Topology inference: escaping the spatial independence straightjacket

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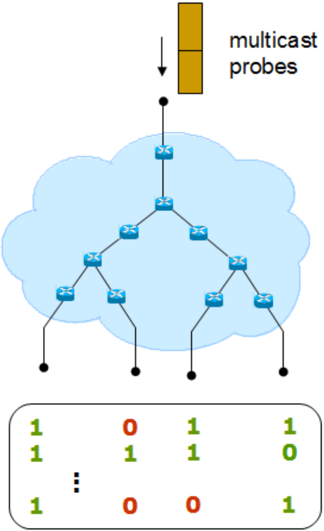
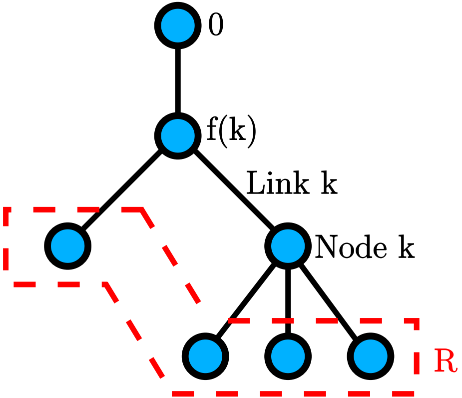
MULTICAST TOMOGRAPHY

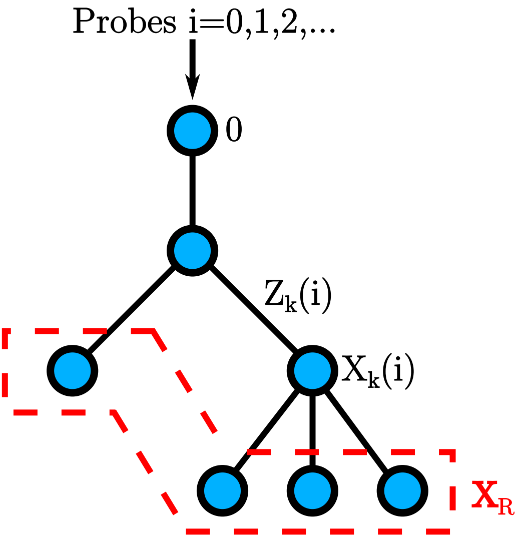


Definitions and Notation

- Tree $T = (V, L)$.
- Nodes V labelled $0, \ldots, n$.
- m receivers R at leaves of tree.



PROBING



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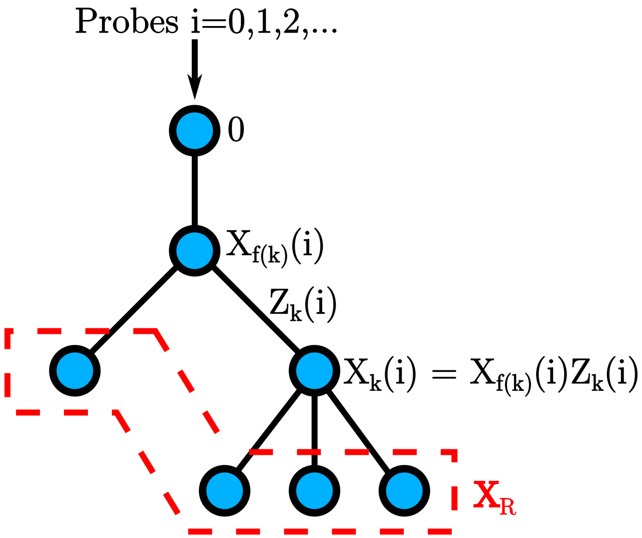

PROBING

- View as vector-valued stochastic process

$$\mathbf{Z}(i) = [Z_1(i), \ldots, Z_n(i)].$$

- Tree-geometry: node/path state fixed by states of ancestor links:

$$X_k(i) = \prod_{j \in 0 \to k} Z_j(i) \; .$$

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GOAL: TOPOLOGY FROM TOMOGRAPHY

- Deduce the topology T from the distribution of $\mathbf{X}_R = (X_k(i))_{k \in R}$.
- First assume infinite data, address identifiability.
- Then consider inference with finite data.

PREVIOUSLY

SPATIAL AND TEMPORAL INDEPENDENCE (CLASSICAL ASSUMPTIONS)

- Link processes $Z_k(i)$ mutually independent.
- Each an i.i.d. random sequence: $\Pr(Z_k(i) = 1) = l_k$.

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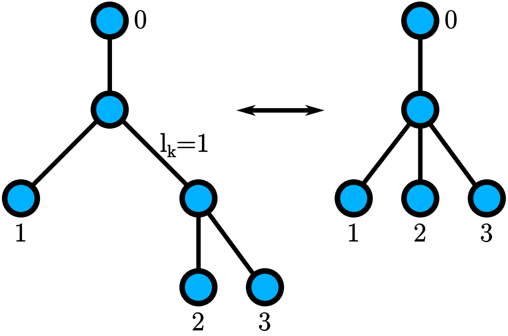
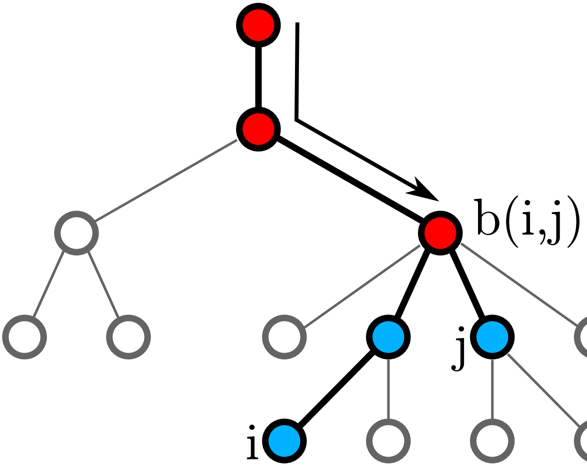
- Each an i.i.d. random sequence: $\Pr(Z_k(i) = 1) = l_k$.
- Assume $l_k < 1$, else unidentifiable.



