Exploring patterns of dependence in financial data.

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We estimate a sample covariance matrix $\Sigma$ from empirical data. . .

- **Objective:** infer **dependence** relationships between variables.
- We only want to isolate **a few key links**.

Elementary solution: look at the magnitude of the covariance coefficients:

$$|\Sigma_{i,j}| > \beta \iff \text{variables } i \text{ and } j \text{ are related,}$$

then simply threshold smaller coefficients to zero (not always psd).
Covariance Selection

Before

After
Covariance Selection

Following Dempster [1972], look for zeros in the inverse covariance matrix:

**Parsimony.** Suppose that we are estimating a Gaussian density:

\[
    f(x, \Sigma) = \left( \frac{1}{2\pi} \right)^{\frac{p}{2}} \left( \frac{1}{\det \Sigma} \right)^{\frac{1}{2}} \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right),
\]

a sparse inverse matrix \( \Sigma^{-1} \) corresponds to a sparse representation of the density \( f \) as a member of an exponential family of distributions:

\[
    f(x, \Sigma) = \exp(\alpha_0 + t(x) + \alpha_{11} t_{11}(x) + \ldots + \alpha_{rs} t_{rs}(x))
\]

with here \( t_{ij}(x) = x_i x_j \) and \( \alpha_{ij} = \Sigma_{ij}^{-1} \). Dempster [1972] calls \( \Sigma_{ij}^{-1} \) a concentration coefficient.
Conditional independence.

- Suppose $X, Y, Z$ have are jointly normal with covariance matrix $\Sigma$, with

\[
\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}
\]

where $\Sigma_{11} \in \mathbb{R}^{2 \times 2}$ and $\Sigma_{22} \in \mathbb{R}$.

- **Conditioned on** $Z$, $X, Y$ are still normally distributed with covariance matrix $C$ satisfying

\[
C = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = (\Sigma^{-1})^{-1}_{11}
\]

- So $X$ and $Y$ are **conditionally independent** iff $(\Sigma^{-1})_{11}$ is diagonal, which is also

\[
\Sigma^{-1}_{xy} = 0
\]
Suppose we have iid noise $\epsilon_i \sim \mathcal{N}(0, 1)$ and the following linear model

\[
x = z + \epsilon_1 \\
y = z + \epsilon_2 \\
z = \epsilon_3
\]

Graphically, this is
The covariance matrix and inverse covariance are given by

$$\Sigma = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \Sigma^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

The inverse covariance matrix has $\Sigma_{12}^{-1}$ clearly showing that the variables $x$ and $y$ are independent conditioned on $z$.

Graphically, this is again

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Let $I \bigoplus J = [1, n]^2$, Dempster [1972] shows:

- **Maximum Entropy.** Among all Gaussian models $\Sigma$ such that $\Sigma_{ij} = S_{ij}$ on $J$, the choice $\hat{\Sigma}_{ij}^{-1} = 0$ on $I$ has **maximum entropy**.

- **Maximum Likelihood.** Among all Gaussian models $\Sigma$ such that $\Sigma_{ij}^{-1} = 0$ on $I$, the choice $\hat{\Sigma}_{ij} = S_{ij}$ on $J$ has **maximum likelihood**.

- **Existence and Uniqueness.** If there is a positive semidefinite matrix $\hat{\Sigma}_{ij}$ satisfying $\hat{\Sigma}_{ij} = S_{ij}$ on $J$, then **there is only one** such matrix satisfying $\hat{\Sigma}_{ij}^{-1} = 0$ on $I$. 

Applications & Related Work

- **Gene expression data.** The sample data is composed of gene expression vectors and we want to isolate links in the expression of various genes. See Dobra et al. [2004], Dobra and West [2004] for example.

- **Speech Recognition.** See Bilmes [1999], Bilmes [2000] or Chen and Gopinath [1999].

- Related work by Dahl et al. [2005]: interior point methods for sparse MLE.
Financial data

Estimating covariance matrices from financial data.

