# Power spectra of random spike fields & related processes

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**Abstract**

This paper presents general methods for obtaining power spectra of a large class of signals and random fields driven by an underlying point processes, in particular spatial shot noises with random impulse response and arbitrary basic stationary point processes described by their Bartlett spectrum, and signals or fields sampled at random times or points, where again the sampling point process is quite general. The formulas obtained clearly show the interaction between the underlying counting process, the sampled process or the impulse response. We also obtain the Bartlett spectrum for the general linear Hawkes spatial branching point process with random fertility rate and general immigrant process described by its Bartlett spectrum. Finally we obtain the Cramér spectrum of general spatial birth and death processes.

**Index Terms**

Shot noises, ultra-wide band communication, random sampling, power spectrum, point processes, filtered point processes, Cramér spectral measure, Bartlett spectral measure, pulse-interval modulation, pulse position modulation, Hawkes processes, random fields.

## I. Introduction

This article is concerned with the second-order properties of signals related to random spike fields, also called random Dirac brushes. In mathematical terms, the latter are best described as point processes. More specifically, we shall consider three types of signals:

(a) the random *spike fields* themselves;

(b) the *filtered* random spike fields;

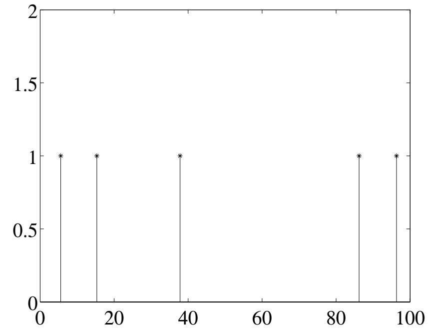
(c) the *modulated* random spike fields.

These types of signals are depicted in Figure 1 in the unidimensional case (spike fields are then called spike trains or Dirac combs). The second category of signals is also known under the name of *shot noises*, and the third category arises in particular in *random sampling*. As for the first category, that is point processes, they form the basic element on which are constructed the other signals in this study.

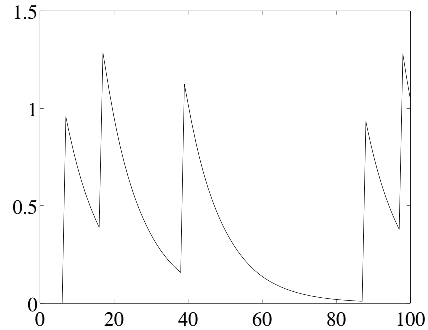
In this article, we derive formulas for the power spectrum (sometimes in a generalized sense) for the above three categories of signals. We do this in very general cases, in particular, concerning shot noises, when the basic point process is not a homogeneous Poisson process.

### *Shot noises*

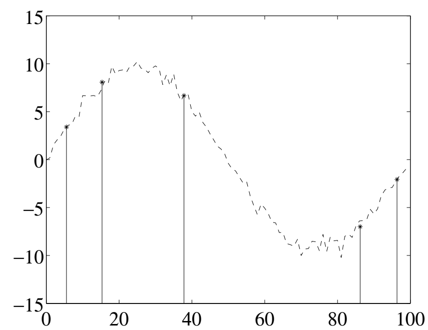
Shot noises have received much attention in the applied literature, whether in physics or in electrical engineering. They owe their name to the fact that they model, at the fine level,



(a) random spike train (point process)



(b) filtered spike train (shot noise)



(c) modulated spike train (randomly located samples)

Fig. 1. Random spike train and related processes

thermoionic noise in conductors (Campbell [1], Schottky [2]), and they have been studied by Rice [3] who contributed to their popularity (see also the references in Bondesson [4] and Gubner [5]).

The article of Lowen and Teich [6] gives a number of application in physics (for instance, Cherenkov radiation) of the so-called *power–law* shot noise.

Shot noise also arises in queuing and teletraffic theory, for instance under the form of $M/GI/\infty$ pure delay system, which is a Poisson shot noise with random impulse function. For the use of $M/GI/\infty$ model in traffic theory, we refer to Parulekar and Makowski [7] and the references therein.

Shot noise is also of interest in insurance risk theory, where they represent delayed claims; see Klüppelberg and Mikosch [8], Samorodnitsky [9]. The signals arising in neurophysiology are typically non-Poisson shot noises and the interference field in a mobile communication system is aptly modeled as a spatial shot noise (Baccelli and Blaszczyszyn [10]).

Shot noises also arise naturally in wavelet signal analysis when the analyzed signal is a point process, since the wavelet coefficients are in this case samples of shot noises. Wavelet statistical analysis has been proposed to detect and compute the Hurst parameter in classical signals (Abry and Veitch [11]) and the method applies equally well to random Dirac combs with long-range dependence properties. The accuracy of the statistical analysis depends very much on the second order properties of the shot noises resulting from the wavelet decomposition.

The first result of this article is the formula giving the power spectrum of a *spatial* shot noise with *random impulse function* (see equation (21) below), when the underlying point process is stationary and its Bartlett spectrum is known. First we consider the case when the random impulses are independent and equally distributed and *independent* of the basic point process. Section IV is devoted to shot noises (filtered spike fields) and the main result given in this section (formula (21)) does not appear in the literature in this general form where the basic spike train is not a standard point process (Poisson, or renewal, or Cox). Note that these results are of special interest not only to biology, but also to ultra-wide band communications (UWB) where the models are of the shot noise type with random impulses, and a basic point process which is a renewal process; see [12] for an example of this type, where the authors derive exact formulas for a family of digital pulse interval modulated (DPIM) signals.

*Modulated random spike fields*

A modulated Dirac comb is a Dirac comb with pulses of varying height. In random sampling, the height of a pulse is equal to the value of the signal sampled at this time. Random sampling has been extensively studied in view of spectral analysis, the object being to recover the power spectrum of the signal from the modulated sample comb, or even from the sample sequence (without timing information); a specific domain of application is laser velocimetry (see Gaster and Roberts [13]), where the samples are collected only at the passage of a reflecting particle through the laser beam.

Two theoretical questions arise. The first one is related to spectral analysis, the second one to signal reconstruction.

1. What is the relation between the spectrum of the modulated Dirac comb (or the sample sequence) to that of the signal?

2. To what extent can we recover the signal from the modulated Dirac comb (or the sample sequence)?

Several works have contributed with answers to such questions. Early investigation on random sampling (Shapiro and Silverman [14], Beutler [15]) was mostly motivated by the search for alias-free sampling schemes, that is, sampling schemes leading to a one-to-one relation between the spectrum of the sample comb to that of the sampled signal.

The first detailed analyses of randomly sampled signals were based on the modeling of the sample comb using the Dirac (pseudo) process $\delta$. Beutler and Leneman [16], [17], [18] obtained formulas for the moments of the sample comb that lead to the expression of the correlation of the sample comb as a function of the correlation of the sampled signal. Leneman and Lewis [19] investigated the reconstruction error for several interpolators of the random samples. Such results depend on the sampling scheme through statistics related to the intervals between successive points of the sampler.

The spectrum of randomly sampled signals has been obtained by Masry [20], [21], using a point process approach. The spectrum of the sample sequence was expressed as a function of the spectrum of the sampled signal and of the second order quantities of the point process, and then, by reformulating the alias-free concept, alias-free sampling schemes werere proved to lead to a consistent spectral estimator. This work is closest to ours, and our method of proof is the same as in [24], our contribution being to give more details for the proof in [24], and to extend these formulas to the spatial case. Also, we give the power spectra of modulated spike fields when the sampler is possibly dependent from the signal. In the independent case, we also give the expression of the error when the signal is approximated by a filtered version of the samples, that is, the reconstruction error.

Modulated random spike fields are studied in Section VI.

*Random spike fields*

As for the spike trains themselves, we recall the basic theory of Bartlett spectra in terms that clearly show their link to the usual power spectral measure of *bona fide* wide-sense stationary processes. The basic definitions, including that of a Bartlett spectrum, are given in Sections II and III.

A particular class of spike fields, the *Hawkes branching point processes*, are studied in Section V. Hawkes processes were introduced, under the name of self-exciting point processes, by Hawkes [22], and further studied in Hawkes and Oakes [23]; see also Daley and Vere-Jones [24, page 367].

Such branching point processes are of interest in epidemics, and also in seismology, where they are known as ETAS models (see Ogata [25]). As we shall see, Hawkes processes are the stochastic equivalent of Hopfield networks. They are also used to model neuronal activity in the brain (see Johnson [26]).

A very important class of processes, the generalized birth and death process (not necessarily Markovian), are shot noises where the basic point process is a Hawkes process and the sequence of pulses is *not* independent of the basic point process. We obtain the power spectrum of such signals. We include in our generalization a non-Poisson "ancestor process", random impulse train, and our general method of proof applies to spatial Hawkes processes. Earlier results in this direction are in Brémaud and Massoulié [27], where however the study is restricted to the line, and the generalized birth and death is not considered. See also Brémaud and Massoulié [34], where the critical case, leading to long-range dependence, was considered.

## II. Random spike fields and their transformations

We now give the formal description of random spike fields (spatial point processes). We begin with the more familiar unidimensional case.

Let $\{T_n\}_{n\in\mathbb{Z}}$ be a non-decreasing sequence of random times in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ such that if we define for all measurable sets $C$ of the real line

$$N(C) = \sum_{n\in\mathbb{Z}} 1_C(T_n) \tag{1}$$

then

- (simplicity) $\mathrm{P}(N(\{a\}) \leq 1, \forall a \in \mathbb{R}) = 1$.
- (local finiteness) $\mathrm{P}(N(C) < \infty) = 1$, for all bounded measurable set $C$.

The family $N = \{N(C)\}$ indexed by the measurable sets of the line and the sequence $\{T_n\}$ then contain the same information and are called a (simple and boundedly finite) point process on the real line. It will be convenient to represent this point process by a random Dirac comb, or a random spike train

$$\Delta_N(t) := \sum_{n\in\mathbb{Z}} \delta(t - T_n)$$

where $\delta$ is the Dirac pseudo-function. In all sums involving the $T_n$'s, it will be assumed that the sum extends to only those $n \in \mathbb{Z}$ such that $|T_n| < \infty$. The three expressions below

$$\sum_{n\in\mathbb{Z}} \varphi(T_n), \quad \int_{\mathbb{R}} \varphi(t) N(dt), \quad \int_{\mathbb{R}} \varphi(t)\Delta_N(t)dt$$

represent the same object. The second expression is an integral with respect to the measure $N$; the last one is symbolic, and uses the symbolic rule $\varphi(0) = \int_{\mathbb{R}} \varphi(t)\delta(t)dt$. These quantities are well-defined only under certain circumstances, for instance when $\varphi \geq 0$ (in which case it may be infinite). Let $h$ be a function, called the impulse response (of some convolutional filter). Passing a Dirac comb $\Delta_N(t)$ through this filter yields the output

$$X(t) = \sum_{n\in\mathbb{Z}} h(t - T_n)$$

This is called a *shot noise*, also denoted

$$X(t) = (h * N)(t) = \int_{\mathbb{R}} h(t-s)N(ds). \tag{2}$$

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EXAMPLE 2.1: Let $N$ be any point process and consider the exponential impulse response

$$h(t) = \gamma e^{-\alpha t} 1_{\{t\geq 0\}},$$

where $\gamma, \alpha$ are positive real numbers. The corresponding shot noise is

$$X(t) = \gamma \int_{-\infty}^{t} e^{-\alpha(t-s)} N(ds)$$

and verifies the differential system

$$dX(t) = -\alpha X(t)dt + \gamma dN(t)$$

(where $dN(t) = N(dt)$).

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Let now $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of random variables with values in the measurable space $(E, \mathcal{E})$, and let $h : \mathbb{R} \times E \to \mathbb{R}$ be some measurable function. The process

$$X(t) = \sum_{n\in\mathbb{Z}} h(t - T_n, Z_n), \tag{3}$$

when it is properly defined, is called a *shot noise with random excitation*.

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EXAMPLE 2.2: Let $N$ is a homogeneous Poisson process with intensity $\lambda$, independent of $\{Z_n\}_{n\in\mathbb{Z}}$, the latter being an i.i.d. sequence of non-negative random variables. With

$$h(t,z) = \begin{cases} 1 & \text{if } 0 \leq t \leq z, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

the process defined by (3) is called $M/GI/\infty$ process, and has the following interpretation. Customer $n$ enters a "system" at time $T_n$, and departs at time $T_n + Z_n$, in which case $X(t)$ is the number of customers in the system at time $t$.

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The above notions are now extended to the case where $N$ is a spatial point process. Let $(K,\mathcal{K})$ be some measurable space, and let $\overline{N}=\{(t,z)\in\mathbb{R}^m\times K\}$ be a random set in $\mathbb{R}^m\times K$ such that the random set in $\mathbb{R}^m$, $N=\left\{t\in\mathbb{R}^m; \exists\ z\in K \text{ s.t. } (t,z)\in\overline{N}\right\}$ is a locally finite and simple point process. $N$ is called the basic point process of $\overline{N}$, and the latter is called a *point process* on $\mathbb{R}^m\times K$ *with locally finite and simple basic point process*.

Define a measure on $\mathbb{R}^m\times K$, also denoted by $\overline{N}$, as follows

$$\overline{N}(C\times L)=\sum_{(t,z)\in\overline{N}}1_C(t)1_L(z), \tag{5}$$

with a similar definition for the random measure $N$:

$$N(C)=\sum_{t\in N}1_C(t)=\overline{N}(C\times K).$$

We have for all measurable function $\varphi:\mathbb{R}^m\times K\to\mathbb{R}$,

$$\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{N}(dt\times dz)=\sum_{(t,z)\in\overline{N}}\varphi(t,z)$$

provided the sum in the right-hand side is well-defined.

In order to define the spatial shot noise with random excitation, we introduce the *marked point process with i.i.i.d. marks* (note the triple *i*).

*Definition 2.1:* Let $N$ be a simple locally finite point process on $\mathbb{R}^m$ and let $\{Z\,(t)\}$, $t\in\mathbb{R}^m$, be an i.i.d. family of $K$-valued random variables, that are independent of $N$. Let

$$\overline{N}=\{(t,Z\,(t))\,;\ t\in N\}$$

or

$$\overline{N}(C\times L)=\sum_{t\in N}1_C\,(t)\,1_L\,(Z\,(t))\,.$$

Then $\overline{N}$ is called a (simple, locally finite) marked point process, with i.i.d. marks in $K$ independent of $N$. For short, we shall say that $\overline{N}$ is a marked point process on $\mathbb{R}^m$ with i.i.i.d. marks in $K$. For brevity, we shall omit to mention that it is simple and locally finite. We call $\{Z\,(t)\}$, $t\in\mathbb{R}^m$ its mark process. $\Diamond$

In particular, in the situation considered at the beginning of the section ($\mathbb{R}^m=\mathbb{R}$), we have

$$\overline{N}(C\times L)=\sum_{n\in\mathbb{Z}}1_C(T_n)1_L(Z_n),$$

$$\int\limits_{\mathbb{R}\times E}\varphi(t,z)\overline{N}(dt\times dz)=\sum_{n\in\mathbb{Z}}\varphi(T_n,Z_n).$$

Let now $h : \mathbb{R}^m \times K \to \mathbb{R}$ be a measurable function. The process $\{X(t)\}_{t\in\mathbb{R}^m}$ defined (when possible) by

$$X(t) = \int_{\mathbb{R}^m \times K} h(t-s,z)\overline{N}(ds \times dz) = \sum_{(s,z)\in\overline{N}} h(t-s,z)$$

is called a *spatial shot noise with random excitation*.

