The purpose of this talk

to honor François for his 60-th birthday
And to opportunistically take advantage of his great popularity and the large number of friends gathered in this occasion to advertise my recently published book:

Fourier Analysis and Stochastic Processes
Consider a point process $N$ on $\mathbb{R}$ with event times $\{T_n\}_{n\in\mathbb{Z}}$. The “random Dirac comb”

$$X(t) := \sum_{n\in\mathbb{Z}} \delta(t - T_n),$$

is not a \textit{bona fide} stochastic process. In particular, one cannot define for the random Dirac comb associated with a stationary point process a power spectral measure as in the case of wide-sense stationary stochastic processes.

The natural extension of the notion of power spectral density is the so-called \textbf{Bartlett spectral measure}.

Here we concentrate on the \textit{computation} of such measures.
Who needs it?

1. biology (spike trains)
2. communications (ultra-wide band)
3. perhaps nobody needs it.
Some contributors


Second-order: for all compact sets $C$, \[ E \left[ N(C)^2 \right] < \infty. \]

\[ M_2(A \times B) := E \left[ N(A) N(B) \right]. \]

$M_2$ is the intensity measure of $N \times N$. By Campbell’s theorem,

\[ E \left[ \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} g(X_n, X_k) \right] = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} g(t, s) M_2(dt \times ds). \]
The collection of functions $\varphi : \mathbb{R}^m \to \mathbb{C}$ such that

$$
\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |\varphi(t)\varphi(s)| M_2(dt \times ds) < \infty,
$$

$\iff E\left[N(|\varphi|)^2\right] < \infty$,

$\Rightarrow E\left[N(|\varphi|)\right] < \infty$, $E\left[N(|\varphi|^2)\right] < \infty$

$\Rightarrow L^2_N(M_2) \subseteq L^1_C(\nu) \cap L^2_C(\nu)$.

(Where $\nu(C) := E[N(C)]$)
Second-order, plus

\[ E[N(C + t)] = E[N(C)], \]

and

\[ E[N(A + t)N(B + t)] = E[N(A)N(B)]. \]

Immediate consequence: for all non-negative \( \varphi, \psi, \)

\[ E \left[ \left( \int_{\mathbb{R}} \varphi(t) \, N(\,dt) \right) \left( \int_{\mathbb{R}} \psi(\tau + t) \, N(\,dt) \right) \right] \]

is independent of \( \tau \in \mathbb{R}. \)
Covariance measure

Basic lemma from measure theory

\((\mathcal{X}, \mathcal{X}), \mu\) loc. fin. measure on \(\mathcal{X}^\otimes k\), invariant by the simultaneous shifts, that is,

\[ \mu((A_1 + h) \times \cdots \times (A_k + h)) = \mu(A_1 \times \cdots \times A_k). \]

Then, there exists a locally finite measure \(\hat{\mu}\) on \(\mathcal{X}^{k-1}\) such that for all non-negative measurable functions \(f\) from \(\mathcal{X}^k\) to \(\mathbb{R}\),

\[
\int_{\mathcal{X}^k} f(x_1, \ldots, x_k) \mu(dx_1 \times \cdots \times dx_k) = \int_{\mathcal{X}} \left\{ \int_{\mathcal{X}^{k-1}} f(x_1, x_1 + x_2, \ldots, x_1 + x_k) \hat{\mu}(dx_2 \times \cdots \times dx_k) \right\} dx_1.
\]
Application to point processes

\[ M_2 \left( (A + t) \times (B + t) \right) = M_2 \left( A \times B \right) \]

Therefore, for all \( \varphi, \psi \in L^2_N(M_2) \),

\[
\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \varphi(t) \psi^*(s) M_2 \left( dt \times ds \right) = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi(t) \psi^* \left( s + t \right) dt \right) \sigma \left( ds \right)
\]

for some locally finite measure \( \sigma \).

