Hydrodynamic Limits of Randomized Load Balancing Networks

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and

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Stochastic Networks and Stochastic Geometry
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A Plethora of Scientific Interests

François Baccelli

- Stochastic Geometry
- Information theory
- Stochastic network calculus
- Simulation
- Performance Evaluation
- Wireless Networks
- ...

“A Mean-Field Model for Multiple TCP Connections through a Buffer Implementing RED”, François Baccelli, David R. Mcdonald, Julien Reynier, 2002.
Model of Interest

Network with

- $N$ identical servers
- an infinite capacity queue for each server
- a common arrival process routed immediately on arrival
- FCFS service discipline within each queue
Load Balancing Algorithm:

- How to assign incoming jobs to servers?
- Aim to achieve good performance with low computational cost

**Goal:** Analysis and comparison of different load balancing algorithms
Model of Interest

Appears in:
- Supermarkets
- Hash tables
- Distributed memory machines
- Path selection in networks
- Web Servers
- etc.
Routing Algorithm: Supermarket Model

Each arriving job

- chooses $d$ queues out of $N$, uniformly at random
- joins the shortest queue among the chosen $d$
- ties broken uniformly at random
Routing Algorithm: Supermarket Model

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Supermarket model for exponential service time

Fluid limit and steady state queue length decay rate is obtained as $N \to \infty$
- case $d = 2$ [Vvedenskaya-Dobrushin-Karpelevich ’96]
- case $d \geq 2$ [Mitzenmacher ’01]

General approach

Using Markovian state descriptor $\{S^N_\ell(t); \ell \geq 1, t \geq 0\}$
- $S^N_\ell(t)$: fraction of stations with at least $\ell$ jobs
- Convergence as $N \to \infty$ proved using an extension of Kurtz’s theorem
- The limit process is a solution to a countable system of coupled ODEs
- Steady state queue length approximated by fixed point of the ODE sequence
Summary of Results:

\( X^{i,N} \) – length of \( i \)th queue in an \( N \)-server network

- \( d = d_N = N \) (Joint the Shortest Queue - JSQ)
  - Performance: \( P(X^{i,N}(\infty) > \ell) \rightarrow 0 \) for \( \ell \geq 1 \)
  - Computational Cost: \( N \) comparisons per routing (not feasible)

Power of two Choices: double-exponential decay for \( d \geq 2 \)
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- $d = 1$ (random routing, decoupled $M/M/1$ queues):
  - Performance: $P(X^{i,N}(\infty) > \ell) \rightarrow c\lambda^\ell$
  - Computational cost: one random flip per routing

Power of two Choices: double-exponential decay for $d \geq 2$
Summary of Results:

\[ X^{i,N} \] – length of \( i \)th queue in an \( N \)-server network

- **Joint the Shortest Queue (JSQ)**
  - Performance: \( P(X^{i,N}(\infty) > \ell) \to 0 \) for \( \ell \geq 1 \)
  - Computational Cost: \( N \) comparison per routing (not feasible)

- \( d \geq 2 \) (supermarket model):
  - Performance: \( P(X^{N}(\infty) > \ell) \to \lambda^{(d^\ell-1)/(d-1)} \)
  - Computational Cost: \( d \) random flips and \( d - 1 \) comparison per routing

- \( d = 1 \) (random routing, decoupled \( M/M/1 \) queues):
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Power of two Choices: double-exponential decay for $d \geq 2$
Prior Work - General Service Distribution

Our Focus: General service time distributions

- almost nothing was known 5 years ago
- Mathematical challenge:
  - $\{S^N_\ell\}$ is no longer Markovian
  - need to keep track of more information
  - No common countable state space for Markovian representations of all $N$-server networks
Recent Progress:

1. When $\lambda < 1$ (proved in a more general setting)
   - Stability of $N$-server networks [Foss-Chernova’98]
   - Tightness of stationary distribution sequence [Bramson’10]