In general, the mark process and the basic counting process are not defined independently. One may start with a point process $\overline{N}$ on $\mathbb{R}^m \times K$ with locally finite and simple basic point process, and define the mark process $\{Z(t)\}_{t\in\mathbb{R}^m}$, $K$-valued as follows:

$$\forall\ t \in \mathbb{R}^m, \quad Z(t) = \begin{cases} z & \text{if } (t,z) \in \overline{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The process $\{Z(t)\}_{t\in\mathbb{R}^m}$ is the mark process associated with $N$. Of course

$$\sum_{(t,z)\in\overline{N}} \varphi(t,z) = \sum_{t\in N} \varphi(t, Z(t)). \tag{6}$$

EXAMPLE 2.3: This is an immediate generalisation to $\mathbb{R}^2$ of the $M/GI/\infty$ process Example 2.2 to spatial situation. Take for instance $\mathbb{R}^m = \mathbb{R}^2$, $N$ a simple locally finite point process on $\mathbb{R}^2$ with intensity $\lambda$, and a $K$-valued element is a random set of $\mathbb{R}^2$ (to avoid formal definitions, think of simple shapes, for example disks of random radius centered at $0$). With

$$h(t,z) = \begin{cases} 1 & \text{if } t \in z, \\ 0 & \text{otherwise,} \end{cases}$$

(recall that $z \in K$ is a subset of $\mathbb{R}^2$), then

$$X(t) = \sum_{(s,z)\in\overline{N}} h(t-s,z) = \sum_{s\in N} 1_{\{t\in(s+Z(s))\}}$$

represents the number of "shapes" $s + Z(s)$ centered at $s \in N$ that *cover* $t$.

## III. BARTLETT SPECTRA OF RANDOM SPIKE FIELDS

We shall need a convenient definition of the spectrum of a random Dirac comb, which is not a *bona fide* wide-sense stationary process, and for which therefore we cannot use the usual Cramér power spectral measure. The natural extension of the latter is the Bartlett power spectrum (Bartlett [28], Daley and Vere-Jones [24, chap. 11.2]).

Let $N$ be a simple locally bounded point process on $\mathbb{R}^m$ of first order, that is such that for all bounded Borel sets $C \subset \mathbb{R}^m$,

$$\mathrm{E}\left[N(C)\right] < \infty. \tag{7}$$

In that case,

$$\nu(C) = \mathrm{E}\left[N(C)\right] \tag{8}$$

defines a locally finite measure $\nu$ on $\mathbb{R}^m$, called the *mean measure* of $N$. If $\varphi : \mathbb{R}^m \to \mathbb{R}$ is a measurable function such that $\varphi \in L^1(\nu)$, that is

$$\int_{\mathbb{R}^m} |\varphi(t)|\nu(dt) < \infty,$$

then the sum $\sum_{t\in N} \varphi(t) = \int_{\mathbb{R}^m} \varphi(t)N(dt)$ is well-defined and

$$\mathrm{E}\left[\int_{\mathbb{R}^m} \varphi(t)N(dt)\right] = \int_{\mathbb{R}^m} \varphi(t)\nu(dt).$$

If $N$ is stationary, then

$$\nu\left(C\right) = \lambda\ell\left(C\right)$$

for some $\lambda \in \mathbb{R}_+$, called the *intensity*, where $\ell$ is the Lebesgue measure. Suppose in addition that for all bounded sets $C$

$$\mathrm{E}\left[N\left(C\right)^2\right] < \infty \tag{9}$$

Consider the second moment measure $M_2$ on $\mathbb{R}^m \times \mathbb{R}^m$ defined by

$$M_2\left(A \times B\right) = \mathrm{E}\left[N\left(A\right) N\left(B\right)\right]$$

Under condition (9), it is a $\sigma$-finite measure, and it can be readily checked that for any measurable function $\varphi :\ \mathbb{R} \to \mathbb{C}$ such that $\varphi \in \mathrm{L}^1$ and

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left|\varphi\left(t\right)\right| \left|\varphi\left(s\right)\right| M_2\left(dt \times ds\right) < \infty, \tag{10}$$

(for instance $\varphi$ continuous with compact support) we have that $\int_{\mathbb{R}^m} \varphi\left(t\right) N\left(dt\right) \in \mathrm{L}^2\left(\mathrm{P}\right)$ and

$$\mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi\left(t\right) N\left(dt\right)\right|^2\right] = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \varphi\left(t\right)\varphi^*\left(s\right) M_2\left(dt \times ds\right).$$

If $N$ is stationary, then $M_2$ is *diagonal-shift invariant*, that is for all Borel sets $A,\ B \subset \mathbb{R}^m$, all $t \in \mathbb{R}^m$

$$M_2\left(\left(A+t\right) \times \left(B+t\right)\right) = M_2\left(A \times B\right).$$

It then follows from the moment measure factorization lemma of Daley and Vere-Jones [24] (Lemma 10.4.III, page 356), that for all $\varphi, \psi \in \mathrm{L}^1\left(\nu\right)$ verifying (10), then

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \varphi\left(t\right)\psi^*\left(s\right) M_2\left(dt \times ds\right) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m} \varphi\left(t\right)\psi^*\left(s+t\right) dt\right)\Gamma\left(ds\right)$$

for some $\sigma$-finite measure $\Gamma$. Since

$$\begin{aligned}\mathrm{E}\left[\int_{\mathbb{R}^m}\varphi(t)\,N(dt)\right]\mathrm{E}\left[\int_{\mathbb{R}^m}\psi^*(s)\,N(ds)\right] &= \left(\int_{\mathbb{R}^m}\varphi(t)\,\nu(dt)\right)\left(\int_{\mathbb{R}^m}\psi^*(s)\,\nu(dy)\right)\\ &= \lambda^2\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,\psi^*(t+s)\,dt\right)ds,\end{aligned}$$

therefore if $\varphi,\psi\in\mathrm{L}^1(\nu)$ and verify condition (10),

$$\operatorname{cov}\left(\int_{\mathbb{R}^m}\varphi(t)\,N(dt),\int_{\mathbb{R}^m}\psi(s)\,N(ds)\right)=\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,\psi^*(t+s)\,dt\right)C(ds)$$

where

$$C:=\Gamma-\lambda^2\ell$$

is called the covariance measure. The link with the classical notion of covariance function of a *bona fide* wide-sense stationary (w.s.s.) signal $\{X(t)\}_{t\in\mathbb{R}^m}$ is the following. Let $C_X(\tau)$ be the covariance function of such signal. Then for all $\varphi,\psi\in\mathrm{L}^1$,

$$\begin{aligned}&\operatorname{cov}\left(\int_{\mathbb{R}^m}\varphi(t)\,X(t)\,dt,\int_{\mathbb{R}^m}\psi(s)\,X(s)\,ds\right)\\ &\qquad=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\varphi(t)\,\psi^*(s)\,C_X(t-s)\,dtds\\ &\qquad=\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,\psi^*(t+s)\,dt\right)C_X(s)\,ds\\ &\qquad=\int_{\mathbb{R}^m}\widehat{\varphi}(\nu)\,\widehat{\psi}^*(\nu)\left(\int_{\mathbb{R}^m}e^{-i2\pi\langle s,\nu\rangle}C_X(s)\,ds\right)d\nu.\end{aligned}\tag{11}$$

The power spectral measure $\mu_X$ associated with $\{X(t)\}_{t\in\mathbb{R}^m}$ can be defined by the condition: For all $\varphi,\psi\in\mathrm{L}^1$

$$\operatorname{cov}\left(\int_{\mathbb{R}^m}\varphi(t)\,X(t)\,dt,\int_{\mathbb{R}^m}\psi(s)\,X(s)\,ds\right)=\int_{\mathbb{R}^m}\widehat{\varphi}(\nu)\,\widehat{\psi}^*(\nu)\,\mu_X(d\nu),$$

or (equivalently): For all $\varphi\in\mathrm{L}^1$

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\,X(t)\,dt\right)=\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,\mu_X(d\nu).$$

Similarly

*Definition 3.1:* Let $N$ be a simple locally bounded point process on $\mathbb{R}^m$, stationary, with intensity $\lambda$ and such that $\mathrm{E}\left[N(C)^2\right]<\infty$ for all compact set $C\subset\mathbb{R}^m$. The measure $\mu_N$ on $\mathbb{R}^m$ is called the Bartlett spectral measure of $N$ if

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\,N(dt)\right)=\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,\mu_N(d\nu)\tag{12}$$

for all $\varphi\in\mathrm{L}^1$ such that (10) holds. $\Diamond$

If (12) holds true for all $\varphi\in\mathrm{L}^1\cap\mathrm{L}^2$, then, since $\mathrm{E}\left[\int_{\mathbb{R}^m}\varphi(t)\,N(dt)\right]=\lambda\int_{\mathbb{R}^m}\varphi(t)\,dt=\lambda\widehat{\varphi}(0)$

$$\mathrm{E}\left[\left|\int_{\mathbb{R}^m}\varphi(t)\,N(dt)\right|^2\right]=\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\left(\mu_N(d\nu)+\lambda^2\delta_0(d\nu)\right)$$

and by polarization

$$\mathrm{E}\left[\int_{\mathbb{R}^m} \varphi(t)\, N(dt) \int_{\mathbb{R}^m} \psi^*(s)\, N(ds)\right] = \int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\, \widehat{\psi}^*(\nu) \left(\mu_N(d\nu) + \lambda^2 \delta_0(d\nu)\right)$$

for all $\varphi,\ \psi \in \mathrm{L}^1 \cap \mathrm{L}^2$, for which it follows also that

$$\mathrm{cov}\left(\int_{\mathbb{R}^m} \varphi(x)\, N(dx), \int_{\mathbb{R}^m} \psi(x)\, N(dx)\right) = \int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\mu_N(d\nu).$$

Therefore (in the case $m = 1$ for definiteness), for any $h \in \mathrm{L}^1 \cap \mathrm{L}^2$, taking

$$\varphi(t) = h(u - t), \ \ \psi(t) = h(v - t),$$

yields

$$\mathrm{cov}\left(\int_{\mathbb{R}} h(u-t)N(dt), \int_{\mathbb{R}} h(v-t)N(dt)\right) = \int_{\mathbb{R}} |T(\nu)|^2\, e^{2i\pi\nu(v-u)}\mu_N(d\nu),$$

where $T(\nu) = \int_{\mathbb{R}} h(t)e^{-2i\pi\nu t}dt$. Since

$$X(t) = \int_{\mathbb{R}} h(t-s)N(ds)$$

is the signal obtained by passing the Dirac comb $\sum_n \delta(t - T_n)$ through the filter of impulse response $h$ and transmittance $T(\nu)$, the Cramér spectral measure of $\{X(t)\}$ is

$$\mu_X(d\nu) = |T(\nu)|^2\, \mu_N(d\nu) \tag{13}$$

which indeed corresponds to the filtering formula for *bona fide* w.s.s. signals if we assimilate the Dirac comb $\sum \delta(t - T_n)$ to a *bona fide* w.s.s. signal with spectral measure $\mu_N$. The spectral measure $\mu_N$ is however *not* a finite measure as it would be for ordinary w.s.s. signals.

It is not necessarily true that under the conditions of Definition 3.1, equality (12) holds for all $\varphi \in \mathrm{L}^1 \cap \mathrm{L}^2$. However it holds for all continuous $\varphi$ with bounded support (since $\mathrm{E}\left[N(C)^2\right] < \infty$ for all compact $C$, and in this case,

$$\mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t)\, N(dt)\right|^2\right] = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \left(\mu_N(d\nu) + \lambda^2\delta_0(d\nu)\right).$$

In some situations of interest, $\varphi \in \mathrm{L}^2(\mathbb{R}^m)$ implies that $\varphi \in \mathrm{L}^2(\mu_N)$, and then the above equality as well as (12) hold for all $\varphi \in \mathrm{L}^1(\mathbb{R}^m) \cap \mathrm{L}^2(\mathbb{R}^m)$.

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EXAMPLE 3.1: Consider the point process on $\mathbb{R}^2$ whose points form a regular $(T_1, T_2)$-grid on $\mathbb{R}^2$ with random origin, that is

$$N = \left\{(n_1T_1 + U_1, n_2T_2 + U_2), (n_1, n_2) \in \mathbb{Z}^2\right\}$$

where $T_1 > 0$, $T_2 > 0$, and $U_1$, $U_2$ are independent uniform random variables on $[0, T_1]$, $[0, T_2]$ respectively. The point process is obviously stationary with intensity $\lambda = 1/(T_1 T_2)$. Let $\varphi \in \mathrm{L}^1(\nu) \cap \mathrm{L}^2(\nu) = \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$.