Shared Path To Branch Point



Shared Transmission

- Function of two nodes, i, j :

$$S(i,j) = \frac{\mathsf{Pr}(X_i = 1)\mathsf{Pr}(X_j = 1)}{\mathsf{Pr}(X_i = 1, X_j = 1)}.$$

- Under spatial independence

$$S(i,j) = \mathsf{Pr}(X_b = 1).$$

SHARED TRANSMISSION

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- Under spatial independence

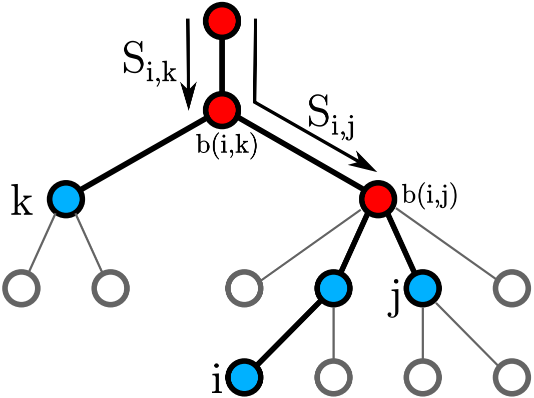
$$S(i,j) = \Pr(X_b = 1).$$

- Use/need pairwise only $\Rightarrow$ still feasible with finite data.

Choosing Siblings

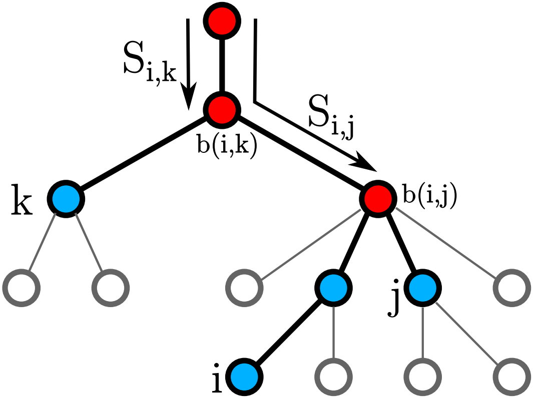
Shared Transmission decreases down the tree

- If $b(i,j)$ under $b(i,k)$ then $S(i,j) < S(i,k)$.



Certain Paternity

- Pair(s) of nodes in B with lowest shared transmission are siblings.
- If $J \subset B$ has $S(i,j)$ minimal for each pair $i,j \in B$ then J are siblings.



Shared Transmission for Virtual Nodes

- Nodes created by merging siblings are "virtual".
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$$\tilde{X}_j = \begin{cases} 1 & \text{if } X_k = 1 \text{ for any } k \in d(j) \cap R \\ 0 & \text{otherwise,} \end{cases}$$

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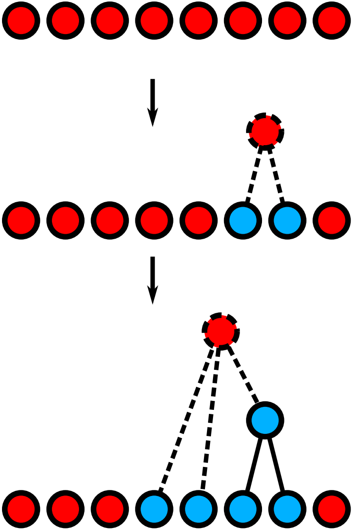
  since know $X_j = 1$ if a transmission seen at any descendant.
- Shared transmission defined analogously:

$$\tilde{S}(i,j) = \frac{\mathsf{Pr}(\tilde{X}_i = 1)\mathsf{Pr}(\tilde{X}_j = 1)}{\mathsf{Pr}(\tilde{X}_i = 1, \tilde{X}_j = 1)}.$$

- $\tilde{S}(i,j) = S(i,j)$ under classical assumptions!

ITERATIVE BOTTOM-UP TOPOLOGY INFERENCE

Red nodes are the working set B.



Shared Loss Topology Discovery – SLTD

Input: Set of receivers R; distribution f_R, $\mathbf{X}_R(i)$.
Variables: Nodes V, Links L, Root nodes B, $\tilde{\mathbf{X}}(i)$.
Initialize: $V \leftarrow R$; $L \leftarrow \emptyset$; $B \leftarrow R$; $\tilde{\mathbf{X}}_R(i) \leftarrow \mathbf{X}_R(i)$.
while $|B| > 1$ **do**
  Calculate $S^* = \max_{\{j,k\} \subset B} \tilde{S}_{j,k}$;
  Find largest $J \subset B$: $\forall \{j,k\} \subset J$, $\tilde{S}_{j,k} = S^*$;
  if exists some $i \notin J, j \in J : \tilde{S}_{i,j} = S^*$ **then**
    return $\emptyset$; # sibling set not transitive!
  else
    Create new node v, set $\tilde{X}_v = \bigvee_{j \in J} \tilde{X}_j$;
    $V \leftarrow V \cup v$;
    $L \leftarrow L \cup \bigcup_{j \in J} (v, j)$;
    $B \leftarrow (B \backslash J) \cup v$;
  end if
end while
Create root node 0;
$V \leftarrow V \cup 0$;
$L \leftarrow L \cup (0, B)$; # $|B| = 1$ here
Output: $T = (V, L)$;

Characterizing the Link Processes

Spatial Structure

- Assume $\mathbf{Z}(i) = [Z_1(i), \ldots, Z_n(i)]$ stationary and ergodic.
- Spatial dependency captured by the marginal $\mathbf{Z} = [Z_1, \ldots, Z_n]$.
- Induces the path-passage marginal $\mathbf{X} = [X_1, \ldots, X_n]$.
- We are interested in $f_{\mathbf{X}_R}$.

