Semidefinite Programming with Applications in Geometry and Machine Learning

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A **linear program** (LP) is written

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

where \( x \geq 0 \) means that the coefficients of the vector \( x \) are nonnegative.

- Starts with Dantzig’s simplex algorithm in the late 40s.
- First efficient algorithm with polynomial complexity derived by Karmarkar [1984], using interior point methods.
A **semidefinite program** (SDP) is written

\[
\begin{align*}
\text{minimize} & \quad \text{Tr}(CX) \\
\text{subject to} & \quad \text{Tr}(A_iX) = b_i, \quad i = 1, \ldots, m \\
& \quad X \succeq 0
\end{align*}
\]

where \(X \succeq 0\) means that the matrix variable \(X \in \mathbb{S}_n\) is **positive semidefinite**.

- Nesterov and Nemirovskii [1994] showed that the **interior point algorithms** used for linear programs could be extended to semidefinite programs.
- Key result: **self-concordance** analysis of Newton’s method (affine invariant smoothness bounds on the Hessian).
Introduction

- Modeling
  - Linear programming started as a toy problem in the 40s, many applications followed.
  - Semidefinite programming has much stronger expressive power, many new applications being investigated today (cf. this talk).
  - Similar conic duality theory.

- Algorithms
  - Robust solvers for solving large-scale linear programs are available today (e.g. MOSEK, CPLEX, GLPK).
  - Not (yet) true for semidefinite programs. Very active work now on first-order methods, motivated by applications in statistical learning (matrix completion, NETFLIX, structured MLE, . . . ).
Outline

- Introduction
- **Semidefinite programming**
  - Conic duality
  - A few words on algorithms
- Recent applications
  - Combinatorial relaxations
  - Ellipsoidal approximations
  - Distortion, embedding
  - Mixing rates for Markov chains & maximum variance unfolding
  - Moment problems & positive polynomials
  - Gordon-Slepian and the maximum of Gaussian processes
- Dictionary metrics
  - Sparse recovery conditions
  - Tractable performance bounds
Semidefinite Programming
Semidefinite programming: conic duality

Direct extension of LP duality results. Start from a semidefinite program

\[
\begin{align*}
\text{minimize} & \quad \text{Tr}(CX) \\
\text{subject to} & \quad \text{Tr}(A_iX) = b_i, \quad i = 1, \ldots, m \\
& \quad X \succeq 0
\end{align*}
\]

which is a convex minimization problem in \( X \in \mathbb{S}_n \). The cone of positive semidefinite matrices is self-dual, i.e.

\[ Z \succeq 0 \iff \text{Tr}(ZX) \geq 0, \text{ for all } X \succeq 0, \]

so we can form the Lagrangian

\[
L(X, y, Z) = \text{Tr}(CX) + \sum_{i=1}^{m} y_i (b_i - \text{Tr}(A_iX)) - \text{Tr}(ZX)
\]

with Lagrange multipliers \( y \in \mathbb{R}^m \) and \( Z \in \mathbb{S}_n \) with \( Z \succeq 0 \).
Rearranging terms, we get

\[ L(X, y, Z) = \text{Tr} \left( X \left( C - \sum_{i=1}^{m} y_i A_i - Z \right) \right) + b^T y \]

hence, after minimizing this affine function in \( X \in \mathbb{S}_n \), the dual can be written

\[
\begin{align*}
\text{maximize} & \quad b^T y \\
\text{subject to} & \quad Z = C - \sum_{i=1}^{m} y_i A_i \\
& \quad Z \succeq 0,
\end{align*}
\]

which is another semidefinite program in the variables \( y, Z \). Of course, the last two constraints can be simplified to

\[ C - \sum_{i=1}^{m} y_i A_i \succeq 0. \]
Semidefinite programming: conic duality

■ Primal dual pair

\[
\begin{align*}
\text{minimize} \quad & \text{Tr}(CX) \\
\text{subject to} \quad & \text{Tr}(A_iX) = b_i \\
& X \succeq 0,
\end{align*}
\]

\[
\begin{align*}
\text{maximize} \quad & b^T y \\
\text{subject to} \quad & C - \sum_{i=1}^m y_i A_i \succeq 0.
\end{align*}
\]

■ Simple constraint qualification conditions guarantee strong duality.

■ We can write a conic version of the KKT optimality conditions

\[
\left\{ \begin{array}{l}
C - \sum_{i=1}^m y_i A_i = Z, \\
\text{Tr}(A_iX) = b_i, \quad i = 1, \ldots, m, \\
\text{Tr}(XZ) = 0, \\
X, Z \succeq 0.
\end{array} \right.
\]
So what?

