Random Graphs and Wireless Ad-hoc Networks

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Journe Graphes alatoires et rseaux sans fil

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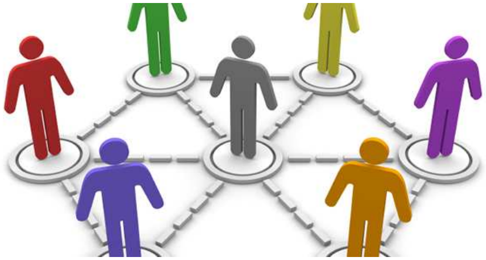
Ad-hoc Network

Network made of nodes “arbitrarily” repartitioned in some region, exchanging packets either transmitting or receiving them on a common frequency, use intermediary retransmissions by nodes lying on the path between the packet source node and its destination nodes.

Ad-hoc Network

In contrast to regular (cellular) networks



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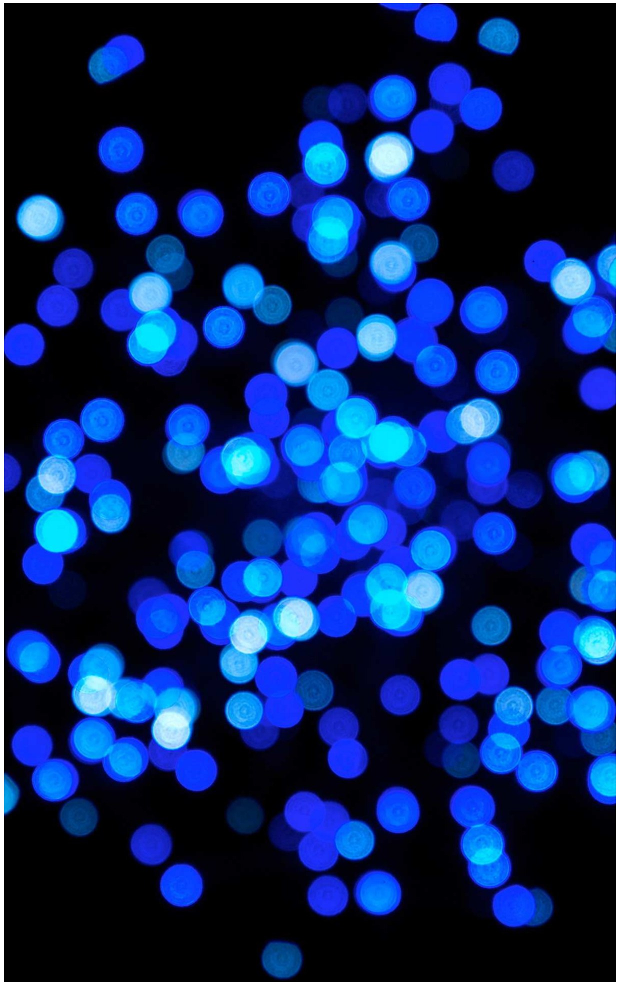

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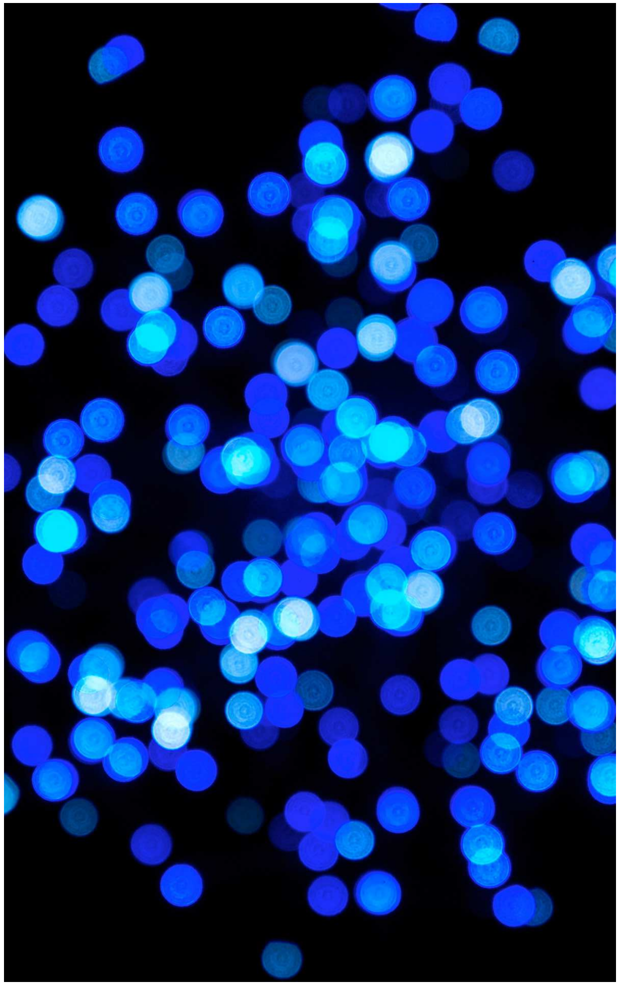

Recall: A random repartition of points Φ is called a (homogeneous) Poisson p.p. of intensity λ (points per unit of surface) if:

- number of points of Φ in any set A, $\Phi(A)$, is Poisson random variable with mean λ times the surface of A.
- numbers of points $\Phi(A_i)$ of Φ in disjoint sets A_i are independent random variables.

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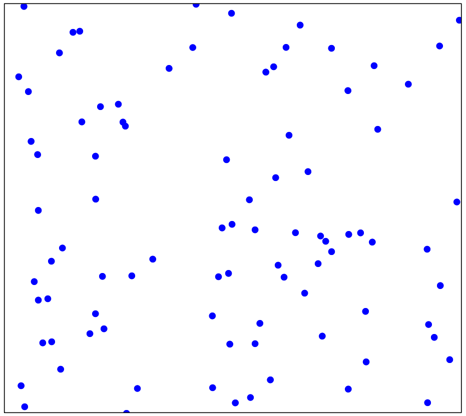

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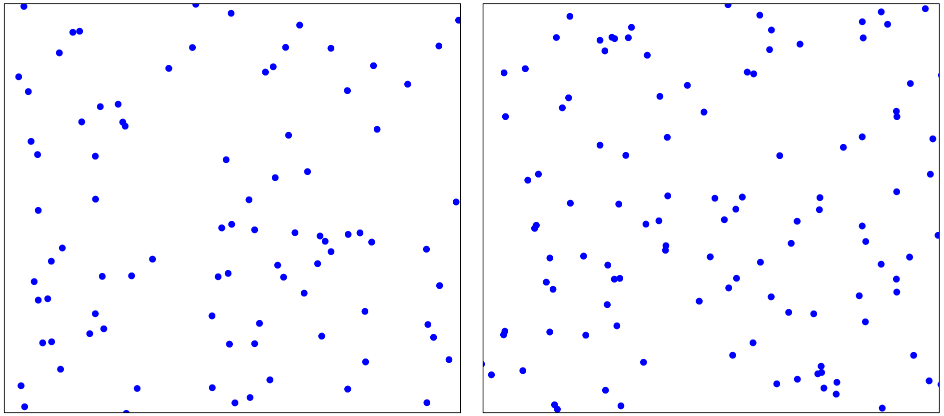

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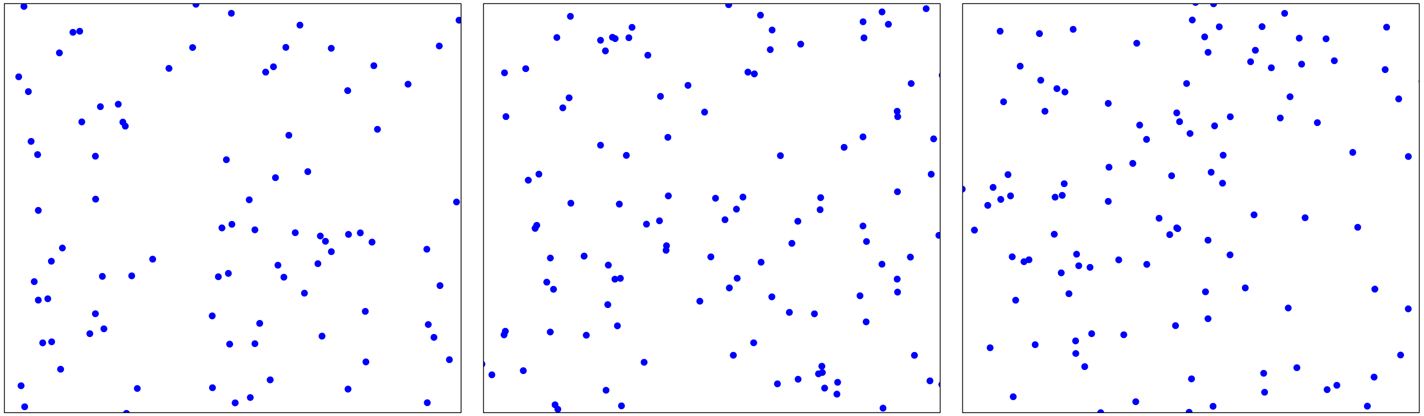

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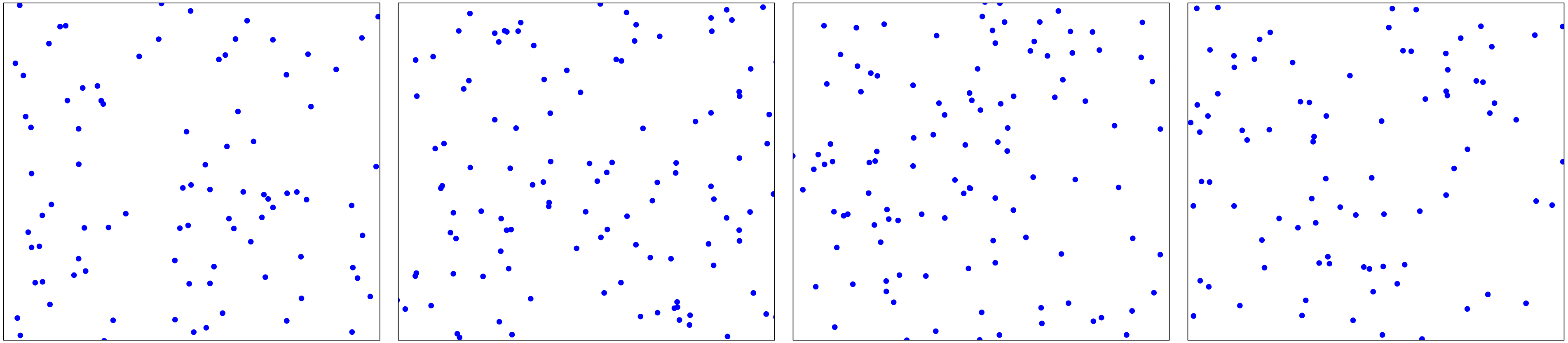

