Segment recombinations and random sharing models

Anton Muratov  Sergei Zuyev

Chalmers University of Technology, Gothenburg, Sweden

January 14th 2015, Paris
Take a homogeneous Poisson point process (PPP) on $\mathbb{R}$ and shift all the points randomly independently by equally distributed amounts. The result is again a homogeneous PPP.
Motivation

Take a homogeneous Poisson point process (PPP) on $\mathbb{R}$ and shift all the points randomly independently by equally distributed amounts. The result is again a homogeneous PPP.

What’s if the shifts depend on the neighbouring points?
Take a homogeneous PPP on the line:
Take the middle points of the segments:
The middle points of each segment do not form a PPP anymore

because the independence of the segment lengths is broken!
Iterate the same procedure

\[ \tau, \tau^1, \tau^2 \]

\[ \tau_1, \tau_2, \tau_3, \tau^1_1, \tau^2_1, \tau^1_2, \tau^2_2 \]
Notice that every second generation moves each point to the centre of its 1D Voronoi cell.
Limiting configuration is a grid (Hasegawa & Tanemura’76)
Recursion

Above we have

\[ \tau_k^n = \frac{1}{2} \tau_{k-1}^n + \frac{1}{2} \tau_{k+1}^{n-1}, \quad n = 1, 2, \ldots, \ (\tau^0 \overset{\text{def}}{=} \tau). \]

More generally, proportions can be independent random variables with a common distribution \( G \) on \([0, 1]\):

\[ \tau_k^n = (1 - b_k^n) \tau_{k-1}^n + b_{k+1}^n \tau_{k+1}^{n-1}, \quad b_k^m \sim G \text{ i.i.d.} \]
Question:

Is there a distribution $G$ such that for $\tau \sim \text{PPP}$,

$$\tau^1 \overset{D}{=} \tau?$$
Yes!

$G$ is Beta distribution $B(r, 1 - r)$ for some $r \in (0, 1)$
A General Question:

Assume that $\tau$ is a renewal process, so that $\tau_k$ are i.i.d. with a common distribution $F$ on $(0, \infty)$. Is there $G$ such that $\tau^1 \overset{D}{=} \tau$?
Theorem

\( \tau^1 \overset{D}{=} \tau \) iff one of the following alternatives is true:

1. \( F \) is degenerate (\( \tau \) is a grid) and \( G \) is degenerate concentrated on some \( b \in [0, 1] \).
2. \( F = \Gamma(\alpha, \gamma) \) and \( G = B(r\alpha, (1 - r)\alpha) \) for some constants \( \alpha > 0, \gamma > 0 \) and \( r \in [0, 1] \).
Theorem

$\tau_1 \overset{D}{=} \tau$ iff one of the following alternatives is true:

1. $F$ is degenerate ($\tau$ is a grid) and $G$ is degenerate concentrated on some $b \in [0, 1]$.

2. $F = \Gamma(\alpha, \gamma)$ and $G = B(\alpha r, (1 - r)\alpha)$ for some constants $\alpha > 0$, $\gamma > 0$ and $r \in [0, 1]$.

Corollary

- $\tau \sim \text{PPP}$ and $G = B(r, 1 - r)$ for some $r \in [0, 1]$
- $\tau$ is renewal process with $\Gamma(2, \lambda)$ inter-point distances (every second point in a PPP) and $G = \text{Unif}(0, 1)$. 
Gamma measure

**Definition**

An independently scattered random measure $\Lambda$ on $\mathbb{R}$ with Gamma-distributed increments:

$$\Lambda([a, b]) \sim \Gamma(\alpha(b - a), \lambda) \quad \text{for some } \alpha, \lambda > 0$$

is called the **Gamma random measure**. $\xi(t) = \Lambda[0, t]$ is called the **Gamma process**.