2. Under further restrictions – namely, service distributions with decreasing hazard rate and time-homogeneous Poisson arrivals
   - Results on decay rate of limiting stationary queue length [Bramson-Lu-Prabhakar’13]
   - Their approach (cavity method) only yields the steady-state distribution – no information on transient behavior
   - Requires showing asymptotic independence on infinite time intervals and the study of a queue in a random environment
   - According to Bramson, extending this asymptotic independence result to more general service distributions is a challenging task
A Phase Transition Result

Theorem (Bramson, Lu, Prabhakar ’12)
Suppose the service distribution is a power law distribution with exponent $-\beta$. Then

- If $\beta > d/(d-1)$, the tail is doubly exponential
- If $\beta < d/(d-1)$, the tail has a power law
- If $\beta = d/(d-1)$ then the tail is exponentially distributed

Observe: The “power of two choices” fails when $\beta \leq 2$

Motivates a better understanding of general service distributions
There is also the need to better understand transient behavior ...
Simulation results for *fraction of busy servers*:
- Poisson arrival with $\lambda = 0.5$
- 1000 servers
- empty initial condition

*Simulation results by Xingjie Li, Brown University*
Simulation results for *fraction of busy servers*

- Poisson arrival with $\lambda = 0.5$
- 1000 servers
- empty initial condition

*Simulation results by Xingjie Li, Brown University*
Our Goal

Observations:
- No existing results on the time scale to reach equilibrium
- Transient behavior is also important
- No result on service distributions without decreasing hazard rate
- Existing results require Poisson arrivals

Our Goal: To develop a framework that
- Allows more general arrival and service distributions
- Sheds insight into the phase transition phenomena for general service distributions
- Captures transient behavior as well
- Can be extended to more general settings, including heterogeneous servers, thresholds, etc.

We introduce a different approach using a particle representation
The age $a_j(t)$ of job $j$ is the time spent up to $t$ in service.
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- $\tau_j$: arrival time of job $j$ to network
- $s_j$: routing (index of chosen queue)
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- $\tau_j$: arrival time
- $S_j$: routing (index of chosen queue)
- $\alpha_j$: service entry time

\[ S_j = i \]
The age $a_j(t)$ of job $j$ is the time spent up to $t$ in service.

- $\tau_j$: arrival time
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- $\beta_j$: departure time
The age $a_j(t)$ of job $j$ is the time spent upto $t$ in service

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- $\beta_j$: departure time
- $\beta_j - \alpha_j$: service time
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- $\tau_j$: arrival time
- $S_j$: routing (index of chosen queue)
- $\alpha_j$: service entry time
- $\beta_j$: departure time
- $\beta_j - \alpha_j$: service time
\( \nu_\ell = \nu_\ell^N : \) unit mass at ages of jobs in servers with queues of length \( \geq \ell \)

\[
\nu_\ell^N (t) = \sum_j \delta_{a_j^N (t)},
\]

where the sum is over indices of job in service at queues of length \( \geq \ell \)
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at least one job
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\]

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\( \nu_1 \) and \( \nu_2 \) represent processes with different ages and at least two jobs.
Interacting Measure-Valued Processes Representation

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\[
\nu_\ell^N(t) = \sum_j \delta_{a_j^N(t)},
\]

where the sum is over indices of job in service at queues of length \( \geq \ell \)

at least three jobs
\( \nu_\ell = \nu_\ell^N \): unit mass at ages of jobs in servers with queues of length \( \geq \ell \)

\[
\nu_\ell^N(t) = \sum_j \delta_{\alpha_j^N(t)},
\]

where the sum is over indices of job in service at queues of length \( \geq \ell \)

---

At least four jobs

\( \nu_1 \)

\( \nu_2 \)

\( \nu_3 \)

\( \nu_4 \)
$\nu_\ell = \nu_\ell^N$: unit mass at ages of jobs in servers with queues of length $\geq \ell$

$$\nu_\ell^N(t) = \sum_j \delta_{a_j^N(t)},$$

where the sum is over indices of job in service at queues of length $\geq \ell$

at least five jobs
$\mathbb{M}_{\leq 1}[0, L]$: space of sub-probability measures on $[0, L)$ with the topology of weak convergence.