The Bartlett spectrum of $N$ has the "pseudo-density" $f_N(\nu_1, \nu_2)$ given by

$$\frac{1}{T_1^2 T_2^2} \sum_{(n_1,n_2) \neq (0,0)} \delta\left(\nu_1 - \frac{n_1}{T_1}\right) \delta\left(\nu_2 - \frac{n_2}{T_2}\right). \tag{14}$$

*Proof:* We have by Poisson's summation formula

$$\int_{\mathbb{R}^2} \varphi(t)\, N(dt) = \sum_{n_1,n_2 \in \mathbb{Z}} \varphi(U_1 + n_1 T_1, U_2 + n_2 T_2) = \frac{1}{T_1 T_2} \sum_{n_1,n_2 \in \mathbb{Z}} \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) e^{2i\pi\left(\frac{n_1}{T_1}U_1 + \frac{n_2}{T_2}U_2\right)}$$

and therefore

$$\begin{aligned} \mathrm{E}\left[\left|\int_{\mathbb{R}^2} \varphi(t)\, N(dt)\right|^2\right] &= \frac{1}{T_1^2 T_2^2} \mathrm{E}\left[\sum_{n_1,n_2 \in \mathbb{Z}} \sum_{k_1,k_2 \in \mathbb{Z}} \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) \widehat{\varphi}^*\left(\frac{k_1}{T_1}, \frac{k_2}{T_2}\right) e^{2i\pi\left(\frac{n_1-k_1}{T_1}U_1 + \frac{n_2-k_2}{T_2}U_2\right)}\right] \\ &= \frac{1}{T_1^2 T_2^2} \sum_{n_1,n_2 \in \mathbb{Z}} \left|\widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right)\right|^2. \end{aligned}$$

Also

$$\begin{aligned} \mathrm{E}\left[\int_{\mathbb{R}^2} \varphi(t)\, N(dt)\right] &= \sum_{n_1,n_2 \in \mathbb{Z}} \mathrm{E}\left[\varphi(U_1 + n_1 T_1, U_2 + n_2 T_2)\right] \\ &= \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \varphi(u_1 + n_1 T_1, u_2 + n_2 T_2)\, du \\ &= \frac{1}{T_1 T_2} \int_{\mathbb{R}^2} \varphi(t)\, dt = \frac{1}{T_1 T_2} \widehat{\varphi}(0,0). \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{Var}\left(\int_{\mathbb{R}^2} \varphi(t)\, N(dt)\right) &= \frac{1}{T_1^2 T_2^2} \sum_{n_1,n_2 \in \mathbb{Z}} \left|\widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right)\right|^2 - \frac{1}{T_1^2 T_2^2} \left|\widehat{\varphi}(0,0)\right|^2 \\ &= \frac{1}{T_1^2 T_2^2} \sum_{(n_1,n_2) \neq (0,0)} \left|\widehat{\varphi}\left(\frac{n_1}{T_1} \frac{n_2}{T_2}\right)\right|^2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\widehat{\varphi}(\nu_1, \nu_2)\right|^2 \left(\frac{1}{T_1^2 T_2^2} \sum_{(n_1,n_2) \neq (0,0)} \delta_{\frac{n_1}{T_1}}(d\nu_1)\, \delta_{\frac{n_2}{T_2}}(d\nu_2)\right). \end{aligned}$$

The Bartlett spectrum of $N$ is therefore

$$\mu_N(d\nu) = \frac{1}{T_1^2 T_2^2} \sum_{(n_1,n_2) \neq (0,0)} \delta_{\frac{n_1}{T_1}}(d\nu_1)\, \delta_{\frac{n_2}{T_2}}(d\nu_2) \tag{15}$$

and, in terms of pseudo spectral density and Dirac pseudo function, we have (14). ■

EXAMPLE 3.2: Suppose that $N$ is obtained by i.i.d. thinning (losses) of the points of the point process $\widetilde{N}$ with Bartlett spectrum $\mu_{\widetilde{N}}$ (and intensity $\widetilde{\lambda}$). Let $p$ the probability of losing a sample, and $q = 1 - p$. Then,

$$\mu_N\left(d\nu\right) = q^2 \mu_{\widetilde{N}}\left(d\nu\right) + \widetilde{\lambda} pq d\nu. \tag{16}$$

*Proof:* We can write the original point process as

$$\widetilde{N}\left(A\right) = \sum_{s\in\widetilde{N}} 1_A\left(s\right), \quad \forall\ A \subseteq \mathcal{B}\left(\mathbb{R}^m\right).$$

Then, the thinned point process can be defined as follows

$$N\left(A\right) = \sum_{s\in\widetilde{N}} 1_A\left(s\right) Z\left(s\right), \quad \text{for all } A \subseteq \mathcal{B}\left(\mathbb{R}^m\right) \tag{17}$$

where, given $N$, $\left\{Z\left(s\right), s \in \widetilde{N}\right\}$ is a sequence of i.i.d. random variables with values in $\{0, 1\}$, with common probability

$$\mathrm{P}\left(Z\left(s\right) = 0\right) = p, \quad \mathrm{P}\left(Z\left(s\right) = 1\right) = q = 1 - p, \quad \forall\ s \in \widetilde{N}.$$

In order to compute its Bartlett spectrum we need to evaluate

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi\left(t\right) N\left(dt\right)\right). \tag{18}$$

We remark that the thinned point process (17) can be seen as a marked point process with i.i.i.d. marks

$$\overline{N}\left(A \times \{1\}\right) = \sum_{s\in\widetilde{N}} 1_A\left(s\right) 1_{\{1\}}\left(Z\left(s\right)\right), \quad \text{for all } A \subseteq \mathcal{B}\left(\mathbb{R}^m\right)$$

where the mark process $\{Z\left(x\right)\}_{x\in\mathbb{R}^m}$ is such that, given $N$, we have the sequence of random variables $\left\{Z\left(s\right), s \in \widetilde{N}\right\}$ described above. Therefore, expression (18) can be computed as a particular case of Theorem 4.1 (fundamental isometry formula), with $K = \{0, 1\}$, $\varphi(x, z) = \psi(x, z) = \varphi(x) 1_{\{1\}}(z)$. Then, the result (16) directly follows from equation (20). ■

EXAMPLE 3.3: Let $N$ be a Cox point process on $\mathbb{R}^m$ with stochastic intensity $\{\lambda(t)\}_{t\in\mathbb{R}^m}$. By this, the following is meant. First $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ is a non-negative a.s. locally integrable process (that is $\mathrm{P}(\int_C \lambda(t)dt < \infty) = 1$, for all bounded Borel set $C$) and conditionally on this process, $N$ is a Poisson process with intensity $\lambda(x)$. We suppose that $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ is a w.s.s. process with mean $\lambda$ and Cramér spectral measure $\mu_\lambda$. Then

$$\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu. \tag{19}$$

*Proof:* By the conditional variance formula,

$$\begin{aligned}
\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(x)N(dx)\right) &= \mathrm{E}\left[\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(x)N(dx)\middle|\,\lambda(\cdot)\right)\right]+\operatorname{Var}\left(\mathrm{E}\left[\int_{\mathbb{R}^m}\varphi(x)N(dx)\middle|\,\lambda(\cdot)\right]\right)\\
&= \mathrm{E}\left[\int_{\mathbb{R}^m}\varphi(x)^2\lambda(x)dx\right]+\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(x)\lambda(x)dx\right)\\
&= \lambda\int_{\mathbb{R}^m}\varphi(x)^2dx+\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(x)\lambda(x)dx\right),
\end{aligned}$$

where the second equality follows from well-known properties of Poisson processes with a not necessarily constant intensity. By definition of the Cramér spectrum

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(x)\lambda(x)dx\right)=\int_{\mathbb{R}^m}\left|\widehat{\varphi}(\nu)\right|^2\mu_\lambda(d\nu),$$

and by the Plancherel-Parseval formula

$$\int_{\mathbb{R}^m}\varphi^2(x)dx=\int_{\mathbb{R}^m}\left|\widehat{\varphi}(\nu)\right|^2d\nu.$$

Therefore

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(x)N(dx)\right)=\int_{\mathbb{R}^m}\left|\widehat{\varphi}(\nu)\right|^2(\mu_\lambda(d\nu)+\lambda d\nu),$$

that is (19). ■

## IV. FILTERED SPIKE FIELDS

Let $\overline{N}$ be a marked point process on $\mathbb{R}^m$ with marks in $K$, let $N$ be the basic point process on $\mathbb{R}^m$, assumed locally finite and simple, and let $\{Z(t)\}_{t\in\mathbb{R}^m}$ be its mark process. Assume that given $N$, the family of random variables $(Z(t),\ t\in N)$ is i.i.d. with common distribution $Q$, and also assume that $N$ is a second order stationary point process with Bartlett spectral measure $\mu_N$.

Let $\varphi:\mathbb{R}^m\times K\to\mathbb{R}$ be a function in $\mathrm{L}^1(\ell\times Q)\cap\mathrm{L}^2(\ell\times Q)$, and define

$$\begin{aligned}
\overline{\varphi}(t) &= \int_K\varphi(t,z)Q(dz),\\
\widehat{\varphi}(\nu,z) &= \int_{\mathbb{R}^m}e^{-2i\pi<\nu,t>}\varphi(t,z)dt,\\
\widehat{\overline{\varphi}}(\nu) &= \int_{\mathbb{R}^m}\overline{\varphi}(t)e^{-2i\pi<\nu,t>}dt=\int_K\widehat{\varphi}(\nu,z)Q(dz)\\
&=\overline{\widehat{\varphi}}(\nu).
\end{aligned}$$

(thus, “bar” denotes expectation with respect to the distribution $Q$).

*Theorem 4.1 (Fundamental isometry formula):* Let $\overline{N}$ be as above, and $\varphi,\psi:\mathbb{R}^m\times K\to\mathbb{R}$ be functions in $\mathrm{L}^1(\ell\times Q)\cap\mathrm{L}^2(\ell\times Q)$. Then

$$\boxed{\begin{aligned}&\operatorname{cov}\Big(\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{N}(dt\times dz),\int\limits_{\mathbb{R}^m\times K}\psi(t,z)\overline{N}(dt\times dz)\Big)\\&\qquad\qquad=\int\limits_{\mathbb{R}^m}\widehat{\overline{\varphi}}(\nu)\widehat{\overline{\psi}}^*(\nu)\mu_N(d\nu)+\lambda\int\limits_{\mathbb{R}^m}\operatorname{cov}\big(\widehat{\varphi}(\nu,Z),\widehat{\psi}^*(\nu,Z)\big)d\nu,\quad(20)\end{aligned}}$$

where $Z$ is a $K$-valued random variable with distribution $Q$.

*Proof:*

$$\begin{aligned}&\mathrm{E}\left[\left(\sum_{(x,z)\in\overline{N}}\varphi(x,z)\right)\left(\sum_{(x,z)\in\overline{N}}\psi^*(x,z)\right)\right]\\&=\mathrm{E}\left[\sum_{(x,z),(x',z')\in\overline{N},x\neq x'}\varphi(x,z)\psi^*(x',z')\right]+\mathrm{E}\left[\sum_{(x,z)\in\overline{N}}\varphi(x,z)\psi^*(x,z)\right]\\&=\mathrm{E}\left[\sum_{x,x'\in N,x\neq x'}\overline{\varphi}(x)\overline{\psi}^*(x')\right]+\mathrm{E}\left[\sum_{x\in N}\varphi(x,Z)\psi^*(x,Z)\right]\\&=\mathrm{E}\left[\left(\sum_{x\in N}\overline{\varphi}(x)\right)\left(\sum_{x'\in N}\overline{\psi}^*(x')\right)\right]-\mathrm{E}\left[\sum_{x\in N}\overline{\varphi}(x)\overline{\psi}^*(x)\right]+\mathrm{E}\left[\sum_{x\in N}\varphi(x,Z)\psi^*(x,Z)\right].\end{aligned}$$

Since $\mathrm{E}\left[\sum_{(x,z)\in\overline{N}}\varphi(x,z)\right]=\mathrm{E}\left[\sum_{x\in N}\overline{\varphi}(x)\right]$,

$$\begin{aligned}&\operatorname{cov}\left(\int_{\mathbb{R}^m\times E}\varphi(x,z)\overline{N}(dx\times dz),\int_{\mathbb{R}^m\times E}\psi(x,z)\overline{N}(dx\times dz)\right)=\\&\operatorname{cov}\left(\int_{\mathbb{R}^m}\overline{\varphi}(x)N(dx),\int_{\mathbb{R}^m}\overline{\psi}(x)N(dx)\right)-\mathrm{E}\left[\sum_{x\in N}\overline{\varphi}(x)\overline{\psi}^*(x)\right]+\mathrm{E}\left[\sum_{x\in N}\varphi(x,Z)\psi^*(x,Z)\right].\end{aligned}$$

Denote by $A$, $B$ and $C$ the three terms in the right-hand side of the above equation, which then reads $A-B+C$. By definition of the Bartlett spectrum,

$$A=\int_{\mathbb{R}^m}\widehat{\overline{\varphi}}(\nu)\widehat{\overline{\psi}}^*(\nu)\mu_N(d\nu).$$

By definition of the intensity $\lambda$,

$$B=\lambda\int_{\mathbb{R}^m}\overline{\varphi}(x)\overline{\psi}(x)^*dx,\quad C=\lambda\int_{\mathbb{R}^m}\mathrm{E}\left[\varphi(x,Z)\psi(x,Z)^*\right]dx.$$

By Plancherel-Parseval's identity,

$$B=\lambda\int_{\mathbb{R}^m}\widehat{\overline{\varphi}}(\nu)\widehat{\overline{\psi}}(\nu)^*d\nu=\lambda\int_{\mathbb{R}^m}\overline{\widehat{\varphi}}(\nu)\overline{\widehat{\psi}}(\nu)^*d\nu=\lambda\int_{\mathbb{R}^m}\mathrm{E}\left[\widehat{\varphi}(\nu,Z)\right]\mathrm{E}\left[\widehat{\psi}(\nu,Z)^*\right]d\nu,$$

and

$$C=\lambda\mathrm{E}\left[\int_{\mathbb{R}^m}\widehat{\varphi}(\nu,Z)\widehat{\psi}(\nu,Z)^*d\nu\right],$$

and the result (20) follows. ■

*Corollary 4.1:* We can now compute the power spectrum of a shot noise. Let $\overline{N}$ be as above and let $h : \mathbb{R}^m \times K \to \mathbb{R}$, $h \in \mathrm{L}^1(\ell \times Q) \cap \mathrm{L}^2(\ell \times Q)$. Define the shot noise $\{X(t)\}_{t\in\mathbb{R}^m}$ by

$$X(t) = \int_{\mathbb{R}^m \times K} h(t-s, z)\overline{N}(ds \times dz).$$

then

$$\mathrm{E}\left[X(t)\right] = \lambda \int_{\mathbb{R}^m} \overline{h}(t)dt,$$

and

$$\mathrm{cov}\left(X(u), X(v)\right) = \int_{\mathbb{R}^m} e^{2i\pi<\nu,u-v>}\mu_X(d\nu),$$

where

$$\boxed{\mu_X(d\nu) = \left|\mathrm{E}\left[T(\nu, Z)\right]\right|^2 \mu_N(d\nu) + \lambda \mathrm{Var}\left(T(\nu, Z)\right)d\nu,} \tag{21}$$

with

$$T(\nu, Z) = \widehat{h}(\nu, Z) = \int_{\mathbb{R}^m} h(t, Z)e^{-2i\pi<\nu,t>}dt.$$

*Proof:* It suffices to apply the fundamental isometry formula to $\varphi(t, z) = h(u-t, z)$, $\psi(t, z) = h(v - t, z)$ to obtain

$$\mathrm{cov}\left(X(u), X(v)\right) = \int_{\mathbb{R}^m} \left|\overline{T}(\nu)\right|^2 e^{-2i\pi<\nu,u-v>}\mu_N(d\nu) + \lambda \int_{\mathbb{R}^m} \mathrm{Var}\left(T(\nu, Z)\right)e^{-2i\pi<\nu,u-v>}d\nu.$$

■

Knowing the Bartlett spectrum $\mu_N$ of a w.s.s. point process $N$, what is the Bartlett spectrum $\mu_{\widetilde{N}}$ of the point process obtained by independent and identically distributed displacements of the points of $N$? The setting is the following.

