SPECTRAL MEASURES OF POINT PROCESSES

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The purpose of this talk

to honor François
for his 60-th birthday

And to opportunistically take advantage of his great popularity and the large number of friends gathered in this occasion to advertise my recently published book:

Fourier Analysis and
Stochastic Processes

What is it about?

Consider a point process N on $\mathbb{R}$ with event times $\{T_n\}_{n\in\mathbb{Z}}$. The "random Dirac comb"

$$X(t) := \sum_{n\in\mathbb{Z}} \delta(t - T_n),$$

is not a *bona fide* stochastic process. In particular, one cannot define for the random Dirac comb associated with a stationary point process a power spectral measure as in the case of wide-sense stationary stochastic processes.

The natural extension of the notion of power spectral density is the so-called Bartlett spectral measure

Here we concentrate on the computation of such measures.

Who needs it?

1. biology (spike trains)
2. communications (ultra-wide band)
3. perhaps nobody needs it.

Some contributors

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J. Neveu, Processus ponctuels, in *École d'été de Saint Flour*, Lect. Notes in Math. **598**, 249-445, Springer (1976).

D.J. Daley, D. Vere–Jones, *An Introduction to the Theory of Point Processes*, Springer, NY (1988, 2003).

P. B. and L. Massoulié, Power Spectra of Generalized Shot Noises and Hawkes Point Processes with a random excitation, *Adv. Appl. Proba.*, 205-222 (2002)

P. B, L. Massoulié, and A. Ridolfi, "Power spectra of random spike fields and related processes", *Adv. in Appl. Probab.*, **37**, 4, 1116-1146 (2005).

Second moment measure

Second-order: for all compact sets C,

$$E\left[N(C)^2\right] < \infty\,.$$

$$M_2(A \times B) := E\left[N(A)\,N(B)\right]\,.$$

M_2 is the intensity measure of $N \times N$. By Campbell's theorem,

$$E\left[\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} g(X_n, X_k)\right] = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} g(t,s)\,M_2(dt \times ds)\,.$$

$L^2_N(M_2)$

The collection of functions $\varphi : \mathbb{R}^m \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |\varphi(t)\varphi(s)|\, M_2(dt \times ds) < \infty\,,$$

$$\Leftrightarrow E\left[N(|\varphi|)^2\right] < \infty\,,$$

$$\Rightarrow E\left[N(|\varphi|)\right] < \infty,\ E\left[N(|\varphi|^2)\right] < \infty$$

$$\Rightarrow L^2_N(M_2) \subseteq L^1_{\mathbb{C}}(\nu) \cap L^2_{\mathbb{C}}(\nu)\,.$$

(where $\nu(C) := E[N(C)]$)

Wide-sense stationary point process

Second-order, plus

$$E\left[N(C+t)\right] = E\left[N(C)\right],$$

and

$$E\left[N(A+t)N(B+t)\right] = E\left[N(A)N(B)\right].$$

Immediate consequence: for all non-negative φ, ψ,

$$E\left[\left(\int_{\mathbb{R}} \varphi(t)\, N(dt)\right)\left(\int_{\mathbb{R}} \psi(\tau+t)\, N(dt)\right)\right]$$

is independent of $\tau \in \mathbb{R}$.

Covariance measure

Basic lemma from measure theory

$(X, \mathcal{X})$, μ loc. fin. measure on $\mathcal{X}^{\otimes k}$, invariant by the simultaneous shifts, that is,

$$\mu((A_1 + h) \times \cdots \times (A_k + h)) = \mu(A_1 \times \cdots \times A_k)\,.$$

Then, there exists a locally finite measure $\hat{\mu}$ on $\mathcal{X}^{k-1}$ such that for all non-negative measurable functions f from X^k to $\mathbb{R}$,

$$\int_{\mathcal{X}^k} f(x_1, \dots, x_k)\mu(dx_1 \times \cdots \times dx_k)$$
$$= \int_{\mathcal{X}} \left\{ \int_{\mathcal{X}^{k-1}} f(x_1, x_1 + x_2, \dots, x_1 + x_k)\hat{\mu}(dx_2 \times \cdots \times dx_k) \right\} dx_1.$$