In fact, \( \sigma \) can be identified to the intensity measure of the Palm version of a given stationary point process.
Since for $\varphi, \psi \in L^1_c(\mathbb{R}^m)$,

$$E \left[ N(\varphi) \right] E \left[ N(\psi) \right]^*$$

$$= \left( \lambda \int_{\mathbb{R}^m} \varphi(t) \, dt \right) \left( \lambda \int_{\mathbb{R}^m} \psi^*(s) \, ds \right)$$

$$= \lambda^2 \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi(t) \psi^*(t+s) \, dt \right) \, ds ,$$

For $\varphi, \psi \in L^2_N(M_2)$,

$$\text{cov} \left( \int_{\mathbb{R}^m} \varphi(t) \, N(dt), \int_{\mathbb{R}^m} \psi(s) \, N(ds) \right)$$

$$= \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi(t) \psi^*(t+s) \, dt \right) \Gamma_N(ds)$$

where the locally finite measure

$$\Gamma_N := \sigma - \lambda^2 \ell^m$$

is called the covariance measure of the stationary second-order point process $N$. 
Covariance of the renewal process.

Let $N$ be a stationary renewal point process with renewal function $R$.

\[ \Gamma_N(dt) = \lambda(R(dt) - \lambda dt). \]

Homogeneous Poisson process on the line. By the covariance formula,

\[ \text{cov}(N(\varphi), N(\psi)) = \lambda \int_{\mathbb{R}} \varphi(t) \psi^*(t) \, dt. \]

\[ = \lambda \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \varphi(t) \psi^*(t + s) \, dt \right) \varepsilon_0(ds), \]

and therefore,

\[ \Gamma_N = \lambda \varepsilon_0. \]
Bartlett spectral measure

The unique locally finite measure $\mu_N$ such that

$$\text{Var} \left( \int \varphi(t) N(dt) \right) = \int |\hat{\varphi}(\nu)|^2 \mu_N(d\nu)$$

for all $\varphi \in \mathcal{B}_N$, where $\mathcal{B}_N \subseteq L^2_N(M^2)$ is a vector space of functions called the test function space.

By polarization, for all $\varphi, \psi \in \mathcal{B}_N$,

$$\text{cov} (N(\varphi), N(\psi)) = \int \hat{\varphi}(\nu)\hat{\psi}^*(\nu)\mu_N(d\nu).$$
\( \mathcal{B}_N \) should contain a class of functions rich enough to guarantee uniqueness of the measure \( \mu_N \): if the locally finite measures \( \mu_1 \) and \( \mu_2 \) are such that

\[
\int |\hat{\varphi}(\nu)|^2 \mu_1(d\nu) = \int |\hat{\varphi}(\nu)|^2 \mu_2(d\nu)
\]

for all \( \varphi \in \mathcal{B}_N \), then \( \mu_1 \equiv \mu_2 \).

Note that \( \mathcal{B}_N \subseteq L^1_\mathbb{C}(\mathbb{R}^m) \) since, as we observed earlier, \( L^2_N(M^2) \subseteq L^1_\mathbb{C}(\mathbb{R}^m) \).

In particular the Fourier transform of any \( \varphi \in \mathcal{B}_N \) is well-defined.

J. Neveu (1976): \( \mathcal{B}_N \) contains at least the functions that are, together with their Fourier transform, \( O \left( \frac{1}{|x|^2} \right) \) as \( |x| \to \infty \).
Poisson impulsive white noise. The covariance function is $\lambda$ times the Dirac measure at the origin, and therefore its spectral measure is $\lambda$ times the Lebesgue measure, therefore it admits a power spectral density that is a constant:

$$f_N(\nu) = \lambda.$$
Examples

Regular grid.

Regular $T$-grid on $\mathbb{R}$ with random origin, that is $N \equiv \{nT + U \; ; \; n \in \mathbb{Z}\}$ where $T > 0$, and $U$ is uniform random $[0, T]$. Here, $\lambda = 1/T$.

$$\mu_N = \frac{1}{T^2} \sum_{n \neq 0} \varepsilon \frac{n}{T},$$

and we can take $B_N$ specified by the following two conditions

$$\varphi \in L^1_C(\mathbb{R}) \cap L^2_C(\mathbb{R})$$

and

$$\sum_{n \in \mathbb{Z}} \left| \hat{\varphi} \left( \frac{n}{T} \right) \right| < \infty.$$

Note that the latter condition implies ($\ell^1_C(\mathbb{Z}) \subset \ell^2_C(\mathbb{Z})$)

$$\sum_{n \in \mathbb{Z}} \left| \hat{\varphi} \left( \frac{n}{T} u \right) \right|^2 < \infty.$$
Regular grid, proof

Weak Poisson summation formula: Both sides of the following equality

\[
\sum_{n \in \mathbb{Z}} \varphi (u + nT) = \frac{1}{T} \sum_{n \in \mathbb{Z}} \hat{\varphi} \left( \frac{n}{T} \right) e^{2i\pi \frac{n}{T} u}. \tag{\star}
\]

are well-defined, and the equality holds for almost-all \( u \in \mathbb{R} \).