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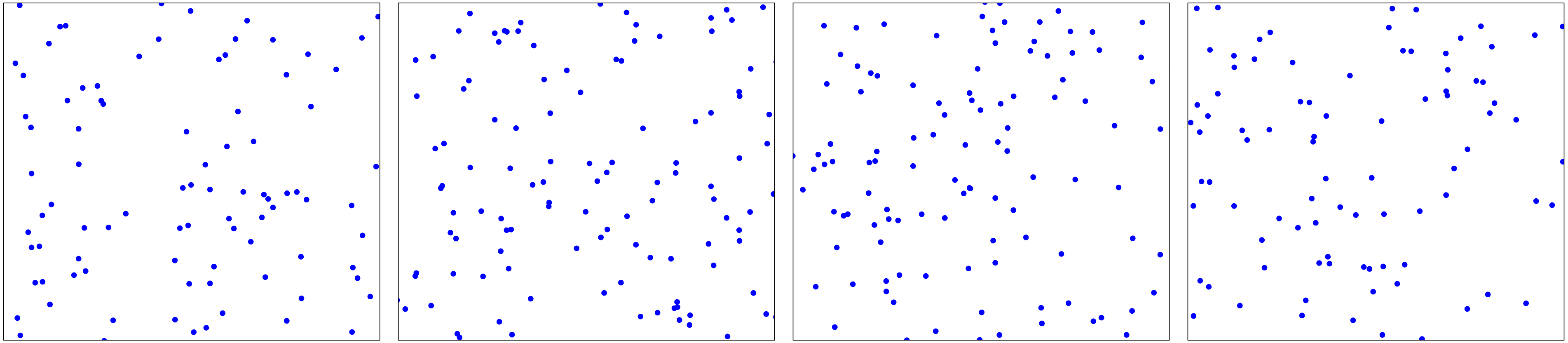

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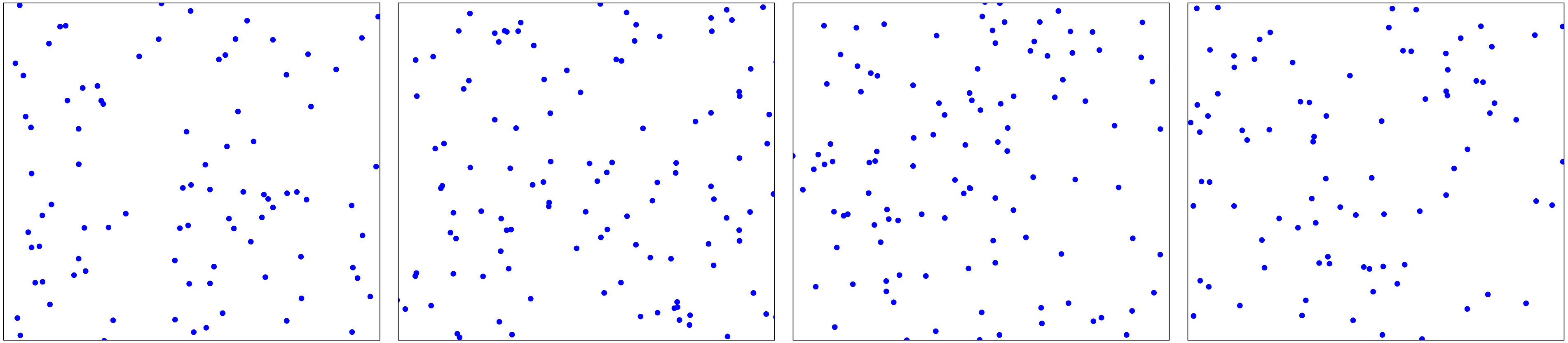

The above patterns are different but somehow “similar”.
In fact, they are

- different realizations (samples)...
- sampled from the same distribution of Poisson p.p. of a given intensity λ — averaged number of nodes per unit of surface.

Why Random Location of Nodes?, cont.

Modeling locations of nodes by a (random) point process, allows one to take into account, in statistical manner, all possible patterns of nodes within some given class (here Poisson patterns of a given intensity).

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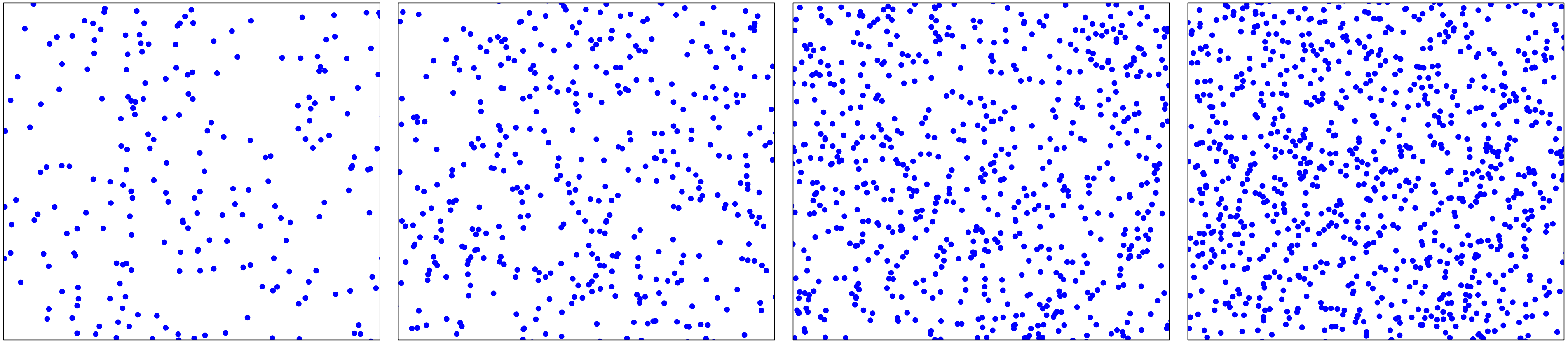
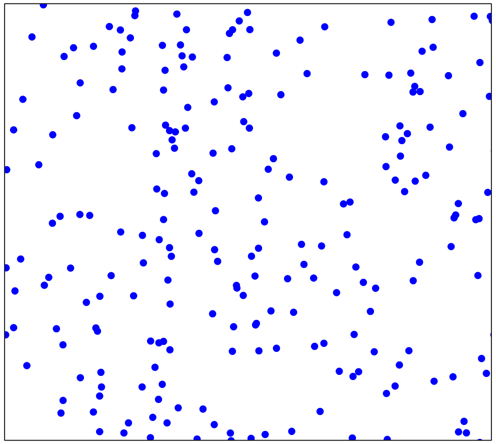


Realizations of Poisson networks of different intensity λ.

Why Poisson Distribution of Nodes?

In principle could be a different distribution.



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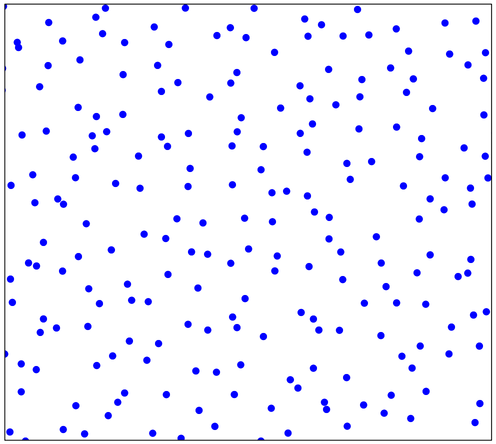
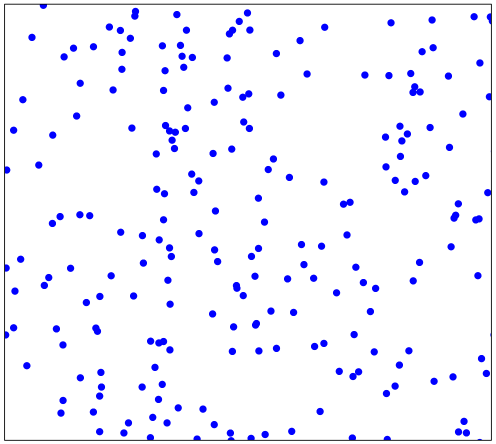


more regular



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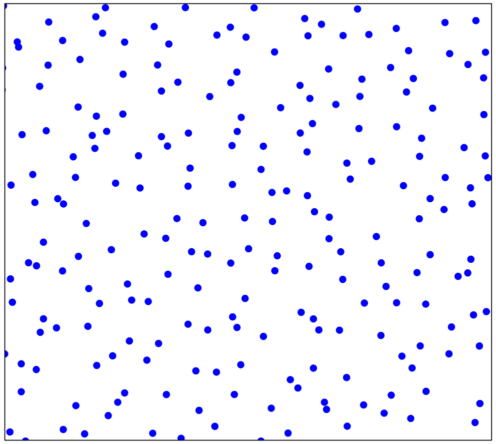

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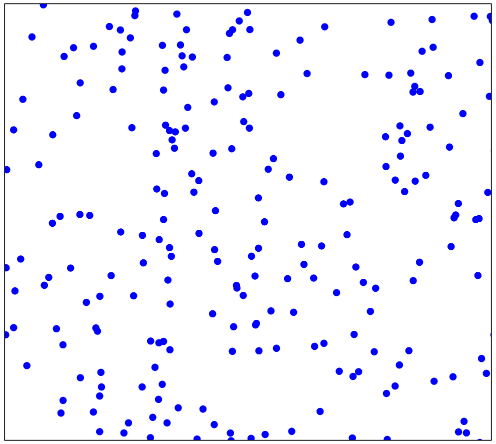

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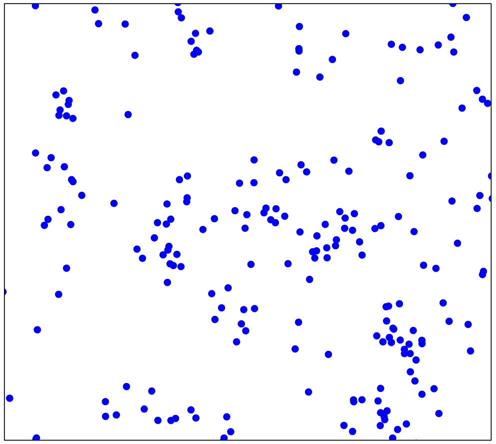