For $\pi \in \mathbb{M}_{\leq 1}[0, L)$ and $f \in C_b[0, L)$, $\langle f, \pi \rangle = \int_{[0, L)} f(x)\pi(dx)$

$S$: space of decreasing sequences of sub-probability measures,

$$S = \{(\pi_\ell)_{\ell \geq 1} \in \mathbb{M}_{\leq 1}[0, L)^\infty | \langle f, \pi_\ell - \pi_{\ell + 1} \rangle \geq 0, \forall \ell \geq 1, f \in C_b[0, L)\}.$$

The $S$-valued process $\{\bar{\nu}^N(t) = \frac{1}{N} (\nu^N_\ell(t))_{\ell \geq 1}; t \geq 0\}$ captures the dynamics

$\{S^N_\ell(t) = \frac{1}{N} \langle 1, \nu^N_\ell(t) \rangle; \ell \geq 1, t \geq 0\}$ is Markovian in exponential case
Theorem 1 (Aghajani-R’14) Markovian Representation

For each $N \in \mathbb{N}$, $\{ (\bar{\nu}_\ell^N(t), \ell \geq 1) : t \geq 0 \}$ is a Markov process on $S$ with respect to a suitable filtration $\{ \mathcal{F}_t^N, t \geq 0 \}$.

Filtration

- $\tilde{\mathcal{F}}_t^N$ : information about all events up to time $t$
  \[
  \tilde{\mathcal{F}}_t = \sigma(\tau_j1(\tau_j \leq s), 1(\alpha_j \leq s), 1(\beta_j \leq s) ; j \leq 1, s \in [0, t]),
  \]

- $\{ \mathcal{F}_t ; t \geq 0 \}$ is the associated right continuous filtration, which is completed.
I. when no arrival/departure is happening, the masses move to the right with unit speed.
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II. Upon departure from a queue with $\ell$ jobs

- the corresponding mass departs from all $\nu_j, j \leq \ell$
- a new mass at zero is added to all $\nu_j, j \leq \ell - 1$ (if $\ell \geq 2$)

$D_\ell$: cumulative departure process from servers with at least $\ell$ jobs before departure.
II. Upon departure from a queue with \( \ell \) jobs

- the corresponding mass departs from all \( \nu_j, j \leq \ell \)
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- \( D_\ell \): cumulative departure process from servers with at least \( \ell \) jobs before departure.
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\( D_\ell \): cumulative departure process from servers with at least \( \ell \) jobs before departure.
II. Form of the cumulative departure process $D_\ell$

The hazard rate function

$$h(x) = \frac{g(x)}{1 - G(x)}$$

- $\langle h, \nu^{(N)}(t) \rangle = \sum_j h(a_j^N(t))$ conditional mean departure rate at time $t$ from queues of length greater than or equal to $\ell$, given ages of jobs
- the compensated departure process

$$D_\ell^N(t) - \int_0^t \langle h, \nu^{(N)}_\ell(s) \rangle \, ds$$

is a martingale (with respect to the filtration $\{\mathcal{F}_t^N\}$).
III. Upon arrival to a queue with $\ell - 1$ jobs right before arrival,

- if $\ell = 1$, a mass at zero joins $\nu_1$
- if $\ell \geq 2$, the mass corresponding to the age of job in that particular server is added to $\nu_\ell$

\[ R_\ell : \text{ routing measure process} \]
III. Upon arrival to a queue with $\ell - 1$ jobs right before arrival,

- if $\ell = 1$, a mass at zero joins $\nu_1$
- if $\ell \geq 2$, the mass corresponding to the age of job in that particular server is added to $\nu_\ell$

- $E_{\ell-1}$: exactly $\ell$ customers

- $R_{\ell-1}$: routing measure process

- $\nu_\ell$,
- $\nu_{\ell-1}$,
- $\nu_{\ell+1}$
Upon arrival of $j^{th}$ job,

- suppose queue $i$ has $\ell$ jobs: $X^i = \ell$.
- $\zeta_j$ is the index of the queue to which job $j$ is routed

what is $\mathbb{P}\{\zeta_j = i | X^i = \ell\}$?