$\Lambda$ is purely atomic and it can be represented as

$$\Lambda([a, b]) \overset{D}{=} \sum_{(x_i, y_i) \in \Phi} y_i \mathbb{I}\{x_i \in [a, b]\},$$

where $\Phi$ is a PPP on $\mathbb{R} \times (0, \infty)$ driven by intensity measure $\alpha dx \int y^{-1} e^{-\lambda y} dy$ [Ferguson & Klass’72].
Proof: sufficiency

\[ \tau_1 + \tau_1 + r_1 = \tau_2 \]

Sergei Zuyev

Segment recombinations and random sharing models
Proof: sufficiency

- \( \tau_k \overset{D}{=} \Lambda([k - 1, k]) \sim \Gamma(\alpha, \lambda) \text{ i.i.d.} \)
- \( \beta_1 = \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \sim \frac{\Gamma(r\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(r\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(r\alpha, (1 - r)\alpha)}{\Gamma(\alpha)}. \)

Similarly other \( b_k \) i.i.d.
Proof: necessity

Formal proof: via a joint ch.f. Idea: shape vs. size result:
\[ b_1 = \frac{\xi_1}{(\xi_1 + \xi_2)} \perp \tau_2 = (\xi_1 + \xi_2) \implies \xi_i \sim \Gamma. \]
The same recursion

\[ \tau_k^n = (1 - b_k^n) \tau_k^{n-1} + b_k^{n+1} \tau_{k+1}^{n-1}, \quad n = 1, 2, \ldots, (\tau_0 \text{ def } \tau). \]

allows for another interpretation: $\tau_k^n$ is a **wealth** of household $k$ at time $n$, each time a random $b$-share of the wealth is passed to the left neighbour.
(1 - b_1^1)\tau_1 - b_2^1\tau_2 - \tau_3
Fix a random probability distribution $\pi$ on $I \subseteq \mathbb{Z}$.
Previously $\pi$ was the pair $(1 - G, G)$ concentrated on $I = \{0, 1\}$, $\pi_1 \overset{D}= b$, $\pi_0 \overset{D}= 1 - b$.

At every moment $n = 1, 2, \ldots$ each household $k$

1. draws independently a realisation $\pi^n(k) = (\pi^n_i(k))_{i \in I}$ of $\pi$ and
Fix a random probability distribution $\pi$ on $I \subseteq \mathbb{Z}$. Previously $\pi$ was the pair $(1 - G, G)$ concentrated on $I = \{0, 1\}$, $\pi_1 \overset{D}{=} b$, $\pi_0 \overset{D}{=} 1 - b$.

At every moment $n = 1, 2, \ldots$ each household $k$

1. draws independently a realisation $\pi^n(k) = (\pi^n_i(k))_{i \in I}$ of $\pi$ and

2. passes proportion $\pi^n_i(k)$ of its wealth $\tau^n_{k-1}$ to household $k - i$ (leaving proportion $\pi_0(k)$ to itself).
Theorem

If one of the following alternatives is true:

1. $F = \Gamma(\alpha, \gamma)$ and $\pi$ is Dirichlet $\text{Dir}((r_i\alpha)_{i \in I})$ for some $\alpha$, $\gamma > 0$ and constants $r_i$ such that $\sum_{i \in I} r_i = 1$.

2. $F$ is degenerate (equal wealths) and $G$ is degenerate (non-random proportions sharing)

then $\tau^1 \overset{D}{=} \tau$. 

