Branching type processes with Stationary Ergodic Immigration

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Dec 2008

1 Background

- Most queueing Theory is Markovian

- Some results are insensitive to correlations, only depend on the the first moment. Example: MG1 PS queue.

- **Objective:** Develop tools for handling non Markovian queues.

- Examples of tools: Stochastic linear difference equations, branching processes.

Background on Branching

•19th centuty: concern among Victorians about possible extinction of aristocratic surnames.

•Galton posed this question in the *Educational Times* of 1873. The Reverand Watson replied with a solution. Joint publication of the solution in 1874.

•The G-W process: $X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)}$.

•The G-W process with immigratioon: $X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)} + B_n$.

Example 1: discrete branching with migration

Queue with Vacations, Gated Regime

• $M/G/1/\infty$ queue,

• Arrival rate λ, i.i.d. service times $\{D_n\}$ with first and second moments d, $d^{(2)}$.

• Sequence of vacations: V_n. Will be assumed stationary ergodic, with first and second moments v, $v^{(2)}$.

• Gated regime: at the nth end of vacation, a gate is closed (nth polling instant). Then the server goes on serving the customers present at the queue at that polling instant:
Then the server leaves on vacation.

• We denote:

- B_n:= the number of arrivals during the nth vacation.
- $\xi_h^{(i)}$:= the number of arrivals during the service time of a customer

• Then:

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_n^{(i)} + B_n, \qquad n \geq n_0.$$

Denote

$$A_n(x) = \sum_{i=1}^{x} \xi_n^{(i)}$$

Then A_n are nonnegative and divisible:

$$A_n(x+y) = A_n^{(1)}(x) + A_n^{(2)}(y)$$

where $A_n^{(i)}$ are i.i.d.

Example 2: continuous branching with migration

Queue with Vacations, Gated Regime

• Define the time to serve N customers as:

$$\tau(N) := \sum_{i=1}^{N} D_i$$

• Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T.

• Denote by $\hat{\mathcal{A}}_n(C_n) = \tau(\mathcal{N}(C_n))$, i.e. the sum of service times of all the arrivals during C_n.

• We obtain

$$C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}. \qquad (1)$$

Example 3: multitype discrete branching

Discrete time infinite server queue

•Service times are considered to be i.i.d. and independent of the arrival process.

•We represent the service time as the discrete time analogous of a phase type distribution: there are N possible service phases.

•The initial phase k is chosen at random according to some probability $p(k)$.

•If at the beginning of slot n a customer is in a service phase i then it will move at the end of the slot to a service phase j with probability P_{ij}.

•With probability $1 - \sum_{j=1}^{N} P_{ij}$ it ends service and leaves the system at the end of the time slot.

•P is a sub-stochastic matrix (it has nonnegative elements and it's largest eigenvalue is strictly smaller than 1), which means that services ends in finite time w.p.1. and that $(I - P)$ is invertible.

• Let $\xi^{(k)}(n)$, $k = 1, 2, 3, ...$, $n = 1, 2, 3, ...$ be i.i.d. random matrices of size $N \times N$. Each of its element can take values of 0 or 1, and the elements are all independent.

• The ijth element of $\xi^{(k)}(n)$ has the interpretation of the indicator that equals one if at time n, the kth customer among those present at service phase i moved to phase j.

• Obviously, $E[\xi_{ij}^{(k)}(n)] = P_{ij}$.

• Let $B_n = (B_n^1, ..., B_n^N)^T$ be a column vector for each integer n, where B_n^i is the number of arrivals at the nth time slot that start their service at phase i.

• B_n is a stationary ergodic sequence and has finite expectation.

• Y_n^i:= number of customers in phase i at time n. Satisfies

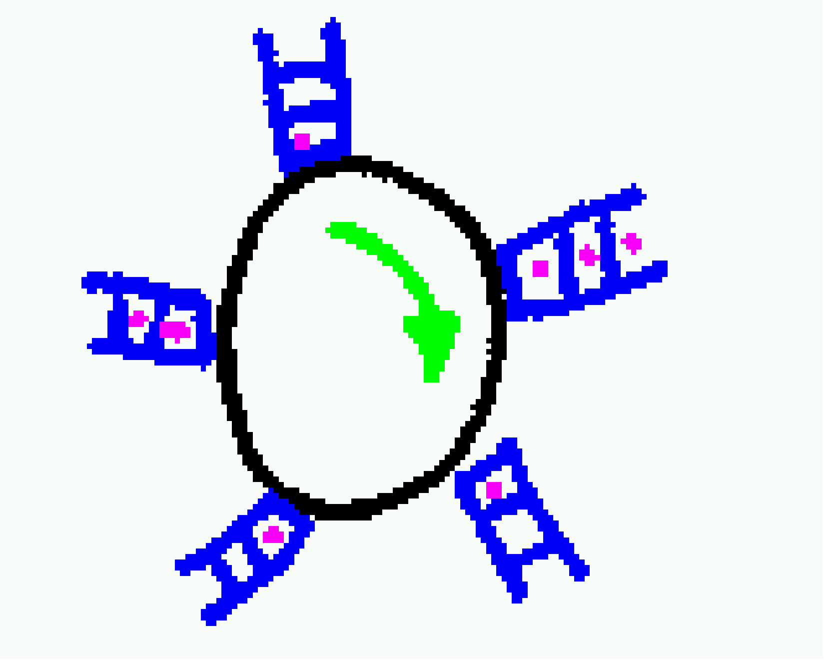
$$Y_{n+1} = A_n(Y_n) + B_n$$

where the ith element of the column vector $A_n(Y_n)$ is given by

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n) \quad (2)$$

Example 4: Polling systems with N queues are special cases!