Application to point processes

$$M_2\left((A+t)\times(B+t)\right)=M_2\left(A\times B\right)$$

Therefore, for all $\varphi, \psi \in L^2_N(M_2)$,

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\varphi(t)\,\psi^*(s)\,M_2(dt\times ds) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,\psi^*(s+t)\,dt\right)\sigma(ds)$$

for some locally finite measure σ.

In fact, σ can be identified to the intensity measure of the Palm version of a given stationary point process.

Since for $\varphi, \psi \in L^1_{\mathbb{C}}(\mathbb{R}^m)$,

$$\begin{aligned} & E\left[N(\varphi)\right] E\left[N(\psi)\right]^* \\ & = \left(\lambda \int_{\mathbb{R}^m} \varphi(t)\, dt\right) \left(\lambda \int_{\mathbb{R}^m} \psi^*(s)\, ds\right) \\ & = \lambda^2 \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \varphi(t)\, \psi^*(t+s)\, dt\right) ds\,, \end{aligned}$$

For $\varphi, \psi \in L^2_N(M_2)$,

$$\begin{aligned} & \operatorname{cov}\left(\int_{\mathbb{R}^m} \varphi(t)\, N(dt)\,, \int_{\mathbb{R}^m} \psi(s)\, N(ds)\right) \\ & \quad = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \varphi(t)\, \psi^*(t+s)\, dt\right) \Gamma_N(ds) \end{aligned}$$

where the locally finite measure

$$\Gamma_N := \sigma - \lambda^2 \ell^m$$

is called the *covariance measure* of the stationary second-order point process N.

Covariance of the renewal process.

Let N be a stationary renewal point process with renewal function R.

$$\Gamma_N(dt) = \lambda(R(dt) - \lambda\, dt)\,.$$

Homogeneous Poisson process on the line. By the covariance formula,

$$\operatorname{cov}\left(N(\varphi), N(\psi)\right) = \lambda \int_{\mathbb{R}} \varphi\left(t\right)\psi^*\left(t\right)\, dt\,.$$

$$= \lambda \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi\left(t\right)\psi^*\left(t+s\right)\, dt \right) \varepsilon_0(ds)\,,$$

and therefore, ,

$$\Gamma_N = \lambda\varepsilon_0\,.$$

Bartlett spectral measure

The unique locally finite measure μ_N such that

$$\mathrm{Var}\left(\int \varphi(t)\, N(dt)\right) = \int |\widehat{\varphi}(\nu)|^2 \mu_N(d\nu)$$

for all $\varphi \in \mathcal{B}_N$, where $\mathcal{B}_N \subseteq L^2_N(M^2)$ is a vector space of functions called the *test function space*.

By polarization, for all $\varphi,\ \psi \in \mathcal{B}_N$,

$$\mathrm{cov}\left(N(\varphi)\,,\, N(\psi)\right) = \int \widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\mu_N(d\nu).$$

$\mathcal{B}_N$ should contain a class of functions rich enough to guarantee uniqueness of the measure μ_N: if the locally finite measures μ_1 and μ_2 are such that

$$\int |\widehat{\varphi}(\nu)|^2 \mu_1(d\nu) = \int |\widehat{\varphi}(\nu)|^2 \mu_2(d\nu)$$

for all $\varphi \in \mathcal{B}_N$, then $\mu_1 \equiv \mu_2$.
Note that $\mathcal{B}_N \subseteq L^1_{\mathbb{C}}(\mathbb{R}^m)$ since, as we observed earlier, $L^2_N(M^2) \subseteq L^1_{\mathbb{C}}(\mathbb{R}^m)$.
In particular the Fourier transform of any $\varphi \in \mathcal{B}_N$ is well-defined.
J. Neveu (1976): $\mathcal{B}_N$ contains at least the functions that are, together with their Fourier transform, $O\left(1/|x|^2\right)$ as $|x| \to \infty$.