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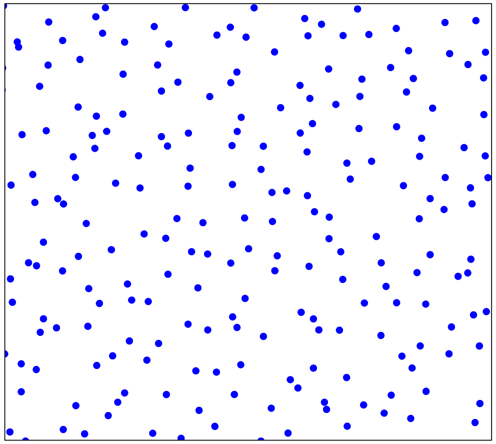

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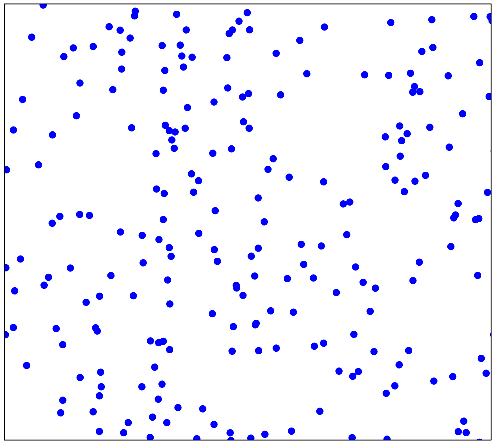

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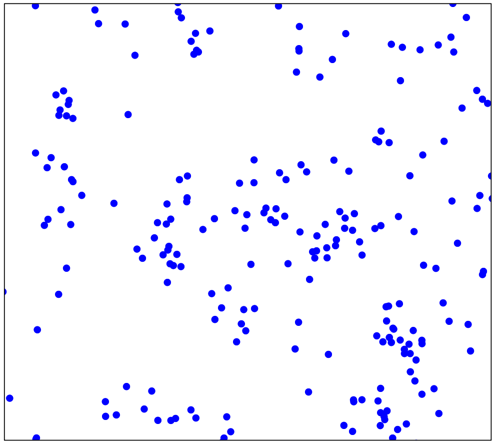

more clustering

However, Poisson distribution is the only one[a] for which the numbers of nodes in disjoint sets are independent!

The above complete independence characterizing Poisson p.p. is a natural “neutral” assumption for node locations, when no other statistical information is available.

[a] no fixed node locations, no multiple nodes

Why Poisson Distribution of Nodes? cont.

If one knows (suspects) that the nodes are not distributed "homogeneously", e.g. there are some "hot-spots", (i.e., the mean number of nodes per unit of surface varies in different regions), then one can model the network by non-homogeneous Poisson pp, with location dependent intensity $\lambda(x)$; in this case $\Phi(A) \sim \text{Poisson}(\int_A \lambda(x)\text{d})x$.

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Poisson distribution of nodes is preserved by various "operations" on the nodes:

- superposition (addition) of independent Poisson processes,
- random independent deletion of points (thinning),
- random independent displacement of points.

Medium Access Control (MAC)

The Medium Access Control (MAC) layer is a part of the data communication protocol organizing simultaneous packet transmissions in the network.

Aloha MAC = Independent Thinning

In our talk we will consider the, perhaps most simple, algorithm used in the MAC layer, called Aloha:

at each time slot (we will consider only slotted; i.e., discrete, time case), each potential transmitter independently tosses a coin with some bias p; it accesses the medium (transmits) if the outcome is heads and it delays its transmission otherwise.

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Thinning is a nice operation on a p.p.

Fact: Thinning of Poisson p.p. of intensity λ leads to Poisson p.p. of intensity $p\lambda$.

Tuning Aloha Parameter p

In Aloha algorithm it is important to tune the value of the Medium Access Probability (MAP) p, so as to realize a compromise between two contradicting types of wishes:

- a "social one" to have as many concurrent transmissions as possible in the network and
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The contradiction between these two wishes stems from the fact that the very nature of the "medium" in which the transmissions take place (Ethernet cable or electromagnetic field in the case of wireless communications) imposes some constraints on the maximal number and configuration of successful concurrent transmissions.

Signal to Interference Ratio (SIR)

A given transmission is successful if the power of the received signal is sufficiently large with respect to the interference, where

interference is the sum of the powers of signals received from all other concurrent transmissions.

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Sometimes we speak about Signal to interference and Noise Ratio (SINR).

Interference as Shot-Noise

Interference created at y when all nodes of Φ transmit a unit power signal can be expressed as the Shot-Noise (SN)

$$I(y) = \sum_{X \in \Phi} \frac{1}{l(|X - y|)},$$

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An important special case consists in taking

$$l(u) = (Au)^{\beta} \quad \text{for } A > 0 \text{ and } \beta > 2.$$

$\beta = 2$ corresponds to the free-space signal energy propagation.

Channel Fading

Signal propagation and interference model can be extended to take into account channel fluctuations due to multi-path signal propagation — the so called fading

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One can reasonably assume that $F_{(X,y)}$ are i.i.d. in X, y. A special case of exponential F is corresponds to the so called Rayleigh fading.

Poisson Shot-Noise

Fact: If Φ is homogeneous Poisson p.p. than the Laplace transform (LT) $\mathcal{L}_I$ of the SN $I(y)$ with i.i.d. fading is

$$\mathcal{L}_I(\xi) := \mathsf{E}[e^{-\xi I}] = \exp\Big[-2\lambda\pi \int_0^\infty r(1 - \mathcal{L}_F(\xi/l(r)))\,dr\Big],$$

where $\mathcal{L}_F(\cdot)$ is the Laplace transform of the fading distribution. Can be extended to joint LT of vectors $(I(y_1), \ldots, I(y_2))$.

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Proof. Follows from the known expression of the LT of (homogeneous) Poisson p.p.

$$\mathcal{L}_\Phi(f) := \mathsf{E}[e^{-\sum_{X\in\Phi} f(X)}] = \mathsf{E}[\exp\{-\lambda\int_{\mathbb{R}^2}(1-e^{f(x)})\,dx\}]\,.$$

Stochastic Geometry for Wireless Networks

Stochastic Geometry (SG) is now a reach branch of applied probability, which allows to study random phenomena on the plane or in higher dimension; it is intrinsically related to the theory of point processes. Initially its development was stimulated by applications to biology, astronomy and material sciences. Nowadays, it is also used in image analysis. See an excellent monograph:
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In communications context, SG allows to capture the non-regular and variable geometry of the network and variability of radio channel conditions in probabilistic manner primarily offering various averaging methods.

A pioneer...

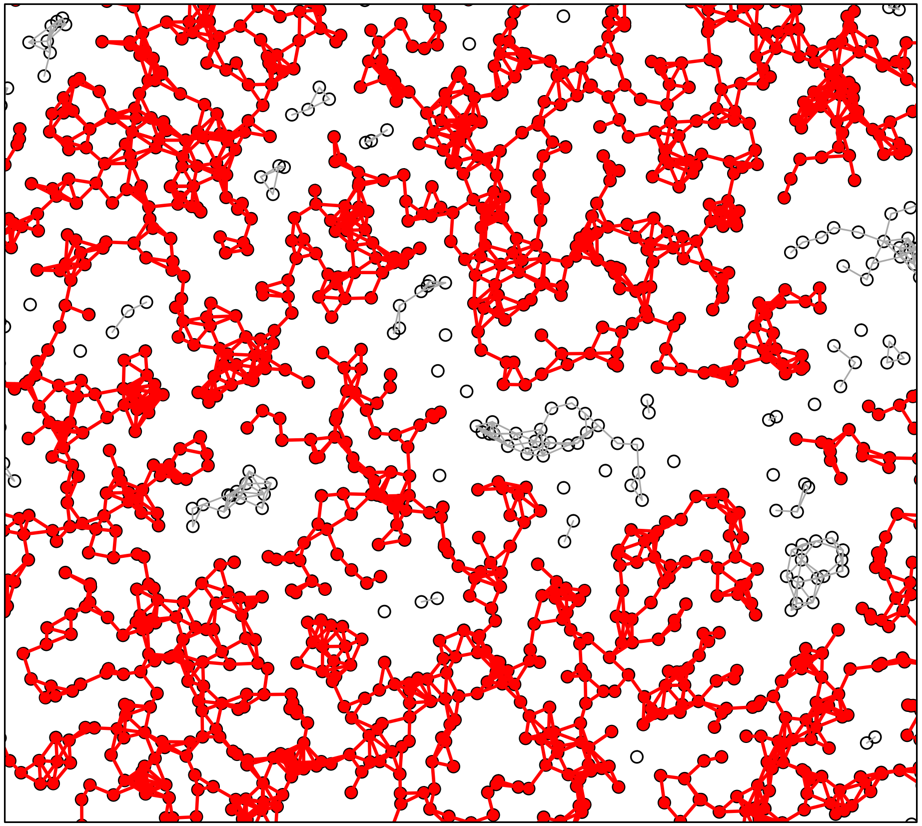
... in using SG for modeling of communication networks

Edgar N. Gilbert (1961) Random plane networks, *SIAM-J*

Edgar N. Gilbert (1962) Random subdivisions of space into crystals, *Ann. Math. Stat.*

Gilbert (1961) proposes continuum percolation model (percolation of the Boolean model) to analyze the connectivity of large wireless networks.

Gilbert (1962) is on Poisson-Voronoi tessellations.



Recent Works

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In a broader sense, many outstanding theoreticians of stochastic geometry, random graphs, percolation theory were and are also interested in communication technology problems ...