1. $\mathbb{P}\{\text{queue } i \text{ has queue length } \geq \ell\} = \mathbb{P}\{\text{all picks have queue length } \geq \ell\} = S^d_\ell$.

   $$S_\ell = S^N_\ell = \frac{1}{N} \langle 1, \nu^N_\ell \rangle = \langle 1, \bar{\nu}^N_\ell \rangle : \text{fraction of queues with at least } \ell \text{ jobs}$$

2. $\mathbb{P}\{\text{queue } \zeta_j \text{ has exactly } \ell \text{ jobs}\} = S^d_\ell - S^d_{\ell+1}$.

3. Number of queues with $\ell$ jobs is $S_\ell - S_{\ell+1}$

4. $\mathbb{P}\{\zeta_j = i | X^i = \ell\} = \frac{1}{N} \frac{S^d_\ell - S^d_{\ell+1}}{S_\ell - S_{\ell+1}}$

5. When $d = 2$, $\mathbb{P}\{\zeta_j = i | X^i = \ell\} = \frac{1}{N} (S_\ell + S_{\ell+1})$
Arrival Process: Belongs to one of the following two classes:

- $E^{(N)}$: (possibly time-inhomogeneous) Poisson Process with rate $\theta_N \lambda(\cdot)$ where $\theta_N/N \to 1$ as $N \to \infty$ and $\lambda(\cdot)$ is locally square integrable.
- $E^{(N)}$ is a renewal process whose interarrival distribution has a density

Service Time

has distribution $G$ with density $g$ and mean 1
Definition  A process $\nu = \{\nu_\ell\}_{\ell \geq 0}$ solves the age equations if for all $f \in \mathbb{C}^1_b[0, \infty)$,

$$\langle f, \nu(t) \rangle = \langle f, \nu(0) \rangle$$

initial jobs
Definition A process $\nu = \{\nu_\ell\}_{\ell \geq 0}$ solves the age equations if for all $f \in C^1_b[0, \infty)$,

$$\langle f, \nu(t) \rangle = \langle f, \nu(0) \rangle + \int_0^t \langle f', \nu(s) \rangle ds$$

linear growth of ages
**Definition** A process $\nu = \{\nu_\ell\}_{\ell \geq 0}$ solves the *age equations* if for all $f \in C^1_b[0, \infty)$,

$$\langle f, \nu_\ell(t) \rangle = \langle f, \nu_\ell(0) \rangle + \int_0^t \langle f', \nu_\ell(s) \rangle ds + f(0)D_{\ell+1}(t)$$
**Definition** A process \( \nu = \{ \nu_\ell \}_{\ell \geq 0} \) solves the *age equations* if for all \( f \in C_b^1[0, \infty) \),

\[
\langle f, \nu_\ell(t) \rangle = \langle f, \nu_\ell(0) \rangle + \int_0^t \langle f', \nu_\ell(s) \rangle ds + f(0)D_{\ell+1}(t) - \int_0^t \langle h, f_\ell(s) \rangle ds
\]
Definition A process $\nu = \{\nu_\ell\}_{\ell \geq 0}$ solves the age equations if for all $f \in C^1_b[0, \infty)$,

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$$
\langle f, \nu_\ell(t) \rangle = \langle f, \nu_\ell(0) \rangle + \int_0^t \langle f', \nu_\ell(s) \rangle ds + f(0) D_{\ell+1}(t) - \int_0^t \langle h, f, \nu_\ell(s) \rangle ds + \int_0^t \langle f, \eta_\ell(s) \rangle ds.
$$

$$
\langle 1, \nu_\ell(t) \rangle - \langle 1, \nu_\ell(0) \rangle = D_{\ell+1}(t) + \int_0^t \langle 1, \eta_\ell(s) \rangle ds - D_\ell(t),
$$

mass balance
**Definition** A process $\nu = \{\nu_\ell\}_{\ell \geq 0}$ solves the *age equations* if for all $f \in C^1_b[0, \infty)$,

\[
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\[
\langle 1, \nu(t) \rangle - \langle 1, \nu(0) \rangle = D_{\ell+1}(t) + \int_0^t \langle 1, \eta(s) \rangle ds - D_\ell(t),
\]

\[
D_\ell(t) = \int_0^t \langle h, \nu(s) \rangle ds,
\]

departure rate
**Definition** A process $\nu = \{\nu_\ell\}_{\ell \geq 0}$ solves the *age equations* if for all $f \in C^1_b[0, \infty)$,