Examples

Poisson impulsive white noise. The covariance function is λ times the Dirac measure at the origin, and therefore its spectral measure is λ times the Lebesgue measure, therefore it admits a power spectral density that is a constant:

$$f_N(\nu) = \lambda.$$

Examples

Regular grid.

Regular T-grid on $\mathbb{R}$ with random origin, that is $N \equiv \{nT + U\,;\, n \in \mathbb{Z}\}$ where $T > 0$, and U is uniform random $[0, T]$. Here, $\lambda = 1/T$.

$$\mu_N = \frac{1}{T^2} \sum_{n \neq 0} \varepsilon_{\frac{n}{T}}\,,$$

and we can take $\mathcal{B}_N$ specified by the following two conditions

$$\varphi \in L^1_{\mathbb{C}}(\mathbb{R}) \cap L^2_{\mathbb{C}}(\mathbb{R})$$

and

$$\sum_{n \in \mathbb{Z}} \left| \hat{\varphi}\left(\frac{n}{T}\right)\right| < \infty\,.$$

Note that the latter condition implies $(\ell^1_{\mathbb{C}}(\mathbb{Z}) \subset \ell^2_{\mathbb{C}}(\mathbb{Z}))$

$$\sum_{n \in \mathbb{Z}} \left| \hat{\varphi}\left(\frac{n}{T}u\right)\right|^2 < \infty.$$

Regular grid, proof

Weak Poisson summation formula: Both sides of the following equality

$$\sum_{n\in\mathbb{Z}} \varphi(u+nT) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \widehat{\varphi}\left(\frac{n}{T}\right) e^{2i\pi\frac{n}{T}u}. \qquad (\star)$$

are well-defined, and the equality holds for almost-all $u \in \mathbb{R}$.

By $(\star)$,

$$\int_{\mathbb{R}} \varphi(t)\, N(dt) = \sum_{n\in\mathbb{Z}} \varphi(U+nT) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \widehat{\varphi}\left(\frac{n}{T}\right) e^{2i\pi\frac{n}{T}U}$$

and therefore

$$\begin{aligned}
E&\left[\left|\int_{\mathbb{R}} \varphi(t)\, N(dt)\right|^2\right] \\
&= \frac{1}{T^2} E\left[\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \widehat{\varphi}\left(\frac{n}{T}\right)\widehat{\varphi}^*\left(\frac{k}{T}\right) e^{2i\pi\left(\frac{n-k}{T}U\right)}\right] \\
&= \frac{1}{T^2}\sum_{n\in\mathbb{Z}} \left|\widehat{\varphi}\left(\frac{n}{T}\right)\right|^2.
\end{aligned}$$

Also

$$E\left[\int_{\mathbb{R}^2}\varphi(t)\,N(dt)\right]=\sum_{n\in\mathbb{Z}}E\left[\varphi(U+nT)\right]$$
$$=\frac{1}{T}\int_0^T\varphi(u+nT)\,du=\frac{1}{T}\int_{\mathbb{R}}\varphi(t)\,dt=\frac{1}{T}\widehat{\varphi}(0)\,.$$

Therefore

$$\mathrm{Var}\,\left(\int_{\mathbb{R}}\varphi(t)\,N(dt)\right)$$
$$=\frac{1}{T^2}\sum_{n\in\mathbb{Z}}\left|\widehat{\varphi}\left(\frac{n}{T}\right)\right|^2-\frac{1}{T^2}\left|\widehat{\varphi}(0)\right|^2$$
$$=\frac{1}{T^2}\sum_{n\neq 0}\left|\widehat{\varphi}\left(\frac{n}{T}\right)\right|^2=\int_{\mathbb{R}}\left|\widehat{\varphi}(\nu)\right|^2\,\mu_N(d\nu)\,.$$