*Corollary 4.2:* Let $\overline{N}$ be as in Theorem 4.1, with $K = \mathbb{R}^m$. The point process $\widetilde{N}$ is defined by

$$\widetilde{N} = \{t + z, (t, z) \in \overline{N}\}.$$

Then, calling $\lambda$ the intensity of $N$, and $\mu_N$ its Bartlett spectrum,

$$\boxed{\mu_{\widetilde{N}}(d\nu) = \left|\psi_Z(\nu)\right|^2 \mu_N(d\nu) + \lambda\left(1 - \left|\psi_Z(\nu)\right|^2\right)d\nu,} \tag{22}$$

where

$$\psi_Z(\nu) = \int_{\mathbb{R}^m} e^{2i\pi<\nu,z>}Q(dz). \tag{23}$$

is the characteristic function of the random displacements distributed as $Q$. This formula is known; see for instance Exercise 11.2.4, p.426 in Daley and Vere-Jones [24].

*Proof:* Let $\widetilde{\varphi} \in \mathrm{L}^1(\mathbb{R}^m) \cap \mathrm{L}^2(\mathbb{R}^m)$.

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \widetilde{\varphi}(t)\widetilde{N}(dt)\right) = \mathrm{Var}\left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \widetilde{\varphi}(t+z)\overline{N}(dt \times dz)\right).$$

Letting $\varphi(t,z)=\widetilde{\varphi}(t+z)$, we have

$$\begin{aligned}
&\widehat{\varphi}(\nu,z)=e^{2i\pi<\nu,z>}\widehat{\widetilde{\varphi}}(\nu),\\
&\widehat{\overline{\varphi}}(\nu)=\overline{\widehat{\varphi}}(\nu)=\psi_Z(\nu)\widehat{\widetilde{\varphi}}(\nu),\\
&\operatorname{cov}\left(\widehat{\varphi}(\nu,Z),\widehat{\varphi}(\nu,Z)^*\right)=\left(1-|\psi_Z(\nu)|^2\right)\left|\widehat{\widetilde{\varphi}}(\nu)\right|^2.
\end{aligned}$$

Therefore, applying formula (20),

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\widetilde{\varphi}(t)\widetilde{N}(dt)\right)=\int_{\mathbb{R}^m}\left|\widehat{\widetilde{\varphi}}(\nu)\right|^2\mu_{\widetilde{N}}(d\nu),$$

where $\mu_{\widetilde{N}}$ is as in (22). ■

---

EXAMPLE 4.1: In UWB communication systems [29], [30], [31], the information is carried by a stream of spikes, more precisely, by the relative position of such spikes with respect to a regular grid (see Figure 2(a)). Then, after filtering in the transmission channel, the signal at the receiver is a shot noise, as depicted in Figure 2(b). In particular, such a stream of spikes can be seen as a regularly spaced point process $\widetilde{N}$ with (positive) i.i.d. jitters, where the jitters encode the symbols to be transmitted. Hence, the computation of the power spectral measure of the stream of spikes is a particular case of (22).

For instance, if we consider the transmission of $M$ independent symbols, each with equal probability $1/M$, the spikes take, with equal probability, the relative positions $\{0,T/M,\ldots,T(M-1)/M\}$. Therefore, the characteristic function (23) of the "jitters" is given by

$$\psi_Z(\nu)=\frac{1}{M}e^{i\pi\nu T\frac{M-1}{M}}\frac{\sin(\pi\nu T)}{\sin(\pi\nu T/M)}.$$

Recalling that a uniform spike train $N$ has the pseudo power spectral density

$$f_N(\nu)=\frac{1}{T^2}\sum_{n\neq 0}\delta\left(\nu-\frac{n}{T}\right)$$

and using formula (22), we obtain the Bartlett pseudo power spectral density of the spike train corresponding to an ultra-wide band transmission

$$f_{\widetilde{N}}(\nu)=\frac{1}{T^2}\sum_{k\neq 0}\delta\left(\nu-\frac{kM}{T}\right)+\frac{1}{T}\left(1-\frac{1}{M^2}\left(\frac{\sin(\pi\nu T)}{\sin(\pi\nu T/M)}\right)^2\right).$$

As an example, we consider a "frame time" $T$ equal to $10$ns and the transmission of $M=5$ symbols: Figure 3 depicts $f_{\widetilde{N}}(\nu)$ (in decibels) for a normalized frequency $\widetilde{\nu}=\nu T$ ranging over $[0,10]$.

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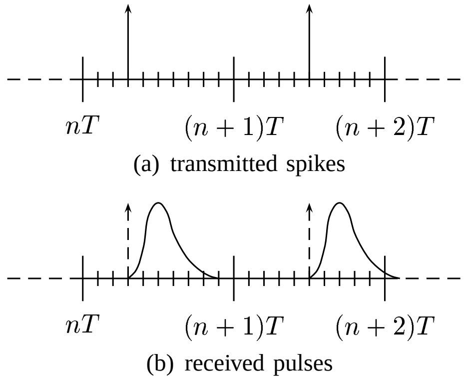


Fig. 2. Ultra-wide band communication.

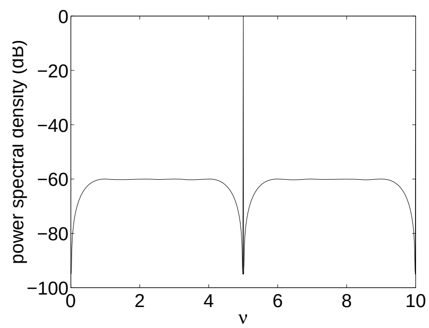


Fig. 3. Power spectra $f_{\widetilde{N}}(\nu)$ (in decibels) of a spike train corresponding to an ultra-wide band communication with $T = 10$ns and $M = 5$, plotted for a normalized frequency $\widetilde{\nu}$ ranging over $[0, 10]$.

## V. Spatial Hawkes processes and birth and death processes

Hawkes processes correspond to the diagram in Figure 4 where the box labeled L.F. represents a linear filter with impulse response $h(t)$. Therefore

$$X(t) = \int_{-\infty}^{t^-} h(t-s)\, N(ds)\,,$$

$$\lambda(t) = \alpha + \int_{-\infty}^{t^-} h(t-s)\, N(ds)\,.$$

The stochastic process $\lambda(t)$ drives the random generator (R.G.) yielding the output a point process with stochastic intensity $\lambda(t)$. This means that $\mathrm{P}(N(dt) = 1 \mid \mathcal{F}_t) = \lambda(t)\, dt$, where $\mathcal{F}_t$ records

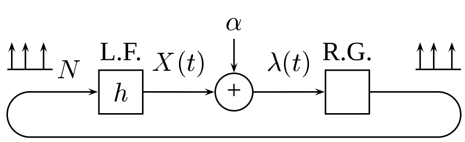


Fig. 4. Diagram of Hawkes processes: L.F. is a linear filter; R.G. is a random generator.

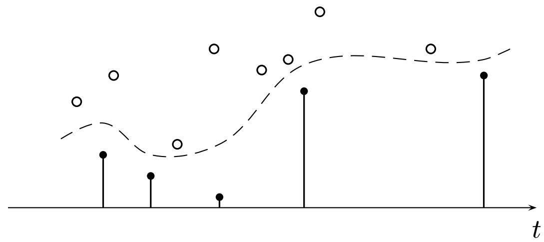


Fig. 5. Intensity $\lambda(t)$.

the information available at time $t$. This information is assumed to increase with $t$ (for details concerning stochastic intensity, see Brémaud [32] or Last and Brandt [33]). For instance, the random generator can in principle construct $N$ by projecting on $\mathbb{R}$ all the points of a homogeneous Poisson process on $\mathbb{R}_+ \times \mathbb{R}_+$ of intensity $1$ that lie below the curve $t \to \lambda(t)$ (see Figure 5).

A generalization of Hawkes processes gives the stochastic analogue of Hopfield networks. The dynamics of a Hopfield network are described by a system of non-linear differential equations

$$\dot{y}_i(t) = \varphi_i\left(\sum_{j=1}^{K}\int_{-\infty}^{t} h_{ji}(t-s)\, y_j(s)\, ds\right)$$

that is

$$\begin{cases} \dot{y}_i(t) = \varphi_i(x_i(t)) \\ x_i(t) = \displaystyle\sum_{j=1}^{K}\int_{-\infty}^{t} h_{ji}(t-s)\, y_j(s)\, ds \end{cases}$$

Here $\varphi_i : \mathbb{R} \to \mathbb{R}$ is a function called the non linearity (N.L.), which often is a sigmoid (a smoothed Heavyside step), and the $h_{ij}$ are impulse responses of filters. In the non linear version of Hawkes processes, we have $K$ nodes, at which we find the spike comb $N_i$ with the stochastic intensity

$$\lambda_i(t) = \varphi_i\left(\sum_{j=1}^{N} X_{ij}(t)\right)$$

with

$$X_{ij}(t) = (h_{ji} * N_j)(t) := \int_{-\infty}^{t^-} h_{ji}(t-s)\, N_j(ds) = \sum_{n\in\mathbb{Z}} h_{ji}\left(t - T_n^{(j)}\right)$$

where $\{T_n^{(j)}\}_{n\in\mathbb{Z}}$ is the sequence of points of $N_j$. The corresponding diagram is shown in Figure 6.

In this article, we are concerned with Hawkes branching point processes on $\mathbb{R}^m$. Such processes are constructed as follows

$$N = \sum_{n\geq 0} N_n,$$

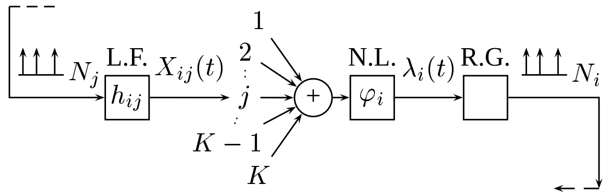


Fig. 6. Diagram of Hopfield networks: L.F. is a linear filter; N.L. is a non linearity; R.G. is a random generator.

where each $N_n$ is the basic point process on $\mathbb{R}^m$ of a marked point process $\overline{N}_n$ on $\mathbb{R}^m \times K$ with i.i.d. marks. All the mark processes $\{Z_n(t)\}$, $n \geq 0$, $t \in \mathbb{R}^m$, are i.i.d., and in particular the distribution $Q$ of $Z_n(t)$ is independent of $n$ and $t$, and independent of $N_0$. $N_0$ is a simple locally finite stationary point process with a Bartlett spectrum $\mu_0$. It is called the "ancestor process". The $N_n$'s, $n \geq 1$, are constructed recursively as follows. First we are given a rate function $h : \mathbb{R}^m \times K \to \mathbb{R}^+$ such that

$$\int_{\mathbb{R}^m} \mathrm{E}\,[h(t,Z)]dt < \infty,$$

where $Z$ is a $K$-valued random variable with distribution $Q$ (the general mark distribution). We denote by $\mathcal{F}_n$ the $\sigma$-field recording all the events relative to $N_0, \ldots, N_n$. Then $N_n$ is, conditionally on $\mathcal{F}_{n-1}$, a Poisson process on $\mathbb{R}^m$ with the intensity

$$\lambda_n(t) = \int\limits_{\mathbb{R}^m \times K} h(t-s,z)\overline{N}_{n-1}(ds \times dz). \tag{24}$$

$N_n$ is called the $n$-th generation point process. The interpretation is the following: each point $a \in N_{n-1}$ of generation $n-1$ creates descendants in the next generation according to a Poisson process of intensity

$$h(t-a, Z_{n-1}(a)).$$

On average this point $a$ creates

$$\mathrm{E}\left[\int_{\mathbb{R}^m} h(t-a, Z_{n-1}(a))dt\right],$$

that is

$$\rho := \int_{\mathbb{R}^m} \mathrm{E}\,[h(t,Z)]dt$$

descendants. A sufficient condition for the process $N$ to be a locally finite point process is that $\rho < 1$, as follows from the branching process interpretation, each ancestor (point of $N_0$) being the ancestor of an eventually extinguishing branching process of average progeny $\rho < 1$.

Denote

$$N' = \sum_{n \geq 1} N_n.$$

From (24) and the Campbell formula [27] we see that, denoting $\lambda_n = \mathrm{E}\left[\lambda_n(t)\right]$,

$$\lambda_n = \lambda_{n-1} \int_{\mathbb{R}^m} \mathrm{E}\left[h(t,Z)\right] dt$$

and therefore the average intensity $\lambda'$ of $N'$ verifies

$$\lambda' = \lambda_0 + \rho\lambda'.$$

Therefore, if $\lambda_0 > 0$, in order for $N'$ to have a finite intensity, we must impose

$$\mathrm{E}\left[\int_{\mathbb{R}^m} h\left(t,Z\right) dt\right] < 1. \tag{25}$$

In this sense $\rho < 1$ is "almost" necessary and sufficient (for the case $\rho = 1$, see [34]).

*Lemma 5.1:* For $\varphi \in \mathrm{L}^1(\ell \times Q) \cap \mathrm{L}^2(\ell \times Q)$,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m \times K} \varphi(t,z)\overline{M}'(dt \times dz)\right) = \lambda' \int_{\mathbb{R}^m} \mathrm{E}\left[\left|\varphi(t,Z)\right|^2\right] dt. \tag{26}$$

where

$$\overline{M}'(dt \times dz) = \overline{N}'(dt \times dz) - \lambda'\left(t\right) dtQ\left(dz\right),$$

$$\lambda'\left(t\right) = \int_{\mathbb{R}^m} h\left(t-s,z'\right) \overline{N}_0\left(dt \times dz'\right) + \int_{\mathbb{R}^m} h\left(t-s,z'\right) \overline{N}'\left(dt \times dz\right).$$

*Proof:* We shall use simplified notation of the kind $\int\int \varphi(t,z)\overline{M}'(dt \times dz) = \int \varphi d\overline{M}'$. We have

$$\int \varphi d\overline{M}' = \sum_{n\geq 1} \int \varphi d\overline{M}_n,$$

where $\overline{M}_n(dt \times dz) = \overline{N}_n(dt \times dz) - \lambda_n(t)dtQ(dz)$. Given $\mathcal{F}_{n-1}$, $\overline{N}_n$ is a Poisson process with mean measure

$$\lambda_{n-1}(t)Q(dz)dt$$

and therefore

$$\begin{aligned}\mathrm{Var}\left(\int \varphi d\overline{M}_n \middle| \mathcal{F}_{n-1}\right) &= \int_{\mathbb{R}^m \times K} \varphi^2(t,z)\lambda_{n-1}(t)Q(dz)dt \\ &= \int_{\mathbb{R}^m} \mathrm{E}\left[\varphi^2(t,Z)\right]\lambda_{n-1}(t)dt,\end{aligned}$$

and

$$\mathrm{E}\left[\int \varphi d\overline{M}_n \middle| \mathcal{F}_{n-1}\right] = 0.$$

