Invariant transports of random measures and the extra head problem

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joint work with
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## 1. Four problems on random shifts

**1. Extra head problem**

Consider a two-sided sequence of independent and fair coin tosses. Find a coin that landed heads so that the other coin tosses are still independent and fair.

**2. Marriage of Lebesgue and Poisson**

Let $\eta$ be a stationary Poisson process in $\mathbb{R}^d$. Find a point $T$ of $\eta$ such that

$$\theta_T \eta - \delta_0 \overset{d}{=} \eta.$$
3. Poisson matching

Let $\eta$ and $\xi$ be two independent stationary Poisson processes with equal intensity. Find a point $T$ of $\xi$ such that

$$\theta_T(\eta + \delta_0, \xi) \overset{d}{=} (\eta, \xi + \delta_0)$$

4. Unbiased shifts of Brownian motion

Let $B = (B_t)_{t \in \mathbb{R}}$ be a two-sided standard Brownian motion. Find a random time $T$ such that the space-time shifted process $(B_{T+t} - B_T)_{t \in \mathbb{R}}$ is a Brownian motion, independent of $B_T$. 
2. Invariant transports of random measures

Setting

$\left( \Omega, \mathcal{F}, \mathbb{P} \right)$ is a $\sigma$-finite measure space. For the first three problems $\mathbb{P}$ can be taken as probability measure.

Definition

A random measure on $\mathbb{R}^d$ is a random element in the space of all locally finite measures on $\mathbb{R}^d$ equipped with the Kolmogorov product $\sigma$-field.
Setting

We consider mappings $\theta_s : \Omega \to \Omega$, $s \in \mathbb{R}^d$, satisfying $\theta_0 = \text{id}_\Omega$ and the flow property

$$\theta_s \circ \theta_t = \theta_{s+t}, \quad s, t \in \mathbb{R}^d.$$ 

The mapping $(\omega, s) \mapsto \theta_s \omega$ is supposed to be measurable. We assume that $\mathbb{P}$ is stationary, that is

$$\mathbb{P} \circ \theta_s = \mathbb{P}, \quad s \in \mathbb{R}^d.$$ 

Definition

A random measure $\xi$ is invariant if

$$\xi(\theta_s \omega, B - s) = \xi(\omega, B), \quad \omega \in \Omega, \quad s \in \mathbb{R}^d, \quad B \in \mathcal{B}^d.$$
Definition

Let $\xi$ be an invariant random measure on $\mathbb{R}^d$. The measure

$$Q_\xi(A) := \int\int \mathbf{1}_{\{\theta_s\omega \in A, s \in B\}} \xi(\omega, ds) \mathbb{P}(d\omega), \quad A \in \mathcal{F},$$

is called the Palm measure of $\xi$ (with respect to $\mathbb{P}$), where $B \in \mathcal{B}^d$ satisfies $0 < \lambda_d(B) < \infty$.

Theorem (Refined Campbell theorem)

Let $\xi$ be an invariant random measure on $\mathbb{R}^d$. Then

$$\mathbb{E}_\mathbb{P} \int f(\theta_s, s) \xi(ds) = \mathbb{E}_{Q_\xi} \int f(\theta_0, s) ds$$

for all measurable $f : \times \mathbb{R}^d \rightarrow [0, \infty)$. 
Definition

An allocation rule is a measurable mapping $\tau : \times \mathbb{R}^d \to \mathbb{R}^d$ that is equivariant in the sense that

$$\tau(\theta_t \omega, s - t) = \tau(\omega, s) - t, \quad s, t \in \mathbb{R}^d, \mathbb{P}\text{-a.e. } \omega \in \mathbb{R}^d.$$

Theorem (L. and Thorisson ’09)

Let $\xi$ and $\eta$ be two invariant random measures with positive and finite intensities. Let $\tau$ be an allocation rule and define $T := \tau(\cdot, 0)$. Then

$$\mathbb{Q}_\xi(\theta_T \in \cdot) = \mathbb{Q}_\eta$$

iff $\tau$ is balancing $\xi$ and $\eta$, that is

$$\int 1\{\tau(s) \in \cdot\} \xi(ds) = \eta \quad \mathbb{P}\text{-a.e.}$$
### Remark

The previous result extends to **weighted transport kernels** and to **LCSC-groups** $G$; see L. and Thorisson ’09 and L. ’10a. It can even be extended to random measures on a space, on which $G$ operates; see L. ’10b and Kallenberg ’11.
Example

Assume that $\xi = \lambda_d$ is Lebesgue measure and that $\eta$ is a simple point process. An allocation rule $\tau$ is balancing $\xi$ and $\eta$, iff $\mathbb{P}$-a.e.

$$\lambda_d(C^\tau(t)) = 1, \quad t \in \eta,$$

where the cell $C^\tau(t)$ is given by

$$C^\tau(t) := \{ s \in \mathbb{R}^d : \tau(s) = t \}.$$

Theorem (Holroyd and Peres '05)

Assume that $\eta$ is a stationary unit-rate Poisson process and let $\tau$ be an allocation rule. Then $\tau$ is balancing Lebesgue measure and $\eta$ iff

$$\mathbb{P}(\theta_{\tau(0)}\eta \in \cdot) = \mathbb{P}(\eta + \delta_0 \in \cdot).$$
Example

Assume that $\xi$ and $\eta$ are simple point processes. An allocation rule $\tau$ is balancing $\xi$ and $\eta$, iff $\tau$ is a perfect matching ($\mathbb{P}$-a.e.) of the points of $\xi$ with the points of $\eta$.