Examples

Cox process.
(on $\mathbb{R}^m$ with stochastic intensity $\{\lambda(t)\}_{t\in\mathbb{R}^m}$.) Suppose that $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ is a WSS process with mean λ and Cramér spectral measure μ_λ. Then the Bartlett spectrum of N is

$$\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu\,,$$

and we can take $\mathcal{B}_N = L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$. Even more, in this case $\mathcal{B}_N = L^2_N(M_2)$

Examples

Renewal point process

Intensity λ and non-lattice renewal distribution F. Define

$$\hat{F}(2i\pi\nu) = \int_{\mathbb{R}_+} e^{-2i\pi\nu t} dF(t)\,.$$

Note that, since F is non-lattice, $\hat{F}(\nu) \neq 1$, except for $\nu = 0$. The covariance measure is given by the formula

$$\Gamma(dx) = \lambda R(dx) - \lambda^2 \ell(dx)\,.$$

The measure $R(dx)$ is the sum of a Dirac measure at 0, $\varepsilon(dx)$, and of a symmetric measure $U(dx)$, given by, for $dx \subset (0, \infty)$,

$$U(dx) = \sum_{n\geq 1} F^{*n}(dx)\,.$$

Assumption: U admits a density u and

$$\int_0^\infty |u(t) - \lambda| dt < \infty\,. \tag{1}$$

Define

$$\hat{g}(\nu) = \int_0^\infty e^{-2i\pi\nu t}(u(t) - \lambda)dt$$

We then have, taking into account the symmetry of u,

$$\int_{\mathbb{R}} e^{-2i\pi\nu t}(u(t) - \lambda)dt = \hat{g}(\nu) + \hat{g}^*(\nu)$$

We shall prove below that

$$\hat{g}(\nu) = \frac{\hat{F}(2i\pi\nu)}{1 - \hat{F}(2i\pi\nu)} + \frac{1}{2i\pi\nu} \tag{2}$$

Combining the above results, we see that the Bartlett spectrum of N admits the density

$$f_N(\nu) = \lambda\left(1 + Re\left(\frac{\hat{F}(2i\pi\nu)}{1 - \hat{F}(2i\pi\nu)}\right)\right)$$

We shall now prove (2). For $\theta > 0$, we have

$$\int_0^\infty e^{-(\theta+2i\pi\nu)t}(u(t)-\lambda)dt$$

$$= \sum_{n\geq 1} \int_0^\infty e^{-(\theta+2i\pi\nu)t} F^{*n}(dt)$$

$$- \int_0^\infty e^{-(\theta+2i\pi\nu)t} \lambda dt$$

$$= \sum_{n\geq 1} \hat{F}(\theta+2i\pi\nu)^n - \frac{\lambda}{\theta+2i\pi\nu}$$

$$= \frac{\hat{F}(\theta+2i\pi\nu)}{1-\hat{F}(\theta+2i\pi\nu)} - \frac{\lambda}{\theta+2i\pi\nu}$$

For $\nu \neq 0$, letting θ tend to 0 in the first term of the above equality, we obtain by dominated convergence $\int_0^\infty e^{-2i\pi\nu t}(u(t)-\lambda)dt$. Letting θ tend to 0 in $\hat{F}(\theta+2i\pi\nu)$, we obtain $\hat{F}(2i\pi\nu)$, again by dominated convergence.