Therefore

$$\begin{aligned}\mathrm{Var}\left(\int \varphi d\overline{M}_n\right) &= \mathrm{E}\left[\mathrm{Var}\left(\int \varphi d\overline{M}_n \middle| \mathcal{F}_{n-1}\right)\right] + \mathrm{Var}\left(\mathrm{E}\left[\int \varphi d\overline{M}_n \middle| \mathcal{F}_{n-1}\right]\right) \\ &= \lambda_{n-1} \int_{\mathbb{R}^m \times K} \mathrm{E}\left[\varphi^2(t,Z)\right] dt.\end{aligned}$$

Also for $j, k \geq 1$,

$$\mathrm{E}\left[\left(\int \varphi d\overline{M}_j\right)\left(\int \varphi d\overline{M}_{j+k}\right)\right] = \mathrm{E}\left[\left(\int \varphi d\overline{M}_j\right)\mathrm{E}\left[\int \varphi d\overline{M}_{j+k}\middle|\mathcal{F}_{j+k-1}\right]\right] = 0.$$

Therefore

$$\begin{aligned}\mathrm{Var}\left(\int \varphi d\overline{M}'\right) &= \sum_{n\geq 1}\mathrm{Var}\left(\int \varphi d\overline{M}_n\right)\\ &= \left(\sum_{n\geq 1}\lambda_{n-1}\right)\int_{\mathbb{R}^m}\mathrm{E}\left[\varphi^2(t,Z)\right]dt\\ &= \lambda'\int_{\mathbb{R}^m}\mathrm{E}\left[\varphi^2(t,Z)\right]dt.\end{aligned}$$

■

*Lemma 5.2:* A. Suppose that

$$\mathrm{E}\left[\left|\int_{\mathbb{R}^m} h\left(t,Z\right)dt\right|^2\right] < \infty. \tag{27}$$

There exists, for given $F:\mathbb{R}^m\times K\to\mathbb{R}$ such that $F\left(t,z\right)\in \mathrm{L}^1\left(\ell\times Q\right)\cap\mathrm{L}^2\left(\ell\times Q\right)$, that is

$$\int_{\mathbb{R}^m\times K}\left|F(t,z)\right|dtQ(dz) < \infty, \tag{28}$$

$$\int_{\mathbb{R}^m\times K}\left|F(t,z)\right|^2 dtQ(dz) < \infty, \tag{29}$$

a unique function $\varphi:\mathbb{R}^m\times K\to\mathbb{R}\in\mathrm{L}^1\left(\ell\times Q\right)\cap\mathrm{L}^2\left(\ell\times Q\right)$, such that

$$\varphi(t,z) - \int_{\mathbb{R}^m} h(s-t,z)\mathrm{E}\left[\varphi(s,Z)\right]ds = F(t,z). \tag{30}$$

B. For given $f:\mathbb{R}^m\to\mathbb{R}$ such that $f\in\mathrm{L}^1\cap\mathrm{L}^2$, there exists a unique $\varphi:\mathbb{R}^m\times K\to\mathbb{R}$ with the same properties as $F$ above and such that

$$\varphi(t,z) - \int_{\mathbb{R}^m} h(s-t,z)\mathrm{E}\left[\varphi(s,Z)\right]ds = f(t). \tag{31}$$

*Proof:* A. For a function $v\left(t,z\right)$, denote $\mathrm{E}\left[v\left(t,Z\right)\right]$ by $\overline{v}\left(t\right)$ and $v\left(-t,z\right)$ by $\check{v}\left(t,z\right)$. Let $h$ and $f$ be as in the statement of the lemma, and consider the renewal equation

$$g = \overline{F} + \check{\overline{h}} * g. \tag{32}$$

Since $\overline{F}\in\mathrm{L}^1\cap\mathrm{L}^2$, and since condition (25) holds, there exists a unique solution $g\in\mathrm{L}^1\cap\mathrm{L}^2$ given by

$$g = \sum_{n\geq 0}\overline{F} * \check{\overline{h}}^{*n} \tag{33}$$

(the convergence of the series in $\mathrm{L}^1$ and in $\mathrm{L}^2$ is guaranteed by the inequalities $\|a*b\|_{\mathrm{L}^1}\leq\|a\|_{\mathrm{L}^1}\|b\|_{\mathrm{L}^1}$ and $\|a*b\|_{\mathrm{L}^2}\leq\|a\|_{\mathrm{L}^1}\|b\|_{\mathrm{L}^2}$; uniqueness follows from the equality $g-g'=\check{\overline{h}}*(g-g')$, where $g'$ is another candidate solution, which implies $\|g-g'\|_{\mathrm{L}^1} \leq \left\|\overline{h}\right\|_{\mathrm{L}^1} \|g-g'\|_{\mathrm{L}^1}$, and hence under condition (25), necessarily $\|g-g'\|_{\mathrm{L}^1}=0$). The Fourier transform of $g$ is therefore

$$\widehat{g}\left(\nu\right)=\frac{\mathrm{E}\left[\widehat{F}\left(\nu,Z\right)\right]}{1-\mathrm{E}\left[T\left(\nu,Z\right)^{*}\right]}. \tag{34}$$

Define now $\varphi\left(t,z\right)$ by

$$\varphi\left(t,z\right)=\int_{\mathbb{R}^m} h\left(s-t,z\right)g\left(s\right)ds+F\left(t,z\right). \tag{35}$$

We have

$$\mathrm{E}\left[\int_{\mathbb{R}^m}\left|\varphi\left(t,Z\right)\right|dt\right] \;\leq\; \mathrm{E}\left[\int_{\mathbb{R}^m}\left|F\left(t,Z\right)\right|dt\right] \;+\; \mathrm{E}\left[\int_{\mathbb{R}^m}\left|h\left(t,Z\right)\right|dt\right]\int_{\mathbb{R}^m}\left|g\left(t\right)\right|dt \;<\; \infty$$

since $g\in\mathrm{L}^1$, and $F,h\in\mathrm{L}^1\left(\ell\times Q\right)$. Therefore $\varphi\in\mathrm{L}^1\left(\ell\times Q\right)$. We now show that $\varphi\in\mathrm{L}^2\left(\ell\times Q\right)$. For this we take for $z$ fixed the Fourier transform of both sides of (35)

$$\widehat{\varphi}\left(\nu,z\right)=T\left(\nu,z\right)^{*}\widehat{g}\left(\nu\right)+\widehat{F}\left(\nu,z\right),$$

which gives in view of (34)

$$\widehat{\varphi}\left(\nu,z\right)=\widehat{F}\left(\nu,z\right)+\frac{T\left(\nu,z\right)^{*}\,\mathrm{E}\left[\widehat{F}\left(\nu,Z\right)\right]}{1-\mathrm{E}\left[T\left(\nu,Z\right)^{*}\right]}. \tag{36}$$

We show that $\widehat{\varphi}\left(\nu,z\right)\in\mathrm{L}^2\left(\ell\times Q\right)$. Since $\widehat{F}\left(\nu,z\right)\in\mathrm{L}^2\left(\ell\times Q\right)$ by hypothesis (29), it remains to show that

$$\frac{T\left(\nu,z\right)^{*}\,\mathrm{E}\left[\widehat{F}\left(\nu,Z\right)\right]}{1-\mathrm{E}\left[T\left(\nu,Z\right)^{*}\right]}\in\mathrm{L}^2\left(\ell\times Q\right). \tag{37}$$

First, we observe that $\left|1-\mathrm{E}\left[T\left(\nu,Z\right)^{*}\right]\right|$ is bounded from $0$ uniformly in $\nu$. Indeed

$$\left|1-\mathrm{E}\left[T\left(\nu,Z\right)^{*}\right]\right|\geq 1-\left|\mathrm{E}\left[T\left(\nu,Z\right)^{*}\right]\right|,$$

and

$$\left|\mathrm{E}\left[T\left(\nu,Z\right)\right]\right|=\left|\mathrm{E}\left[\int_{\mathbb{R}^m}h\left(t,Z\right)e^{2i\pi\nu t}dt\right]\right|\leq\int_{\mathbb{R}^m}\left|\mathrm{E}\left[h\left(t,Z\right)\right]\right|dt<1.$$

Therefore, to prove (37) it suffices to show that

$$T\left(\nu,z\right)^{*}\,\mathrm{E}\left[\widehat{F}\left(\nu,Z\right)\right]\in\mathrm{L}^2\left(\ell\times Q\right).$$

This follows from the fact that $\widehat{g}\in\mathrm{L}^2$ and

$$\mathrm{E}\left[\left|T\left(\nu,Z\right)\right|^2\right]=\mathrm{E}\left[\left|\int_{\mathbb{R}^m}h\left(t,Z\right)e^{2i\pi\nu t}dt\right|^2\right]\leq\mathrm{E}\left[\left|\int_{\mathbb{R}^m}h\left(t,Z\right)dt\right|^2\right]$$

(by hypothesis (27)). Therefore, $\widehat{\varphi}(\nu, z) \in \mathrm{L}^2(\ell \times Q)$ and hence, using the Plancherel-Parseval equality

$$\mathrm{E}\left[\int_{\mathbb{R}^m} |\varphi(t, Z)|^2\, dt\right] = \mathrm{E}\left[\int_{\mathbb{R}^m} |\widehat{\varphi}(\nu, Z)|^2\, d\nu\right] < \infty.$$

B. This is clearly a particular case of A. Indeed, we have

$$\widehat{\varphi}(\nu, z) = \widehat{f}(\nu)\left(1 + \frac{\widehat{h}^*(\nu, z)}{1 - \mathrm{E}\left[\widehat{h}^*(\nu, Z)\right]}\right). \tag{38}$$

■

*Theorem 5.1:* Let $h(t, z)$ verify (25) and (27). The Bartlett spectrum of $N$ defined above is

$$\boxed{\mu_N(d\nu) = \tfrac{1}{|1-\mathrm{E}[T(\nu,Z)]|^2}\left[\mu_0(d\nu) + \lambda' d\nu + \lambda \mathrm{Var}\left(T(\nu, Z) d\nu\right)\right].} \tag{39}$$

*Proof:*

$$\begin{aligned}\int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\overline{M}'(dt\times dz) &= \int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\left(\overline{N}'(dt\times dz) - \lambda'(t)Q(dz)dt\right)\\ &= \int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\overline{N}'(dt\times dz) - \int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\lambda'(t)Q(dz)dt.\end{aligned}$$

Also,

$$\begin{aligned}\int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\lambda'(t)Q(dz)dt &= \int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\left(\int\limits_{\mathbb{R}^m\times K} h(t-s,z')\overline{N}(ds\times dz')\right)Q(dz)dt\\ &= \int_{\mathbb{R}^m\times\mathbb{R}^m\times K} h(t-s,z')\mathrm{E}\left[\varphi(t,Z)\right]dt\overline{N}(ds\times dz')\\ &= \int\limits_{\mathbb{R}^m\times K} \left(\check{h}(s-\cdot,z') * \mathrm{E}\left[\varphi(\cdot,Z)\right]\right)(s)\,\overline{N}(ds\times dz').\end{aligned}$$

Therefore, since $\overline{N} = \overline{N}' + \overline{N}_0$,

$$\begin{aligned}\int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\overline{M}'(dt\times dz) &= \int\limits_{\mathbb{R}^m\times K} \left(\varphi(t,z) - \left(\check{h}(t-\cdot,z) * \mathrm{E}\left[\varphi(\cdot,Z)\right]\right)(t)\right)\overline{N}(dt\times dz)\\ &\quad - \int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\overline{N}_0(dt\times dz).\end{aligned}$$

Take $f \in \mathrm{L}^1 \cap \mathrm{L}^2$, and let $\varphi(t, z)$ be the solution of (31). We have therefore

$$\int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\overline{M}'(dt\times dz) + \int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\overline{N}_0(dt\times dz) = \int_{\mathbb{R}^m} f(t)N(dt).$$

Also, by the isometry lemma,

$$\mathrm{Var}\left(\int\limits_{\mathbb{R}^m\times K} \varphi(t,z)\overline{M}'(dt\times dz)\right) = \lambda' \int_{\mathbb{R}^m} \mathrm{E}\left[|\varphi(t,Z)|^2\right]dt = \lambda' \int_{\mathbb{R}^m} \mathrm{E}\left[|\widehat{\varphi}(\nu,Z)|^2\right]d\nu.$$

Now,

$$\mathrm{E}\left[\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{M}'(dt\times dz)\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{N}_0(dt\times dz)\right]$$
$$=\mathrm{E}\left[\mathrm{E}\left[\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{M}'(dt\times dz)\,\middle|\,\mathcal{F}_0\right]\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{N}_0(dt\times dz)\right]=0.$$

Therefore,

$$\mathrm{Var}\left(\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{M}'(dt\times dz)+\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{N}_0(dt\times dz)\right)=$$
$$\mathrm{Var}\left(\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{M}'(dt\times dz)\right)+\mathrm{Var}\left(\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{N}_0(dt\times dz)\right)=$$
$$\lambda'\int_{\mathbb{R}^m}\mathrm{E}\left[|\varphi(t,Z)|^2\right]dt+\mathrm{Var}\left(\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{N}_0(dt\times dz)\right).$$

On the other hand, by Theorem 4.1,

$$\mathrm{Var}\left(\int\limits_{\mathbb{R}^m\times K}\varphi(t,z)\overline{N}_0(dt\times dz)\right)=\int_{\mathbb{R}^m}\left|\mathrm{E}\left[\widehat{\varphi}(\nu,Z)\right]\right|^2\mu_0(d\nu)+\lambda_0\int_{\mathbb{R}^m}\mathrm{Var}\left(\widehat{\varphi}(\nu,Z)\right)d\nu.$$

Combining the above, we have

$$\begin{aligned}\mathrm{Var}\left(\int_{\mathbb{R}^m}f(t)N(dt)\right)&=\lambda'\int_{\mathbb{R}^m}\mathrm{E}\left[|\varphi(\nu,Z)|^2\right]d\nu+\int_{\mathbb{R}^m}\left|\mathrm{E}\left[\widehat{\varphi}(\nu,Z)\right]\right|^2\mu_0(d\nu)\\&\quad+\lambda_0\int_{\mathbb{R}^m}\mathrm{Var}\left(\widehat{\varphi}(\nu,Z)\right)d\nu\\&=A+B+C.\end{aligned}$$

By Formula (38),

$$\begin{aligned}A&=\lambda'\int_{\mathbb{R}^m}|\widehat{f}(\nu)|^2\frac{1+\mathrm{Var}\left(T(\nu,Z)\right)}{|1-\mathrm{E}\left[T(\nu,Z)\right]|^2}d\nu,\\B&=\int_{\mathbb{R}^m}|\widehat{f}(\nu)|^2\frac{1}{|1-\mathrm{E}\left[T(\nu,Z)\right]|^2}\mu_0(d\nu),\\C&=\int_{\mathbb{R}^m}|\widehat{f}(\nu)|^2\lambda_0\frac{\mathrm{Var}\left(T(\nu,Z)\right)}{|1-\mathrm{E}\left[T(\nu,Z)\right]|^2}d\nu.\end{aligned}$$