By (\star),

\[
\int_{\mathbb{R}} \varphi (t) N (dt) = \sum_{n \in \mathbb{Z}} \varphi (U + nT) = \frac{1}{T} \sum_{n \in \mathbb{Z}} \hat{\varphi} \left( \frac{n}{T} \right) e^{2i\pi \frac{n}{T} U}
\]

and therefore

\[
E \left[ \left| \int_{\mathbb{R}} \varphi (t) N (dt) \right|^2 \right] = \frac{1}{T^2} E \left[ \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \hat{\varphi} \left( \frac{n}{T} \right) \hat{\varphi}^* \left( \frac{k}{T} \right) e^{2i\pi \frac{n-k}{T} U} \right]
\]

\[
= \frac{1}{T^2} \sum_{n \in \mathbb{Z}} \left| \hat{\varphi} \left( \frac{n}{T} \right) \right|^2.
\]
Also

\[
E \left[ \int_{\mathbb{R}^2} \varphi(t) N(dt) \right] = \sum_{n \in \mathbb{Z}} E \left[ \varphi(U + nT) \right]
\]

\[
= \frac{1}{T} \int_0^T \varphi(u + nT) \, du = \frac{1}{T} \int_{\mathbb{R}} \varphi(t) \, dt = \frac{1}{T} \hat{\varphi}(0).
\]

Therefore

\[
\text{Var} \left( \int_{\mathbb{R}} \varphi(t) N(dt) \right)
\]

\[
= \frac{1}{T^2} \sum_{n \in \mathbb{Z}} \left| \hat{\varphi} \left( \frac{n}{T} \right) \right|^2 - \frac{1}{T^2} \left| \hat{\varphi}(0) \right|^2
\]

\[
= \frac{1}{T^2} \sum_{n \neq 0} \left| \hat{\varphi} \left( \frac{n}{T} \right) \right|^2 = \int_{\mathbb{R}} \left| \hat{\varphi}(\nu) \right|^2 \mu_N(d\nu).
\]
Cox process.
(on $\mathbb{R}^m$ with stochastic intensity $\{\lambda(t)\}_{t \in \mathbb{R}^m}$.) Suppose that $\{\lambda(t)\}_{t \in \mathbb{R}^m}$ is a wss process with mean $\lambda$ and Cramér spectral measure $\mu_\lambda$. Then the Bartlett spectrum of $N$ is

$$\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu,$$

and we can take $\mathcal{B}_N = L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$. Even more, in this case $\mathcal{B}_N = L^2_N(M_2)$.
Examples

Renewal point process

Intensity $\lambda$ and non-lattice renewal distribution $F$. Define

$$\hat{F}(2i\pi\nu) = \int_{\mathbb{R}^+} e^{-2i\pi\nu t} dF(t).$$

Note that, since $F$ is non-lattice, $\hat{F}(\nu) \neq 1$, except for $\nu = 0$. The covariance measure is given by the formula

$$\Gamma(dx) = \lambda R(dx) - \lambda^2 \ell(dx).$$

The measure $R(dx)$ is the sum of a Dirac measure at 0, $\varepsilon(dx)$, and of a symmetric measure $U(dx)$, given by, for $dx \subset (0, \infty)$,

$$U(dx) = \sum_{n \geq 1} F^*(dx).$$

Assumption: $U$ admits a density $u$ and

$$\int_0^\infty |u(t) - \lambda| dt < \infty. \quad (1)$$
Define
\[ \hat{g}(\nu) = \int_0^\infty e^{-2i\pi\nu t} (u(t) - \lambda) dt \]

We then have, taking into account the symmetry of \( u \),
\[ \int_{\mathbb{R}} e^{-2i\pi\nu t} (u(t) - \lambda) dt = \hat{g}(\nu) + \hat{g}^*(\nu) \]

We shall prove below that
\[ \hat{g}(\nu) = \frac{\hat{F}(2i\pi\nu)}{1 - \hat{F}(2i\pi\nu)} + \frac{1}{2i\pi\nu} \] (2)