A universal covariance formula

$N \equiv \{X_n\}_{n\in\mathbb{N}}$ p.p. on $\mathbb{R}^m$, locally finite and simple, spectral measure μ_N.
$\{Z_n\}_{n\in\mathbb{N}}$ IID, values in $(K, \mathcal{K})$ and distribution Q, independent of N.
$L^p_{\mathbb{C}}(\ell \times Q) := \{\int E\left[|\varphi(t,Z)|^p\right] dt < \infty\}$
Let $\varphi : \mathbb{R}^m \times K \to \mathbb{R}$ such that

$$\varphi \in L^1_{\mathbb{C}}(\ell \times Q) \cap L^2_{\mathbb{C}}(\ell \times Q)$$

In particular, $\varphi(t,Z) \in L^2_{\mathbb{C}}(P)$ t-a.e. and we can define t-a.e.

$$\bar{\varphi}(t) := E\left[\varphi(t,Z)\right] .$$

Also $\bar{\varphi} \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$ and for Q-almost all $z \in K$,
$\varphi(\cdot, z) \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$.

$$\widehat{\bar{\varphi}}(\nu) = E\left[\widehat{\varphi}(\nu, Z)\right] := \bar{\widehat{\varphi}}(\nu).$$

Finally, suppose that

$$\bar{\varphi} \in \mathcal{B}_N .$$

$$
\begin{aligned}
\operatorname{cov} & \left(\sum_{n \in \mathbb{N}} \varphi(X_n, Z_n) \,, \sum_{n \in \mathbb{N}} \psi(X_n, Z_n) \right) \\
& = \int_{\mathbb{R}^m} \widehat{\overline{\varphi}}(\nu) \, \widehat{\overline{\psi}}^{*}(\nu) \mu_N(d\nu) \\
& \qquad + \lambda \int_{\mathbb{R}^m} \operatorname{cov} \left(\widehat{\varphi}(\nu, Z), \widehat{\psi}^{*}(\nu, Z) \right) d\nu,
\end{aligned}
$$

Thinning

$Z_1 \in 0, 1$, $P(Z_1 = 1) = \alpha$. Let

$$N_\alpha(C) := \sum_{n \geq 1} Z_n 1_{\{X_n \in C\}}.$$

$$\mu_{N_\alpha} := \alpha^2 \mu_N + \lambda\alpha(1-\alpha)\ell^m$$

and $\mathcal{B}_{N_\alpha} := L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m) \cap \mathcal{B}_N$
Must show that for any function $\varphi \in \mathcal{B}_{N_\alpha}$,

$$\mathrm{Var} \int_{\mathbb{R}^m} \varphi(x)\, N_\alpha(dx) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|\; \mu_{N_\alpha}(d\nu).$$

Now

$$\int_{\mathbb{R}^m} \varphi(x)\, N_\alpha(dx) = \sum_{n \geq 1} Z_n \varphi(X_n)\,.$$

Applying the general formula with $\varphi(x, z) = \psi(x, z) = z\varphi(x)$ with $\varphi \in \mathcal{B}_{N_\alpha}$.

Jittering

$\widetilde{N}$ defined by its points

$$\{X_n + Z_n\}_{n\in\mathbb{N}}.$$

$$\mu_{\widetilde{N}}(d\nu) = |\psi_Z(\nu)|^2 \mu_N(d\nu) + \lambda\left(1 - |\psi_Z(\nu)|^2\right) d\nu,$$

where

$$\psi_Z(\nu) = E\left[e^{2i\pi<\nu,Z>}\right].$$

We can take

$$\mathcal{B}_{\tilde{N}} = \{\varphi\,;\ E[\varphi(t+Z)] \in \mathcal{B}_N\} \cap L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m).$$

Jittered regular grid.
We can take

$$B_{\tilde{N}} = \left\{ \varphi\,;\, \sum_{n\in\mathbb{Z}} \left| \hat{\varphi}\left(\frac{n}{T}\right) \right| < \infty \right\} \cap L^1_{\mathbb{C}}(\mathbb{R}) \cap L^2_{\mathbb{C}}(\mathbb{R})$$

Jittered Cox process.
We can take

$$B_{\tilde{N}} = L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$$

Indeed condition $E\left[\varphi(t+Z)\right] \in \mathcal{B}_N$, that is, in this particular case, $E\left[\varphi(t+Z)\right] \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$, is exactly $\varphi \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$.