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Link Joint Distribution

- Characterise joint distribution $f_{\mathbf{Z}}$ using probabilities

$$\mathsf{Pr}(\mathbf{Z} = \mathbf{r}) = \mathsf{Pr}(Z_1 = r_1, Z_2 = r_2, \ldots Z_n = r_n),$$

  one for each link passage pattern $\mathbf{r} = [r_1, \ldots, r_n] \in \{0, 1\}^n$.
- These sum to 1, so $2^n - 1$ degrees of freedom.
- In contrast: classical case is much simpler, n degrees of freedom.

MODELS VERSUS TOPOLOGY

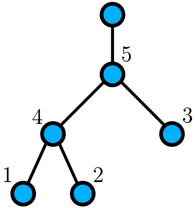
MODELS

- A topology T with a joint distribution $f_{\mathbf{Z}}$ is a *model* $M = (T, f_{\mathbf{Z}})$.
- A model M induces a joint distribution $f_R(M)$ on the vector observable $\mathbf{X}_R$.
- $T(M)$ is the tree component of the model M.
- Goal: to determine $T(M)$ from $f_R(M)$.

Measurement Equivalence

Two models M_1 and M_2 are *measurement equivalent* if $f_R(M_1) = f_R(M_2)$.

Example 1:



Classical with $l_k = 0.9$ for all $k \in V$.

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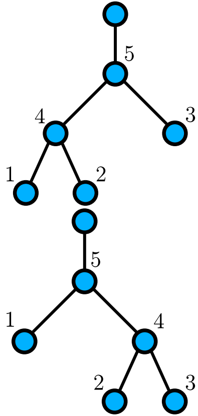

Both models have
$\Pr(X_1 = 1) = 0.9^3$
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$\Pr(\mathbf{X}_R = \mathbf{1}_3) = 0.9^5$.

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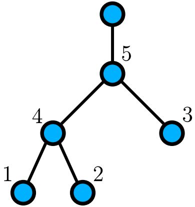

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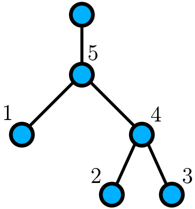

$$\Pr(\mathbf{Z} = \mathbf{z}) = \begin{cases} 0 & [z_1, z_2, z_3] = [1, 1, 0] \\ 0.9^3 0.1^2 + 0.9^2 - 0.9^3 & \mathbf{z} = [1, 0, 1, 0, 1] \\ 0.9^{\sum_i z_i} 0.1^{5 - \sum_i z_i} & \text{otherwise.} \end{cases}$$

Topology Identifiability

Example 1 lessons

- Example 1 gave two models with same $f_R(M)$, different $T(M)$.
- So in that case, T is not identifiable.

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Topological Determinism

- A class $\mathcal{M}$ is Topologically Determinate if $\nexists M_1, M_2 \in \mathcal{M}$ with
  $f_R(M_1) = f_R(M_2)$, and
  $T(M_1) \neq T(M_2)$.
- *i.e.*, models with same f_R have same T.

Goals (Infinite Data Case)

- Find "large", natural Topologically Determinate class(es) $\mathcal{M}$.
- Find algorithm guaranteed to recover $T(M)$ for all $M \in \mathcal{M}$.

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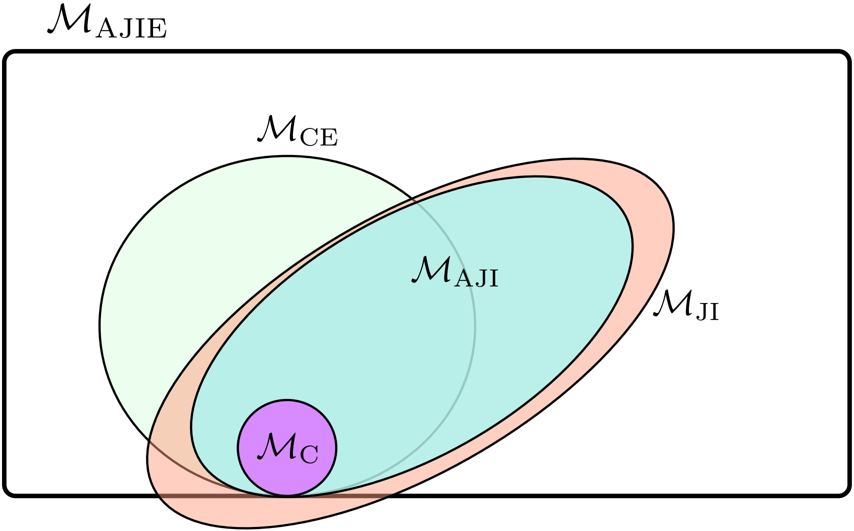
Example: Classical Models $\mathcal{M}_C$

- Classical models are Topologically Determinate.
- SLTD works for them.
- In fact, one model per $f_R(M)$, so one model per T.

New Classes


$\mathcal{M}_{\text{AJIE}}$
$\mathcal{M}_{\text{CE}}$
$\mathcal{M}_{\text{AJI}}$
$\mathcal{M}_{\text{JI}}$
$\mathcal{M}_{\text{C}}$

DIMENSIONS OF NEW CLASSES

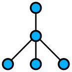
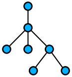
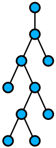
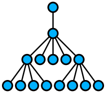
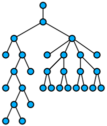
T					
$\dim(\mathcal{M}_{\mathrm{C,T}})$	4	6	9	14	29
$\dim(\mathcal{M}_{\mathrm{CE,T}})$	12	54	489	14350	536805405
$\dim(\mathcal{M}_{\mathrm{JI,T}})$	15	56	478	14133	536613988
$\dim(\mathcal{M}_{\mathrm{AJI,T}})$	15	56	478	14133	536613988
$\dim(\mathcal{M}_{\mathrm{AJIE,T}})$	15	57	489	14395	536805415
$\dim(\mathcal{M}_{T})$	15	63	511	16383	536870911

TABLE : Examples of model class dimensions.

