

Automated Security Proofs with Sequences of Games

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Proofs of cryptographic protocols

There are two main frameworks for analyzing security protocols:

- The **Dolev-Yao model**: a formal, abstract model.

The cryptographic primitives are **ideal blackboxes**.

The adversary uses only those primitives.

Proofs can be done automatically.

- The **computational model**: a realistic model.

The cryptographic primitives are functions on bit-strings.

The adversary is a polynomial-time Turing machine.

Proofs are done manually.

Our goal: achieve **automatic provability** under the realistic **computational** assumptions.

We have implemented an **automatic prover** sound in the **computational model**:

- proves **secrecy** properties and that **events** can be executed only with negligible probability.
- handles various **cryptographic primitives**: MACs (message authentication codes), stream and block ciphers, public-key encryption, signatures, hash functions, ...
- works for **N sessions** with an **active adversary**.
- gives a bound on the **probability** of an attack (exact security).

As in Shoup's or Bellare and Rogaway's method, the proof is a **sequence of games**:

- In the first game, the adversary plays against the **real protocol**.
- One goes from one game to the next by syntactic transformations or by applying security assumptions on cryptographic primitives.

The difference of probability between consecutive games is bounded.

- The last game is **"ideal"**: the security property can be read directly on it.

(The advantage of the adversary is 0 for this game.)

Games are formalized in a process calculus.

Case study: Full Domain Hash signature scheme

$hash$ hash function (in the random oracle model)

$f(pk, m)$ one-way trapdoor permutation, with inverse $invf(sk, m)$

We define a **signature scheme** as follows:

- signature: $sign(m, sk) = invf(sk, hash(m))$
- verification: $verify(m, pk, s) = (f(pk, s) = hash(m))$

Our goal is to show that this signature scheme is UF-CMA (satisfies unforgeability under chosen message attacks).

Formalizing the security of a signature scheme (1)

Key generation oracle:

$$Ogen() := r \xleftarrow{R} \text{seed}; pk \leftarrow pkgen(r); sk \leftarrow skgen(r); \mathbf{return}(pk)$$

Chooses a random seed uniformly in the set of bit-strings $seed$ (consisting of all bit-strings of a certain length), generates a public key pk , a secret key sk , and returns the public key.

Formalizing the security of a signature scheme (2)

Signature oracle:

$$OS(m : \textit{bitstring}) := \mathbf{return}(sign(sk, m))$$

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In fact, this is an abbreviation for:

$$\mathbf{foreach} \ i_S \leq q_S \ \mathbf{do} \ OS[i_S](m[i_S] : \textit{bitstring}) := \mathbf{return}(sign(sk, m[i_S]))$$

The variables in repeated oracles are arrays, with one cell for each call, to remember the values used in each oracle call.

These arrays are indexed with the call number i_S .

Formalizing the security of a signature scheme (3)

Test oracle:

$OT(m' : \text{bitstring}, s : D) :=$ **if** $\text{verify}(m', pk, s)$ **then**
 find $u \leq q_S$ **suchthat** $(\text{defined}(m[u]) \wedge m' = m[u])$
 then end else event *forge*

If s is a signature for m' and the signed message m' is not contained in the array m of messages passed to signing oracle, then the signature is a **forgery**, so we execute **event** *forge*.

Formalizing the security of a signature scheme (summary)

The signature and test oracles make sense only **after** the key generation oracle has been called, hence a **sequential composition**.

The signature and test oracles are **simultaneously** available, hence a **parallel composition**.

```
Ogen() := r  $\stackrel{R}{\leftarrow}$  seed; pk  $\leftarrow$  pkgen(r); sk  $\leftarrow$  skgen(r); return(pk);  
(foreach  $i_S \leq q_S$  do OS( $m : \text{bitstring}$ ) := return(sign(sk, m))  
| OT( $m' : \text{bitstring}, s : D$ ) := if verify( $m', pk, s$ ) then  
  find  $u \leq q_S$  suchthat (defined( $m[u]$ )  $\wedge m' = m[u]$ )  
  then end else event forge)
```

The probability of executing **event** *forge* in this game is the probability of forging a signature.