Combining the above results, we see that the Bartlett spectrum of \( N \) admits the density
\[ f_N(\nu) = \lambda \left( 1 + \text{Re} \left( \frac{\hat{F}(2i\pi\nu)}{1 - \hat{F}(2i\pi\nu)} \right) \right) \]
We shall now prove (2). For $\theta > 0$, we have

$$
\int_0^\infty e^{-(\theta + 2i\pi \nu)t} (u(t) - \lambda) dt
= \sum_{n \geq 1} \int_0^\infty e^{-(\theta + 2i\pi \nu)t} F^n(dt)
- \int_0^\infty e^{-(\theta + 2i\pi \nu)t} \lambda dt
= \sum_{n \geq 1} \hat{F}(\theta + 2i\pi \nu)^n - \frac{\lambda}{\theta + 2i\pi \nu}
= \frac{\hat{F}(\theta + 2i\pi \nu)}{1 - \hat{F}(\theta + 2i\pi \nu)} - \frac{\lambda}{\theta + 2i\pi \nu}
$$

For $\nu \neq 0$, letting $\theta$ tend to 0 in the first term of the above equality, we obtain by dominated convergence $\int_0^\infty e^{-2i\pi \nu t} (u(t) - \lambda) dt$. Letting $\theta$ tend to 0 in $\hat{F}(\theta + 2i\pi \nu)$, we obtain $\hat{F}(2i\pi \nu)$, again by dominated convergence.
A universal covariance formula

\[ N \equiv \{X_n\}_{n \in \mathbb{N}} \text{ p.p. on } \mathbb{R}^m, \text{ locally finite and simple, spectral measure } \mu_N. \]
\[ \{Z_n\}_{n \in \mathbb{N}} \text{ IID, values in } (K, \mathcal{K}) \text{ and distribution } Q, \text{ independent of } N. \]
\[ L^p_{\mathbb{C}}(\ell \times Q) := \left\{ \int E \left[ |\varphi(t, Z)|^p \right] \, dt < \infty \right\} \]
Let \( \varphi : \mathbb{R}^m \times K \to \mathbb{R} \) such that
\[ \varphi \in L^1_\mathbb{C}(\ell \times Q) \cap L^2_\mathbb{C}(\ell \times Q) \]
In particular, \( \varphi(t, Z) \in L^2_\mathbb{C}(P) \) t-a.e. and we can define t-a.e.
\[ \bar{\varphi}(t) := E[\varphi(t, Z)]. \]
Also \( \bar{\varphi} \in L^1_\mathbb{C}(\mathbb{R}^m) \cap L^2_\mathbb{C}(\mathbb{R}^m) \) and for \( Q \)-almost all \( z \in K \),
\[ \varphi(\cdot, z) \in L^1_\mathbb{C}(\mathbb{R}^m) \cap L^2_\mathbb{C}(\mathbb{R}^m). \]
\[ \hat{\bar{\varphi}}(\nu) = E[\hat{\varphi}(\nu, Z)] := \bar{\varphi}(\nu). \]
Finally, suppose that
\[ \bar{\varphi} \in \mathcal{B}_N. \]
\[
\text{cov} \left( \sum_{n \in \mathbb{N}} \varphi(X_n, Z_n), \sum_{n \in \mathbb{N}} \psi(X_n, Z_n) \right) \\
= \int_{\mathbb{R}^m} \widehat{\varphi}(\nu) \widehat{\psi}^*(\nu) \mu_N(d\nu) \\
+ \lambda \int_{\mathbb{R}^m} \text{cov} \left( \widehat{\varphi}(\nu, Z), \widehat{\psi}^*(\nu, Z) \right) d\nu,
\]
Thinning

$Z_1 \in 0, 1, \ P(Z_1 = 1) = \alpha$. Let

$$N_\alpha(C) := \sum_{n \geq 1} Z_n 1\{X_n \in C\}.$$

$$\mu_{N_\alpha} := \alpha^2 \mu_N + \lambda \alpha (1 - \alpha) \ell^m$$

and $B_{N_\alpha} := L^1_\mathbb{C}(\mathbb{R}^m) \cap L^2_\mathbb{C}(\mathbb{R}^m) \cap B_N$

Must show that for any function $\varphi \in B_{N_\alpha}$,

$$\text{Var} \int_{\mathbb{R}^m} \varphi(x) \ N_\alpha(dx) = \int_{\mathbb{R}^m} |\hat{\varphi}(\nu)| \ \mu_{N_\alpha}(d\nu).$$

Now

$$\int_{\mathbb{R}^m} \varphi(x) \ N_\alpha(dx) = \sum_{n \geq 1} Z_n \varphi(X_n).$$

Applying the general formula with $\varphi(x, z) = \psi(x, z) = z \varphi(x)$ with $\varphi \in B_{N_\alpha}$. 