Clustering.

$\{Z_n\}_{n\geq 1}$ is an IID collection of point processes on $\mathbb{R}^m$, independent of N. Let Z be a point process on $\mathbb{R}^m$ with the same distribution as the Z_n's. Define

$$\psi_Z(\nu) := E\left[\int_{\mathbb{R}^m} e^{2i\pi\langle\nu,t\rangle} Z(dt)\right]$$

The function ψ_Z is well defined under the assumption

$$E[Z(\mathbb{R}^m)] < \infty.$$

(In particular, Z is almost surely a finite point process.)

We now define

$$\widetilde{N}(C) = N(C) + \sum_{n\geq 1} Z_n(C - X_n),$$

$$\widehat{N}(C) = \sum_{n\geq 1} Z_n(C - X_n),$$

Formally

$$\begin{aligned}
&\mathrm{Var}\ \left(\int_{\mathbb{R}^m} \varphi(t)\widetilde{N}(dt)\right) \\
&= \mathrm{Var}\ \left(\sum_{n\geq 1}\left\{\varphi(X_n) + \int_{\mathbb{R}^m} \varphi(X_n + s)\, Z_n(ds)\right\}\right) \\
&= \mathrm{Var}\ \left(\sum_{n\geq 1} \varphi(X_n, Z_n)\right),
\end{aligned}$$

where

$$\varphi(x, z) = \varphi(x) + \int_{\mathbb{R}^m} \varphi(x + s)\, z(ds).$$

We have

$$E\left[\varphi\left(x,Z\right)\right]=\varphi\left(x\right)+E\left[\int_{\mathbb{R}^m}\varphi\left(x+s\right)Z\left(ds\right)\right]$$

$$\begin{aligned}
&\widehat{\varphi}\left(\nu,z\right)\\
&=\widehat{\varphi}\left(\nu\right)+\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi\left(t+s\right)z\left(ds\right)\right)e^{-2i\pi\langle\nu,t\rangle}dt\\
&=\widehat{\varphi}\left(\nu\right)+\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi\left(t+s\right)e^{-2i\pi\langle\nu,t\rangle}dt\right)z\left(ds\right)\\
&=\widehat{\varphi}\left(\nu\right)+\int_{\mathbb{R}^m}\widehat{\varphi}\left(\nu\right)e^{2i\pi\langle\nu,s\rangle}z\left(ds\right)\\
&=\widehat{\varphi}\left(\nu\right)\left(1+\int_{\mathbb{R}^m}e^{2i\pi\langle\nu,s\rangle}z\left(ds\right)\right)
\end{aligned}$$

Also

$$\widehat{\overline{\varphi}}\left(\nu\right)=\widehat{\varphi}\left(\nu\right)\left(1+\psi_Z\left(\nu\right)\right)$$

Applying the general covariance formula, we obtain

$$\mu_{\widetilde{N}}(d\nu) = |1 + \psi_Z(\nu)|^2 \mu_N(d\nu) \\ + \lambda \mathrm{Var}\left(\int_{\mathbb{R}^m} e^{2i\pi\langle \nu, s\rangle} Z(ds)\right) d\nu.$$

Similarly

$$\mu_{\widehat{N}}(d\nu) = |\psi_Z(\nu)|^2 \mu_N(d\nu) \\ + \lambda \mathrm{Var}\left(\int_{\mathbb{R}^m} e^{2i\pi\langle \nu, s\rangle} Z(ds)\right) d\nu.$$