Classically Equivalent Models: $\mathcal{M}_{\mathrm{CE}}$

Definition

$M_1 \in \mathcal{M}_{\mathrm{CE}}$ iff $\exists M_2 \in \mathcal{M}_C$ with $f_R(M_1) = f_R(M_2)$ and $T(M_1) = T(M_2)$.

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SLTD still works!

- SLTD returns $T(M)$ correctly for every $M \in \mathcal{M}_{\text{CE}}$.
  - Returns topology as though M is classical.
  - $\therefore$ Returns correct topology.
- So $\mathcal{M}_{\text{CE}}$ is Topologically Determinate.

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- So $\mathcal{M}_{\text{CE}}$ is Topologically Determinate.

Extension trick works in general

- Can apply for any algorithm and class it works on.

Comments on $\mathcal{M}_{\text{CE}}$

Strengths

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Drawbacks

- Not constructive.
- Depends on receiver positions.

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STRENGTHS

- $\mathcal{M}_C \subset \mathcal{M}_{\text{CE}}$.
- Much larger than $\mathcal{M}_C$.
- Can contain complex spatial dependencies.

DRAWBACKS

- Not constructive.
- Depends on receiver positions.
- Need a model class that:
    - Is not based on receiver positions.
    - Reflects properties of real networks.

Painless Generality

Recall

- $X_k = \prod_{i \in (0 \to k)} Z_k.$

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Dependency of hidden Z

- If $X_i = 0$ then for all k below i, $X_k = 0$.

PAINLESS GENERALITY

RECALL

- $X_k = \prod_{i \in (0 \to k)} Z_k$.

DEPENDENCY OF HIDDEN Z

- If $X_i = 0$ then for all k below i, $X_k = 0$.
- If $X_{f(i)} = 0$ then changing the value of Z_i won't change the output.
- This suggests a way of adding dependency without affecting $f_R(M)$.

Model Principles

How does dependency arise?

- Links touch at routers, influenced by router traffic and dynamics
  – suggests dependencies between siblings.
- Distant links unlikely to affect each other except via tree.
  – suggests ruling out 'action at a distance'.

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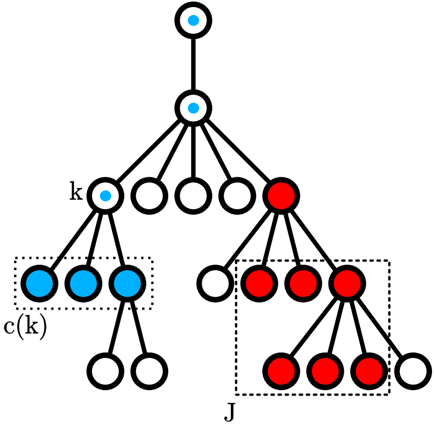
Translation to Model Principles

- Locally: most general possible dependency between adjacent links.
- Globally: only necessary dependency over non-adjacent links.

Jump Independence

Definition (Jump Independent Models)

A model with links L and receivers R is *Jump Independent* if $\forall k \in V \backslash R$, $\forall J \subset V$ with $J \cap d(k) = \emptyset$, $\mathbf{X}_{c(k)}$ is conditionally independent of $\mathbf{X}_J$ given $X_k = 1$.



Definitions

Definition (Subtree Induced by U)

Let $M(T, f_{\mathbf{Z}}) \in \mathcal{M}_{\text{JI}}$ with $T = (V, L)$. Let $U \subset V$. Then define the subtree induced by U as

$$T(U) = \bigcup_{i \in U} \{0 \to i\}$$

and $R(U)$ as the leaves of $T(U)$.

Definition (ρ-values)

Define sibling passage probabilities:

$$\rho_J = \mathsf{Pr}(\cap_{j \in D}\{X_j = 1\} | X_{f(D)} = 1)$$

for each set of siblings D.

FUNDAMENTAL PROPERTY OF JI MODELS

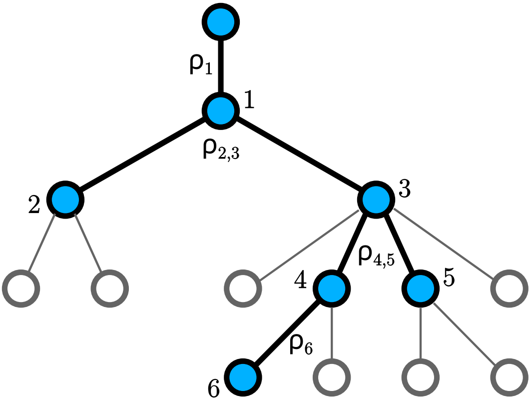
LEMMA (FUNDAMENTAL PROPERTY OF JI MODELS)

Let $M(T, f_{\mathbf{Z}}) \in \mathcal{M}_{\mathrm{JI}}$. *Then*

$$\mathsf{Pr}\Big(\bigcap_{k \in U} \{X_k = 1\}\Big) = \prod_{i \in T(U) \setminus R(U)} \rho_{c(i) \cap T(U)}$$

for every $U \subset V$.

Fundamental Property of JI models



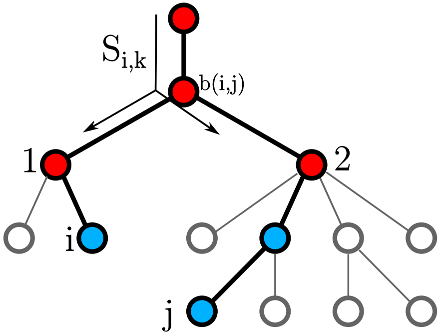
Example : $U = \{2, 5, 6\}$

$\mathsf{Pr}(X_2 = 1, X_5 = 1, X_6 = 1) = \rho_1 \cdot \rho_{2,3} \cdot \rho_{4,5} \cdot \rho_6$

Shared Transmission in JI models

- For $i, j \in V$,

$$S_{i,j} = \Pr(X_b = 1) \cdot \frac{\rho_1 \rho_2}{\rho_{1,2}}$$
$$= \Big(\prod_{k \in 0 \to b} \rho_k \Big) \cdot \frac{\rho_1 \rho_2}{\rho_{1,2}}$$



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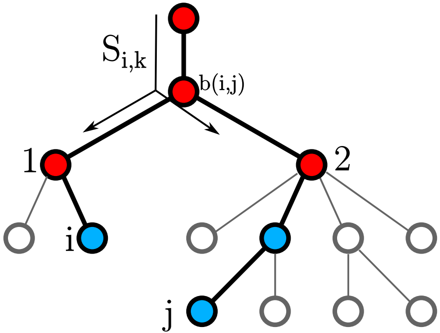
- Shared Transmission a function of the shared path and the two children at the branch point.