- Asset returns are given by (schematically)

\[ \Delta S_t = \Delta M_t + \epsilon_t \]

where

- \( M_t \) is the market return
- \( \epsilon_t \) is an idiosyncratic component

- All assets are usually highly correlated: \( M_t \) dominates the picture. We are only interested in the correlation between \( \epsilon_t \) for various assets.

- The inverse matrix is also used to computed portfolios on the efficient frontier for CAPM.
Outline

- Introduction
- Penalized maximum likelihood estimation
- Algorithms & complexity
- Consistency
- Graph layout
- Numerical experiments
Penalized Maximum Likelihood Estimation
Akaike [1973]: penalize the likelihood function:

$$\max_{X \in \mathbb{S}^n} \log \det X - \text{Tr}(SX) - \rho \text{Card}(X)$$

where $\text{Card}(X)$ is the number of nonzero elements in $X$.

- Set $\rho = 2/(m + 1)$ for the Akaike Information Criterion (AIC).
- Set $\rho = \frac{\log(m+1)}{(m+1)}$ for the Bayesian Information Criterion (BIC).

Of course, this is a (NP-Hard) combinatorial problem...
We can form a **convex relaxation** of AIC or BIC penalized MLE

\[
\max_{X \in \mathbb{S}^n} \log \det X - \text{Tr}(SX) - \rho \text{Card}(X)
\]

replacing \(\text{Card}(X)\) by \(\|X\|_1 = \sum_{i,j} |X_{ij}|\) to solve

\[
\max_{X \in \mathbb{S}^n} \log \det X - \text{Tr}(SX) - \rho \|X\|_1
\]

**Classic \(l_1\) heuristic:** \(\|X\|_1\) is a **convex lower bound** on \(\text{Card}(X)\).

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The problem
\[
\max_{X \in S^n} \log \det X - \text{Tr}(SX) - \rho \|X\|_1
\]
is convex in the variable $X \in S_n$. This means that we can get explicit complexity bounds and efficient algorithms.

- Standard convex optimization algorithms easily solve small instances. (see Boyd and Vandenberghe [2004])
- Specialized techniques solve larger problems with complexity $O(n^{4.5})$. We can exploit the block structure of the dual. Cost per iteration comparable to that of a penalized regression (LASSO).
- In practice, we can get a good solution with complexity $O(n^{3.5})$. A bit harder than computing a matrix inverse. . .
## Algorithms

Complexity options...

<table>
<thead>
<tr>
<th>Complexity</th>
<th>Memory</th>
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<tbody>
<tr>
<td>$O(n)$</td>
<td>$O(1/\epsilon^2)$</td>
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<tr>
<td>$O(n)$</td>
<td>$O(1/\epsilon)$</td>
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<tr>
<td>$O(n^2)$</td>
<td>$O(\log(1/\epsilon))$</td>
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The convex relaxation of the covariance selection problem has a particular \textbf{min-max} structure

\[
\max_{X \in \mathbb{S}_+^n} \min_{|U_{ij}| \leq \rho} \log \det X - \text{Tr}((S + U)X)
\]

This min-max representation means that we use prox function algorithms by Nesterov [2005] (see also Nemirovski [2004]) to solve large, dense problem instances.

We also detail a “greedy” block-coordinate descent method with good empirical performance.
Nesterov’s method

Assuming that a problem can be written according to this min-max model, the algorithm works as follows. . .

- **Regularization.** Add strongly convex penalty inside the min-max representation to produce an $\epsilon$-approximation of $f$ with Lipschitz continuous gradient (generalized Moreau-Yosida regularization step, see Lemaréchal and Sagastizábal [1997] for example).