Multivariate point process

N_1 and N_2 are WSS and moreover *jointly* WSS, that is if

$$E\left[N_1(A+t)N_2(B+t)\right] = E\left[N_1(A)N_2(B)\right].$$

One says that N_1 and N_2 admit the *cross-spectral measure* μ_{N_1,N_2}, sigma-finite signed, if for all $\varphi_1 \in \mathcal{B}_{N_1}$, $\varphi_2 \in \mathcal{B}_{N_2}$

$$\begin{aligned} &\text{cov}\ (N_1(\varphi_1), N_2(\varphi_2)) \\ &= \int_{\mathbb{R}^m} \widehat{\varphi_1}(\nu)\widehat{\varphi_2}(\nu)^*\,\mu_{N_1,N_2}(d\nu). \end{aligned}$$

Bivariate WSS Cox processes. Let N_1 and N_2 be WSS Cox processes with stochastic intensities $\{\lambda_1(t)\}_{t\in\mathbb{R}}$ and $\{\lambda_2(t)\}_{t\in\mathbb{R}}$, jointly stationary WSS stochastic processes with cross-spectral measure $\mu_{\lambda_1,\lambda_2}$.

$$\mu_{N_1,N_2} = \mu_{\lambda_1,\lambda_2}.$$

Cross-spectrum of a point process and its jittered version.

$$\mathrm{cov}\left(\sum_{n\geq 1}\varphi(X_n)\,,\,\sum_{n\geq 1}\psi(X_n+Z_n)\right)$$
$$=\int_{\mathbb{R}^m}\widehat{\varphi}(\nu)E\left[\widehat{\psi}(\nu+Z)^*\right]\mu_N(d\nu).$$

But

$$\begin{aligned}\widehat{\psi}(\nu+Z) &= \int_{\mathbb{R}^m}\psi(t+Z)\,e^{-2i\pi\nu t}\,dt \\ &= \int_{\mathbb{R}^m}\psi(t)\,e^{-2i\pi\nu(t-Z)}\,dt \\ &= \widehat{\psi}(\nu)E\left[e^{+2i\pi\nu Z}\right]\end{aligned}$$

where the expectation is with respect to Z a random variable with the common probability distribution of the marks.

Finally

$$\mathrm{cov}\left(\sum_{n\in\mathbb{Z}}\varphi(X_n),\sum_{n\in\mathbb{Z}}\psi(X_n+Z_n)\right)$$
$$=\int_{\mathbb{R}^m}\widehat{\varphi}(\nu)\widehat{\psi}(\nu)^* E\left[e^{-2i\pi\nu Z}\right]\mu_N(d\nu)\,,$$

and therefore

$$\mu_{N_1,N_2}(d\nu)=E\left[e^{-2i\pi\nu Z}\right]\mu_N(d\nu)\,.$$

Random sampling

The sampler: A WSS point process on $\mathbb{R}^m$ with intensity λ, point sequence $\{V_n\}_{n\geq 1}$.
The sampled process is WSS

$$X(t) = \int_{\mathbb{R}^m} e^{2i\pi\langle \nu,t\rangle} Z_X\,(d\nu) + m_X$$

The sampled process and the sampler are independent.
The sample brush

$$Y\,(t) = \sum_{n\geq 1} X(V_n)\delta(t - V_n)$$

is identified with the signed measure

$$\sum_{n\geq 1} X(V_n)\varepsilon_{V_n}\,.$$

.

The extended spectral measure of the sample brush: A locally finite measure μ_Y such that, for any $\varphi \in B_Y$,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)\right)$$
$$= \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) ,$$

where $\mathcal{B}_Y$ is a large enough vector space of functions, here also called the "*test functions*".