Theorem (Holroyd, Pemantle, Peres, Schramm ’09)

Assume that $\xi$ and $\eta$ are independent stationary unit-rate Poisson processes (defined on their canonical probability space) and let $\tau$ be an allocation rule. Then $\tau$ is balancing $\xi$ and $\eta$ iff

$$\theta_T(\xi + \delta_0, \eta) \overset{d}{=} (\xi, \eta + \delta_0),$$

where $T := \tau((\xi + \delta_0, \eta), 0)$. 
3. Local time of Brownian motion

Setting

$B = (B_t)_{t \in \mathbb{R}}$ is a two-sided standard Brownian motion starting in 0 ($B_0 = 0$) defined on its canonical probability space $(\mathcal{F}, \mathbb{P}_0)$.

Definition

An unbiased shift (of $B$) is a random time $T$ (negative values are allowed) such that:

- $B^{(T)} := (B_{T+t} - B_T)_{t \in \mathbb{R}}$ is a Brownian motion,
- $B^{(T)}$ is independent of $B_T$. 
Example

If $T \geq 0$ is a stopping time, then $(B_{T+t} - B_T)_{t \geq 0}$ is a one-sided Brownian motion independent of $B_T$. However, the example

$$T := \inf \{ t \geq 0 : B_t = a \}$$

shows that $(B_{T+t} - B_T)_{t \in \mathbb{R}}$ need not be a two-sided Brownian motion.

Example

Consider a deterministic $T \equiv t_0$. Then $B^{(T)} = (B_{t_0 + t} - B_{t_0})_{t \in \mathbb{R}}$ is a two-sided Brownian. However, since $B^{(T)}_{-t_0} = -B_{t_0}$, this two-sided motion is not independent of $B_T = B_{t_0}$. 
Remark

An unbiased shift with $B_T = 0$ is characterized by

$$(B_{T+t})_{t \in \mathbb{R}} \overset{d}{=} B.$$ 

According to Mandelbrot (The Fractal Geometry of Nature) „...the process of Brownian zeros is stationary in a weakened form.“ He is using the (non-rigorous) concept of conditional stationarity.

However, the stopping time

$$T := \inf\{ t \geq 1 : B_t = 0 \}$$

has the property $B_T = 0$. But clearly $B^{(T)}$ is not a Brownian motion. The missing link will be provided by balancing local times at different levels.
Definition

Let $\ell^0$ be the local time (random measure) at zero. Its right-continuous (generalised) inverse is defined as

$$T_r := \begin{cases} 
\sup\{t \geq 0 : \ell^0[0, t] = r\}, & r \geq 0, \\
\sup\{t < 0 : \ell^0[t, 0] = -r\}, & r < 0.
\end{cases}$$

Theorem

Let $r \in \mathbb{R}$. Then $T_r$ is an unbiased shift.

Idea of the proof: The intervals $[T_n, T_{n+1}]$, $n \in \mathbb{Z}$, split $B$ into iid-cycles. The distribution of these cycles is time-reversible.
Definition

The **local time** measure $\ell^x$ at $x \in \mathbb{R}$ can be defined by

$$\ell^x(C) := \lim_{h \to 0} \frac{1}{h} \int \mathbf{1}\{s \in C, x \leq B_s \leq x + h\} ds.$$ 

Hence

$$\int f(B_s, s) ds = \int \int f(x, s) \ell^x(ds) dx \quad \mathbb{P}_0\text{-a.s.}$$

for all measurable $f : \mathbb{R}^2 \to [0, \infty)$.
Definition

For $t \in \mathbb{R}$ the shift $\theta_t : \omega \rightarrow \omega_{t+s}$, $s \in \mathbb{R}$, is given by

$$(\theta_t \omega)_s := \omega_{t+s}, \quad s \in \mathbb{R}.$$ 

For $x \in \mathbb{R}$ let

$$P_x := P_0(B + x \in \cdot), \quad x \in \mathbb{R},$$

where $B$ is the identity on $\mathbb{R}$.

Remark

It is possible to choose a perfect version of local times, that is, a (measurable) kernel satisfying for all $x \in \mathbb{R}$ and $P_x$-a.e. that $\ell^x$ is diffuse and

$$\ell^x(\theta_t \omega, C - t) = \ell^x(\omega, C), \quad C \in \mathcal{B}, \quad t \in \mathbb{R},$$

$$\ell^x(B, \cdot) = \ell^0(B - x, \cdot).$$
Definition

Let

\[ P := \int P_x \, dx \]

be the distribution of a Brownian motion with a „uniformly distributed“ starting value.