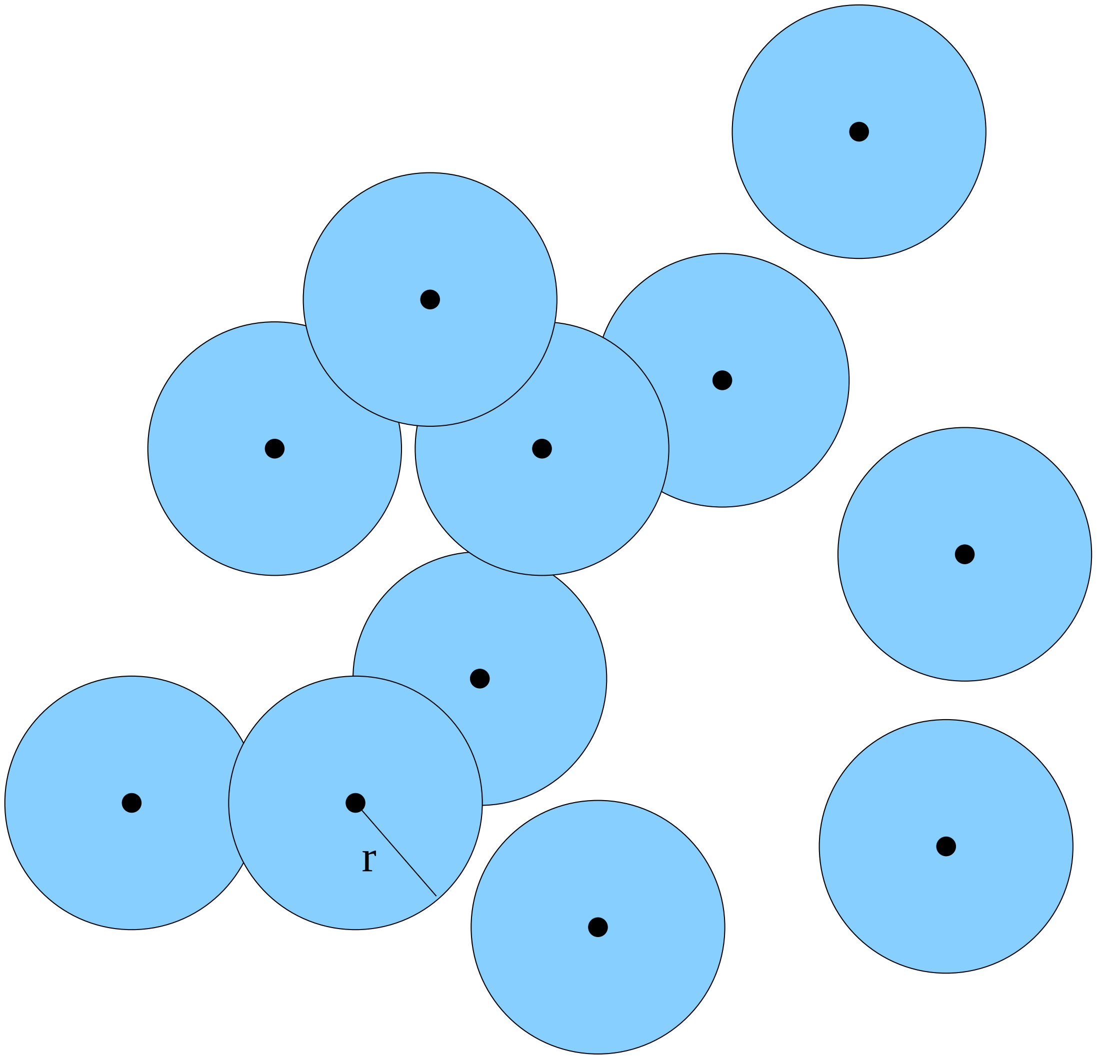
I will not be able to pay tribute to the work they have done ...

Outline of the talk

- **RANDOM GEOMETRIC GRAPH**
- **SINR GRAPH**
- **SPACE-TIME SINR GRAPH**

Continuum percolation

Boolean model $C(\Phi, r)$:
germs in Φ,
spherical grains of given
radius r.



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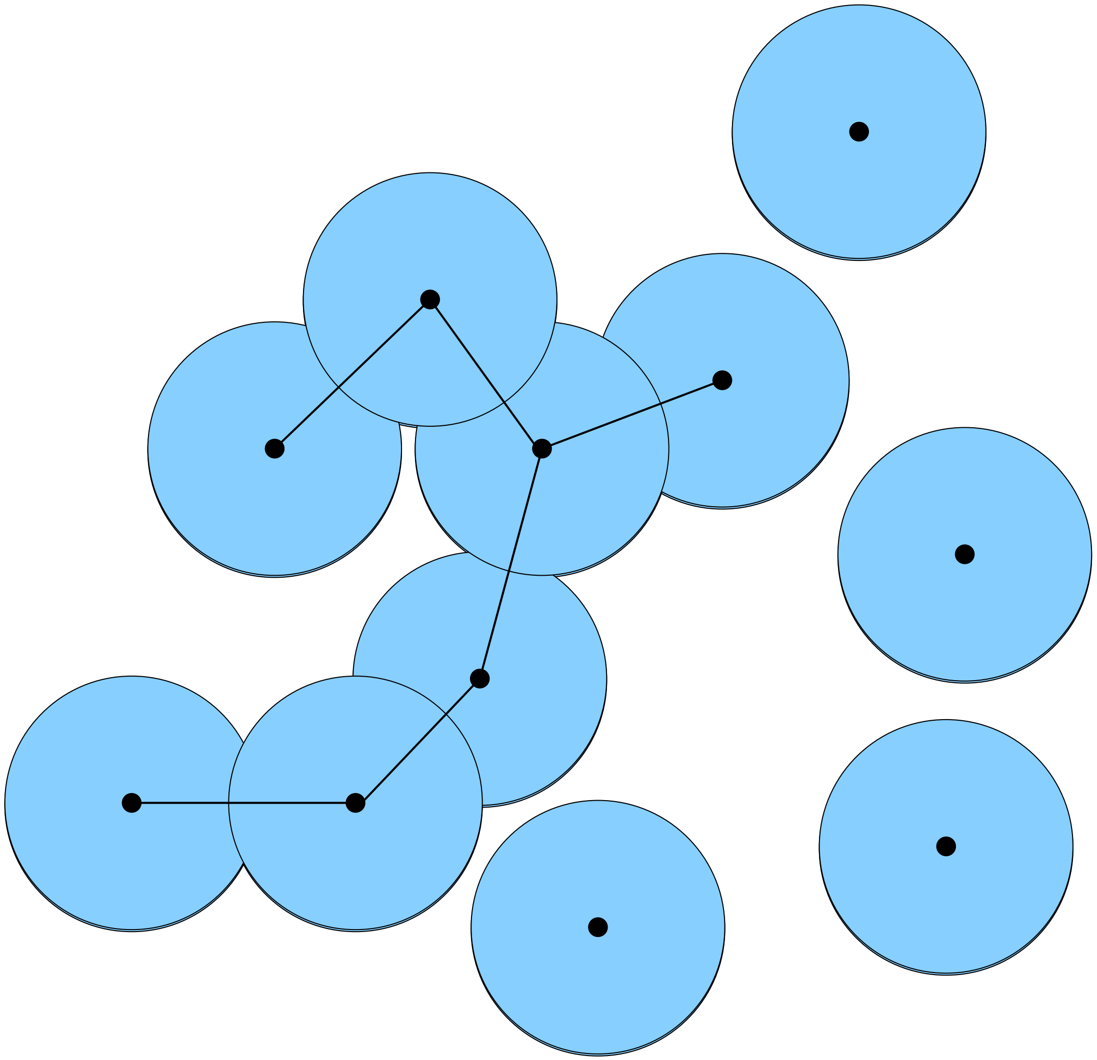

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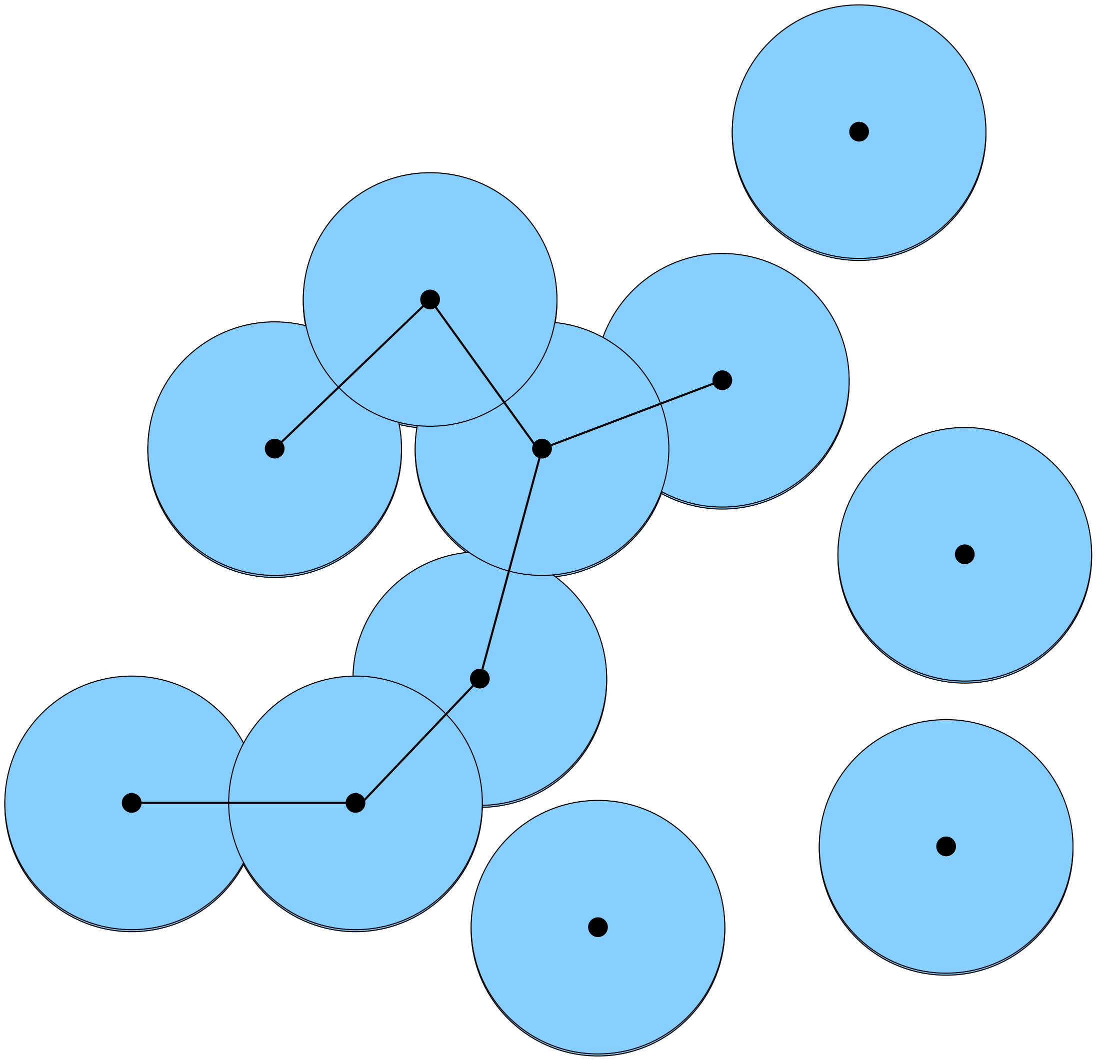

percolation $\equiv$ existence of an infinite connected subset (component).