•The server moves cyclically (fixed order) between the queues $1, ..., M$.
It requires walking times (vacations) for moving from one queue to another.



•Upon arrival at a queue, some customers are served. The number to be served is determined by the "polling regime":

Globally Gated (GG) regime (Boxma, Levy, Yechiali 1992):
The cycle time satisfies a **one dimensional recursion**.
We obtained the first two moments of the cycle and the expected waiting times at all queues.

Gated and Exhaustive regimes [see e.g. book by Takagi 1986]:
satisfy M-dimensional recursive equations.

No explicit expression for 2nd moments of buffer occupancy or cycle times.

No explicit expression for the expected waiting times.

2 Introduction and Background on Lévy fields

Introduction

•Consider the stochastic recursive equation:

$$Y_{n+1} = A_n(Y_n) + B_n, \qquad n \geq n_0. \tag{3}$$

•Y_n is a vector in R_+^m

•$\{A_n\}_n$ are

- i.i.d., independent of B_n.
- Increasing in the arg for all n.
- nonnegative **Additive Lévy field taking values in** $\mathbb{R}_+^m$

•$\{B_n\}$ stationary ergodic taking values in $\mathbb{R}_+^m$

(3) defines a **Continuous Multitype Branching Process (BP) with Migration**

Background: Lévy processes

Lévy process taking values in $\mathbb{R}_+$:

- Example: Poisson Point Process with intensity λ,
- **Expectation** and **variance** are linear: $E[A(t)] = t\mathcal{A}$ and cov$[A(t)] = t\Gamma$.
- For random time τ independent of A,

$$E[A(\tau)] = E[\tau]\mathcal{A}\,, \qquad \mathrm{var}[A(\tau)] = \mathrm{E}[\tau]\Gamma + \mathrm{var}[\tau]\mathcal{A}^2\,,$$

- **Divisibility:** $A(\cdot)$ is divisible if the following holds.
  For any k, there exist $A^{(i)}(\cdot)$, $i = 0, ..., k$ such that for any non-negative $x_i, i = 0, ..., k$,

$$A\left(\sum_{i=0}^{k} x_i\right) = \sum_{i=0}^{k} A^{(i)}\left(x_i\right) \tag{4}$$

  where $\{A^{(i)}(\cdot)\}_{i=0,1,2,...,k}$ are i.i.d. with the same distribution as $A(\cdot)$.

Lévy process taking values in $\mathbb{R}_+^m$ (subordinators):

- Example: Poisson arrival process where the nth arrival brings a batch $B_n = (B_n^1, ..., B_n^m)$. B_n^i customers go to queue i.
- For $A(t)$ in $\mathbb{R}_+^m$, $E[A(t)] = \mathcal{A}t$ where $\mathcal{A}$ is of dimension m.
- $cov[A(t)] = \Gamma t$, where Γ is a matrix of dimension $m \times m$.

Example of Random fields

Random field taking values in $\mathbb{R}_+$

- Example: Black and white picture.
- The level of grey is a function of two parameters: x and y.

Random field taking values in $\mathbb{R}_+^d$

- Example: color picture.
- The level of the green, red and blue as a function of the location x and y.

Background: Additive Lévy Fields

Let $A^{(1)}, ..., A^{(d)}$ be d indep. Lévy proc. on $\mathbb{R}^m$ with scalar "time" parameters.

Additive Lévy field: $A(y) = A^{(1)}(y_1) + ... + A^{(d)}(y_d)\,, \quad \forall y = (y_1, ..., y_d) \in \mathbb{R}_+^d$.

The expectation: $\mathrm{E}[A(y)] = \sum_{j=1}^d y^j \mathcal{A}^{(j)} = \mathcal{A}y\,,$
$\mathcal{A}$ is a matrix whose jth column equals $\mathcal{A}^{(j)}$,
$\mathcal{A}^{(j)} = \mathrm{E}[A^{(j)}(1)]$,

The covariance matrix: $\mathrm{cov}[A(y)] = \sum_{j=1}^d y_j \Gamma^{(j)}\,,$
where $\Gamma^{(j)} = \mathrm{cov}[A^{(j)}(1)]$ is the corresponding covariance matrix of $A^{(j)}(1)$.

Composition: If A_n and A_{n+1} are Additive Lévy processes in $\mathbb{R}_+^m$ then their composition is also an Additive Lévy process.