Recalling that $\lambda'=\lambda_0/(1-\rho)$, we obtain finally that for all $f\in\mathrm{L}^1\cap\mathrm{L}^2$,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m}f(t)N(dt)\right)=\int_{\mathbb{R}^m}|\widehat{f}(\nu)|^2\left(\frac{1}{|1-\mathrm{E}\left[T(\nu,Z)\right]|^2}\right)\left(\mu_0(d\nu)+\lambda'd\nu+\lambda\mathrm{Var}\left(T(\nu,Z)\right)d\nu\right),$$

and this allows us to identify $\mu_N$ as (39). ■

In the particular case

$$h(t, Z) = h(t),$$

$T(\nu, Z) = T(\nu)$, and we have

$$\mu_N(d\nu) = \frac{1}{|1 - T(\nu)|^2} \left[\mu_0(d\nu) + \lambda' d\nu\right].$$

If in addition $N_0$ is a Poisson process with average intensity $\alpha$, since $\alpha + \lambda' = \lambda$, we have the original formula of Hawkes [35]

$$\mu_N(d\nu) = \frac{\lambda d\nu}{|1 - T(\nu)|^2}.$$

We now consider a shot noise based on the Hawkes branching point process $N$ of the previous section,

$$X(t) = \int_{\mathbb{R}^m \times K} \alpha(t - s, z) \overline{N}(ds \times dz). \tag{40}$$

---

EXAMPLE 5.1: In the univariate case $\mathbb{R}^m = \mathbb{R}$,

$$X(t) = \sum_{n \in \mathbb{Z}} \alpha(t - T_n, Z_n).$$

We observe that this situation is not covered by the results presented in Section IV, because the sequence $\{Z_n\}_{n \in \mathbb{Z}}$ of marks is not independent of $\{T_n\}_{n \in \mathbb{Z}}$. To further specialize this example, take

$$h(t, z) = \beta 1_{[0,Z]}(t),$$
$$\lambda(t) = \alpha + \int\limits_{\mathbb{R} \times K} h(t - s, z) N(ds \times dz),$$

and

$$\alpha(t, z) = 1_{[0,z]}(t).$$

Therefore interpreting $T_n$ as the birth time of individual $n$ in colony, and $Z_n$ as its lifetime,

$$X(t) = \sum_{n \in \mathbb{Z}} 1_{(-\infty,t]}(T_n) 1_{(t,+\infty)}(T_n + Z_n)$$

is the number of individuals in the colony, and

$$\lambda(t) = \alpha + \beta X(t).$$

If moreover we assume that $Z_n$ is exponentially distributed with parameter $\gamma$, the process $\{X\,(t)\}$ is a Markov birth and death process with infinitesimal generator $Q$ given by its non-null terms $q_{i,i+1} = \alpha + \beta i$, $q_{i,i-1} = \gamma i$.

---

*Theorem 5.2:* For the process $\{X(t)\}$ defined by (40), under the conditions stated in Theorem 5.1, to which we add

$$\mathrm{E}\left[\left|\int_{\mathbb{R}^m} |\alpha(t,Z)|\, dt\right|^2\right] < \infty$$

the Cramér spectral measure $\mu_X$ is given by the expression

$$|1-\mathrm{E}[T(\nu,Z)]|^2 \mu_X(d\nu) = |\mathrm{E}[\widehat{\alpha}(\nu,Z)]|^2 \left(\mu_0(d\nu) + \frac{\lambda_0}{1-\rho} d\nu\right) + \frac{\lambda_0\rho}{1-\rho} \mathrm{Var}\{\widehat{\alpha}(\nu,Z)(1-\mathrm{E}[T(\nu,Z)]) + T(\nu,Z)\,\mathrm{E}[\widehat{\alpha}(\nu,Z)]\}\, d\nu. \quad (41)$$

*Proof:* We seek to find a measure $\mu_X$ such that for all $f \in \mathrm{L}^1 \cap \mathrm{L}^2$

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} f(t)\, X(t)\, dt\right) = \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \mu_X(d\nu). \quad (42)$$

But

$$\int_{\mathbb{R}^m} f(t)\, X(t)\, dt = \int_{\mathbb{R}^m} f(t) \left(\int_{\mathbb{R}^m\times K} \alpha(t-s,z)\overline{N}(ds\times dz)\right) dt = \int_{\mathbb{R}^m\times K} F(s,z)\, \overline{N}(ds\times dz),$$

where

$$F(s,z) = \int_{\mathbb{R}^m} \alpha(t-s,z) f(t)\, dt = (\check{\alpha}(\cdot,z) * f)(s)$$

is a function in $\mathrm{L}^1(\ell\times Q) \cap \mathrm{L}^2(\ell\times Q)$ (use the hypothesis $\alpha \in \mathrm{L}^1(\ell\times Q)$, $\mathrm{E}\left[\int_{\mathbb{R}} |\alpha(t,Z)|^2 dt\right] < \infty$ and $f \in \mathrm{L}^1 \cap \mathrm{L}^2$). Therefore we seek $\mu_X$ such that

$$\mathrm{Var}\left(\int_{\mathbb{R}^m\times K} F(s,z)\, \overline{N}(ds\times dz)\right) = \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \mu_X(d\nu).$$

Following the same calculations as in the proof of Theorem 5.1 up to the 3rd displayed equation thereof, and letting $\varphi$ be the unique solution in $\mathrm{L}^1(\ell\times Q) \cap \mathrm{L}^2(\ell\times Q)$ of equation (30) of Lemma 5.2, we have

$$\int_{\mathbb{R}^m\times K} F(s,z)\, \overline{N}(ds\times dz) = \int_{\mathbb{R}^m\times K} \varphi(t,z)\, \overline{M}'(dt\times dz) + \int_{\mathbb{R}^m\times K} \varphi(t,z)\, \overline{N}_0(dt\times dz).$$

Resuming the proof of Theorem 5.1 after the 4th displayed equation thereof, we obtain

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} f(t)\, X(t)\, dt\right) = \lambda' \int_{\mathbb{R}^m} \mathrm{E}\left[|\widehat{\varphi}(\nu,Z)|^2\right] d\nu + \int_{\mathbb{R}^m} |\mathrm{E}[\widehat{\varphi}(\nu,Z)]|^2 \mu_0(\nu) + \lambda_0 \int_{\mathbb{R}^m} \mathrm{Var}(\widehat{\varphi}(\nu,Z)) d\nu = A + B + C$$

where, using the expression for $\widehat{\varphi}(\nu, z)$

$$\begin{aligned}\widehat{\varphi}(\nu, z) &= \widehat{F}(\nu, z) + \frac{T(\nu, z)^* \, \mathrm{E}\left[\widehat{F}(\nu, Z)\right]}{1 - \mathrm{E}\left[T(\nu, Z)^*\right]} \\ &= \widehat{f}(\nu)\left[\widehat{\alpha}(\nu, z)^* + \frac{T(\nu, z)^* \, \mathrm{E}\left[\widehat{\alpha}(\nu, Z)^*\right]}{1 - \mathrm{E}\left[T(\nu, Z)^*\right]}\right]\end{aligned}$$

we find that

$$\begin{aligned}A &= \lambda' \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \frac{\mathrm{E}\left[\left|\widehat{\alpha}(\nu, Z)(1 - \mathrm{E}[T(\nu, Z)]) + T(\nu, Z)\,\mathrm{E}[\widehat{\alpha}(\nu, Z)]\right|^2\right]}{\left|1 - \mathrm{E}[T(\nu, Z)]\right|^2} d\nu, \\ B &= \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \frac{\left|\mathrm{E}[\widehat{\alpha}(\nu, Z)]\right|^2}{\left|1 - \mathrm{E}[T(\nu, Z)]\right|^2} \mu_0(d\nu), \\ C &= \lambda_0 \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \frac{\mathrm{Var}\left(\widehat{\alpha}(\nu, Z)(1 - \mathrm{E}[T(\nu, Z)]) + T(\nu, Z)\,\mathrm{E}[\widehat{\alpha}(\nu, Z)]\right)}{\left|1 - \mathrm{E}[T(\nu, Z)]\right|^2} d\nu.\end{aligned}$$

Therefore, using the expression $\lambda' = \lambda_0 / 1 - \rho$, we find after rearrangement (42) with $\mu_X(d\nu)$ given by (41). ∎

## VI. MODULATED RANDOM SPIKE FIELDS

Random sampling of a continuous time random signal $X(t)$, $t \in \mathbb{R}$, yields a sequence of samples

$$X(T_n), \quad n \in \mathbb{Z} \tag{43}$$

where $T_n$, $n \in \mathbb{Z}$, is the sequence of points (times of events) of a point process.

At the extremities of the spectrum of randomness, we find the completely random sampling, or Poisson sampling, where $T_n$, $n \in \mathbb{Z}$, is a homogeneous Poisson process [24], and the regular sampling, where $T_n = nT$, $T^{-1}$ being the sampling frequency.

Random sampling is in most cases not deliberate. For instance: In regular sampling, some samples may be lost. If the probability of loss of a sample is $p$, and the losses are independent, then the times between samples form an i.i.d. sequence with geometric distribution

$$P(T_{n+1} - T_n = rT) = p^{r-1}(1 - p), \quad r \geq 1;$$

Regular sampling can become random due to *jitter* (lack of synchronization).

$$T_n = nT + \Delta_n;$$

Random sampling may be inherent to the sampling procedure. For instance, in laser velocimetry, a sample is collected only at the passage of a “grain” of matter through the laser beam [13].

The signal $X(t)$, $t \in \mathbb{R}$, is called the *sampled signal*, the point process $T_n$, $n \in \mathbb{Z}$, the *sampler*, the sequence (43) is the *sample sequence*, and the process

$$Y(t) = \sum_{n \in \mathbb{Z}} X(T_n)\,\delta(t - T_n) \tag{44}$$

where $\delta(t)$ is the Dirac pulse, is called the *sample comb*.

The sampled signal and the sampler are assumed independent, and stationary (or at least wide-sense stationary for the sampled process). However, we shall also consider in this article dependent sampling.

The average intensity $\lambda$ of the sampler is, by definition, the average number of sampling time per unit time, and the sampling frequency is then

$$\nu_s = \lambda.$$

Two well know results concern regular sampling and Poisson sampling, the two extremal cases.

EXAMPLE 6.1: In regular sampling, the spectrum of the sample comb is an aliased version of that of the sampled signal. For instance, in the case of a power spectral density

$$f_Y(\nu) = \sum_{n \in \mathbb{Z}} f_X\left(\nu - \frac{n}{T}\right)$$

and the sampled signal can be entirely recovered from the sample comb provided the former is band limited, with band width $2B < \nu_s = \frac{1}{T}$. It suffices to filter the sample comb with a low-pass of cutoff frequency $B$ ([36]).

EXAMPLE 6.2: In Poisson sampling, the spectrum of the sample comb is (density case)

$$f_Y(\nu) = \lambda^2 f_X(\nu) + \lambda \sigma_X^2$$

where $\sigma_X^2 = \mathrm{Var}(X(t))$ is the power of the sampled signal. Therefore, whatever the sampling frequency $\nu_s = \lambda$, there is no aliasing, and the spectrum of the sampled signal can be recovered from that of the sample comb. However, if we apply the sample comb to a low-pass of cutoff frequency $\nu_s = \lambda$, the output signal, $Z(t)$, $t \in \mathbb{R}$, is the **worse** reconstruction of the sampled signal, assumed band-limited with bandwidth 2B, in the sense that

$$\mathrm{E}\left[|Z(t) - X(t)|^2\right] = \sigma_X^2 \frac{2B}{\lambda}\cdot$$

We formulate random sampling in the general spatial case. Here the sampled signal is a wide-sense stationary process

$$X(t), \quad t = (t_1, \ldots, t_m) \in \mathbb{R}^m$$

with mean $m_X$, autocovariance function $C_X(\tau)$ (and therefore $R_X(\tau) = \mathrm{E}[X(t+\tau)X(t)^*] = C_X(\tau) + |m_X|^2$), and power spectral measure $\mu_X$, where

$$C_X(\tau) = \int_{\mathbb{R}^m} e^{2i\pi\langle\nu,\tau\rangle}\mu_X(d\nu)$$

with $\nu = (\nu_1, \ldots, \nu_m)$ and $<\nu, \tau> = \nu_1\tau_1 + \ldots + \nu_m\tau_m$. Moreover, $\{X(t)\}_{t\in\mathbb{R}^m}$ admits the Cramér-Khinchin's decomposition $\{Z_X(A)\}$, $A \subseteq \mathcal{B}(\mathbb{R}^m)$ (see Dacunha and Duflo [37]). Recall that the latter is a complex-valued stochastic process, with centered and orthogonal increments

$$\mathrm{E}[Z_X(C)] = 0$$

$$\mathrm{E}[Z_X(C)Z_X(D)^*] = 0$$

for all $C, D \subseteq \mathcal{B}(\mathbb{R}^m)$ that are disjoint. Moreover

$$\mathrm{E}\left[|Z_X(C)|^2\right] = \mu_X(C),$$

where $\mu_X$ is the Cramér spectral measure of $\{X(t)\}_{t\in\mathbb{R}^m}$.

Finally, we have the Cramér-Khinchin decomposition

$$X(t) = \int_{\mathbb{R}^m} e^{2i\pi\langle\nu,t\rangle}Z_X(d\nu) + m_X \tag{45}$$

where the integral thereof is a Wiener integral. Note that for all functions $g \in \mathrm{L}^2_{\mathbb{C}}(\mu_X)$, the Wiener integral $\int_{\mathbb{R}^m} g(\nu)Z_X(d\nu)$ is well defined, and it is in $\mathrm{L}^2_{\mathbb{C}}(\mathrm{P})$; moreover

$$\mathrm{E}\left[\left|\int_{\mathbb{R}^m} g(\nu)Z_X(d\nu)\right|^2\right] = \int_{\mathbb{R}^m} |g(\nu)|^2\mu_X(d\nu). \tag{46}$$

The sample “brush” is

$$Y(t) = \sum_{s\in N} X(s)\delta(t-s) \tag{47}$$

and can be identified with the measure

$$\sum_{s\in N} X(s)\delta_s(\cdot).$$

For the sample brush we consider the generalized Cramér-Khinchin spectrum ([37]), that is, a measure $\mu_X(d\nu)$ such that, for any $\varphi(t) \in \mathrm{L}^1 \cap \mathrm{L}^2$,

$$\boxed{\mathrm{Var}\left(\int_{\mathbb{R}^m}\varphi(t)X(t)\,dt\right) = \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\mu_X(d\nu)} \tag{48}$$

A first result concerns the spectrum when the sampler is independent from the signal. Let $N$ be a wide-sense stationary simple point process on $\mathbb{R}^m$ with intensity $\lambda < \infty$, and Bartlett spectrum $\mu_N$.