Critical radius for percolation

Critical radius for the percolation in the Boolean Model with germs in $\mathbf{\Phi}$

$$r_c(\Phi) = \inf\{r > 0 : \mathsf{P}(C(\Phi, r)\text{percolates}) > 0\}$$

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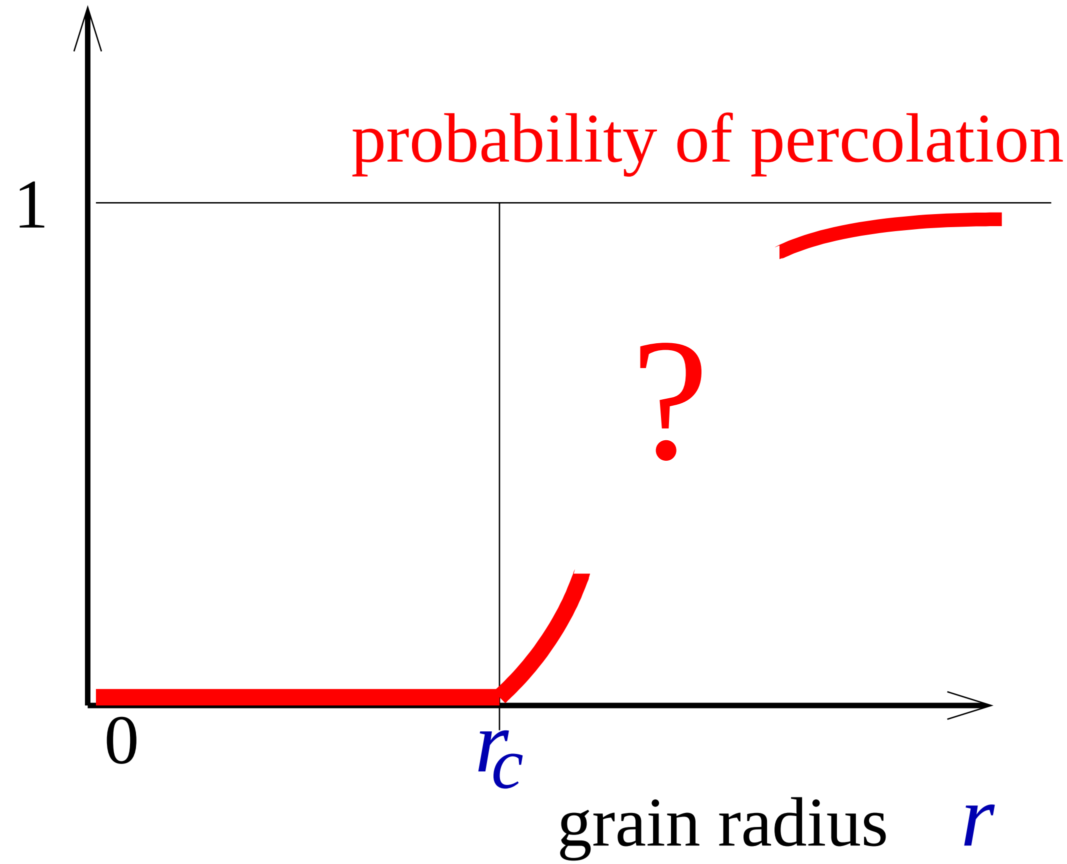
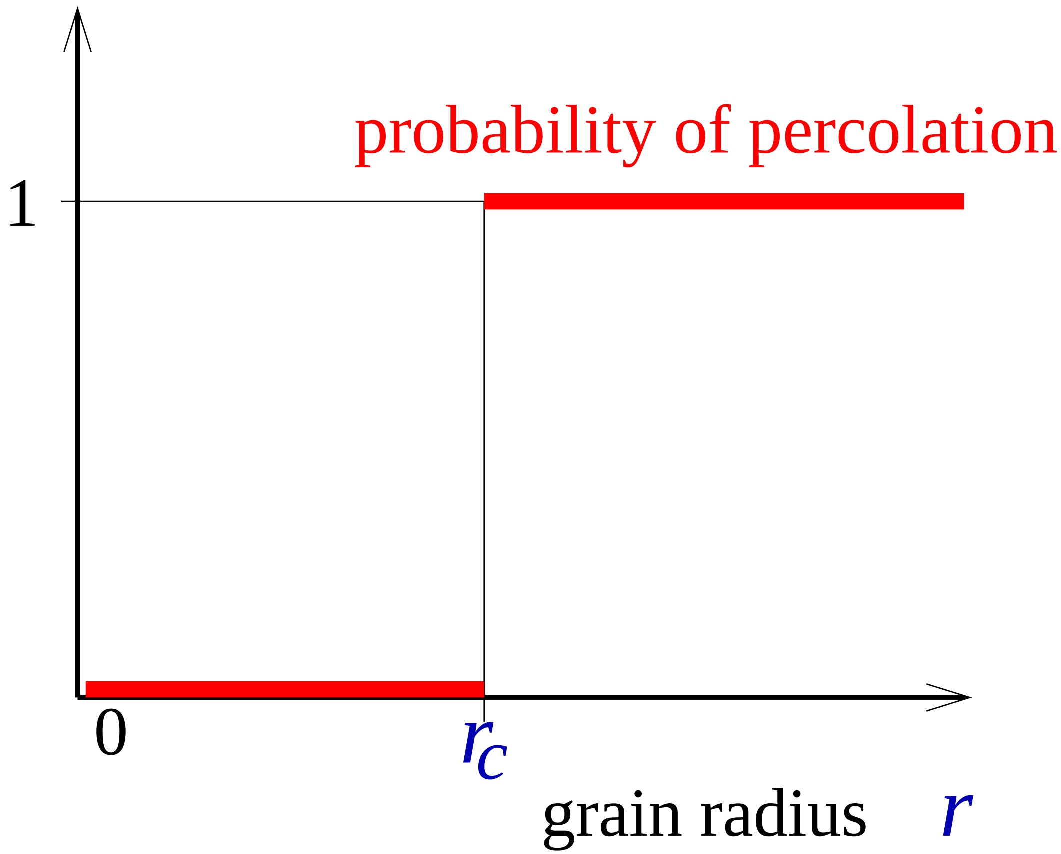


Phase transition in ergodic case

In the case when Φ is stationary and ergodic



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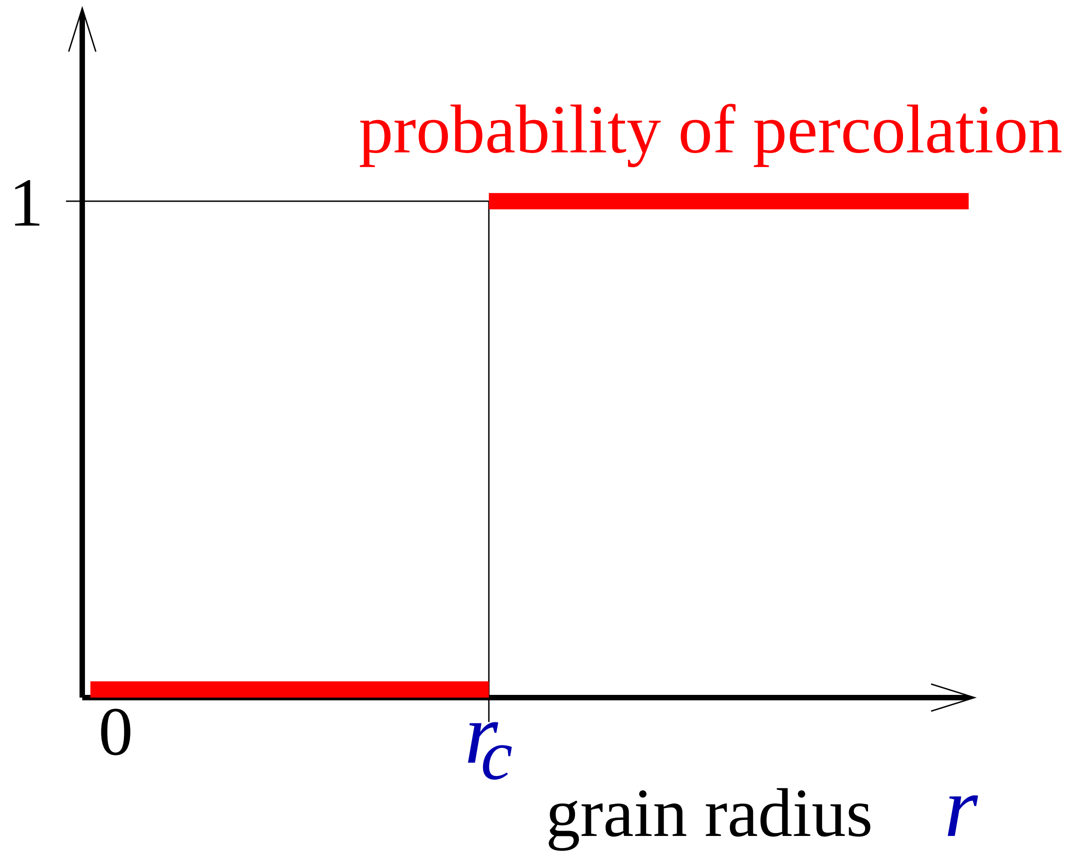

If $0 < r_c < \infty$ we say that the phase transition is non-trivial.

Phase transition in Poisson RGG

Proposition 1 (Gilbert 1961) *For Poisson RGG we have* $0 < r_c(\Phi) < \infty$.