Properties of Lévy Fields

- Expectation and Covariance are linear in y,
- Let τ be a non-negative random variable in $\mathbb{R}_+^d$, independent of A and represented as a column vector. Then

$$E[A(\tau)] = \sum_{j=1}^{m} \mathcal{A}^{(j)} E[\tau_j] \,,$$

and,

$$\mathrm{cov}[A(\tau)] = \sum_{j=1}^{d} \mathrm{E}[\tau_j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[\tau]\mathcal{A}^T \,, \qquad (5)$$

where τ_j is the jth entry of the vector τ.

Result 1: Steady State Probabilities of CBP

Iterating $Y_{n+1} = A_n(Y_n) + B_n$, we obtain from A1:

$$\begin{aligned} Y_2 &= A_1(Y_1) + B_1 \\ &= A_1(A_0(Y_0) + B_0) + B_1 \\ &= A_1^{(0)}(A_0(Y_0)) + A_1^{(1)}(B_0) + B_1 \\ &= A_1^{(0)}A_0^{(0)}(Y_0) + A_1^{(1)}(B_0) + B_1. \end{aligned}$$

$$\begin{aligned} Y_3 &= A_2(Y_2) + B_2 \\ &= A_2(A_1(Y_1) + B_1) + B_2 \\ &= A_2(A_1(A_0(Y_0) + B_0) + B_1) + B_2 \\ &= A_2^{(0)}A_1^{(0)}A_0^{(0)}(Y_0) + A_2^{(1)}A_1^{(1)}(B_0) + A_2^{(2)}(B_1) + B_2 \end{aligned}$$

In general:

$$Y_n = \sum_{j=0}^{n-1}\left(\prod_{i=n-j}^{n-1} A_i^{(n-j)}\right)(B_{n-j-1}) + \left(\prod_{i=0}^{n-1} A_i^{(0)}\right)(Y_0), \quad n > 0 \tag{6}$$

(we understand $\prod_{i=n}^{k} A_i(x) = x$ whenever $k < n$, and $\prod_{i=n}^{k} A_i(x) = A_k A_{k-1} \ldots A_n$ whenever $k > n$).

• Under fairly general assumptions, $\lim_{n\to\infty}\left(\prod_{i=0}^{n-1} A_i^{(0)}\right)(y) = 0$, so Y_n has a limit as $n \to \infty$ distributed like

$$Y_n^* =_d \sum_{j=0}^{\infty}\left(\prod_{i=n-j}^{n-1} A_i^{(n-j)}\right)(B_{n-j-1}), \qquad n \in Z, \tag{7}$$

where for each integer i, $\{A_i^{(j)}(\cdot)\}_j$ have the same distribution as $A_i(\cdot)$.

• Sufficient condition: stationarity plus $\|\mathcal{A}\| < 1$.

• Branching processes: $\{A_i^{(j)}(\cdot)\}_j$ are i.i.d.

• Stochastic differential equations: they are equal.

• The representation holds for general dependence: Semi linear processes.

Application: Expected waiting time for a gated queue with vacations

Consider an arbitrary customer. Upon arrival, it has to wait for

1. The residual cycle time C_{res},
2. The service time of all the customers that arrived during C_{past} which is the past cycle time: $d(\lambda E[C_{past}]) = \rho E[C_{past}]$

We have from [Baccelli & Brémaud, 1994]

$$E[C_{res}] = E[C_{past}] = \frac{E[C_0^2]}{2E[C_0]}.$$

Thus the expected waiting time of an arbitrary customer is given by

$$E[W_n] = (1+\rho)\frac{E[C_0^2]}{2E[C_0]},$$

The expected number of customers in queue in stationary regime (not including service) is obtained using Little's Theorem: $\lambda E[W_n]$.

Conclusion: we need to compute $E[C_0]$ and $E[C_0^2]$!

Computing $E[C_0]$ and $E[C_0^2]$

- Dynamics: $C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}$.

- $\hat{\mathcal{A}}_n(c)$ is the workload that arrives during duration $[0, c)$.

- Introduce the correlation function: $r(n) = E[V_0 V_n]$.

- The first and second moments of C_n in stationary regime are given by

$$E[C_n] = \frac{v}{1-\rho},$$

$$E[C_n^2] = \frac{1}{(1-\rho^2)} \left(\frac{\lambda v d^{(2)}}{1-\rho} + r(0) + 2 \sum_{j=1}^{\infty} \rho^j r(j) \right). \quad (8)$$

Proof of expressions for $E[C_0^2]$

Useful relations: 2nd moment of workload arriving during T

• If N is a random variable independent of the sequence D_n, and $\tau(N) := \sum_{i=1}^{N} D_i$ then

$$E[\tau(N)^2] = E[N^2]d^2 + E[N](d^{(2)} - d^2). \tag{9}$$

• Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T. Then

$$E[\mathcal{N}(T)^2] = \lambda^2 E[T^2] + \lambda E[T]. \tag{10}$$

• If we take an arbitrary T and choose $N = \mathcal{N}(T)$, then we get from (9)-(10)

$$\begin{aligned} E[(\hat{\mathcal{A}}(T))^2] &= E[\tau(\mathcal{N}(T))^2] \\ &= d^2(\lambda^2 E[T^2] + \lambda E[T]) + \lambda E[T](d^{(2)} - d^2) \\ &= d^2\lambda^2 E[T^2] + \lambda E[T] d^{(2)}. \end{aligned} \tag{11}$$

