA\xi + r_0^4 m}\right],\end{aligned}$$

for $\xi \in \mathbb{R}$, where the branch of the complex square root function is chosen with positive real part.

Special case — Rayleigh fading

Res. [Baccelli&BB&Muhlethaler (2004)]: Assume that S (power of the given antenna) is exponential r.v. with par. μ (which corresponds e.g. to constant emitted power and Rayleigh fading in the channel). Then

$$p_R = \mathcal{L}_W\Big(\mu T/L(R)\Big) \times \exp\left\{ -2\pi\lambda \int_0^\infty r(1 - \mathsf{E}[e^{-S'TL(r)/L(R)}])\,\mathsf{d}r \right\},$$

where S' is the generic power emitted by an interferer.

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Proof:

$$\begin{aligned} p_R &= \mathsf{P}(S > (W + I_{\tilde\Phi})T/L(R)) \\ &= \int_0^\infty e^{-\mu s T/L(R)}\,\mathsf{dP}(W + I_{\tilde\Phi} \le s) \\ &= \mathcal{L}_W(\mu T/L(R)) \times \mathcal{L}_{I_{\tilde\Phi}}(\mu T/L(R)). \end{aligned}$$

Cor. When the generic power S' emitted by interferers is also exponential (Rayleigh fading) and assuming the simplified attenuation function $L(r) = (Ar)^{-\beta}$ and $W = 0$

$$p_R = e^{-\lambda R^2 T^{2/\beta} C} \,,$$

where $C = C(\beta) = \Big(2\pi\Gamma(2/\beta)\Gamma(1-2/\beta)\Big)/\beta$.

Some useful optimizations

One can study the following optimization problems for the expected effective transmission range $r \times p_r$:

- given the density of stations λ find the targeted range r that optimizes the expected effective transmission range

$$\rho = \rho(p) = \max_{r \geq 0}\{r p_r(p)\} = \frac{1}{T^{1/\beta}\sqrt{2\lambda p C}}$$

$$r_{\max} = r_{\max}(p) = \text{argmax}_{r\geq 0}\{r p_r(\lambda)\} = \frac{1}{T^{1/\beta}\sqrt{2\lambda C}}$$

- given the targeted range R find the density of emitters λ that optimize the expected effective transmission range $R \times p_R$:

$$\lambda_{\max} = \lambda_{\max}(R) = \text{argmax}_{\lambda \geq 0}\{Rp_R(\lambda)\} = \frac{1}{R^2 T^{2/\beta} C}$$

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More can be done. See optimal tuning of Aloha MAC for transport capacity in *IEEE Tran. Inf. Theory* 2004.