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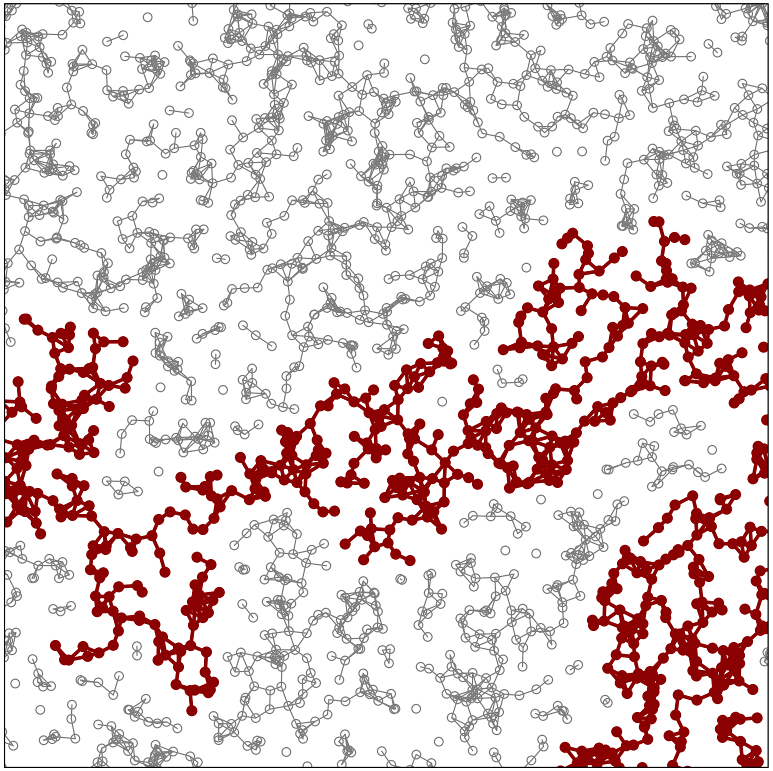
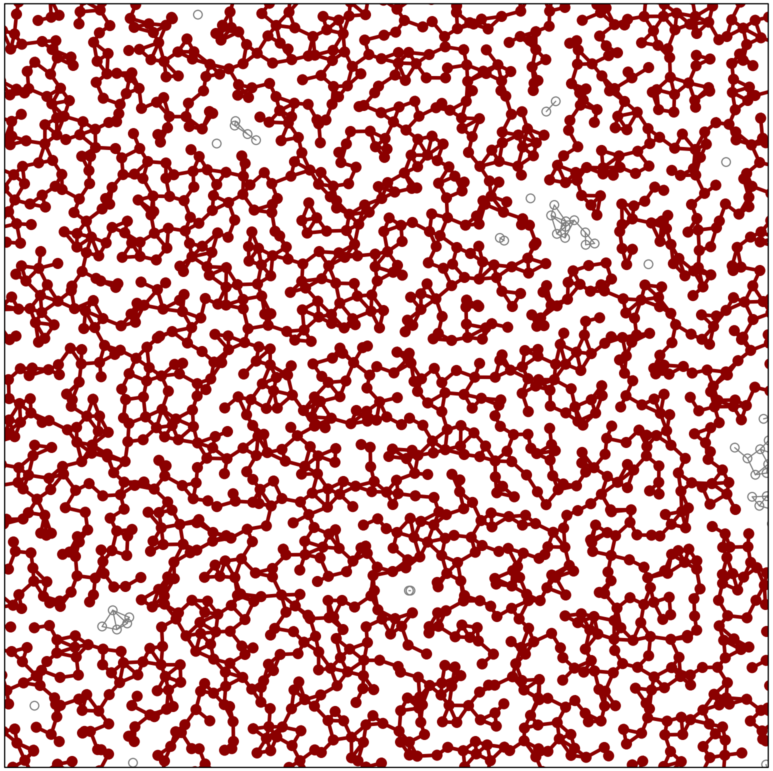




communication range $r \leq r_c$
network disconnected

$r > r_c$
well connected network

SINR MODEL

$\Phi = \{X_i, (S_i, T_i)\}$ marked point process

$\{X_i\}$ points of the p.p. on $\mathbb{R}^2$

$(S_i, T_i) \in (\mathbb{R}^+)^2$ possibly random mark of point X_i — (emitted power,SINR threshold)

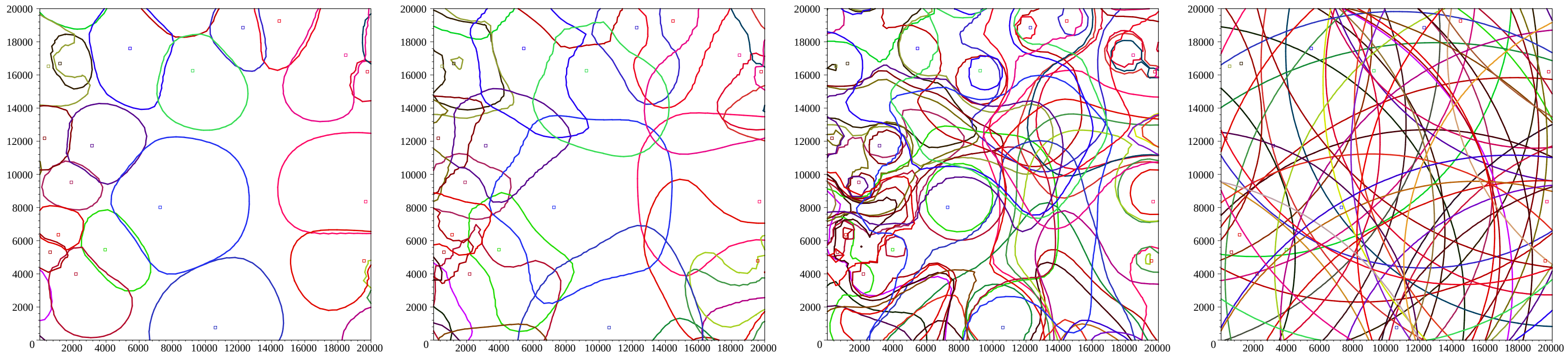
Cell attached to point X_i:

$$C_i(\Phi, W) = \left\{ y : \frac{S_i/l(y - X_i)}{W + \kappa I_\Phi(y)} \geq T_i \right\}$$

where $I_\phi(y) = \sum_{i \neq 0} S_i/l(y - X_i)$ (interference) $\kappa \geq 0$ (interference cancellation factor), $W \geq 0$ (external noise).
We call C_i SINR cell and $\Xi(\Phi; W) = \bigcup_{i \in \mathbb{N}} C_i(\Phi, W)$ the SINR coverage process.

SINR COVERAGE MODEL

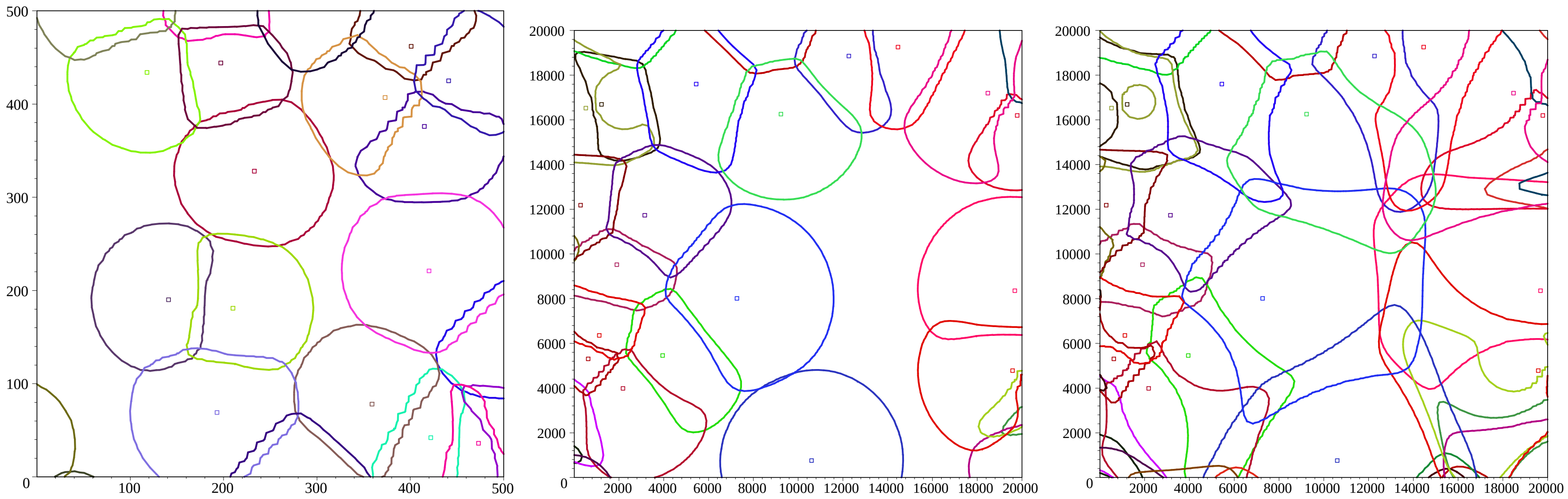
Coverage properties of the SINR Model studied in [BB, Baccelli 2001].



interference cancellation factor $\boldsymbol{\kappa} \longrightarrow \mathbf{0}$

Small interference factor allows one to approximate SINR cells by a Boolean model (quantitative results via perturbation methods)