P. Brémaud (Inria and EPFL)
\( \tilde{N} \) defined by its points

\[ \{X_n + Z_n\}_{n \in \mathbb{N}}. \]

\[
\mu_{\tilde{N}}(d\nu) = |\psi_Z(\nu)|^2 \mu_N(d\nu) + \lambda \left(1 - |\psi_Z(\nu)|^2\right) d\nu,
\]

where

\[
\psi_Z(\nu) = E \left[ e^{2i\pi <\nu, Z>} \right].
\]

We can take

\[
\mathcal{B}_{\tilde{N}} = \{ \varphi; E[\varphi(t + Z)] \in \mathcal{B}_N \} \cap L^1_C(\mathbb{R}^m) \cap L^2_C(\mathbb{R}^m).
\]
Jittered regular grid.

We can take

\[ B_{\tilde{N}} = \left\{ \phi ; \sum_{n \in \mathbb{Z}} \left| \hat{\phi} \left( \frac{n}{T} \right) \right| < \infty \right\} \cap L^1_{\mathbb{C}}(\mathbb{R}) \cap L^2_{\mathbb{C}}(\mathbb{R}) \]

Jittered Cox process.

We can take

\[ B_{\tilde{N}} = L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m) \]

Indeed condition \( E \left[ \phi(t + Z) \right] \in B_N \), that is, in this particular case, \( E \left[ \phi(t + Z) \right] \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m) \), is exactly \( \phi \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m) \).
\(\{Z_n\}_{n \geq 1}\) is an IID collection of point processes on \(\mathbb{R}^m\), independent of \(N\). Let \(Z\) be a point process on \(\mathbb{R}^m\) with the same distribution as the \(Z_n\)'s. Define

\[
\psi_Z(\nu) := E \left[ \int_{\mathbb{R}^m} e^{2i\pi \langle \nu, t \rangle} Z(dt) \right]
\]

The function \(\psi_Z\) is well defined under the assumption

\[E[Z(\mathbb{R}^m)] < \infty.\]

(In particular, \(Z\) is almost surely a finite point process.)
We now define

\[ \tilde{N}(C) = N(C) + \sum_{n \geq 1} Z_n(C - X_n), \]

\[ \hat{N}(C) = \sum_{n \geq 1} Z_n(C - X_n), \]

Formally

\[
\text{Var} \left( \int_{\mathbb{R}^m} \varphi(t) \tilde{N}(dt) \right) \\
= \text{Var} \left( \sum_{n \geq 1} \left\{ \varphi(X_n) + \int_{\mathbb{R}^m} \varphi(X_n + s) Z_n(ds) \right\} \right) \\
= \text{Var} \left( \sum_{n \geq 1} \varphi(X_n, Z_n) \right),
\]

where

\[ \varphi(x, z) = \varphi(x) + \int_{\mathbb{R}^m} \varphi(x + s) z(ds). \]
We have

$$E [\varphi (x, Z)] = \varphi (x) + E \left[ \int_{\mathbb{R}^m} \varphi (x + s) Z (ds) \right]$$

$$\hat{\varphi} (\nu, z)$$

$$= \hat{\varphi} (\nu) + \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi (t + s) z (ds) \right) e^{-2i\pi \langle \nu, t \rangle} dt$$

$$= \hat{\varphi} (\nu) + \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi (t + s) e^{-2i\pi \langle \nu, t \rangle} dt \right) z (ds)$$

$$= \hat{\varphi} (\nu) + \int_{\mathbb{R}^m} \hat{\varphi} (\nu) e^{2i\pi \langle \nu, s \rangle} z (ds)$$

$$= \hat{\varphi} (\nu) \left( 1 + \int_{\mathbb{R}^m} e^{2i\pi \langle \nu, s \rangle} z (ds) \right)$$

Also

$$\hat{\varphi} (\nu) = \hat{\varphi} (\nu) (1 + \psi Z (\nu))$$
Applying the general covariance formula, we obtain

\[ \mu_{\tilde{N}} (d\nu) = |1 + \psi_Z (\nu)|^2 \mu_N (d\nu) \]

\[ + \lambda \text{Var} \left( \int_{\mathbb{R}^m} e^{2i\pi \langle \nu, s \rangle} Z (ds) \right) d\nu. \]