- **Optimal first order minimization.** Use optimal first order scheme for Lipschitz continuous functions detailed in Nesterov [1983] to solve the regularized problem.
Nesterov’s method

**Regularization.** The objective is first smoothed by penalization. We solve the following (modified) problem

\[
\begin{align*}
\max_{X \in S^n: \alpha I_n \preceq X \preceq \beta I_n} & \quad \min_{U \in S^n: |U_{ij}| \leq \rho} \log \det X - \text{Tr}((S - U)X) - \frac{\epsilon}{2D_2}d_2(U) \\
\end{align*}
\]

an \(\epsilon\) approximation of the original problem if \(\alpha \leq 1/(\|S\| + n\rho)\) and \(\beta \geq n/\rho\).

- **Prox on** \(Q_2 := \{U \in S^n : \|U\|_\infty \leq 1\}\) is \(d_2(U) = \frac{1}{2} \text{Tr}(U^TU) = \frac{1}{2}\|U\|^2\)
- **Prox** \(d_1(X)\) for the set \(\{\alpha I_n \preceq X \preceq \beta I_n\}\) given by

\[
\begin{align*}
\end{align*}
\]

This corresponds to a classic Moreau-Yosida regularization of the penalty \(\|X\|_1\) and the function \(f_\epsilon\) has a Lipschitz continuous gradient with constant

\[
L_\epsilon := M + D_2\rho^2/(2\epsilon)
\]
Nesterov’s method

Optimal first-order minimization. The minimization algorithm in Nesterov [1983] then involves the following steps

Choose $\epsilon > 0$ and set $X_0 = \beta I_n$, For $k = 0, \ldots, N(\epsilon)$ do

1. Compute $\nabla f_{\epsilon}(X_k) = -X^{-1} + \Sigma + U^*(X_k)$

2. Find $Y_k = \arg \min_Y \{ \text{Tr}(\nabla f_{\epsilon}(X_k)(Y - X_k)) + \frac{1}{2}L_{\epsilon}\|Y - X_k\|^2_F : Y \in Q_1 \}$.

3. Find $Z_k = \arg \min_X \left\{ L_{\epsilon}\beta^2 d_1(X) + \sum_{i=0}^{k} \frac{i+1}{2} \text{Tr}(\nabla f_{\epsilon}(X_i)(X - X_i)) : X \in Q_1 \right\}$.

4. Update $X_k = \frac{2}{k+3}Z_k + \frac{k+1}{k+3}Y_k$. 
Nesterov’s method

At each iteration

- **Step 1:** only amounts to computing the inverse of $X$ and the (explicit) solution to the regularized subproblem on $Q_2$.

- **Steps 2 and 3:** are both projections on $Q_1 = \{ \alpha I_n \preceq X \preceq \beta I_n \}$ and require an eigenvalue decomposition.

This means that the total complexity estimate of the method is

$$O \left( \frac{\kappa \sqrt{(\log \kappa)}}{\epsilon} n^{4.5} \alpha \rho \right)$$

where $\log \kappa = \log(\beta/\alpha)$ bounds the solution’s condition number.
Dual block-coordinate descent

- Here we consider the dual of the original problem

\[
\begin{align*}
\text{maximize} & \quad \log \det (S + U) \\
\text{subject to} & \quad \|U\|_\infty \leq \rho \\
& \quad S + U \succeq 0
\end{align*}
\]

- Let \(C = S + U\) be the current iterate, after permutation we can always assume that we optimize over the last column

\[
\begin{align*}
\text{maximize} & \quad \log \det \begin{pmatrix} C^{11} & C^{12} + u \\ C^{21} + u^T & C^{22} \end{pmatrix} \\
\text{subject to} & \quad \|u\|_\infty \leq \rho
\end{align*}
\]

where \(C^{12}\) is the last column of \(C\) (off-diag.).