Binary JI models

Measurement Equivalence

- Assume $M_1 \in \mathcal{M}_{\text{JI}}$ and $M_2 \in \mathcal{M}_C$ with $T(M_1) = T(M_2)$.
- Solve for l_i from M_2 in terms of ρ_J from M_1.

$$l_i = \begin{cases} \dfrac{\rho_{i,s(i)}}{\rho_{s(i)}}, & \text{if } i \in R \\ \rho_i \cdot \dfrac{\rho_{c_1(i)}\rho_{c_2(i)}}{\rho_{c_1(i),c_2(i)}} & \text{if } i = 1 \\ \dfrac{\rho_{i,s(i)}}{\rho_{s(i)}} \cdot \dfrac{\rho_{c_1(i)}\rho_{c_2(i)}}{\rho_{c_1(i),c_2(i)}} & \text{otherwise.} \end{cases}$$

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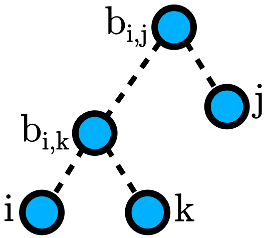
Obtain (Binary) Examples of models in CE

- If $l_i < 1$, must be the marginal link passage parameter of the CE model.
- Insight: siblings dependencies compensated by change in transmission on path to father.

Identifiability failure: invisible paths

Lemma

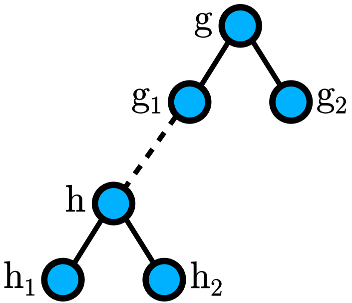
Let i, j, k be three distinct receivers in a Jump Independent model such that $b(i,k)$ is below $b(i,j)$. Then $S(i,k) = S(j,k)$ if and only if $b(i,j) \rightarrow b(i,k)$ is invisible.



Identifiability failure: invisible paths

Augmented Path

- An augmented path $g(g_1, g_2) \to h(h_1, h_2)$ is a path $g \to h$ together with $g_1, g_2 \in c(g)$, $h_1, h_2 \in c(h)$ such that $g_1 \in g \to h$.



IDENTIFIABILITY FAILURE: INVISIBLE PATHS

INVISIBLE PATH

- An augmented path is invisible if

$$\frac{\rho_{g_1}\rho_{g_2}}{\rho_{g_1,g_2}} = \Big(\prod_{i\in g\to h}\rho_i\Big)\frac{\rho_{h_1}\rho_{h_2}}{\rho_{h_1,h_2}}.$$

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Invisible path

- An augmented path is invisible if

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- For Binary models this reduces to:

$$\prod_{i \in g \to h} l_i = 1.$$

- Analogue of $l_k \neq 1$ from classical.

IDENTIFIABILITY FAILURE: LOCAL STRUCTURE

LOCAL LIMITATIONS ON ANY SIBLING SET J

- *Internally agreeing* if $S_{i,j} = S_{k,l}\ \forall i,j,k,l \in J$ with $i \neq j, k \neq l$.
- *Internally disagreeing* if $S_{i,j} \neq S_{k,l}\ \forall i,j,k,l \in J$ with $\{i,j\} \neq \{k,l\}$.

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ROLES

- Disagreeing is the generic/general case.
- Agreeing includes classical.

Agreeable JI models

Definition (Agreeable JI models ($\mathcal{M}_{\text{AJI}}$))

An AJI model is a model $M \in \mathcal{M}_{\text{JI}}$ which satisfies :

i) (internally consistent) Each sibling set J is agreeing or disagreeing.

ii) (no invisible paths) No augmented paths in M are invisible.

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ROLE OF RESTRICTIONS

- Condition (i) prevents sibling sets from looking like they aren't.
- Condition (ii) prevents groups of non-siblings from looking like they are siblings.

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Role of restrictions

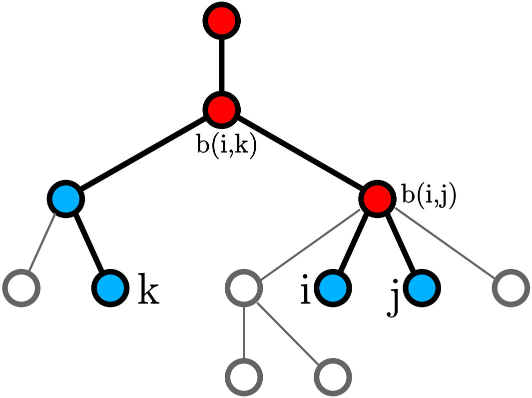
- Condition (i) prevents sibling sets from looking like they aren't.
- Condition (ii) prevents groups of non-siblings from looking like they are siblings.

Including 'agreeing' in (i) a big headache, but important!

A PROPERTY OF SIBLINGS IN JI MODELS

LEMMA (SIBLINGS AGREE EXTERNALLY)

Let $M \in \mathcal{M}_{\text{JI}}$. If two nodes i, j are members of a sibling set J, and $k \in R$ such that $(0 \to k) \cap J = \emptyset$, then $S_{i,k} = S_{j,k}$.