• Also, if we take $T = \tau(N)$, then

$$E[\mathcal{N}(\tau(N))]^2 = \lambda^2 \Big[E[N^2]d^2 + E[N](d^{(2)} - d^2) \Big] + \lambda d E[N]. \tag{12}$$

- From $C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}$ we have

$$\begin{aligned} E[C_{n+1}^2] &= E[\hat{\mathcal{A}}_n(C_n)^2] + v^{(2)} + 2E[\hat{\mathcal{A}}_n(C_n)V_{n+1}] \\ &= \left(\rho^2 E[C_n^2] + \lambda E[C_n] d^{(2)}\right) + v^{(2)} + 2E[\hat{\mathcal{A}}_n(C_n)V_{n+1}]. \end{aligned}$$

- To compute the last term, we now use the explicit form of C_0:

$$C_0 = \sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} \hat{\mathcal{A}}_i^{(-j)} \right) (V_{-j}).$$

- We use the fact that the processes $\{\hat{\mathcal{A}}_i^{(j)}\}$ are independent of $\{V_n\}$. We get:

$$\begin{aligned} E[\hat{\mathcal{A}}_n(C_n)V_{n+1}] &= E[\hat{\mathcal{A}}_0(C_0)V_1] = E\left[\hat{\mathcal{A}}_0 \left(\sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} \hat{\mathcal{A}}_i^{(-j)} \right) (V_{-j}) \right) V_1 \right] \\ &= \rho \sum_{j=0}^{\infty} \rho^j E[V_{-j} V_1] = \sum_{j=1}^{\infty} \rho^j r(j). \end{aligned}$$

Substituting this, we obtain the second moment.

3 2nd order moments in continuous B.P.

Joint work with Dieter Fiems

Notation: •Auto-correlations: $\mathcal{B}(k) =_{def} \mathrm{E}[B_0(B_k)^T]$, where k is an integer
•$\hat{\mathcal{B}}(k) =_{def} \mathcal{B}(k) - \mathrm{E}[B_0]\,\mathrm{E}[B_0]^T$. (Note: $\hat{\mathcal{B}}(0)$ equals $\mathrm{cov}[B_0]$.)

Assumptions: Consider $Y_{n+1} = A_n(Y_n) + B_n,\ n \geq n_0$, where

- A_n are i.i.d. additive Lévy fields,
- A_n independent of $\{B_n\}$,
- $\{B_n\}$ are stationary ergodic,
- All eigenvalues of $\mathcal{A}$ are within the unit disk,
- the elements of B_0 have finite second order moments.

Theorem: Consider $Y_{n+1} = A_n(Y_n) + B_n$ in stationary regime. Then

(i) $\mathrm{E}[Y_0] = (\mathcal{I} - \mathcal{A})^{-1}\,\mathrm{E}[B_0]$,

(ii) $\mathrm{cov}(Y_0)$ is the unique solution of the linear equations:

$$\mathrm{cov}[Y_0] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T + \mathrm{cov}[B_0] + \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + (\mathcal{A}^j \hat{\mathcal{B}}(j))^T\,, \tag{13}$$

where $\mathrm{E}[Y_0^j]$ denotes the jth element of $\mathrm{E}[Y_0]$.

Proof for first moments:

Taking expectation in $Y_{n+1} = A_n(Y_n) + B_n$ we get

$$\mathrm{E}[Y_0] = \mathcal{A}\,\mathrm{E}[Y_0] + \mathrm{E}[B_0],$$

Since the eigenvalues of $\mathcal{A}$ are within the unit disk, $(\mathcal{I} - \mathcal{A})$ is inverible.
Hence we obtain (i).

Proof of uniqueness for the second moments

- Let Z_1 and Z_2 be two solutions of

$$\mathrm{cov}[Y_0] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T + \mathrm{cov}[B_0] + \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + (\mathcal{A}^j \hat{\mathcal{B}}(j))^T \,.$$

- Define $Z = Z_1 - Z_2$. Then Z satisfies $Z = \mathcal{A}^T Z \mathcal{A}$.

- Iterating, we obtain,

$$Z = \lim_{n\to\infty} \mathcal{A}^n Z (\mathcal{A}^T)^n = 0$$

where the last equality follows from the fact that all the eigenvalues of $\mathcal{A}$ are within the unit disk.

- This implies uniqueness.

Proof for expression of second moments

•Consider $Y_{n+1} = A_n(Y_n) + B_n$.

- Multiply both sides by their transpose,
- take expectation and
- use the stationarity

we get:

$$\mathrm{E}[Y_0 Y_0^T] = \mathrm{E}[A_0(Y_0)A_0^T(Y_0)] + \mathrm{E}[B_0 B_0^T] + \mathrm{E}[A_0(Y_0)B_0^T] + \mathrm{E}[B_0 A_0^T(Y_0)]\,.$$

The covariance matrix $\mathrm{cov}[Y_0]$ therefore equals,

$$\begin{aligned} \mathrm{cov}[Y_0] \quad &= \quad \mathrm{cov}[A_0(Y_0)] + \mathrm{cov}[B_0] + \mathrm{E}\left[A_0(Y_0)B_0^T\right] \\ &\quad -\mathcal{A}\,\mathrm{E}[Y_0]\,\mathrm{E}[B_0]^T + \mathrm{E}\left[B_0 A_0(Y_0)^T\right] - \mathrm{E}[B_0](\mathcal{A}\,\mathrm{E}[Y_0])^T\,. \end{aligned} \tag{14}$$

It remains to compute the red and the blue expressions.