Similarly

\[ \mu_{\hat{N}} (d\nu) = |\psi_Z (\nu)|^2 \mu_N (d\nu) \]

\[ + \lambda \text{Var} \left( \int_{\mathbb{R}^m} e^{2i\pi \langle \nu, s \rangle} Z (ds) \right) d\nu. \]
Multivariate point process

\( N_1 \) and \( N_2 \) are \textit{wss} and moreover \textit{jointly} \textit{wss}, that is if

\[
E \left[ N_1(A + t)N_2(B + t) \right] = E \left[ N_1(A)N_2(B) \right].
\]

One says that \( N_1 \) and \( N_2 \) admit the \textit{cross-spectral measure} \( \mu_{N_1,N_2} \), sigma-finite signed, if for all \( \varphi_1 \in B_{N_1}, \varphi_2 \in B_{N_2} \)

\[
\text{cov} \left( N_1(\varphi_1), N_2(\varphi_2) \right) = \int_{\mathbb{R}^m} \hat{\varphi}_1(\nu)\hat{\varphi}_2(\nu)^* \mu_{N_1,N_2}(d\nu).
\]

\textbf{Bivariate wss Cox processes.} Let \( N_1 \) and \( N_2 \) be \textit{wss} Cox processes with stochastic intensities \( \{\lambda_1(t)\}_{t \in \mathbb{R}} \) and \( \{\lambda_2(t)\}_{t \in \mathbb{R}} \), jointly stationary \textit{wss} stochastic processes with cross-spectral measure \( \mu_{\lambda_1,\lambda_2} \).

\[
\mu_{N_1,N_2} = \mu_{\lambda_1,\lambda_2}.
\]
Cross-spectrum of a point process and its jittered version.

\[
\text{cov} \left( \sum_{n \geq 1} \varphi(X_n), \sum_{n \geq 1} \psi(X_n + Z_n) \right) \\
= \int_{\mathbb{R}^m} \hat{\varphi}(\nu) E \left[ \hat{\psi}(\nu + Z)^* \right] \mu_N(d\nu).
\]

But

\[
\hat{\psi}(\nu + Z) = \int_{\mathbb{R}^m} \psi(t + Z) e^{-2i\pi\nu t} dt \\
= \int_{\mathbb{R}^m} \psi(t) e^{-2i\pi\nu(t - Z)} dt \\
= \hat{\psi}(\nu) E \left[ e^{2i\pi\nu Z} \right]
\]

where the expectation is with respect to \( Z \) a random variable with the common probability distribution of the marks.
Finally

\[
\text{cov} \left( \sum_{n \in \mathbb{Z}} \varphi(X_n), \sum_{n \in \mathbb{Z}} \psi(X_n + Z_n) \right) = \int_{\mathbb{R}^m} \hat{\varphi}(\nu) \hat{\psi}(\nu)^* E \left[ e^{-2i\pi \nu Z} \right] \mu_N(d\nu),
\]

and therefore

\[
\mu_{N_1, N_2}(d\nu) = E \left[ e^{-2i\pi \nu Z} \right] \mu_N(d\nu).
\]
The **sampler**: A wss point process on $\mathbb{R}^m$ with intensity $\lambda$, point sequence $\{V_n\}_{n \geq 1}$.

The **sampled process** is wss

$$X(t) = \int_{\mathbb{R}^m} e^{2i\pi \langle \nu, t \rangle} Z_X(d\nu) + mX$$

The sampled process and the sampler are independent.

The **sample brush**

$$Y(t) = \sum_{n \geq 1} X(V_n) \delta(t - V_n)$$

is identified with the signed measure

$$\sum_{n \geq 1} X(V_n) \varepsilon_{V_n}.$$
The extended spectral measure of the sample brush: A locally finite measure $\mu_Y$ such that, for any $\varphi \in B_Y$,

$$\text{Var} \left( \int_{\mathbb{R}^m} \varphi(t) X(t) N(dt) \right) = \int_{\mathbb{R}^m} |\hat{\varphi}(\nu)|^2 \mu_Y(d\nu),$$

where $B_Y$ is a large enough vector space of functions, here also called the “test functions”.