*Theorem 6.1:* Suppose that $\{X(t)\}$ and $N$ are independent. Then, the generalized process

$$Y(t) = \sum_{s \in N} X(s)\, \delta(t - s)$$

admits the extended Cramér-Khinchin power spectral measure

$$\boxed{\mu_Y = \mu_N * \mu_X + \lambda^2 \mu_X + |m_X|^2\, \mu_N.} \tag{49}$$

*Proof:* Let us consider the definition of the generalized Cramér-Khinchin spectrum (48)) with $\varphi \in \mathrm{L}^1_{\mathbb{R}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{R}}(\mathbb{R}^m)$. Here

$$\begin{aligned}
\int_{\mathbb{R}^m} \varphi(t)\, Y(t)\, dt &= \int_{\mathbb{R}^m} \varphi(t) \left( \sum_{s \in N} X(s)\, \delta(t-s) \right) dt \\
&= \sum_{s \in N} \varphi(s)\, X(s) \\
&= \int_{\mathbb{R}^m} \varphi(t)\, X(t)\, N(dt)\, .
\end{aligned}$$

Recall from (45) that $X(t) = \int_{\mathbb{R}^m} e^{2i\pi\langle \nu, t\rangle} Z_X(d\nu) + m_X$ and therefore

$$\begin{aligned}
\int_{\mathbb{R}^m} \varphi(t)\, X(t)\, N(dt) &= \int_{\mathbb{R}^m} \varphi(t) \left( \int_{\mathbb{R}^m} e^{2i\pi\langle \nu, t\rangle} Z_X(d\nu) + m_X \right) N(dt) \\
&= \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi\langle \nu, t\rangle} N(dt) \right) Z_X(d\nu) \\
&\quad + m_X \int_{\mathbb{R}^m} \varphi(t)\, N(dt)\, ,
\end{aligned}$$

where we have formally exchanged the order of integration. Since the integrals with respect to $N(dt)$ and with respect to $Z_X(d\nu)$ are of a different nature (one is a usual infinite sum, the other is a Wiener integral), this exchange must be formally justified, which we do in Appendix A.

Using the conditional variance formula, we have

$$\begin{aligned}
&\mathrm{Var}\left( \int_{\mathbb{R}^m} \varphi(t)\, X(t)\, N(dt) \right) \\
&\quad = \mathrm{E}\left[ \mathrm{Var}\left( \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi\langle \nu, t\rangle} N(dt) \right) Z_X(d\nu) + m_X \int_{\mathbb{R}^m} \varphi(t)\, N(dt) \,\middle|\, \mathcal{F}^N_\infty \right) \right] \\
&+ \mathrm{Var}\left( \mathrm{E}\left[ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi\langle \nu, t\rangle} N(dt) \right) Z_X(d\nu) + m_X \int_{\mathbb{R}^m} \varphi(t)\, N(dt) \,\middle|\, \mathcal{F}^N_\infty \right] \right) = \alpha + \beta.
\end{aligned}$$

Using the fact that, when $N$ is fixed, $m_X \int_{\mathbb{R}^m} \varphi(t) N(dt)$ is deterministic,

$$\begin{aligned}
\alpha &= \mathrm{E}\left[\mathrm{Var}\left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\, e^{2i\pi\langle\nu,t\rangle}N(dt)\right)Z_X(d\nu)\middle|\,\mathcal{F}_\infty^N\right)\right] \\
&= \mathrm{E}\left[\int_{\mathbb{R}^m}\left|\int_{\mathbb{R}^m}\varphi(t)\, e^{2i\pi\langle\nu,t\rangle}N(dt)\right|^2\mu_X(d\nu)\right] \text{(by eq. (46))} \\
&= \int_{\mathbb{R}^m}\mathrm{E}\left[\left|\int_{\mathbb{R}^m}\varphi(t)\, e^{2i\pi\langle\nu,t\rangle}N(dt)\right|^2\right]\mu_X(d\nu) \\
&= \int_{\mathbb{R}^m}\left(\mathrm{Var}\int_{\mathbb{R}^m}\varphi(t)\, e^{2i\pi\langle\nu,t\rangle}N(dt) + \left|\mathrm{E}\int_{\mathbb{R}^m}\varphi(t)\, e^{2i\pi\langle\nu,t\rangle}N(dt)\right|^2\right)\mu_X(d\nu) \\
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\widehat{\varphi}(x-\nu)|^2\mu_N(dx) + \left|\int_{\mathbb{R}^m}\varphi(t)\, e^{2i\pi\langle\nu,t\rangle}\lambda dt\right|^2\right)\mu_X(d\nu) \\
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\widehat{\varphi}(x-\nu)|^2\mu_N(dx)\right)\mu_X(d\nu) + \lambda^2\int_{\mathbb{R}^m}|\widehat{\varphi}(-\nu)|^2\mu_X(d\nu) \\
&= \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2(\mu_N * \mu_X)(d\nu) + \lambda^2\int_{\mathbb{R}^m}|\widehat{\varphi}(-\nu)|^2\mu_X(d\nu)
\end{aligned}$$

and, since $\mathrm{E}\left[\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\, e^{2i\pi\langle\nu,t\rangle}N(dt)\right)Z_X(d\nu)\middle|\,\mathcal{F}_\infty^N\right] = 0$,

$$\beta = \mathrm{Var}\left(m_X\int_{\mathbb{R}^m}\varphi(t)\, N(dt)\right) = |m_X|^2\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\mu_N(d\nu).$$

Finally,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\, Y(t)\, dt\right) = \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\left(\mu_N * \mu_X + \lambda^2\mu_X + |m_X|^2\mu_N\right)(d\nu),$$

that is, $\{Y(t)\}_{t\in\mathbb{R}^m}$ admits an extended Cramér spectral measure given by equation (49). ∎

Here are some examples of the power spectral measure for some of the spike fields $N$ with spectrum $\mu_N$ presented in Section III.

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EXAMPLE 6.3: Let $N$ be a Cox process with a wide-sense stationary intensity $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ with Cramér spectrum $\mu_\lambda$, then $\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu$ and

$$\mu_Y = \mu_\lambda * \mu_X + \lambda^2\mu_X + |m_X|^2\mu_\lambda + \lambda R_X(0)\,\ell^m \tag{50}$$

where $\ell^m$ is the Lebesgue measure and we have used $(\ell^m * \mu_X) = \mu_X(\mathbb{R}^m)\,\ell^m$; indeed

$$\begin{aligned}
(\ell^m * \mu_X)(C) &= \int_{\mathbb{R}^m}\ell^m(C-x)\,\mu_X(dx) \\
&= \int_{\mathbb{R}^m}\ell^m(C)\,\mu_X(dx) = \mu_X(\mathbb{R}^m)\,\ell^m(C).
\end{aligned}$$

The spectrum when the spike field is a homogeneous Poisson process is obtained as a particular case with $\mu_\lambda \equiv 0$:

$$\mu_Y = \lambda^2\mu_X + \lambda R_X(0)\,\ell^m. \tag{51}$$

EXAMPLE 6.4: Let $\mu_{\widetilde{Y}}$ denote the spectrum of the sample comb obtained by sampling the signal with the sampler $\widetilde{N}$. When the former is "randomly thinned" (as described in Example 3.2), an application of Formula (49) shows that the spectrum of the resulting sample comb is

$$\mu_Y(d\nu) = q^2 \mu_{\widetilde{Y}}(d\nu) + \widetilde{\lambda} pq R_X(0)\, d\nu. \tag{52}$$

EXAMPLE 6.5: Let $N$ correspond to uniform sampling (regular grid). For simplicity of notation, we develop the computations in the univariate case. However, similar formulas, with more complicated notation, hold in the multivariate case. Then, $\mu_X$ is a particular case of the Bartlett spectrum of the regular grid (15) and from (49) we have

$$\mu_Y(d\nu) = \frac{1}{T^2} \sum_{n \in \mathbb{Z}} \mu_X\left(d\nu - \frac{n}{T}\right) + \frac{|m_X|^2}{T^2} \sum_{n \neq 0} \delta_{n/T}(d\nu) \tag{53}$$

where here $\lambda = 1/T$. In particular, if $m_X = 0$, we obtain the *aliased spectrum*

$$\mu_Y(d\nu) = \frac{1}{T^2} \sum_{n \in \mathbb{Z}} \mu_X\left(d\nu - \frac{n}{T}\right). \tag{54}$$

If now we consider random displacements (jitter) of the points of the uniform spike train, the spectrum of the uniformly sampled signal becomes

$$\mu_Y(d\nu) = \frac{1}{T^2} \sum_{n \in \mathbb{Z}} \left|\psi_Z\left(\frac{n}{T}\right)\right|^2 \mu_X\left(d\nu - \frac{n}{T}\right) + \frac{1}{T}\left(1 - |\psi_Z(\cdot)|^2\right) * \mu_X(d\nu) \tag{55}$$

where $\psi_Z(\nu)$ is the characteristic function of the random displacements: this is a consequence of Corollary 4.2.

Concerning the effect of thinning on the spectrum, assuming for simplicity that $m_X = 0$, as a particular case of Example 6.4, using (52) we obtain

$$\mu_Y(d\nu) = \frac{1}{T^2} q^2 \sum_{n \in \mathbb{Z}} \mu_X\left(d\nu - \frac{n}{T}\right) + \frac{1}{T} pq R_X(0)\, d\nu \tag{56}$$

The second result is relative to the spectrum when the sampling rate depends on the process. The model for the sampler is now a Cox process [24] on $\mathbb{R}^m$ with the conditional (w.r.t. $X$) intensity of the form

$$\lambda(t) = \lambda(t, X).$$

For instance, in the univariate case, $\lambda(t) = |X(t)|^2$, $\lambda(t) = \left|\dot{X}(t)\right|^2$ where $\dot{X}$ is the derivative at $t$ of $t \to X(t)$. More complicated functionals can be considered.

*Theorem 6.2:* Assume that $\mathrm{E}\left[X(t)^2 \lambda(t,X)^2\right] < \infty$, $\forall t \in \mathbb{R}^m$, and that $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ is a locally integrable process, that is

$$\int_C \lambda(t) < \infty, \quad a.s.$$

for all bounded Borel sets $C$. Let $\mu_Z$ be the power spectrum of the stationary process

$$Z(t) = X(t)\lambda(t).$$

Then,

$$\mu_Y(d\nu) = \mu_Z(d\nu) + \overline{X^2\lambda}d\nu \tag{57}$$

where we have denoted $\overline{X^2\lambda} = \mathrm{E}\left[X(t)^2\lambda(t)\right]$ (independent of $t$).

*Proof:* In order to compute the Bartlett spectrum of $Y(t)$, we have, as in the independent case, to evaluate the variance of

$$\int_{\mathbb{R}^m} \varphi(t) Y(t)\, dt = \int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)$$

for all $\varphi \in \mathrm{L}^1 \cap \mathrm{L}^2$.

It holds that

$$\begin{aligned}
&\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)\right) = \\
&\qquad \mathrm{E}\left[\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt) \,|X\right)\right] + \mathrm{Var}\left(\mathrm{E}\left[\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)\,|X\right]\right) \\
&\qquad\qquad = \mathrm{E}\left[\int_{\mathbb{R}^m} |\varphi(t)|^2 |X(t)|^2 \lambda(t,X)\, dt\right] + \mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) X(t) \lambda(t,X)\, dt\right).
\end{aligned}$$

By the definition of $\mu_Z$, we have

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(u) X(u) \lambda(u)\, du\right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Z(d\nu).$$

Therefore, recalling the notation $\mathrm{E}\left[X(t)^2\lambda(t)\right] = \overline{X^2\lambda}$ (independent of $t$), we have

$$\begin{aligned}
\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)\right) &= \mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(u) X(u) \lambda(u)\, du\right) + \overline{X^2\lambda}\int_{\mathbb{R}^m} |\varphi(t)|^2\, dt \\
&= \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Z(d\nu) + \overline{X^2\lambda}\int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2\, d\nu \\
&= \int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\left(\mu_Z(d\nu) + \overline{X^2\lambda}d\nu\right)
\end{aligned}$$

and the result (57) follows. ■

As particular cases of the above result, for $X(t) \equiv 1$, we recover the formula

$$\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu$$

of the Bartlett spectrum of the Cox process, and for $X(t) = \lambda(t)$, we have

$$\mu_Y(d\nu) = \mu_{\lambda^2}(d\nu) + \mathrm{E}\left[\lambda^3(0)\right] d\nu.$$

The third result concerns the problem of approximating $X(t)$ by a filtered version of $\{Y(t)\}$

$$\int_{\mathbb{R}^m} \varphi(t-s)\, Y(s)\, ds$$

where $\varphi \in \mathrm{L}^1 \cap \mathrm{L}^2$. The difference between $X(t)$ and its approximation, that is, the reconstruction error, is measured by

$$\epsilon = \mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t-u)\, Y(u)\, du - X(t)\right|^2\right].$$

Then, we have the following expression for the reconstruction error.

*Theorem 6.3:* Reconstructing the signal $\{X(t)\}_{t\in\mathbb{R}}$ by filtering the sample comb $\{Y(t)\}_{t\in\mathbb{R}}$ with a filter $\varphi \in \mathrm{L}^1 \cap \mathrm{L}^2$ gives the following error

$$\boxed{\epsilon = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + \int_{\mathbb{R}^m} \mu_X(d\nu) - \lambda \int_{\mathbb{R}^m} \left(\widehat{\varphi}(\nu) + \widehat{\varphi}^*(\nu)\right) \mu_X(d\nu) + |m_X|^2 \left|1 - \lambda\widehat{\varphi}(0)\right|^2.} \tag{58}$$

*Proof:* We have

$$\begin{aligned}
\epsilon &= \mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t-u)\, X(u)\, N(du) - X(t)\right|^2\right] \\
&= \mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t-u)\, X(u)\, N(du)\right|^2\right] - 2\Re\left\{\mathrm{E}\left[\int_{\mathbb{R}} \varphi(t-u)\, X(t)\, X(u)\, N(du)\right]\right\} + \mathrm{E}\left[|X(t)|^2\right] \\
&= A - 2\Re\{B\} + C.
\end{aligned}$$

In this expression,

$$A = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + \lambda^2 |m_X|^2 \left|\int_{\mathbb{R}^m} \varphi(t)\, dt\right|^2 = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + \lambda^2 |m_X|^2 |\widehat{\varphi}(0)|^2,$$

$$\begin{aligned}
B = \mathrm{E}\left[\int_{\mathbb{R}^m} \varphi(t-u)\, X(t)\, X(u)\, N(du)\right] &= \lambda \int_{\mathbb{R}^m} \varphi(t-u)\, R_X(t-u)\, du \\
&= \lambda \int_{\mathbb{R}^m} \varphi(t)\, R_X(t)\, dt \\
&= \lambda \int_{\mathbb{R}^m} \varphi(t)\, C_X(t)\, dt + \lambda |m_X|^2 \int_{\mathbb{R}^m} \varphi(t)\, dt \\
&= \lambda \int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\, \mu_X(d\nu) + \lambda |m_X|^2\, \widehat{\varphi}(0),
\end{aligned}$$

and

$$C = \int_{\mathbb{R}^m} \mu_X(d\nu) + |m_X|^2.$$

Therefore

$$\mathrm{E}\left[\left|\int_{\mathbb{R}^m}\varphi(t-u)\,X(u)\,N(du)-X(t)\right|^2\right]=\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,\mu_Y(d\nu)$$

$$-\lambda\int_{\mathbb{R}^m}\left(\widehat{\varphi}(\nu)+\widehat{\varphi}(\nu)^*\right)\mu_X(d\nu)$$

$$+|m_X|^2\left(1-\lambda\left(\widehat{\varphi}(0)+\widehat{\varphi}(0)^*\right)+\lambda^2\,|\widehat{\varphi}(0)|^2\right)+\int_{\mathbb{R}^m}\mu_X(d\nu)$$

■

In particular, in the case $m_X = 0$ the error is

$$\epsilon=\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,\mu_Y(d\nu)-2\lambda\Re\left\{\int_{\mathbb{R}^m}\widehat{\varphi}(\nu)\,\mu_X(d\nu)\right\}+\mu_X(\mathbb{R}^m)\,. \tag{59}$$

We now give some examples of reconstruction error for different sampling schemes. For notation ease, we consider that the signal is centered, that is, $m_X = 0$, so as to apply the simpler form of the error formula (59). Moreover, some parts of the examples are developed in the univariate case. However, similar formulas, with more complicated notation, hold in the multivariate case. We develop the computations in the "classical" situation of a band-limited signal $X(t)$, filtered with a band-limited (low-pass) filter $\varphi(\nu)$. More precisely, let $S$ be the support of $\mu_X$, with length $2B = \ell(S)$, then, we consider

$$\widehat{\varphi}(\nu)=\begin{cases}\frac{1}{\lambda} & \text{on } S\\ 0 & \text{otherwise}\end{cases}$$

where $\lambda$ is the intensity of the spike comb.