Application to the FDH signature scheme

We add a hash oracle because the adversary must be able to call the random oracle (even though it cannot be implemented).

```
foreach  $i_H \leq q_H$  do  $OH(x : \text{bitstring}) := \text{return}(\text{hash}(x))$   
|  $Ogen() := r \xleftarrow{R} \text{seed}; pk \leftarrow pkgen(r); sk \leftarrow skgen(r); \text{return}(pk);$   
  (foreach  $i_S \leq q_S$  do  $OS(m : \text{bitstring}) := \text{return}(\text{invf}(sk, \text{hash}(m)))$ )  
|  $OT(m' : \text{bitstring}, s : D) := \text{if } f(pk, s) = \text{hash}(m') \text{ then}$   
  find  $u \leq q_S$  suchthat ( $\text{defined}(m[u]) \wedge m' = m[u]$ )  
  then end else event forge)
```

Our goal is to bound the probability that event *forge* is executed in this game.

This game is given as input to the prover in the syntax above.

Indistinguishability as observational equivalence

Two processes (games) Q_1, Q_2 are **observationally equivalent** up to probability p when an adversary running in time t has probability at most $p(t)$ of distinguishing them:

$$Q_1 \approx_p Q_2$$

The adversary is represented by an acceptable **evaluation context** C (essentially a process put in parallel with the considered games).

- Observational equivalence is reflexive and symmetric.
- If $Q_1 \approx_p Q_2$ and $Q_2 \approx_{p'} Q_3$ then $Q_1 \approx_{p+p'} Q_3$.
- It is **contextual**: $Q_1 \approx_p Q_2$ implies $C[Q_1] \approx_{p'} C[Q_2]$ where C is any acceptable evaluation context running in time t_C and $p'(t) = p(t + t_C)$.

We transform a game G_0 into an observationally equivalent one using:

- **observational equivalences** $L \approx_p R$ given as **axioms** and that come from security properties of primitives. These equivalences are used inside a context:

$$G_1 \approx_0 C[L] \approx_{p'} C[R] \approx_0 G_2$$

- **syntactic transformations**: simplification, expansion of assignments, ...

We obtain a **sequence of games** $G_0 \approx_{p_1} G_1 \approx_{p_2} \dots \approx_{p_m} G_m$, which implies $G_0 \approx_{p_1 + \dots + p_m} G_m$.

If **event** *forge* cannot be executed in G_m , it can be executed with probability at most $p_1 + \dots + p_m$ in G_0 .

The adversary inverts f when, given the public key $pk = pkgen(r_0)$ and the image of some x_0 by f_{pk} , it manages to find x_0 (without having the trapdoor).

The function f is **one-way** when the adversary has negligible probability of inverting f , say at most probability $\text{Succ}_{\mathcal{P}}^{\text{ow}}(t)$ when the adversary runs in time t .

One-wayness as an observational equivalence

Security assumptions have to be given as **equivalences**. The following equivalence formalizes one-wayness:

$$\begin{aligned} & \text{foreach } i_k \leq n_k \text{ do } r \stackrel{R}{\leftarrow} \text{seed}; (Opk() := \text{return}(pkgen(r))) \\ & \quad | \text{foreach } i_f \leq n_f \text{ do } x \stackrel{R}{\leftarrow} D; (Oy() := \text{return}(f(pkgen(r), x))) \\ & \quad \quad | \text{foreach } i_1 \leq n_1 \text{ do } Oeq(x' : D) := \text{return}(x' = x) \\ & \quad \quad | OX() := \text{return}(x))) \\ & \approx_{n_k \times n_f \times \text{ Succ}_P^{\text{ow}}(t + (n_k n_f - 1)t_f + (n_k - 1)t_{pkgen}} \\ & \text{foreach } i_k \leq n_k \text{ do } r \stackrel{R}{\leftarrow} \text{seed}; (Opk() := \text{return}(pkgen'(r))) \\ & \quad | \text{foreach } i_f \leq n_f \text{ do } x \stackrel{R}{\leftarrow} D; (Oy() := \text{return}(f'(pkgen'(r), x))) \\ & \quad \quad | \text{foreach } i_1 \leq n_1 \text{ do } Oeq(x' : D) := \\ & \quad \quad \quad \text{if defined}(k) \text{ then return}(x' = x) \text{ else return}(false) \\ & \quad \quad | OX() := k \leftarrow \text{mark}; \text{return}(x))) \end{aligned}$$

This equivalence is proved manually, **once** for one-way trapdoor permutations. It can be reused in the proof of many schemes.