Seeking Certain Paternity

Try to invert sibling property

- Define agreement set of $i, j \in V$

$$A_{i,j} = \{k \in R : S(i,k) = S(j,k), k \neq i,j\}.$$

- Agreement sets used to compare ‘world view’ of candidate siblings.

Finding Complete Sibling Sets

Definition (Externally-agreeing sets)

Call $D \subset R$ an *externally-agreeing* set (EAS) if $|D| \geq 3$ and $A_{i,j} = R \backslash D$ for all $i, j \in D$.

Finding Complete Sibling Sets

Definition (Externally-agreeing sets)

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Definition (All-agreeing sets)

Call $D \subset R$ with $|D| \geq 2$ an *all-agreeing set* (AAS) if $A_{i,j} = R \backslash \{i, j\}$ for all $i, j \in D$.

Subsets of an all-agreeing set are also all-agreeing. Call an all-agreeing set D a *maximal all-agreeing set* (MAAS) if it is not a proper subset of another one.

Finding Complete Sibling Sets

Lemma (Finding disagreeing sibling sets)

Consider $M \in \mathcal{M}_{\text{AJI}}$ with receiver nodes R. A set $D \subset R$ with $|D| \geq 3$ is an disagreeing sibling set if and only if it is an EAS.

Finding Complete Sibling Sets

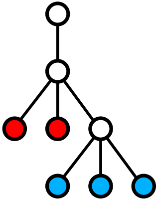
Lemma (Finding disagreeing sibling sets)

Consider $M \in \mathcal{M}_{\mathrm{AJI}}$ with receiver nodes R. A set $D \subset R$ with $|D| \geq 3$ is an disagreeing sibling set if and only if it is an EAS.

Lemma (Finding agreeing sibling subsets)

Consider $M \in \mathcal{M}_{\mathrm{AJI}}$ with receiver nodes R. A set $D \subset R$ with $|D| \geq 2$ is a subset of an agreeing sibling set if and only if it is an AAS.

- The MAAS are the maximal agreeing sibling subsets.
- Some/all of these may still have hidden siblings.



Proposition (Certain Paternity II)

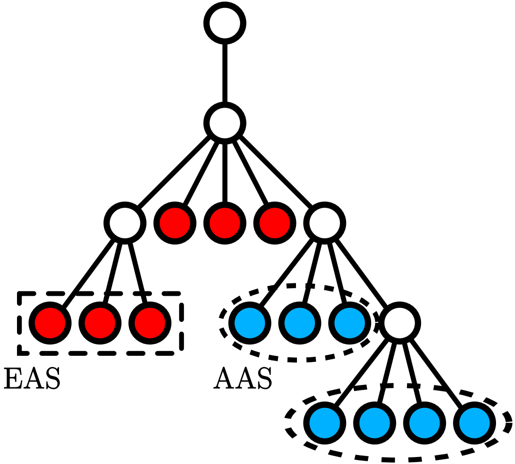
Assume an $M \in \mathcal{M}_{\mathrm{AJI}}$ model. Then at least one available sibling set can be identified without error.

Proof

- Find all the EAS and AASes

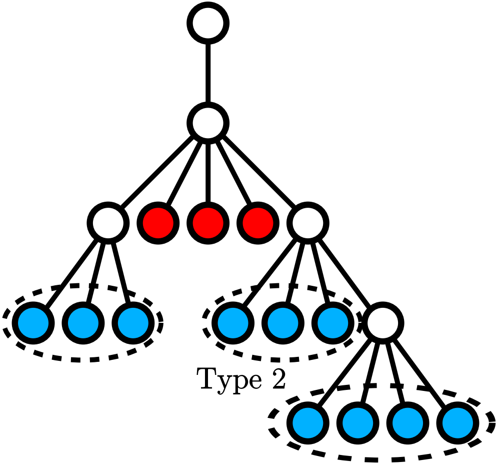
Case 1: at least one EAS exists

Select any of them.



Case 2: no EAS exists

Select a MAAS which is a sibling set (can test if one below another).



SLTD2

Similar to SLTD, but agreement set based.

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Let $M = (T, f_{\mathbf{Z}}) \in \mathcal{M}_{\mathrm{AJI}}$. Then SLTD2 returns T.

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- $S(i,j) = \tilde{S}(i,j)$ for $M \in \mathcal{M}_{\text{JI}}$.

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- $S(i,j) = \tilde{S}(i,j)$ for $M \in \mathcal{M}_{\text{JI}}$.
- So each iteration will be correct.
- Hence recover T at termination.

AJIE Models

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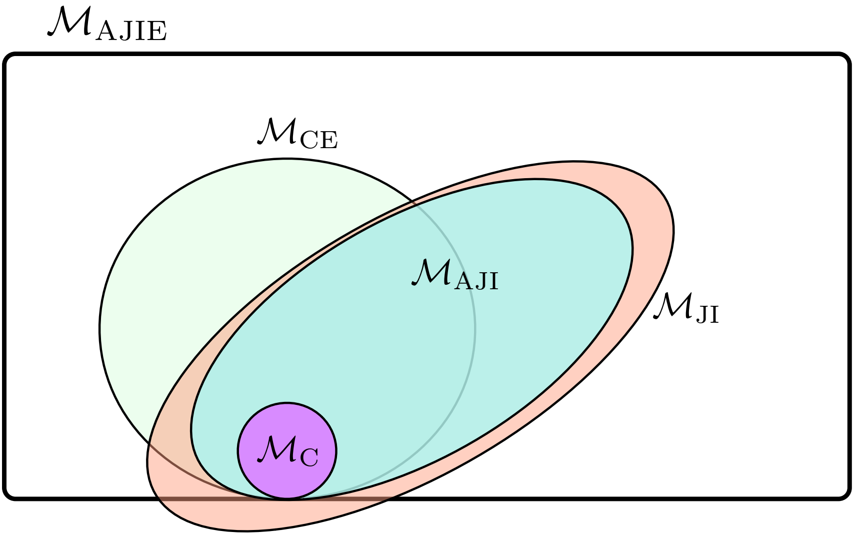
- $\mathcal{M}_{\text{CE}} \subset \mathcal{M}_{\text{AJIE}}$, since $\mathcal{M}_{\text{C}} \subset \mathcal{M}_{\text{AJI}}$.
- SLTD2 succeeds on all topologies in $\mathcal{M}_{\text{AJIE}}$.