---

EXAMPLE 6.6: When $N$ is a homogeneous Poisson process with intensity $\lambda$, $\mu_Y$ is given by (51) and then the error is

$$\epsilon=\int_{\mathbb{R}^m}|\lambda\widehat{\varphi}(\nu)-1|^2\,\mu_X(d\nu)+\lambda C_X(0)\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,(d\nu)\,. \tag{60}$$

In the "classical" band-limited case described above, we have

$$\begin{aligned}\epsilon &= \lambda C_X(0)\int_{\mathbb{R}}|\widehat{\varphi}(\nu)|^2\,(d\nu)\\ &= \lambda C_X(0)\int_{\mathbb{R}}\frac{1}{\lambda^2}1_S(\nu)\,d\nu,\end{aligned}$$

that is

$$\epsilon = C_X(0)\,\frac{2B}{\lambda}\cdot$$

Therefore, sampling at the Nyquist rate $\lambda = 2B$ gives very poor performances, not better than the estimate based on no observation at all.

This does not mean, however, that below the rate $\lambda = 2B$, there is no information (or in a sense as the result suggests "negative information") concerning the process itself contained in its samples. A better choice of a filter would indeed give a linear estimate with error less than $\sigma^2 = C_X(0)$. For instance, if we let $\widehat{\varphi}$ be real, we find for the error

$$\epsilon = \int_{\mathbb{R}} \left[ (\lambda \widehat{\varphi}(\nu) - 1)^2 f_X(\nu) + \lambda \sigma^2 \widehat{\varphi}(\nu)^2 \right] d\nu,$$

where it is assumed that $\{X(t)\}_{t \in \mathbb{R}}$ has the power spectral density $f_X(\nu)$. The minimum occurs for

$$\widehat{\varphi}(\nu) = \frac{\lambda f_X(\nu)}{\lambda^2 f_X(\nu) + \lambda \sigma^2}$$

and then

$$\epsilon = \sigma^2 \left( 1 - \int_{\mathbb{R}} \frac{\lambda \widetilde{f}_X(\nu)}{1 + \lambda \widetilde{f}_X(\nu)} \widetilde{f}_X(\nu)\, d\nu \right) \cdot$$

where $\widetilde{f}_X(\nu)$ is the normalized power spectral density

$$\widetilde{f}_X(\nu) = \frac{f_X(\nu)}{\int_{\mathbb{R}} f_X(\nu')\, d\nu'} = \frac{f_X(\nu)}{\sigma^2}.$$

Therefore

$$\epsilon = \sigma^2 (1 - \rho)$$

where $\rho = \int_{\mathbb{R}} \frac{\lambda \widetilde{f}_X(\nu)}{1 + \lambda \widetilde{f}_X(\nu)} \widetilde{f}_X(\nu)\, d\nu$ can be interpreted as the correlation coefficient between $X(t)$ and $N$ for fixed $t$.

EXAMPLE 6.7: When the sampled comb is derived by $T$-uniform sampling, with an extended Cramér-Khinchin spectral measure given by (53), the reconstruction error (59) reads

$$\begin{aligned} \epsilon &= \frac{1}{T^2} \int_{\mathbb{R}} |\widehat{\varphi}(\nu)|^2 \mu_X(d\nu) - \frac{2}{T} \Re \left( \int_{\mathbb{R}} \widehat{\varphi}(\nu) \mu_X(d\nu) \right) + \int_{\mathbb{R}} \mu_X(d\nu) \\ &= \int_{\mathbb{R}} \left| \frac{1}{T} \widehat{\varphi}(\nu) - 1 \right|^2 \mu_X(d\nu) . \end{aligned} \tag{61}$$

In the band-limited case, if we consider $T = 1/2B$, that is, $\lambda = 2B$, equation (61) gives an error equal to zero. Therefore, the signal is perfectly reconstructed by

$$\begin{aligned} X(t) &= \int_{\mathbb{R}} \varphi(t - s) X(s) N(ds) \\ &= \sum_{n \in \mathbb{Z}} X(T_n) \operatorname{sinc}(t - T_n), \end{aligned}$$

where $\operatorname{sinc}(t) = \sin(2\pi B t) / (2\pi B t)$, which is the usual reconstruction formula ([36]).

EXAMPLE 6.8: The reconstruction error from uniform samples in the presence of jitter is obtained plugging $\mu_Y$ given by (55) into the error formula (59). The previous example showed that within the "classical" sampling framework the signal may be perfectly reconstructed. Now, in the presence of jitter this is not possible and the reconstruction error is given by

$$\epsilon = \frac{1}{2B}\left(\int_{-B}^{B}\sigma^2\left(1-\left(|\psi_Z|^2 * \widetilde{f}_X\right)(\nu)\right)d\nu\right) \tag{62}$$

where $\widetilde{f}_X$ is the normalized power spectral density of the signal $X(t)$.

EXAMPLE 6.9: Let $\widetilde{\epsilon}$ be the reconstruction error when the sample brush is characterized by $\mu_{\widetilde{Y}}$ and the sampler by $\mu_{\widetilde{N}}$. Suppose now that the sampler is randomly thinned (as described by (16)). From the effect of thinning on the sampler (16) and, then, on the sample brush (52), the reconstruction error is now

$$\epsilon = q^2\widetilde{\epsilon}+\left(1-q^2\right)\mu_X\left(\mathbb{R}^m\right)-q\widetilde{\lambda}\left(1-q\right)\int_{\mathbb{R}^m}\left(\widehat{\varphi}(\nu)+\widehat{\varphi}(\nu)^*\right)\mu_X(d\nu)+\widetilde{\lambda}pqC_X(0)\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,d\nu.$$

Remark that when there the probability of a loss is zero, that is, $q=1$, we have $\epsilon=\widetilde{\epsilon}$.

When we are in the "classical" sampling framework, that is, uniform sampling of band-limited signal, due to the loss of samples the reconstruction is not perfect and the error is

$$\epsilon = pC_X(0).$$

## VII. CONCLUSION

This article is devoted to the derivation of the power spectra, sometimes in a generalized sense, of a large class of signals and random fields connected with point processes and of interest to communications, biology, seismology, and wavelet spectral analysis, among other domain of applications. It unifies the results scattered in the litterature and provides a rigorous and at the same time systematic method, based on the theory of point processes, for obtaining them. It gives extensions of the previous results for time–indexed signals to spatial random fields.

Concerning shot noise processes, the main contribution is to take for the the underlying spike field a *general* stationary point process described by its Bartlett spectrum. The formulas obtained are a prerequisite for the spectral analysis of many signals where the classical Poisson model is not realistic as is certainly the case for the biological signals in neurophysiology.

Concerning random sampling, the results presented in the present paper extend to random fields the previous studies mentioned in the introduction. Some related results are given, in particular concerning *signal–dependent* sampling rates.

The result giving the power spectrum of Hawkes branching random *fields* with *random impulse function* and *general* stationary immigration processes is also new, as well as the result concerning the generalized *linear spatial birth and death* process. Hawkes processes provide the most important source of models in seismology, and the results in the present paper are a prerequisite to identification methods based on second-order properties.

Theoretical research in progress concerns other classes of signals related to point process, in particular semi-Markov point processes, of interest to pulse interval modulation as arising for instance in ultra-wide band communications, and space–time point process models of interest to mobile communications.

## APPENDIX A

## EXCHANGE OF INTEGRALS

*Lemma 1.1:* Let $N$ be a simple locally bounded stationary point process defined on $\mathbb{R}^m$ and admitting a Bartlett spectrum $\mu_N$. Let $M_2$ be its second moment measure. Let $\{X(t)\}_{t\in\mathbb{R}^m}$ be a WSS random field with Cramér decomposition $Z_X$ and power spectral measure $\mu_X$. Then, for all $\varphi \in \mathrm{L}^1$ such that (10) holds

$$\int_{\mathbb{R}^m} \varphi(t)\, X(t)\, N(dt) = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi\langle \nu, t\rangle} N(dt) \right) Z_X(d\nu)\,. \tag{63}$$

*Proof:* We do the proof in the univariate case. The multivariate case follows the same lines, with more notation. The left-hand side of (63) is

$$A = \sum_{n\in\mathbb{Z}} \varphi(T_n)\, X(T_n) = \lim_{c\uparrow\infty} \sum_{n\in\mathbb{Z}} \varphi(T_n)\, X(T_n)\, 1_{[-c,+c]}(T_n) = \lim_{c\uparrow\infty} A(c)$$

where the limit is in $\mathrm{L}^1(\mathbb{P})$. Indeed

$$\begin{aligned} \mathrm{E}\left[|A - A(c)|\right] &\le \mathrm{E}\left[ \int_{\overline{[-c,+c]}} |\varphi(t)\, X(t)|\, N(dt) \right] \\ &= \int_{\overline{[-c,+c]}} |\varphi(t)|\, \mathrm{E}\left[|X(t)|\right] \lambda dt \le \lambda K \int_{\overline{[-c,+c]}} |\varphi(t)|\, dt \end{aligned}$$

where $K = \sup_t \mathrm{E}\left[|X(t)|\right] < \infty$ (by Schwarz's inequality, $\mathrm{E}\left[|X(t)|\right] \le \mathrm{E}\left[|X(t)|^2\right]^{\frac{1}{2}} = \mathrm{E}\left[|X(0)|^2\right]^{\frac{1}{2}}$). Therefore, since $\varphi \in \mathrm{L}^1$, $\lim_{c\uparrow\infty} \mathrm{E}\left[|A - A(c)|\right] = 0$.

The right-hand side is

$$B = \lim_{c\uparrow\infty} \int_{\mathbb{R}} \left( \int_{[-c,+c]} \varphi(t)\, e^{2i\pi\langle \nu, t\rangle} N(dt) \right) Z_X(d\nu) = \lim_{c\uparrow\infty} B(c)$$

where the limit is in $L^2(\mathbb{P})$. Indeed

$$\begin{aligned}
\mathrm{E}\left[|B-B(c)|^2\right] &= \mathrm{E}\left[\left|\int_{\mathbb{R}}\left(\int_{\overline{[-c,+c]}}\varphi(t)\,e^{2i\pi\nu t}N(dt)\right)Z_X(d\nu)\right|^2\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[\left|\int_{\mathbb{R}}\left(\int_{\overline{[-c,+c]}}\varphi(t)\,e^{2i\pi\nu t}N(dt)\right)Z_X(d\nu)\right|^2\middle|\mathcal{F}_\infty^N\right]\right] \\
&= \mathrm{E}\left[\int_{\mathbb{R}}\left|\int_{\overline{[-c,+c]}}\varphi(t)\,e^{2i\pi\nu t}N(dt)\right|^2\mu_X(d\nu)\right].
\end{aligned}$$

Denote $\varphi_c(t)=\varphi(t)\,1_{\overline{[-c,c]}}(t)$. Then

$$\mathrm{E}\left[\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\varphi_c(t)\,e^{2i\pi\nu t}N(dt)\right|^2\mu_X(d\nu)\right]=\int_{\mathbb{R}}\mathrm{E}\left[\left|\int_{\mathbb{R}}\varphi_c(t)\,e^{2i\pi\nu t}N(dt)\right|^2\right]\mu_X(d\nu).$$

But

$$\begin{aligned}
\mathrm{E}\left[\left|\int_{\mathbb{R}}\varphi_c(t)\,e^{2i\pi\nu t}N(dt)\right|^2\right] &= \int_{\mathbb{R}\times\mathbb{R}}\varphi_c(t)\,\varphi_c^*(s)\,e^{2i\pi\nu(t-s)}M_2(dt\times ds) \\
&\le \int_{\mathbb{R}\times\mathbb{R}}|\varphi_c(t)|\,|\varphi_c(s)|\,M_2(dt\times ds),
\end{aligned}$$

a quantity that tends to $0$ as $c\uparrow\infty$, by Lebesgue dominated convergence theorem, which is applicable to the measure $M$ in view of (10). Lebesgue dominated convergence applied to the finite measure $\mu_X$ then yields the desired $L^2$ convergence.

But

$$\begin{aligned}
A(c) &= \sum_{n\in\mathbb{Z}}\varphi(T_n)\,X(T_n)\,1_{[-c,+c]} \\
&= \sum_{n\in\mathbb{Z}}\varphi(T_n)\left(\int_{\mathbb{R}}e^{2i\pi\nu T_n}Z_X(d\nu)\right)1_{[-c,+c]}(T_n) \\
&= \int_{\mathbb{R}}\left(\sum_{n\in\mathbb{Z}}\varphi(T_n)\,e^{2i\pi\nu T_n}1_{[-c,+c]}(T_n)\right)Z_X(d\nu) \\
&= B(c),
\end{aligned}$$

where we have used the fact that the sums involved are finite. Thus

$$\lim_{c\uparrow\infty}A(c)=\begin{cases}A & \text{in } L^1\\ B & \text{in } L^2\end{cases}$$

from which it follows that $A=B$, a.s. (use the fact that if a sequence of random variables converges in $L^1$ or $L^2$ to some r.v., one can extract a subsequence that converges a.s. to the same r.v.). ■

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