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \varphi(t)\, Y(t)\, dt \\
&= \int_{\mathbb{R}^m} \varphi(t) \left(\sum_{n\geq 1} X(V_n)\, \delta(t - V_n) \right) dt \\
&= \sum_{n\geq 1} \varphi(V_n)\, X(V_n) = \int_{\mathbb{R}^m} \varphi(t)\, X(t) N(dt) \ ,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Var}\ \left(\int_{\mathbb{R}^m} \varphi(t)\, Y(t)\, dt \right) \\
&= \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2\, \mu_Y(d\nu)\ .
\end{aligned}
$$

$$\mu_Y = \mu_N * \mu_X + \lambda^2 \mu_X + |m_X|^2 \mu_N .$$

If $\mathcal{B}_N$ is stable with respect to multiplications by complex exponential functions, we can take for test function space $\mathcal{B}_Y = \mathcal{B}_N$.
To be compared with that giving the spectral measure μ_Y of the product of two independent WSS stochastic processes, $Y(t) = Z(t)X(t)$:
$\mu_Y = \mu_Z * \mu_X$.)

Examples

Cox sampling.

$$\mu_Y = \mu_\lambda * \mu_X + \lambda^2 \mu_X + |m_X|^2 \mu_\lambda + \lambda \left(\sigma_X^2 + |m_X|^2 \right) \ell^m$$

where ℓ^m is the Lebesgue measure.

$$\mathcal{B}_N = L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m) = \mathcal{B}_Y .$$

Examples

Regular sampling.

$$f_Y(\nu) = \left(\frac{1}{T}\right)^2 \sum_{n \in \mathbb{Z}} f_X\left(\nu - \frac{n}{T}\right) .$$

The spectral density can recovered from that of the sample comb provided the former is band-limited, with band width $2B < \frac{1}{T}$.

Examples

Poisson sampling.

$$f_Y(\nu) = \lambda^2 f_X(\nu) + \lambda\sigma_X^2 \,.$$

Whatever the sampling frequency $\nu_s = \lambda$, there is no aliasing.

Reconstruction

Approximate the sampled process by a filtered version of the sample comb:

$$\int_{\mathbb{R}^m} \varphi(t-s)\, Y(s)\, ds$$

reconstruction error:

$$\epsilon = E\left[\left|\int_{\mathbb{R}^m} \varphi(t-u)\, Y(u)\, du - X(t)\right|^2\right].$$

The reconstruction error is, when the sampled process is centered:

$$\epsilon = \int_{\mathbb{R}^m} \left|\lambda\widehat{\varphi}(\nu) - 1\right|^2 \mu_X(d\nu) + \lambda \int_{\mathbb{R}^m} \left|\widehat{\varphi}(\nu)\right|^2 (\mu_X * \mu_\lambda)(d\nu).$$

Denoting by S the support (assumed of Lebesgue measure $2B < \infty$) of the spectral measure μ_X,

$$\widehat{\varphi}(\nu) = \frac{1}{\lambda} 1_S(\nu).$$

Examples

Poisson sampling, bad news

$$\epsilon = \sigma_X^2 \frac{2B}{\lambda}.$$

Therefore, sampling at the "Nyquist rate" $\lambda = 2B$ gives a very poor performance, not better than the estimate based on no observation at all.

Examples

Regular sampling

$$\epsilon = \int_{\mathbb{R}} \left| \frac{1}{T} \widehat{\varphi}(\nu) - 1 \right|^2 \mu_X(d\nu) + \frac{1}{T} \int_{\mathbb{R}} \left| \widehat{\varphi}(\nu) - 1 \right|^2 d\nu$$

In the band-limited case, $T = 1/2B$ (that is, $\lambda = 2B$) the error is null. Therefore, the process is perfectly reconstructed by

$$X(t) = \int_{\mathbb{R}} \varphi(t-s) X(s) N(ds) = \sum_{n \in \mathbb{Z}} X(T_n) \operatorname{sinc}(2B(t - T_n)),$$

Examples

Effects of jitter in Nyquist sampling

$$\epsilon = \frac{1}{2B}\left(\int_{-B}^{B}\sigma_X^2\left(1-\left(|\psi_Z|^2 * \widetilde{f}_X\right)(\nu)\right)d\nu\right),$$

where $\widetilde{f}_X$ is the normalized power spectral density of the process $X(t)$.

THE END
(for the time being)