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$$

$$
\langle 1, \nu_\ell(t) \rangle - \langle 1, \nu_\ell(0) \rangle = D_{\ell+1}(t) + \int_0^t \langle 1, \eta_\ell(s) \rangle ds - D_\ell(t),
$$

where

$$
D_\ell(t) = \int_0^t \langle h, \nu_\ell(s) \rangle ds,
$$

$$
\eta_\ell(t) = \begin{cases} 
\lambda(1 - \langle 1, \nu_1(t) \rangle^2)\delta_0 & \text{if } \ell = 1, \\
\lambda\langle 1, \nu_{\ell-1}(t) + \nu_\ell(t) \rangle(\nu_{\ell-1}(t) - \nu_\ell(t)) & \text{if } \ell \geq 2.
\end{cases}
$$
**Definition** A process $\nu = \{\nu_\ell\}_{\ell \geq 0}$ solves the *age equations* if for all $f \in C^1_b[0, \infty)$,

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\]

\[
\langle 1, \nu_\ell(t) \rangle - \langle 1, \nu_\ell(0) \rangle = D_{\ell+1}(t) + \int_0^t \langle 1, \eta_\ell(s) \rangle ds - D_\ell(t),
\]

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\lambda (1 - \langle 1, \nu_1(t) \rangle)^2 \delta_0 & \text{if } \ell = 1, \\
\lambda (1, \nu_{\ell-1}(t) + \nu_\ell(t))(\nu_{\ell-1}(t) - \nu_\ell(t)) & \text{if } \ell \geq 2.
\end{cases}
\]
Theorem 2 (Aghajani-R’14) Age Equations

Given any $\nu(0) = (\nu_\ell(0), \ell \geq 1) \in S$ there exists a unique solution $\nu(\cdot) = \{(\nu_\ell(t), \ell \geq 1); t \geq 0\}$ to the age equations with initial condition $\nu(0)$.
• Let $\{\nu^{(N)}(t) = (\nu^{(N)}_{\ell}(t), \ell \geq 1); t \geq 0\}$ be the measure-valued representation for the $N$-server system with initial condition $\nu^{(N)}(0)$.

**Theorem 3 (Aghajani-R’14) Hydrodynamic Limit**

If for every $\ell \geq 1$, $\nu^{(N)}_{\ell}(0)/N \to \nu_{\ell}(0)$, then

$$\frac{1}{N}\nu^{(N)}(\cdot) \Rightarrow \nu(\cdot)$$

in $S$, where $\nu$ is the unique solution to the age equation corresponding to $\nu(0)$. 

Kavita Ramanan and Mohammadreza Aghajani

Hydrodynamic Limits of Randomized Load Balancing Networks
Informal statement

The evolution of any subset of $k$ queues are asymptotically independent on finite time intervals with marginal queue lengths given by the hydrodynamic equations. Let $X^{N,i}(\cdot)$ be the process that tracks the length of the $i$th queue.

**Theorem 4 (Aghajani-R’14) Propagation of Chaos**

Suppose for each $N$, $\{X^{N,i}(0), i = 1, \ldots, N\}$ is exchangeable, let $\nu^N(0) \rightarrow \nu(0)$ as $N \rightarrow \infty$ and let $\nu = (\nu_{\ell}, \ell \geq 1)$ be the solution to the age equations associated with $\nu(0)$. Then

$$\lim_{N \rightarrow \infty} P\{X^{N,1}(t) \geq \ell\} = S_{\ell}(t) = \langle 1, \nu_{\ell}(t) \rangle,$$

and for any $\ell_1, \ldots, \ell_k \in \mathbb{N}^k$,

$$\lim_{N \rightarrow \infty} P\{X^{N,1}(t) \geq \ell_1, \ldots, X^{N,k}(t) \geq \ell_k\} = \prod_{m=1}^{k} S_{\ell_m}(t).$$
Use (weak-sense) PDE techniques to partially solve the age equation:

Lemma (Aghajani–’R ’14) Partial Solution of the Age Equations

Under suitable assumptions on $D_\ell$ and $\eta_\ell$, for every $f \in C_b[0, \infty)$,

$$\langle f, \nu_\ell(t) \rangle = \langle f, \nu_\ell(0) \rangle + \int_0^t \langle f', \nu_\ell(s) \rangle ds + f(0)D_{\ell+1}(t)$$

$$- \int_0^t \langle hf, \nu_\ell(s) \rangle ds + \int_0^t \langle f, \eta_\ell(s) \rangle ds$$

(1)

holds if and only if

$$\langle f, \nu_\ell(t) \rangle = \langle f(\cdot + t)\frac{\bar{G}(\cdot + t)}{\bar{G}(\cdot)}, \nu_\ell(0) \rangle + \int_{[0,t]} f(t-s)\bar{G}(t-s)dD_{\ell+1}(s)$$

$$+ \int_0^t \langle f(\cdot + t - s)\frac{\bar{G}(\cdot + t - s)}{\bar{G}(\cdot)}, \eta_\ell(s) \rangle ds$$

(2)
Definition. We refer to equation (2):

\[
\langle f, \nu_\ell(t) \rangle = \langle f(\cdot + t) \frac{\tilde{G}(\cdot + t)}{\tilde{G}(\cdot)}, \nu_\ell(0) \rangle + \int_{[0,t]} f(t - s) \tilde{G}(t - s) dD_{\ell+1}(s)
\]

\[
+ \int_{0}^{t} \langle f(\cdot + t - s) \frac{\tilde{G}(\cdot + t - s)}{\tilde{G}(\cdot)}, \eta_\ell(s) \rangle ds
\]

and the remaining age equations, (3)–(5) below, as the Hydrodynamics Equations.

\[
\langle 1, \nu_\ell(t) \rangle - \langle 1, \nu_\ell(0) \rangle = D_{\ell+1}(t) + \int_{0}^{t} \langle 1, \eta_\ell(s) \rangle ds - D_\ell(t), \quad (3)
\]

with

\[
D_\ell(t) = \int_{0}^{t} \langle h, \nu_\ell(s) \rangle ds \quad (4)
\]

and

\[
\eta_\ell(t) = \left\{ \begin{array}{ll}
\lambda(1 - \langle 1, \nu_1(t) \rangle^2) \delta_0 & \text{if } \ell = 1, \\
\lambda\langle 1, \nu_{\ell-1}(t) + \nu_\ell(t) \rangle (\nu_{\ell-1}(t) - \nu_\ell(t)) & \text{if } \ell \geq 2.
\end{array} \right. \quad (5)
\]
Step 2: Show that these hydrodynamic equations have a unique solution.

- Consider the special class of functions $\mathcal{F}$

$\mathcal{F} = \left\{ \frac{\tilde{G}(\cdot + r)}{\bar{G}(\cdot)} : r \geq 0 \right\}$.

- Show that the class of functions is (in a suitable sense) invariant under the hydrodynamic equation (2)

$$\langle f, \nu_\ell(t) \rangle = \langle f(\cdot + t) \frac{\tilde{G}(\cdot + t)}{\bar{G}(\cdot)}, \nu_\ell(0) \rangle + \int_{[0,t]} f(t - s)\tilde{G}(t - s) dD_{\ell+1}(s)$$

$$+ \int_0^t \langle f(\cdot + t - s) \frac{\tilde{G}(\cdot + t - s)}{\bar{G}(\cdot)}, \eta_\ell(s) \rangle ds$$

- Show uniqueness first for this class of functions $f \in \mathcal{F}$ and then show that this implies uniqueness for all $f \in C_b[0, L]$.
Skipping details and some subtleties ...