Sergei Zuyev
Segment recombinations and random sharing models
Redistribution of the Gamma-measure:
The same proof works for lattices with one type of vertices.
Is the ballance condition enough?
Half-wealth sharing leads to equal wealths (socialism)
Fundamental implication

- Half-wealth sharing leads to equal wealths (socialism)
- Beta-wealth sharing of gamma-distributed wealth leaves its distribution unchanged (stable capitalism)
Fundamental implication

- Half-wealth sharing leads to equal wealths (socialism)
- Beta-wealth sharing of gamma-distributed wealth leaves its distribution unchanged (stable capitalism)
- **Question:** How attractive is capitalism? (e.g. does beta-sharing of somehow distributed wealths leads to gamma-distributed independent wealths?)
The recursion can be written as a linear operator acting on sequences: $\tau^n = B_n \tau^{n-1}$, where $B_n$ is a double-infinite matrix with elements $\pi^n_{i+k}(k)$ at the place $(k, i)$ (the proportion of the wealth of the household $i$ which $k$ receives at time $n$, $(k, i \in \mathbb{Z})$. E.g.,

$$B_n = \begin{pmatrix} \ldots & 1 - b_1^n & b_2^n & 0 & 0 & \ldots \\ \ldots & 0 & 1 - b_2^n & b_3^n & 0 & \ldots \\ \ldots & 0 & 0 & 1 - b_3^n & b_4^n & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \quad n = 1, 2, \ldots$$
Let \( h = (h_i)_{i \in \mathbb{Z}} \) be a non-negative sequence with a compact support: \( \sum_i \mathbb{I}_{h_i > 0} < \infty \).

\[
L_n[h] = \mathbb{E} e^{-\langle h, \tau^n \rangle} = \mathbb{E} e^{-\langle h, B_n \ldots B_1 \tau \rangle} = \mathbb{E} e^{-\langle h, B_1 \ldots B_n \tau \rangle}.
\]
Assume $\tau = 1$ and consider $\xi_n = \langle h, B_1 \ldots B_n 1 \rangle$. Let $\mathcal{B}_n = \sigma\{B_1, \ldots, B_n\}$.

$$
\mathbb{E}[\xi_n \mid \mathcal{B}_{n-1}] = \langle h, B_1, \ldots, B_{n-1}(\mathbb{E}B_n)1 \rangle
$$

The $k$-th element of $(\mathbb{E}B_n)1$ is

$$
\mathbb{E}_i \pi_i^n(k) = (\text{average total prop.'ns received by } k) \\
= (\text{average total prop.'ns sent by } k) = 1
$$

so that $(\mathbb{E}B_n)1 = 1$ and thus $\xi_n$ is a martingale.
Thus $\xi_n = \xi_n(h)$ converges a.s. to some $\xi_\infty(h)$ which is a linear function of a finite-dimensional $h$. Hence it is $\langle h, \tau_\infty \rangle$ for $\tau_k^\infty = \xi_\infty(\delta_k)$. Finally, $\exp\{-\xi_n\}$ is a bounded submartingale so it converges in $\mathcal{L}_1$:

$$\mathbb{E} \exp\{-\langle h, \tau^n \rangle\} \to \mathbb{E} \exp\{-\langle h, \tau_\infty \rangle\}$$

so that $\tau_\infty$ is a weak limit of $\tau^n$. 
More generally,

**Theorem**

If $\tau^0$ is an i.i.d. sequence of realisations of $\tau$ with $\text{var} \tau < \infty$, then there exists a random sequence $\tau^\infty$ (not necessarily i.i.d.) such that $\tau^\infty$ is a **weak limit of** $\tau^n$. 
If $\pi \sim \text{Dir}$ then $\{\tau_k^\infty\}$ are independent Gamma-distributed, so the stable capitalism is attractive for i.i.d. starting wealths with finite variance and Dirichlet sharing!
Beyond 1D

$\mathbf{GRID}$

$x \mapsto x + \frac{1}{2}(r_1 + r_2)$
Beyond 1D

\[ x \mapsto x + 2(d_1 r_1 + d_3 r_2) \]

\((d_1, d_2, d_3) \sim \text{Dir}(1/4, 1/2, 1/4)\)
Lloyd algorithm

\[ x \mapsto x + \frac{1}{n} \sum_{i} r_i \]

Sergei Zuyev
Segment recombinations and random sharing models
How to preserve Poisson?

\[ x \mapsto x + \, ? \]

PPP


Thank you!

Questions?