SINR COVERAGE MODEL cont'd



noise $W = 0$ and attenuation exponent $\beta \to \infty$

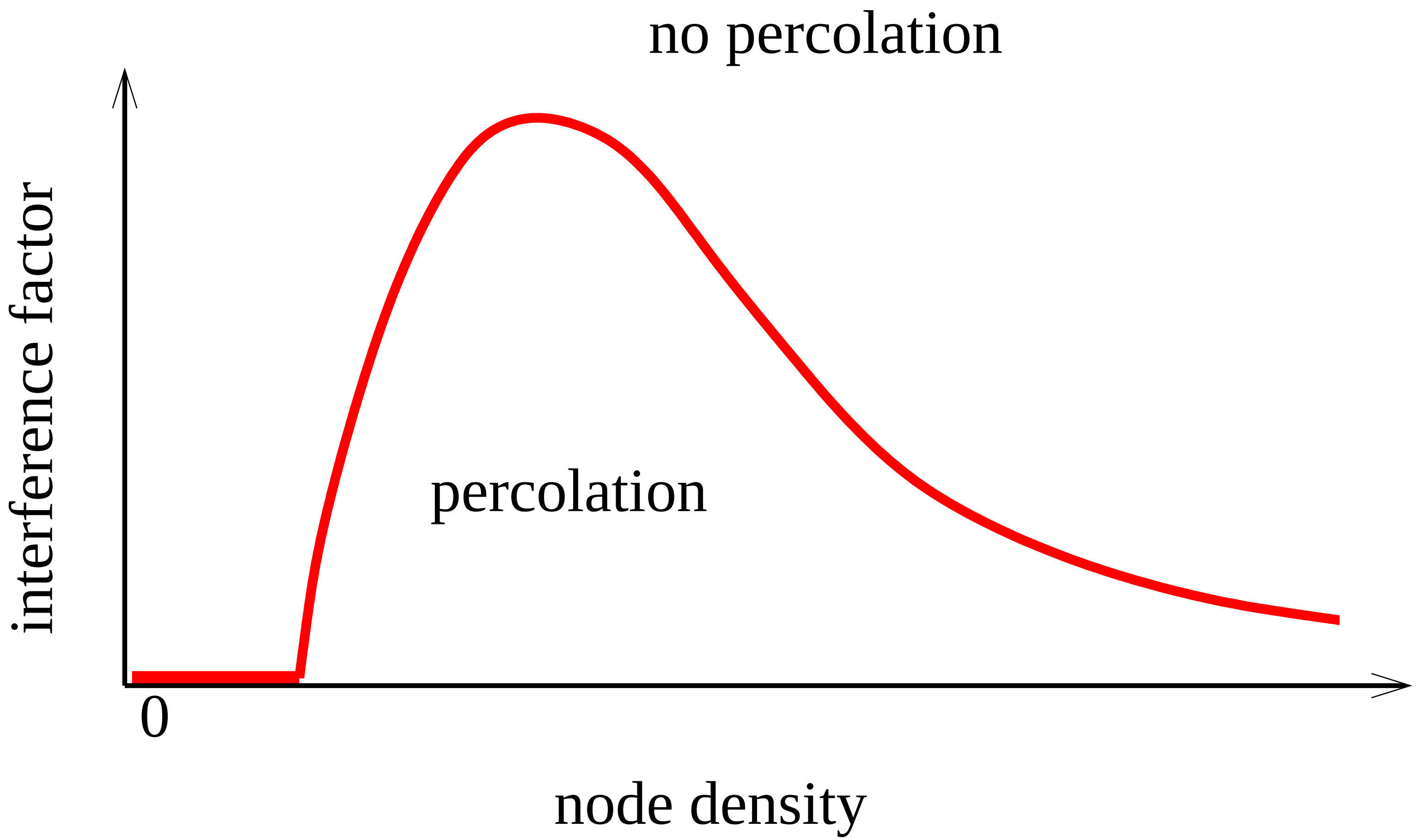
SIR cells tend to Voronoi cells whenever attenuation is stronger, e.g. in urban areas.

SINR Graph

Connect nodes $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$ by an edge when $\boldsymbol{X}_i \in \boldsymbol{C}_j$ and $\boldsymbol{X}_j \in \boldsymbol{C}_i$, i.e.; when $\boldsymbol{X}_i$ is in the SINR cell of $\boldsymbol{X}_j$ and vice-versa.

Phase transition in SINR Graph

Proposition 2 (Dousse etal 2006) *In Poisson SINR graph we observe a non-trivial phase transition for the percolation.*



In contrast to the Boolean model an increase of the node density (or signal power) can disconnect the network.

Beyond Poisson assumption

We say that Φ is sub(super)-Poisson if it is dcx smaller (larger) than Poisson pp (of the same mean measure).

We say that Φ is weakly sub(super)-Poisson if it has void probabilities and moment measures smaller than Poisson pp of the same mean measure.

Sub-Poisson pp cluster their points less than Poisson.
Super-Poisson pp cluster their points more.

Conjecture: Clustering worsens percolation.

Conjecture for perturbed lattices

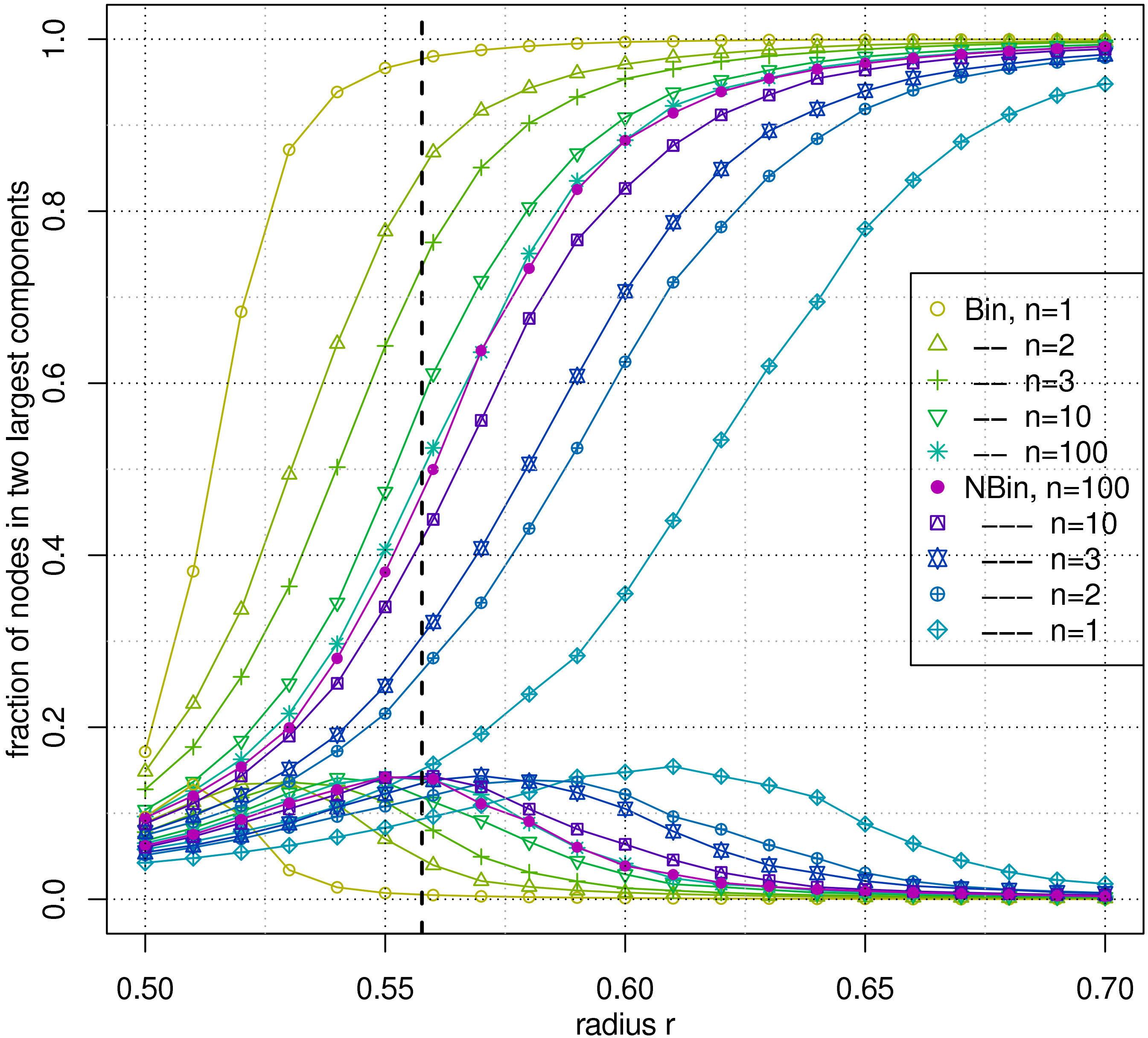
$$\Phi_1 \;\le_{dcx}\; \Phi_2$$
$$\Downarrow$$
$$r_c(\Phi_1) \;\le\; r_c(\Phi_2)$$

$Bin(1,1) =$ $const$

$Bin(1,1/n) \nearrow_{cx}$ $Poi(1)$

$NBin(n,1/(1+$ $n))\searrow_{cx} Poi(1)$

$NBin(1,1/2) =$ $Geo(1/2)$



Phase transitions for sub-Poisson pp

Proposition 3 (BB. Yogeshwaran 2011) *Let* Φ *be a stationary pp on* $\mathbb{R}^d$*, weakly sub-Poisson (void probabilities and moment measures smaller than for the Poisson pp of some intensity* λ*). Then*

$$0 < \frac{1}{(2^d\lambda(3^d-1))^{1/d}} \le r_c(\Phi) \le \frac{\sqrt{d}(\log(3^d-2))^{1/d}}{\lambda^{1/d}} < \infty.$$

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Similar results for SINR-graph percolation.

SPACE-TIME SINR GRAPH

Previous Model, Time Dimension Added

- Static Poisson network of density λ nodes/km^2.

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- I.i.d. point-to-point fading F, constant in a given time slot, may or may-not vary across times slots:
  - slow fading (shadowing): channel conditions do not change in time,
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- External noise power W, may or may-not vary in time ($\boxed{\text{slow}}$ or $\boxed{\text{fast noise}}$ scenario, respectively).

Space-Time Network Model, cont'd

We restrict ourselves to Poisson p.p. and to the fast fading and fast noise scenario (most favorable for reducing local delays).

As before, we consider SINR condition for the successful transmission.

Successful Transmission

We will say that transmitter $\{X_i\}$ covers its receiver y_i in the reference time slot if

$$\text{SINR}_i = \frac{F_i^i / l(|X_i - y_i|)}{W + I_i^1} \geq T\,, \tag{1}$$

where

- $I_i^1 = \sum_{X_j \in \widetilde{\Phi}^1,\, j \neq i} F_j^i / l(|X_j - y_i|)$ is the SN of $\widetilde{\Phi}^1 = \{X_i : e_i = 1\}$ and models the interference,
- $W > 0$ is the external (thermal) noise — a r. v. independent of everything else.
- and where T is some SINR threshold.

We say equivalently that x_i is successfully received by y_i.

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• Broadcast model allows us to consider and compare different routing schemes and show some universal bounds on the performance (end-to-end delay) of these schemes.