- Identify compensators of various processes \(\text{à la Baccelli-Bremaud}\)
- Establish tightness
- Show convergence
We have obtained a general convergence result and characterized the limit.

So what?
What can one do with this measure-valued hydrodynamic limit?
Can one use it to compute anything?
A PDE representation

- If one is only interested in $S_\ell(t) = \langle 1, \nu_\ell(t) \rangle$, one can get a simpler representation. Define
  \[
  f^r(x) = \frac{\bar{G}(x + r)}{G(x)} \quad \xi_\ell(t, r) = \langle f^r, \nu_\ell(t) \rangle
  \]
  and note that
  \[
  \xi_\ell(t) = x \iota_\ell(t, 0) \quad \text{and} \quad \langle h, \nu_\ell(t) \rangle = -\partial_r \xi_\ell(t, 0).
  \]

**Theorem 5 (Aghajani-R ’15)**

Suppose, in addition, we assume time-varying Poisson arrivals and bounded hazard rate function. If $\nu$ solves the age equations associated with $\nu(0)$, then $\xi(\cdot, \cdot) = \{\xi_\ell(\cdot, \cdot), \ell \geq 1\}$ is the unique solution to a certain system of PDEs.
Details of the PDE representation

Recall

\[ f^r(x) = \frac{\bar{G}(x + r)}{\bar{G}(x)} \quad \xi_\ell(t, r) = \langle f^r, \nu_\ell(t) \rangle \]

and

\[ S_\ell(t) = \xi_\ell(t, 0) \quad \text{and} \quad \langle h, \nu_\ell(t) \rangle = -\partial_r \xi_\ell(t, 0). \]

Then (for \( d = 2 \)) the “PDE” takes the following form: for \( t > 0 \)

\[ \xi_\ell(t, r) = \xi_\ell(0, t + r) - \int_0^t \bar{G}(t + r - u) \partial_r \xi_{\ell+1}(u, 0) du, \]

\[ + \lambda \int_0^t (\xi_{\ell-1}(u, 0) + \xi_\ell(u, 0)) (\xi_{\ell-1}(u, t + r - u) - \xi_\ell(u, t + r - u)) du \]

with boundary condition

\[ \xi_\ell(t, 0) - \xi_\ell(0, 0) = \int_0^t \left( \lambda(u) (\xi_{\ell-1}(u, 0)^2 - \xi_\ell(u, 0)^2) - (\partial_r \xi_{\ell-1}(u, 0) - \partial_r \xi_\ell(u, 0)) \right) du \]

- This system of PDEs can be numerically solved to provide approximations to performance measures of the network.
- The class of functionals represented by \( \{\xi_\ell(\cdot, \cdot), \ell \geq 1\} \) is rich enough to include both the queue length and the virtual waiting time.
We can numerically solve the PDE and compare the results to simulations.
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Summary of Results

We introduced a framework for the analysis of load balancing algorithms, featuring

- Hydrodynamic limit which captures transient behavior
- Applicable to general service distributions
- Incorporates more general time varying arrival processes
- Propagation of chaos on the finite interval was established

For **Exponential service distribution:**

- limit process is characterized by the solution to a sequence of ODEs

For **General service distribution:**

- limit process is characterized by the solution to a sequence of PDEs
- Equilibrium distributions are characterized by the fixed point of the PDEs
- We can also show that uniqueness of fixed points of the PDE imply propagation of chaos on the infinite interval
Concluding Remarks

Interacting measure-valued processes framework

- Obtained a PDE that provides more efficient alternative to simulations in order to address network optimization and design questions
- Applicable for modifications of this randomized load balancing algorithm
- Can be applied to the analysis of the Serve the Longest Queue (SLQ)-type service disciplines [Ramanan, Ganguly, Robert]
- The framework can be used for other non-queueing models arising in materials science

Other Questions

- **Ongoing**: Analysis of fixed points of the PDE to gain insight into the stationary distribution and phase transition (ongoing)
- Implications for rate of convergence to stationary distribution
- More on Numerical solution for the PDEs