Other properties of one-way trapdoor permutations

- $x \mapsto f(pkgen(r), x)$ and $x \mapsto invf(skgen(r), x)$ are **inverse permutations**:

$$invf(skgen(r), f(pkgen(r), x)) = x$$

- $x \mapsto f(pk, x)$ is **injective**:

$$(f(pk, x) = f(pk, x')) \implies (x = x')$$

- When x is a uniformly distributed random number, we can **replace x with $f(pk, x)$** without changing the probability distribution ($x \mapsto f(pk, x)$ is a permutation).

This is again expressed by an observational equivalence.

Hash function in the random oracle model

A **random oracle** is formalized by the following equivalence:

```
foreach  $i_h \leq n_h$  do  $OH(x : \text{bitstring}) := \text{return}(\text{hash}(x))$   
 $\approx_0$  foreach  $i_h \leq n_h$  do  $OH(x : \text{bitstring}) :=$   
  find  $u \leq n_h$  suchthat  $(\text{defined}(x[u], r[u]) \wedge x = x[u])$   
  then  $\text{return}(r[u])$  else  $r \stackrel{R}{\leftarrow} D; \text{return}(r)$ 
```

The hash function is equivalent to a function that looks up the argument x in the array of previous arguments given to OH .

- If x is not found, it returns a **fresh random number** r .
- If x is found, it returns the same result as in the previous call.

Automatic proof of FDH: result

Our prover is given as input

- the **security assumptions** of one-way trapdoor permutations and of the hash function,
- the **initial game** that formalizes the security of FDH.

It **automatically** produces a proof that this game executes **event** *forge* with probability at most

$$(q_H + q_S + 1)\text{Succ}_{\mathcal{P}}^{\text{ow}}(t + (q_H + q_S)t_f)$$

where t_f is the time of one evaluation of f (ignoring the time of bit-string comparisons).

This is the standard upper-bound [Bellare, Rogaway, CCS'93].

The prover also outputs

- the **sequence of games** that leads to this proof,
- a **succinct explanation** of the transformations performed between games.

- The prover tries to apply **all equivalences** given as axioms, which represent security assumptions.

It transforms the left-hand side into the right-hand side of the equivalence.

- If such a **transformation succeeds**, the obtained game is then simplified, using in particular equations given as axioms.
- When these **transformations fail**, they may return syntactic transformations to apply in order to make them succeed, called **advised transformations**.

The prover then applies the advised transformations, and retries the initial transformation.

Proof steps for the proof of FDH

- Apply security of the hash function.
- Simplify.
- Remove assignments to sk , substituting $skgen(r)$ for sk .
(This transformation is advised by the next transformation.)
- Apply invariance of uniform distributions by permutations.
- Simplify.
- Apply one-wayness.
- Simplify.
- Success: the obtained game never executes **event** *forge*.

Other examples: Encryption schemes

Our prover has successfully proved many protocols and the following properties of encryption schemes:

- $\mathcal{E}(m, r) = f(r) \| \text{hash}(r) \text{ xor } m$ is IND-CPA.
- $\mathcal{E}(m, r) = f(r) \| \text{hash}(r) \text{ xor } m \| \text{hash}'(\text{hash}(r) \text{ xor } m, r)$ is IND-CCA2.
- With an improved treatment of the equational theory of *xor*, we believe that it could also show that $\mathcal{E}(m, r) = f(r) \| \text{hash}(r) \text{ xor } m \| \text{hash}'(m, r)$ is IND-CCA2.

The proofs of these encryption schemes use a **manual mode** of the prover, in which the user indicates the main game transformations to perform.

We have presented a **new tool** that can prove automatically the security of cryptographic primitives and protocols.

- The **security assumptions** are given as **observational equivalences**.

The manual proof of these equivalences is done **once** for all proofs, and has already been done for many primitives.

- The **protocol or scheme** to prove is specified using a syntax close to the notations classically used in cryptography.
- The prover provides a **sequence of indistinguishable games** which leads to a final game in which the adversary has advantage 0.
- The user is allowed (but does not have) to interact with the prover to make it follow a specific sequence of games.

Details at <http://www.di.ens.fr/~blanchet/cryptoc/FDH/>