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Directed edges of this oriented graph connect

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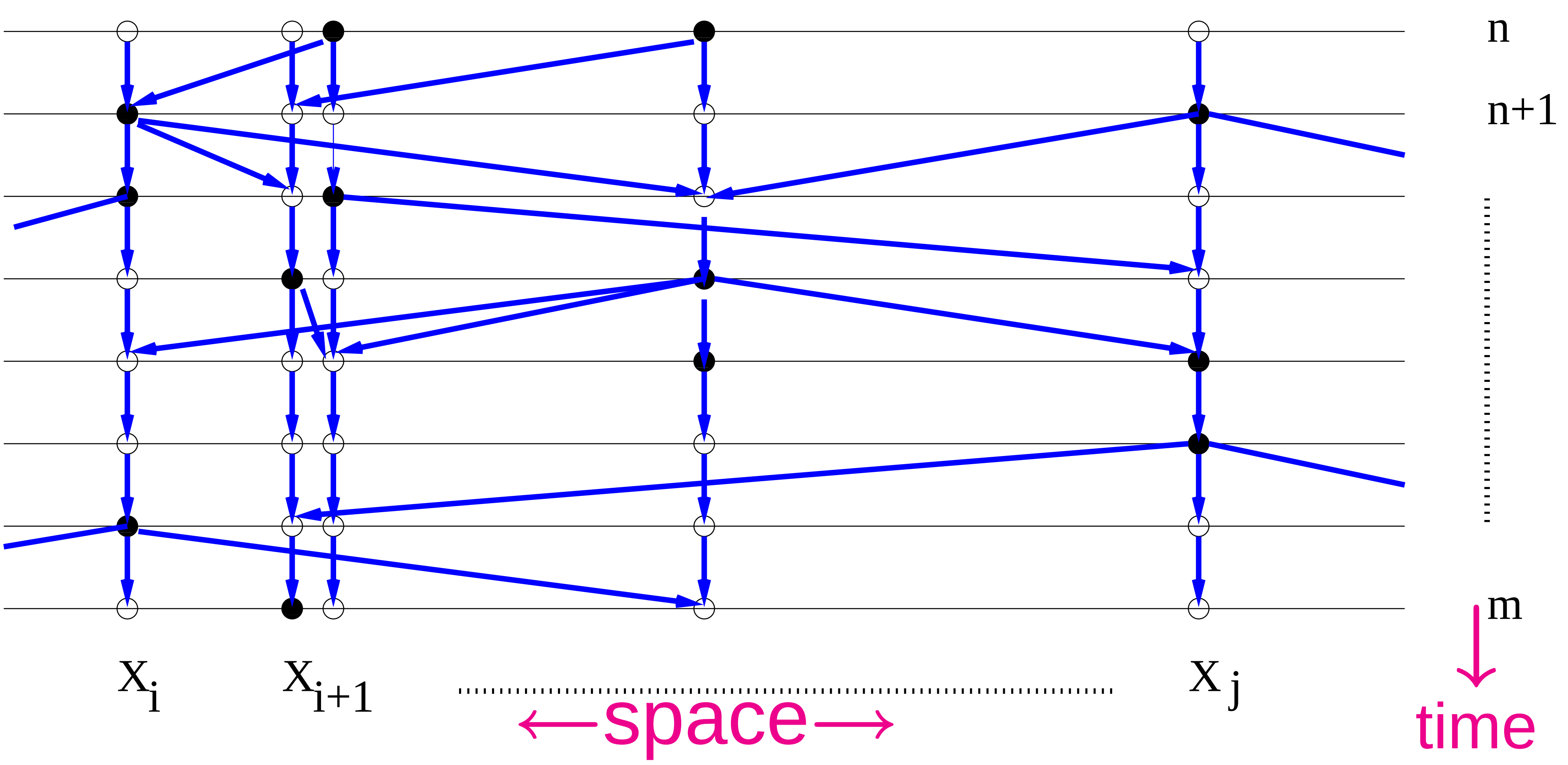
i.e. all possible moves of a tagged packet from X_i at time n.

SINR Graph $\mathcal{G}$



● emitting nodes, ○ non-emitting nodes (receives)

↘ ↙ successful packet transmissions

↓ packet stays at the given node.

Node Degree

Trivial observation: $\mathcal{G}$ has no isolated nodes.

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In particular: $\mathcal{H}_i^{out}(n) = \mathcal{H}_i^{out,1}(n)$ out-degree of the node (X_i, n) and $\mathcal{H}_i^{in}(n) = \mathcal{H}_i^{in,1}(n)$ in-degree.

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Proof: Simple SINR algebra shows that no node can simultaneously, successfully receive more than $1/T + 1$ transmissions. ←so called “pole capacity” of down-link channel

Node Degree, cont.

Denote:

$h^{out,k} = \mathsf{E}^0[\mathcal{H}_0^{out,k}(n)]$ the expected numbers of paths of (graph) length k originating or from the typical node,

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Proof: In a stationary graph (with shift invariant distribution) on average "what flows in" to a given region is equal to "what flows out" from it. ←so called "mass-transport principle"

Node Degree, cont.

Corollary 1 *Under the assumptions of Facts 1 and 2*

- $\mathcal{G}$ *is locally finite (both on in- and out-degrees of all nodes are* P*-a.s. finite).*
- $\mathcal{H}_i^{in,k}(n) \leq \xi^k$ P*-a.s for all* i, n, k*.*
- $h^{in,k} = h^{out,k} \leq \xi^k$ *for all* k*.*

In particular

- In-degrees are a.s. bounded by a constant $\xi < \infty$.
- Out-degrees are bounded by ξ in mean.

Connectivity

Denote $L_{i,j}(n) = \inf\{k \geq n : e_i \delta_{i,j}(k) = 1\}$ number of time slots (hops on the graph $\mathcal{G}$) after time n, necessary to go from X_i directly to X_j; i.e., local delay from X_i to X_j at time n.

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Corollary 2 *Under the assumptions of Fact 3, $\mathcal{G}$ is* P*-a.s. connected in the following weak sense: for all $X_i, X_j \in \Phi$ and all n, there exists a path from (X_i, n) to the set $\{(X_j, n+l) : l \geq 1\}$.*

Mean Exit Time from the Typical Point of Φ

Denote $L_i(n) = \inf_{j \neq i} L_{i,j}$ the length of a shortest path from (X_i, n) to $(\{\Phi \setminus X_i\}) \times \{n+1, n+2, \ldots\}$; i.e., exit time from point $X_i \in \Phi$ after time slot n. Denote by $\ell = \mathsf{E}^0[L_0(n)] = \mathsf{E}^0[L_0(0)]$ the mean exit time from the typical point of Φ.

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Corollary 3 *Under the assumptions of Fact 4 the mean exit time from the typical node is infinite; $\ell = \infty$. Moreover, the fraction of points of Φ which have exit delays larger than q decreases not faster than $1/q$ asymptotically for large q (heavy tailed distribution!).*

Optimal Paths

Denote by $P_{i,j}(n)$ the (graph) length of a shortest path on $\mathcal{G}$ from (X_i, n) to $\{X_j\} \times \{n+1, n+2, \ldots\}$; i.e., the least possible end-to-end delay from X_i to X_j starting at time n.

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Use upper or lower bound to prove, “positive” or “negative” result for the asymptotic end-to-end delay.

Optimal Paths—Poisson Case; Negative Result

Proposition 4 *Assume $\mathbf{\Phi}$ to be a Poisson p.p., $\boldsymbol{F}$ to be exponential and the noise $\boldsymbol{W}$ to have a general distribution. Then for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{\mathbf{2}}$, the mean local delay from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is finite given the existence of these two points in $\mathbf{\Phi}$. More precisely,*

$$\mathsf{E}^{\boldsymbol{X},\boldsymbol{Y}}[\boldsymbol{L}_{\boldsymbol{X},\boldsymbol{Y}}(\mathbf{0})] < \infty\,,$$

where $\mathsf{E}^{\boldsymbol{X},\boldsymbol{Y}}$ is two-point Palm expectation.

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Proposition 4 *Assume Φ to be a Poisson p.p., F to be exponential and the noise W to have a general distribution. Then for all $X, Y \in \mathbb{R}^2$, the mean local delay from X to Y is finite given the existence of these two points in Φ. More precisely,*

$$\mathsf{E}^{X,Y}[L_{X,Y}(0)] < \infty\,,$$

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Proposition 5 *[BB. Baccelli, Mirsadeghi (2011)] Under the assumptions of Proposition 4*

$$\lim_{|X-Y|\to\infty} \frac{\mathsf{E}^{X,Y}[P_{X,Y}(0)]}{|X-Y|} = \infty\,;$$

i.e., the mean least end-to-end delay in Poisson network grows faster than the distance! (Because of Poisson voids.)

Filling-in Poisson Voids; Toward a Positive Result

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Extend the definition of the shortest path on $\boldsymbol{\mathcal{G}}$: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\mathbf{2}}$, define $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{n}) = \boldsymbol{P}_{\boldsymbol{X}(\boldsymbol{x}), \boldsymbol{X}(\boldsymbol{y})}(\boldsymbol{n})$.

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Extend the definition of the shortest path on $\mathcal{G}$: for all $x, y \in \mathbb{R}^2$, define $P(x, y, n) = P_{X(x),X(y)}(n)$.

Let $p(x, y, \Phi) = \mathsf{E}[P(x, y, 0) \mid \Phi]$ be expected conditional shortest end-to-end delay from x to y given locations of network nodes Φ.

Filling-in Poisson Voids; a Positive Result

Proposition 6 *Consider Poisson+Grid p.p., with remaining assumptions as in Prop. 4. Then, for all unit vectors $\mathbf{d} \in \mathbb{R}^2$, the non-negative limit*

$$\kappa_{\mathrm{d}} = \lim_{t\to\infty} \frac{p(0, t\mathbf{d}, \Phi)}{t}$$

exists and is P*-a.s. finite. The convergence also holds in L_1.*

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Superposing an arbitrarily sparse, periodic infrastructure of nodes with Poisson p.p. makes the least end-to-end delay scale linearly with the distance.

The Positive Result — Proof Idea

Note that

$$P(x, z, n) \leq P(x, y, n) + P\Big(y, z, n + P(x, y, n)\Big)\,.$$

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One has to work (rather hard) to prove that this limit is positive and finite. For this we use our previously developed machinery to analyze mean local delays, this time in the broadcast receiver model. This completes the proof.

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Our main performance characteristic is the limit of the ratio

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when $|O - D| \to \infty$, called time constant.

The number of hops on $\mathcal{G}$ in the numerator above, corresponds to the end-to-end delay (from O to D); it is the sum of the local delays at all nodes visited on the shortest-time path by some tagged packet, which does not experience any queuing at nodes before being scheduled for transmission.

Summary...; Two Main Results

1. In Poisson network the end-to-end delay grows faster than the distance $|O - D|$ (time constant is infinite) (principally due to large voids in the repartition of nodes).

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1. In Poisson network the end-to-end delay grows faster than the distance $|O - D|$ (time constant is infinite) (principally due to large voids in the repartition of nodes).
2. Adding an arbitrarily sparse, periodic infrastructure of nodes (superposing it with Poisson p.p.) makes end-to-end delay scale linearly with $|O - D|$ (time constant positive and finite).

THANK YOU