- Each iteration reduces to a simple box-constrained QP

\[
\begin{align*}
\text{minimize} & \quad u^T (C^{11})^{-1} u \\
\text{subject to} & \quad \|u\|_\infty \leq \rho
\end{align*}
\]
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■ Consistency
■ Graph layout
■ Numerical experiments
Consistency

**Proposition 1**

**Consistency.** Let $\hat{C}_k^\lambda$ denote our estimate of the connectivity component of node $k$. Let $\alpha$ be a given level in $[0, 1]$. Consider the following choice for the penalty parameter

$$
\lambda(\alpha) := \left( \max_{i > j} \hat{\sigma}_i \hat{\sigma}_j \right) \frac{t_{n-2}(\alpha/2p^2)}{\sqrt{n - 2 + t_{n-2}^2(\alpha/2p^2)}}
$$

where $t_{n-2}(\alpha)$ denotes the $(100 - \alpha)\%$ point of the Student’s t-distribution for $n - 2$ degrees of freedom, and $\hat{\sigma}_i$ is the empirical variance of variable $i$. Then

$$
\text{Prob}(\exists k \in \{1, \ldots, p\} : \hat{C}_k^\lambda \not\subseteq C_k) \leq \alpha.
$$

**Proof.** Argument similar to Meinshausen and Buhlmann [2006].
Cross-validation

In practice, we can use **cross-validation**

- Remove a random subset of the variables and compute the inverse covariance matrix.
- Compute the pattern of zeros.
- Repeat the procedure for various variable subsets and various values of the penalty $\rho$.

How do we pick the value of the penalty parameter $\rho$?

- We pick the $\rho$ minimizing the variability of these dependence relationships across samples.
- Also, dependence relationships which show up in most subsampled networks are considered more reliable.
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- **Graph layout**
- Numerical experiments
How do we represent these results?

- Turn the pattern of zeros in the inverse covariance into a graph.
- Use graph visualization algorithms to layout this graph.

Trickier than it sounds...

- Graph layout problems are usually very hard. Again, good approximation algorithms exist.
- Many possible representations.
- Some coefficients are close to zero (numerical noise): threshold.
Many characteristics of the graph have a statistical interpretation.

- if the graph is **chordal**, then there is a linear/Gaussian model with the same sparsity pattern (see Wermuth [1980] for an early reference on linear recursive models and path analysis).

**Left:** a chordal graphical model: no cycles of length greater than three.

**Right:** a non-chordal graphical model of U.S. swap rates.
If there is a **path** between two nodes on a graph, then the corresponding variables have nonzero covariance (see Gilbert [1994] for a survey of graph theory/sparse linear algebra).

**Left:** connected model of U.S. swap rates, with dense covariance matrix. **Right:** disconnected model, the covariance matrix is block-diagonal.
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Sparse covariance model. **Left:** ROC curves for both thresholding and covariance selection using 20 samples to compute the covariance. **Right:** Binary dependence classification performance of inverse sample covariance thresholding (THRES) and covariance selection (COVSEL) for various sample sizes, measured by area under ROC curve.

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Covariance Selection

Forward rates covariance matrix for maturities ranging from 0.5 to 10 years.

\[ \rho = 0 \] \hspace{1cm} \rho = .01
Graph of conditional covariance among a cluster of U.S. dollar exchange rates. Positive dependencies are plotted as green links, negative ones in red, thickness reflects the magnitude of the covariance.
Hedge fund returns

- We track 116 hedge funds between January 1995 and December 2005.
- Monthly hedge fund returns from the Center for International Securities and Derivatives Markets hedge fund database, via WRDS.
- Hedge fund nodes are colored to represent their primary strategy.
Hedge fund returns

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Hedge fund returns: strategies

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Hedge fund returns: markets
Conclusion

- Covariance selection highlights key dependence structure.
- Very good statistical performance compared to thresholding techniques.
- Results are often intuitive.

- Slides, papers and MATLAB software available at:
  
  http://www.cmap.polytechnique.fr/~aspremon

- R package using a pathwise algorithm at
  
  http://cran.r-project.org/web/packages/Covpath/index.html

- A free network layout software called cytoscape:
  
  http://www.cytoscape.org
References