\[
\int_{\mathbb{R}^m} \varphi(t) Y(t) \, dt \\
= \int_{\mathbb{R}^m} \varphi(t) \left( \sum_{n \geq 1} X(V_n) \delta(t - V_n) \right) \, dt \\
= \sum_{n \geq 1} \varphi(V_n) X(V_n) = \int_{\mathbb{R}^m} \varphi(t) X(t) N(dt),
\]

\[
\text{Var} \left( \int_{\mathbb{R}^m} \varphi(t) Y(t) \, dt \right) \\
= \int_{\mathbb{R}^m} |\hat{\varphi}(\nu)|^2 \mu_Y(d\nu).
\]
\[ \mu_Y = \mu_N \ast \mu_X + \lambda^2 \mu_X + |m_X|^2 \mu_N. \]

If \( B_N \) is stable with respect to multiplications by complex exponential functions, we can take for test function space \( B_Y = B_N \).

To be compared with that giving the spectral measure \( \mu_Y \) of the product of two independent wss stochastic processes, \( Y(t) = Z(t)X(t) \):
\[ \mu_Y = \mu_Z \ast \mu_X. \)
Examples

Cox sampling.

\[ \mu_Y = \mu_\lambda \ast \mu_X + \lambda^2 \mu_X + |m_X|^2 \mu_\lambda \]
\[ + \lambda \left( \sigma_X^2 + |m_X|^2 \right) \ell^m \]

where \( \ell^m \) is the Lebesgue measure.

\[ \mathcal{B}_N = L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m) = \mathcal{B}_Y. \]
Examples

Regular sampling.

\[ f_Y(\nu) = \left(\frac{1}{T}\right)^2 \sum_{n \in \mathbb{Z}} f_X(\nu - \frac{n}{T}) . \]

The spectral density can recovered from that of the sample comb provided the former is band-limited, with band width \(2B < \frac{1}{T}\).
Poisson sampling.

\[ f_Y (\nu) = \lambda^2 f_X (\nu) + \lambda \sigma_X^2. \]

Whatever the sampling frequency \( \nu_s = \lambda \), there is no aliasing.
Reconstruction

Approximate the sampled process by a filtered version of the sample comb:

$$\int_{\mathbb{R}^m} \varphi(t-s)Y(s)\,ds$$

reconstruction error:

$$\epsilon = E \left[ \left| \int_{\mathbb{R}^m} \varphi(t-u)Y(u)\,du - X(t) \right|^2 \right].$$

The reconstruction error is, when the sampled process is centered:

$$\epsilon = \int_{\mathbb{R}^m} |\lambda \hat{\varphi}(\nu) - 1|^2 \mu_X(\,d\nu)$$

$$+ \lambda \int_{\mathbb{R}^m} |\hat{\varphi}(\nu)|^2 (\mu_X * \mu_\lambda)(\,d\nu).$$

Denoting by $S$ the support (assumed of Lebesgue measure $2B < \infty$) of the spectral measure $\mu_X$,

$$\hat{\varphi}(\nu) = \frac{1}{\lambda} 1_S(\nu).$$
Poisson sampling, bad news

\[ \epsilon = \sigma^2 \chi \frac{2B}{\lambda}. \]

Therefore, sampling at the “Nyquist rate” \( \lambda = 2B \) gives a very poor performance, not better than the estimate based on no observation at all.
Examples

Regular sampling

\[ \epsilon = \int_{\mathbb{R}} \left| \frac{1}{T} \hat{\varphi}(\nu) - 1 \right|^2 \mu_X(d\nu) \]
\[ + \frac{1}{T} \int_{\mathbb{R}} |\hat{\varphi}(\nu) - 1|^2 d\nu \]

In the band-limited case, \( T = 1/2B \) (that is, \( \lambda = 2B \)) the error is null. Therefore, the process is perfectly reconstructed by

\[ X(t) = \int_{\mathbb{R}} \varphi(t-s)X(s)N(ds) \]
\[ = \sum_{n \in \mathbb{Z}} X(T_n) \text{sinc}(2B(t - T_n)) , \]
Effects of jitter in Nyquist sampling

\[ \epsilon = \frac{1}{2B} \left( \int_{-B}^{B} \sigma_x^2 \left( 1 - \left| \psi_Z \right|^2 * \tilde{f}_x (\nu) \right) d\nu \right) , \]

where \( \tilde{f}_x \) is the normalized power spectral density of the process \( X(t) \).
THE END
(for the time being)