Relationships between classes



DIMENSIONS OF CLASSES

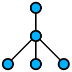
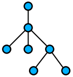
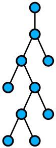
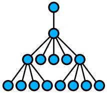
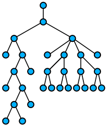
T					
$\dim(\mathcal{M}_{\mathrm{C,T}})$	4	6	9	14	29
$\dim(\mathcal{M}_{\mathrm{CE,T}})$	12	54	489	14350	536805405
$\dim(\mathcal{M}_{\mathrm{JI,T}})$	15	56	478	14133	536613988
$\dim(\mathcal{M}_{\mathrm{AJI,T}})$	15	56	478	14133	536613988
$\dim(\mathcal{M}_{\mathrm{AJIE,T}})$	15	57	489	14395	536805415
$\dim(\mathcal{M}_T)$	15	63	511	16383	536870911

TABLE : Examples of model class dimensions.

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Previous work

- Classical model: full spatial independence of tree loss process.
- Algorithm SLTD to recover topology in this case.

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- Define class $\mathcal{M}_{\text{JI}}$ with physically motivated structure.
- Find TD class $\mathcal{M}_{\text{AJI}}$ with $\dim(\mathcal{M}_{\text{AJI}}) = \dim(\mathcal{M}_{\text{JI}})$.
- New algorithm SLTD2 recovers topology for all $M \in \mathcal{M}_{\text{AJI}}$.
- Also recovers topology for all $M \in \mathcal{M}_{\text{AJIE}}$.

Challenges for Finite Data

- Underlying S_{ij} not known, only estimated.
- Failure of exact S_{ij} equality underlying agreement set definition.
- Random topology selection in $\mathcal{M}_{\text{AJI}}$, with degree constraints.
- Random model selection, with loss constraints.
- Sensible error metric on trees.

A SLTD Based Algorithm

Modified Iteration

- Estimate shared transmission over all pairs

$$\widehat{S}_{ij} = \frac{\sum \mathbf{X}_i/n_p \sum \mathbf{X}_j/n_p}{\sum \mathbf{X}_i \mathbf{X}_j/n_p}.$$

- Merge i, j into $J^* = (ij)$ with minimal $\widehat{S}_{ij}$.
- Merge additional receivers k in J^* obeying (we use $\beta = 0.002$)

$$\widehat{S}_{(ij)k} \leq (1+\beta)\widehat{S}^*.$$

Straightforward because key steps based on inequality of $\widehat{S}_{ij}$.

MEASURING APPROXIMATE AGREEMENT

THREE STEPS TO MEASURE AGREEMENT OF J TO A

(i) shared passage measure $p_{k;ij}$ ($|J| = 2$ and $|A| = 1$);
(ii) agreement set measure $g_{ij}(A)$ ($|J| = 2$ and $|A| \geq 1$);
(iii) sibling set measure $r_A(J)$ ($|J| \geq 2$ and $|A| \geq 1$).

Measuring Approximate Agreement

Step (i): shared passage measure $p_{k;ij}$ $(|J| = 2$ and $|A| = 1)$

Let $p_{k|i} = \mathsf{Pr}(X_k = 1|X_i = 1)$.

From the definition, $S_{ik} = S_{jk}$ equivalent to $p_{k|i} = p_{k|j}$.

MEASURING APPROXIMATE AGREEMENT

STEP (I): SHARED PASSAGE MEASURE $p_{k;ij}$ ($|J| = 2$ AND $|A| = 1$)

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Estimate $p_{k|i}$ by

$$\hat{p}_{k|i} = \sum(\mathbf{X}_k\mathbf{X}_i)/n_i \ .$$

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$$\hat{p}_{k|i} = \sum (\mathbf{X}_k \mathbf{X}_i)/n_i \ .$$

Null hypothesis: $p_{k|i} = p_{k|j}$.

Under H0 $\hat{p}_{k|} = (n_i \hat{p}_{k|i} + n_j \hat{p}_{k|j})/(n_i + n_j)$

$$\text{Test statistic: } T_{ij}(k) = \frac{\hat{p}_{k|i} - \hat{p}_{k|j}}{\sqrt{\frac{n_i+n_j}{n_i n_j} \hat{p}_{k|}(1 - \hat{p}_{k|})}}$$

with corresponding (Gaussian based) p-value $p_{ij} \in [0, 1]$.

Higher $p_{ij} \implies$ higher agreeement.

Measuring Approximate Agreement

Step (ii): agreement set measure $g_{ij}(A)$ $(|J| = 2$ and $|A| \geq 1)$

Let $A \subset B \backslash \{i, j\}$, and select a significance level α.

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(Avoids using p-value as a weight – bad idea)

Note worst agreement: $g_w = \min_{k \in A} p(k)$.

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Define $g_{ij}(A) = g_p$, using g_w to break ties.

In other words, agreement follows the worst case in A.

Measuring Approximate Agreement

Step (iii): sibling set measure $r_A(J)$ ($|J| \geq 2$ and $|A| \geq 1$)

Assume $A \subset B \setminus J$.

To define $r_A(J)$, must combine the values of $g_{ij}(A)$ for all $\{i, j\} \in J$.

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Notes:
– $r_A(J) = g(A)$ whenever $|J| = 2$ such as in binary trees.
– Typically $A = B \setminus J$ in which case we write simply $r(J)$.

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Inspired by SLTD2, tries to use $r(J)$ to identify the MAAS and EAS.

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- Set seed $J_1 = \{i,j\}$, record $r(J_1)$.
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- For each of $\binom{m}{2}$ seeds, get a sequence of $|B| - 1$ candidates EAS J sets.
- Select J^* with the highest agreement $r(J)$.
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LOCATING A COMPLETE MAAS

Try to assemble set with highest internal agreement based on sibling transitivity.