Red Expression: Using the convariance expression (5) of Additive Lévy processes at random "time":

$$\mathrm{cov}[A_0(Y_0)] = \sum_{j=1}^{m} \mathrm{E}[Y_0^j]\Gamma^{(j)} + \mathcal{A}\,\mathrm{cov}[Y_0]\mathcal{A}^T\,. \tag{15}$$

Blue Expression: We use the explicit expression (7) for the stationary state process to obtain

$$\begin{aligned}
\mathrm{E}[Y_0 B_0^T] &= \sum_{j=0}^{\infty} \mathrm{E}\left\{ \bigotimes_{i=-j}^{-1} A_{-j,i}(B_{-j-1}) B_0^T \right\} \\
&= \sum_{j=0}^{\infty} \mathrm{E}\left(\mathrm{E}\left\{ \bigotimes_{i=-j}^{-1} A_{-j,i}(B_{-j-1}) B_0^T \right\} \middle| \mathbf{B_0^-} \right) \\
&= \sum_{j=0}^{\infty} \mathrm{E}\left(\mathcal{A}^j B_{-j-1} B_0^T \right) = \sum_{j=0}^{\infty} \mathcal{A}^j \mathcal{B}(j+1)\,,
\end{aligned} \tag{16}$$

with $\mathbf{B_0^-} := (B_0, B_{-1}, B_{-2}, ...)$.

Substituting the last expression, we compute,

$$\mathrm{E}[A_0(Y_0)B_0^T] = \mathrm{E}\left[\mathrm{E}\left[A_0(Y_0)B_0^T \,\middle|\, Y_0, B_0\right]\right] = \mathcal{A}\,\mathrm{E}\left[Y_0 B_0^T\right] = \sum_{j=1}^{\infty} \mathcal{A}^j \mathcal{B}(j)\,,$$

or equivalently,

$$\begin{aligned} \mathrm{E}[A_0(Y_0)B_0^T] &= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \sum_{j=1}^{\infty} \mathcal{A}^j\,\mathrm{E}[B_0]\,\mathrm{E}[B_0]^T \\ &= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \mathcal{A}(\mathcal{I} - \mathcal{A})^{-1}\,\mathrm{E}[B_0]^T \\ &= \sum_{j=1}^{\infty} \mathcal{A}^j \hat{\mathcal{B}}(j) + \mathcal{A}\,\mathrm{E}[Y_0]\,\mathrm{E}[B_0]^T\,. \end{aligned} \tag{17}$$

Substitution of expressions RED and BLUE provides the covariance equation.

4 Symmetric gated polling systems

m gated queues.

Arrivals:

- Arrival processes $\rho^i(t)$ to queue i are i.i.d. Levy processes, distributed as some $\rho(t)$, $t \in \mathbb{R}_+$.
- $\overline{\rho} = \mathrm{E}[\rho(1)]$ and $\sigma^2 = \mathrm{var}[\rho(1)]$

Walking times:

- $\{V_n\}$: Stationary ergodic series of walking times, $v := \mathrm{E}[V_0]$.
- $\mathcal{V}(j) := \mathrm{E}[V_0 V_j]$ for some integer j and $\hat{\mathcal{V}}(j) := \mathrm{E}[V_0 V_j] - v^2$.

Notation:

- $I(n)$:= the queue visited at the nth polling instant
- $S(n)$:= nth polling instant (time at which the server arrives at the nth queue)
- $Y_n^i := S(n) - S(n-i), \quad (i = 1, 2, ..., m)$ is the time between the $(n-i)$th and the nth polling instant.
- In particular, Y_n^m is the duration of the nth cycle.
- Let ρ_n^i be i.i.d. copies of the process ρ^i, $n = 1, 2, 3, ...$

The dynamics:

$$
\begin{aligned}
Y_{n+1}^1 &= S(n+1) - S(n) = \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^2 &= S(n+1) - S(n-1) = Y_n^1 + \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n \,, \\
&\vdots \\
Y_{n+1}^m &= S(n+1) - S(n-m+1) = Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n \,.
\end{aligned}
\tag{18}
$$

•(18) states that the time between $S(n)$ and $S(n+1)$ is the sum of the busy period at queue $I(n)$ plus the nth vacation time;

•The busy period = the workload that arrived at queue $I(n)$ during the nth cycle.

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Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n \,,\\
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$$

•(18) states that the time between $S(n)$ and $S(n+1)$ is the sum of the busy period at queue $I(n)$ plus **the nth vacation time;**

•The busy period = the workload that arrived at queue $I(n)$ during the nth cycle.