- Weak duality produces simple bounds on e.g. combinatorial problems.
- Consider the MAXCUT relaxation

\[
\begin{align*}
\max & \quad x^T C x \\
\text{s.t.} & \quad x^2_i = 1
\end{align*}
\]

is bounded by

\[
\begin{align*}
\max & \quad \text{Tr}(X C') \\
\text{s.t.} & \quad \text{diag}(X) = 1, \quad X \succeq 0,
\end{align*}
\]

in the variables \( x \in \mathbb{R}^n \) and \( X \in \mathbb{S}_n \) (more later on these relaxations).

- The dual of the SDP on the right is written

\[
\min_y n\lambda_{\max}(C - \text{diag}(y)) + 1^T y
\]

in the variable \( y \in \mathbb{R}^n \).

- By **weak duality**, plugging any value \( y \) in this problem will produce an upper bound on the optimal value of the combinatorial problem above.
Semidefinite programming: algorithms

Algorithms for semidefinite programming

- Following [Nesterov and Nemirovskii, 1994], most of the attention was focused on interior point methods.
- Newton’s method, with efficient linear algebra solving for the search direction.
- Fast, and robust on small problems \((n \sim 500)\).
- Computing the Hessian is too hard on larger problems.

Solvers

- Open source solvers: SDPT3, SEDUMI, SDPA, CSDP, . . .
- Very powerful modeling systems: CVX
Solving the maxcut relaxation

\[
\begin{align*}
\text{max.} & \quad \text{Tr}(X C) \\
\text{s.t.} & \quad \text{diag}(X) = 1 \\
& \quad X \succeq 0,
\end{align*}
\]

is written as follows in CVX/MATLAB

```
cvx_begin
variable X(n,n) symmetric
maximize trace(C*X)
subject to
diag(X)==1
X==semidefinite(n)
cvx_end
```
Semidefinite programming: large-scale

Solving large-scale problems is a bit more problematic.

- No universal algorithm known yet. No CVX like modeling system.
- Performance and algorithmic choices heavily depends on problem structure.
- Very basic codes only require computing one leading eigenvalue per iteration, with complexity $O(n^2 \log n)$ using e.g. Lanczos.
- Each iteration requires about 300 matrix vector products, but making progress may require many iterations. Typically $O(1/\epsilon^2)$ or $O(1/\epsilon)$ in some cases.
- In general, most optimization algorithms are purely sequential, so only the linear algebra subproblems benefit from the multiplication of CPU cores.
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- Introduction

- Semidefinite programming
  - Conic duality
  - A few words on algorithms

- Recent applications
  - Combinatorial relaxations
  - Distortion, embedding
  - Ellipsoidal approximations
  - Mixing rates for Markov chains & maximum variance unfolding
  - Moment problems & positive polynomials
  - Gordon-Slepian and the maximum of Gaussian processes

- Dictionary metrics
  - Sparse recovery conditions
  - Tractable performance bounds
Many classical problems can be cast as or approximated by semidefinite programs.

Recognizing this is not always obvious.

At reasonable scales, numerical solutions often significantly improve on classical closed-form bounds.

A few examples follow...
Combinatorial relaxations
Combinatorial relaxations


Semidefinite programs with constant trace often arise in convex relaxations of combinatorial problems. Use MAXCUT as an example here.

The problem is written

\[
\begin{align*}
\text{max.} & \quad x^T C x \\
\text{s.t.} & \quad x \in \{-1, 1\}^n
\end{align*}
\]

in the binary variables \( x \in \{-1, 1\}^n \), with parameter \( C \in S_n \) (usually \( C \succeq 0 \)). This problem is known to be \textbf{NP-Hard}. Using

\[
x \in \{-1, 1\}^n \iff x_i^2 = 1, \quad i = 1, \ldots, n
\]

we get

\[
\begin{align*}
\text{max.} & \quad x^T C x \\
\text{s.t.} & \quad x_i^2 = 1, \quad i = 1, \ldots, n
\end{align*}
\]

which is a nonconvex quadratic program in the variable \( x \in \mathbb{R}^n \).
We now do a simple change of variables, setting $X = xx^T$, with

$$X = xx^T \iff X \in S_n, \ X \succeq 0, \ \text{Rank}(X) = 1$$

and we also get

$$\text{Tr}(CX) = x^TCx$$
$$\text{diag}(X) = 1 \iff x_i^2 = 1, \ i = 1, \ldots, n$$

so the original combinatorial problem is equivalent to

$$\max. \ \text{Tr}(CX)$$
$$\text{s.t.} \ \text{diag}(X) = 1$$
$$X \succeq 0, \ \text{Rank}(X) = 1$$

which is now a nonconvex problem in $X \in S_n$. 
Combinatorial relaxations

- If we simply drop the rank constraint, we get the following relaxation

\[
\begin{align*}
\text{max.} & \quad x^T C x \\
\text{s.t.} & \quad x \in \{-1, 1\}^n
\end{align*}
\]

is bounded by

\[
\begin{align*}
\text{max.} & \quad \text{Tr}(CX) \\
\text{s.t.} & \quad \text{diag}(X) = 1 \\
& \quad X \succeq 0,
\end{align*}
\]

which is a semidefinite program in \(X \in S_n\).