- Order all seed J's according to their $r(J)$: $r(J_1) \leq r(J_2) \leq r(J_3) \leq$ etc..
- Initialize $J^* = J_1$.
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Finally: from candidate EAS and MAAS, select one with highest $r(J)$.

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- includes binary special case.

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Test cases

$d_{\max} =$	2	3	4	5	6	7	8	9
$m = 3$	✓	✓	—	—	—	—	—	—
$m = 5$	✓	✓		✓	—	—	—	—
$m = 9$	✓	✓		✓				✓

Table : The $(m, d_{\max})$ used in model generation.

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Compose sibling set samples according to global JI model rules.

Resulting model-sample is in $\mathcal{M}_{\text{AJI}}(T)$ with probability 1.

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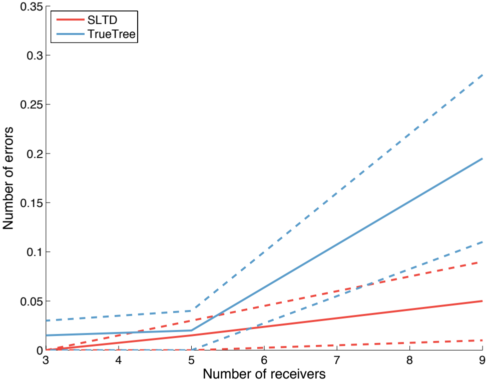
Error:

$$e_T = \text{dist}(T, \widehat{T})$$

PERFORMANCE UNDER 'GENTLE MODELS'

Low Loss Regime: $\rho_i \in [0.9, 0.99]$ for each node i.



(errors averaged over 200 random models for each fixed T, and 6000 probes)

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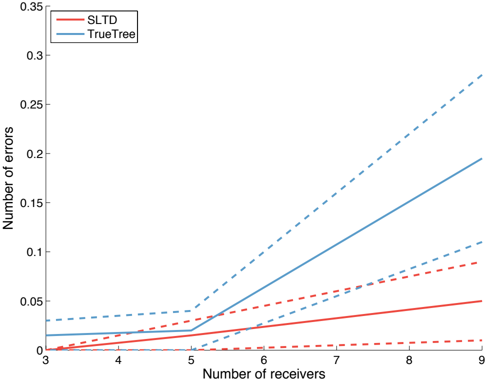

(errors averaged over 200 random models for each fixed T, and 6000 probes)

Binary trees: samples Classically Equivalent $\Longrightarrow$ SLTD, TrueTree legal.
If $d_{\max} > 2$: TrueTree legal (model in $\mathcal{M}_{\mathrm{AJI}}$), but SLTD behaviour undefined.

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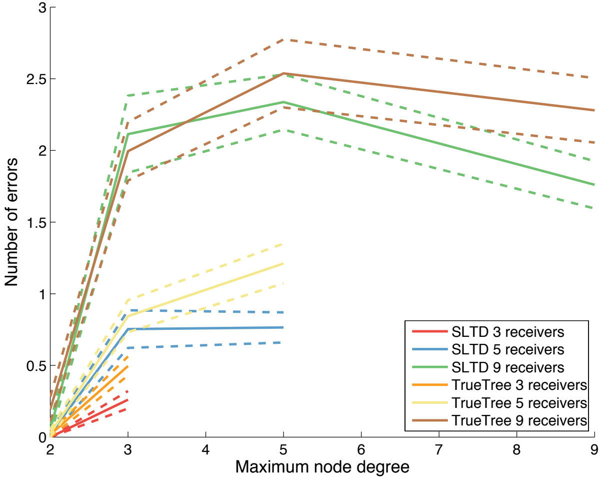
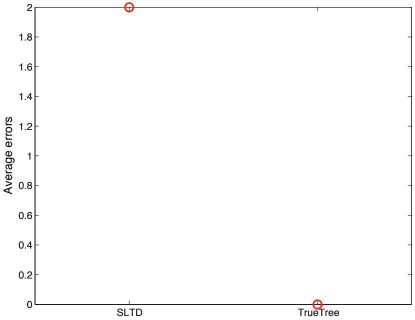
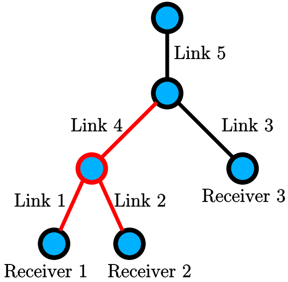


PERFORMANCE ON DISRUPTIVE MODELS

Low Hot Spot scenario: single model with negative dependency.





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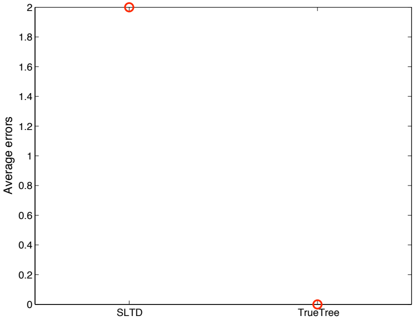
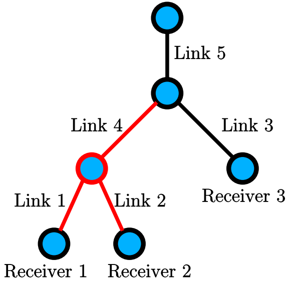
Performance on Disruptive Models

Low Hot Spot scenario: single model with negative dependency.





(errors averaged over 200 random models for each fixed T, and 6000 probes)

SLTD has worst possible $e_T = 2$ in 100% of cases.
TrueTree has $e_T = 0$ in 100% of cases.

Finite Conclusion

- Agreement sets great in theory, tricky in practice, but can be done.
- *TrueTree*
  - gives comparable results to SLTD on gentle loss.
  - can handle disruptive loss.
  - outperforms SLTD when loss higher.
- More work to be done, but promise of SLTD2 on rich class of spatial models can be realized.