Notation:

- $I(n)$:= the queue visited at the nth polling instant
- $S(n)$:= nth polling instant (time at which the server arrives at the nth queue)
- $Y_n^i := S(n) - S(n-i), \quad (i = 1, 2, ..., m)$ is the time between the $(n-i)$th and the nth polling instant.
- In particular, Y_n^m is the duration of the nth cycle.
- Let ρ_n^i be i.i.d. copies of the process ρ^i, $n = 1, 2, 3, ...$.

The dynamics:

$$\begin{aligned} Y_{n+1}^1 &= S(n+1) - S(n) = \boldsymbol{\rho_n^m(Y_n^m)} + V_n \,, \\ Y_{n+1}^2 &= S(n+1) - S(n-1) = Y_n^1 + \rho_n^m(Y_n^m) + V_n \,, \\ Y_{n+1}^3 &= S(n+1) - S(n-2) = Y_n^2 + \rho_n^m(Y_n^m) + V_n \,, \\ &\vdots \\ Y_{n+1}^m &= S(n+1) - S(n-m+1) = Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n \,. \end{aligned} \tag{18}$$

•(18) states that the time between $S(n)$ and $S(n+1)$ is the sum of the busy period at queue $I(n)$ plus the nth vacation time;

•The busy period = **workload that arrived at queue $I(n)$ during the nth cycle.**

Interpretation of the other equations:

For $i > 0$, we have

$$Y_{n+1}^{i+1} = S(n+1) - S(n-i) = S(n+1) - S(n) + S(n) - S(n-i)$$

where

- by definition, $S(n) - S(n-i) = Y_n^i$, and
- $S(n+1) - S(n) = \rho_n^m(Y_n^m) + V_n$ (see previous slide).

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n \,, \text{ with}$$

$$\begin{aligned}
Y_{n+1}^1 &= S(n+1) - S(n) = & \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^2 &= S(n+1) - S(n-1) = & Y_n^1 + \rho_n^m(Y_n^m) + V_n \,, \\
Y_{n+1}^3 &= S(n+1) - S(n-2) = & Y_n^2 + \rho_n^m(Y_n^m) + V_n \,, \\
&\vdots & \\
Y_{n+1}^m &= S(n+1) - S(n-m+1) = & Y_n^{m-1} + \rho_n^m(Y_n^m) + V_n \,.
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&\vdots &\\
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\end{aligned}$$

where $Y_{n+1} = (Y_{n+1}^1, ..., Y_{n+1}^m)^T$,

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n \,, \text{ with}$$

$$\begin{aligned}
Y^1_{n+1} &= S(n+1) - S(n) = & \rho^m_n(Y^m_n) + V_n \,, \\
Y^2_{n+1} &= S(n+1) - S(n-1) = & Y^1_n + \rho^m_n(Y^m_n) + V_n \,, \\
Y^3_{n+1} &= S(n+1) - S(n-2) = & Y^2_n + \rho^m_n(Y^m_n) + V_n \,, \\
&\vdots & \\
Y^m_{n+1} &= S(n+1) - S(n-m+1) = & Y^{m-1}_n + \rho^m_n(Y^m_n) + V_n \,.
\end{aligned}$$

where $B_n = V_n(1,1,1,...,1)^T$,

•in the special case that $\{B_n\}$ is i.i.d. Y_n is a Markov chain

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n\,, \text{ with}$$

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&\vdots & \\
Y^m_{n+1} &= S(n+1) - S(n-m+1) = & Y^{m-1}_n + \rho^m_n(Y^m_n) + V_n\,.
\end{aligned}$$

$$Y_{n+1} = Y^1_n \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + Y^2_n \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + Y^{m-1}_n \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + \rho^m_n(Y^m_n) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + B_n$$

Vector notation:

$$Y_{n+1} = A_n(Y_n) + B_n\,, \text{ where}$$

$$A_n(y) = A_n^{(1)}(y_1) + ... + A_n^{(m)}(y_m)\,, \tag{19}$$

where $y = (y_1, ..., y_m)^T \in \mathbb{R}_+^m$, $t \in \mathbb{R}_+$ and

$$\begin{aligned} A_n^{(1)}(t) &= (0, t, 0, 0, ..., 0)^T, \\ A_n^{(2)}(t) &= (0, 0, t, 0, ..., 0)^T, \\ &\vdots \\ A_n^{(m-1)}(t) &= (0, 0, 0, ..., 0, t)^T, \\ A_n^{(m)}(t) &= \rho_n^m(t)(1, 1, \ldots, 1)^T\,, \end{aligned} \tag{20}$$

•For each i, $A_n^{(i)}$ is a Lévy **process** taking values in $\mathbb{R}_+^m$.

•A_n are Additive Lévy fields

Checking the stability condnition

Taking expectation we get:

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 1 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 1 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 0 & 1 & 0 & \dots & 0 & \overline{\rho} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \overline{\rho} \\ 0 & 0 & 0 & 0 & \dots & 1 & \overline{\rho} \end{pmatrix}. \tag{21}$$

$\mathcal{A}$ is known as the *Companion matrix*.