Remark

Stationary increments of \( B \) imply that \( P \) is stationary, that is

\[ P = P \circ \theta_s, \quad s \in \mathbb{R}. \]
Theorem (Geman and Horowitz ’73)

The Palm (probability) measure of the local time $\ell^x$ is $P_x$.

Definition

Let $\nu$ be a probability measure on $\mathbb{R}$. Define

$$P_\nu := \int P_x \nu(dx), \quad \ell^\nu := \int \ell^x \nu(dx).$$

Corollary

$P_\nu$ is the Palm probability measure of $\ell^\nu$.

Remark

In the language of stochastic analysis $\ell^\nu$ is a continuous additive functional with Revuz measure $\nu$. 
4. Existence of unbiased shifts

Definition (Skorokhod embedding problem)

Let $\nu$ be a probability measure on $\mathbb{R}$. A random time $T$ embeds $\nu$ if $B_T$ has distribution $\nu$.

Theorem

Let $T$ be a random time and $\nu$ be a probability measure on $\mathbb{R}$. Then $T$ is an unbiased shift embedding $\nu$ if and only if the allocation rule $\tau$ defined by $\tau_T(s) := T \circ \theta_s + s$ is balancing $\ell^0$ and $\ell^\nu$. 
Example

Let $r > 0$. Then

$$\tau(s) := \inf\{ t > s : \ell^0([s, t]) = r \}, \quad s \in \mathbb{R}.$$  

Then $\tau$ is an allocation rule balancing $\ell^0$ with itself. Hence $T_r = \tau(\cdot, 0)$ is an unbiased shift (embedding $\delta_0$).
Theorem

Let $\nu$ be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then the stopping time

$$T := \inf\{t > 0: \ell^0[0, t] = \ell^\nu[0, t]\}$$

embeds $\nu$ and is an unbiased shift.

Remark

The above stopping time above was introduced in Bertoin and Le Jan (1992) as a solution of the Skorokhod embedding problem.

Theorem (L., Mörters and Thorisson ’14)

Let $\nu$ be a probability measure on $\mathbb{R}$. Then there is a non-negative stopping time that is an unbiased shift embedding $\nu$. 
**Theorem (L., Mörters and Thorisson ’14)**

Let $\xi$ and $\eta$ be jointly stationary orthogonal diffuse random measures on $\mathbb{R}$ with finite and equal intensities. Then the mapping $\tau : \times \mathbb{R} \to \mathbb{R}$, defined by

$$
\tau(s) := \inf\{ t > s : \xi[s, t] = \eta[s, t] \}, \quad s \in \mathbb{R},
$$

is an allocation rule balancing $\xi$ and $\eta$.

**Remark**

The previous theorem holds in a more general stationary setting. The assumption of equal intensities has to be replaced by

$$
\mathbb{E}[\xi[0, 1] | \mathcal{I}] = \mathbb{E}[\eta[0, 1] | \mathcal{I}] \quad \mathbb{P}\text{-a.e.},
$$

where $\mathcal{I}$ is the invariant $\sigma$-field. In the Brownian setting, $\mathbb{P}$ is trivial on $\mathcal{I}$. (If $A \in \mathcal{I}$ then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A^c) = 0$.)
5. Moment properties of unbiased shifts

Theorem (L., Mörters and Thorisson ’14)

If $T$ is an unbiased shift embedding a probability measure $\nu \neq \delta_0$, then

$$\mathbb{E}_0 \sqrt{|T|} = \infty.$$

Idea of the proof:

- Take an $x > 0$ such that $\nu[x, \infty) = \mathbb{P}(B_T > x) > 0$.
- On the event $\{B_T > x\}$, $T$ can be bounded from below by the minimum of two independent hitting times for $-x$, independent of $B_T$.
- Use the moment properties of hitting times.
Suppose $\nu$ is a distribution with $\nu\{0\} = 0$. If the stopping time $T \geq 0$ is an unbiased shift embedding $\nu$, then

$$\mathbb{E}_0 T^{1/4} = \infty.$$ 

Suppose $\nu$ is a distribution with a finite first moment and let $T$ be the Bertoin/Le Jan stopping time. Then, for all $\beta \in [0, 1/4)$,

$$\mathbb{E}_0 T^\beta < \infty.$$
Idea of the proof: Recall that

\[ T = \inf\{ t > 0 : X(t) = 0 \} \]

where \( X_t := \ell^0[0, t] - \ell^\nu[0, t] \). Define a time-change

\[ U_r := \inf\{ t > 0 : \ell^0[0, t] + \ell^\nu[0, t] = r \}, \quad r > 0, \]

with respect to a clock which does not tick during the flat pieces of \( X \). Then

\[ \Xi(r) := X(U_r), \quad r > 0 \]

resembles a random walk whose return times have tails of order \( t^{-\frac{1}{2}} \). As \( U_r \sim r^2 \) by Brownian scaling, the return times for the original \( X \) have tails of order \( t^{-\frac{1}{4}} \).
6. References


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