Theorem: A sufficient and necessary condition for all eigenvalues of $\mathcal{A}$ to be in the interior of the unit circle is

$$\overline{\rho} < \frac{1}{m}.$$

Conclusions and Discussion

- We use neither the "buffer occupancy" nor the "station times" approaches.

- Advantage: one component of the state is the cycle time;
its two first moments provide the expected waiting time.

- A very similar structure is obtained in the exhaustive case.

5 Semi linear processes

We shall assume that A_n satisfy the following conditions:

A1: $A_n(y)$ has the following **divisibility property**: if for some k, $y = y^0 + y^1 + ... + y^k$ where y^m are vectors, then $A_n(y)$ can be represented as

$$A_n(y) = \sum_{i=0}^{k} \widehat{A}_n^{(i)}(y^i)$$

where $\{\widehat{A}_n^{(i)}\}_{i=0,1,2,...,k}$ are identically distributed with the same distribution as $A_n(\cdot)$.

A2: (i) There is some matrix $\mathcal{A}$ such that for every y,

$$E[A_n(y)] = \mathcal{A}y.$$

(ii) The correlation matrix of $A_n(y)$ is linear in yy^T and in y. We shall represent it as

$$E[A_n(y)A_n(y)^T] = F(yy^T) + \sum_{j=1}^{d} y_j \Gamma^{(j)}\,, \tag{22}$$

where F is a linear operator that maps $d \times d$ nonnegative definite matrices to other $d \times d$ nonnegative definite matrices and satisfies $F(0) = 0$.

Moments:

•(i) The first moment of X_n^* is given by

$$E[X_0^*] = (I - \mathcal{A})^{-1} b. \tag{23}$$

•(ii) Assume that the first and second moments b_i and $b_i^{(2)}$'s are finite and that F satisfies

$$\lim_{n \to \infty} F^n = 0. \tag{24}$$

Define Q to be the matrix whose ijth entry is $Q_{ij} = \sum_{k=1}^{d} \overline{y}_k \Gamma^{(k)}$. Then the matrix $cov(X^*)$ is the unique solution of the set of linear equations:

$$cov(X) \quad = \quad cov(B) + \sum_{r=1}^{\infty} \left(\mathcal{A}^r \widehat{\mathcal{B}}(r) + \left[\mathcal{A}^r \widehat{\mathcal{B}}(r) \right]^T \right) + F(cov[X]) + Q. \tag{25}$$

The second moment matrix $E[XX^T]$ in steady state is the unique solution of the set of linear equations:

$$E[XX^T] \quad = \quad E[B_0 B_0^T] + \sum_{r=1}^{\infty} \left(\mathcal{A}^r \mathcal{B}(r) + \left[\mathcal{A}^r \mathcal{B}(r) \right]^T \right) + F(E[XX^T]) + Q \tag{26}$$

6 Example: Discrete time infinite server queue

Example 5: Discrete time infinite server queue

- Service times are considered to be i.i.d. and independent of the arrival process.
- We represent the service time as the discrete time analogous of a phase type distribution: there are N possible service phases.
- The initial phase k is chosen at random according to some probability $p(k)$.
- If at the beginning of slot n a customer is in a service phase i then it will move at the end of the slot to a service phase j with probability P_{ij}.
- With probability $1 - \sum_{j=1}^{N} P_{ij}$ it ends service and leaves the system at the end of the time slot.
- P is a sub-stochastic matrix (it has nonnegative elements and it's largest eigenvalue is strictly smaller than 1), which means that services ends in finite time w.p.1. and that $(I - P)$ is invertible.

•Let $\xi^{(k)}(n)$, $k = 1, 2, 3, ...$, $n = 1, 2, 3, ...$ be i.i.d. random matrices of size $N \times N$. Each of its element can take values of 0 or 1, and the elements are all independent.

•The ijth element of $\xi^{(k)}(n)$ has the interpretation of the indicator that equals one if at time n, the kth customer among those present at service phase i moved to phase j.

•Obviously, $E[\xi_{ij}^{(k)}(n)] = P_{ij}$.

•Let $B_n = (B_n^1, ..., B_n^N)^T$ be a column vector for each integer n, where B_n^i is the number of arrivals at the nth time slot that start their service at phase i.

•B_n is a stationary ergodic sequence and has finite expectation.

•Y_n^i:= number of customers in phase i at time n. Satisfies

$$Y_{n+1} = A_n(Y_n) + B_n$$

where the ith element of the column vector $A_n(Y_n)$ is given by

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n) \tag{27}$$

- Numerical example: Service times are geometrically distributed,
- The SRE becomes one dimensional. Y_n denotes the number of customers in the system.
- $\xi_n^{(k)}$ is the indicator that the kth customer present at the beginning of time-slot n will still be there at the end of the time-slot.
- The probability that a customer in the system finishes its service within a time slot is precisely $p = 1 - \mathsf{A} = 1 - E[\xi_n]$.
- We consider a Markov chain with two states $\{\gamma, \delta\}$ with transition probabilities given by

$$\mathcal{P} = \begin{pmatrix} 1 - \epsilon p & \epsilon p \\ \epsilon q & 1 - \epsilon q \end{pmatrix}$$

•As an example, consider the following parameters: $p = q = 1$, at a given state there is at most one arrival with prob. $p_\gamma = 1, p_\delta = 0.5$. This gives:

$$var[Y^*] = \frac{1}{(1 - \mathsf{A}^2)} \left(\frac{3}{16} + \frac{2\mathsf{A}}{1 - \mathsf{A} + 2\epsilon\mathsf{A}} + \frac{3}{4}\mathsf{A} \right).$$

In Fig. 1 we plot the variance of the steady state number of customers, $var[Y^*]$, while varying ϵ and A.

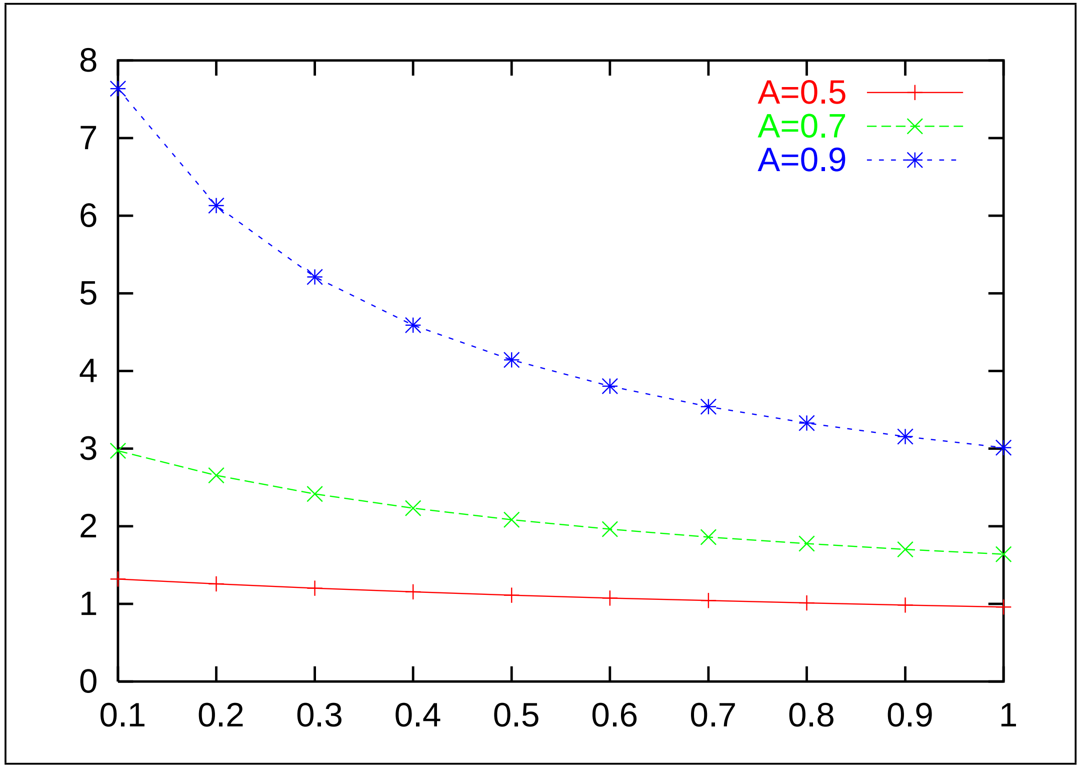

Figure 1: var$[Y*]$ as a function of ϵ and of A

7 Example: Delay Tolerant Ad-hoc Networks

- Delay tolerant Ad-hoc Networks make use of nodes' mobility to compensate for lack of instantaneous connectivity.

- Information sent by a source to a disconnected destination can be forwarded and relayed by other mobile nodes.

- Let X_n^+ be the number of nodes that have a copy of the packet at time n,

- Let X_n^- be the number of nodes that do not have a copy of the packet at time n.

- Mobility: a mobile present at time n may leave and other may arrive. Let B_n be the number of new arrivals.

- Let $\rho_n^{(i)}$ and $\hat{\rho}_n^{(i)}$ be the indicator that node i remains in the system for the next slot. ρ is used for nodes that have the packet and $\hat{\rho}$ for the others.

- Let $\xi_n^{(i)}$ be the indicator that the source meats mobile i at time slot n. These are i.i.d. Then

$$X_{n+1}^{+} = \sum_{i=1}^{X_n^+} \rho_n^{(i)} + \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^{-} = \sum_{i=1}^{X_n^-} \hat{\rho}_n^{(i)} (1 - \xi_n^{(i)}) + B_n$$

- Assume that the source limits the transmissions in order to save energy
- Let ζ_n be the indicator that the source intends to transmit a packet at time n. Assume ζ_n are i.i.d.

$$X_{n+1}^{+} = \sum_{i=1}^{X_n^{+}} \rho_n^{(i)} + \zeta_n \sum_{i=1}^{X_n^{-}} \hat{\rho}_n^{(i)} \xi_n^{(i)}$$

$$X_{n+1}^{-} = \sum_{i=1}^{X_n^{-}} \hat{\rho}_n^{(i)} (1 - \zeta_n \xi_n^{(i)}) + B_n$$

- This is a semi-linear process, not a branching process

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