- **Rank constraints** in semidefinite programs are usually hard. All semi-algebraic optimization problems can be formulated as rank constrained SDPs.

- Randomization techniques produce bounds on the approximation ratio. When \(C \succeq 0\) for example, we have

\[
\frac{2}{\pi} \text{SDP} \leq \text{OPT} \leq \text{SDP}
\]

for the MAXCUT relaxation (more details in [Ben-Tal and Nemirovski, 2001]).

- Applications in graph, matrix approximations (CUT-Norm, \(\| \cdot \|_{2 \rightarrow 1}\)) [Frieze and Kannan, 1999, Alon and Naor, 2004, Nemirovski, 2005]
Distortion, embedding problems, . . .
We cannot hope to always get low rank solutions, unless we are willing to admit some **distortion**. . . The following result from [Ben-Tal, Nemirovski, and Roos, 2003] gives some guarantees.

**Theorem**

**Approximate $S$-lemma.** Let $A_1, \ldots, A_N \in S_n$, $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ and a matrix $X \in S_n$ such that

$$A_i, X \succeq 0, \quad \text{Tr}(A_i X) = \alpha_i, \quad i = 1, \ldots, N$$

Let $\epsilon > 0$, there exists a matrix $X_0$ such that

$$\alpha_i (1 - \epsilon) \leq \text{Tr}(A_i X_0) \leq \alpha_i (1 + \epsilon) \quad \text{and} \quad \text{Rank}(X_0) \leq 8 \frac{\log 4N}{\epsilon^2}$$

**Proof.** Randomization, concentration results on Gaussian quadratic forms.

A particular case: Given $N$ vectors $v_i \in \mathbb{R}^d$, construct their Gram matrix $X \in \mathbb{S}_N$, with

$$X \succeq 0, \quad X_{ii} - 2X_{ij} + X_{jj} = \|v_i - v_j\|_2^2, \quad i, j = 1, \ldots, N.$$ 

The matrices $D_{ij} \in \mathbb{S}_n$ such that

$$\text{Tr}(D_{ij}X) = X_{ii} - 2X_{ij} + X_{jj}, \quad i, j = 1, \ldots, N$$

satisfy $D_{ij} \succeq 0$. Let $\epsilon > 0$, there exists a matrix $X_0$ with

$$m = \text{Rank}(X_0) \leq 16 \frac{\log 2N}{\epsilon^2},$$

from which we can extract vectors $u_i \in \mathbb{R}^m$ such that

$$\|v_i - v_j\|_2^2 (1 - \epsilon) \leq \|u_i - u_j\|_2^2 \leq \|v_i - v_j\|_2^2 (1 + \epsilon).$$

In this setting, the Johnson-Lindenstrauss lemma is a particular case of the approximate $S$ lemma...
The problem of reconstructing an $N$-point Euclidean metric, given partial information on pairwise distances between points $v_i, i = 1, \ldots, N$ can also be cast as an SDP, known as and Euclidean Distance Matrix Completion problem.

\[
\begin{align*}
\text{find} & \quad D \\
\text{subject to} & \quad 1v^T + v1^T - D \succeq 0 \\
& \quad D_{ij} = \|v_i - v_j\|_2^2, \quad (i, j) \in S \\
& \quad v \geq 0
\end{align*}
\]

in the variables $D \in \mathbb{S}_n$ and $v \in \mathbb{R}^n$, on a subset $S \subset [1, N]^2$.

We can add further constraints to this problem given additional structural info on the configuration.

Applications in sensor networks, molecular conformation reconstruction etc. . .
[Dattorro, 2005] 3D map of the USA reconstructed from pairwise distances on 5000 points. Distances reconstructed from Latitude/Longitude data.
Distortion, embedding problems, 

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**Theorem**

**Embedding. [Bourgain, 1985]** Every $n$-point metric space $(X, d)$ can be embedded in an $O(\log n)$-dimensional Euclidean space with an $O(\log n)$ distortion.

Let $(X = \{x_1, \ldots, x_n\}, d)$ be a finite metric space, we can find the **minimum distortion embedding** by solving

\[
\begin{align*}
\text{minimize} & \quad z \\
\text{subject to} & \quad X_{ii} + X_{jj} - 2X_{ij} \geq d(x_i, x_j)^2 \\
& \quad X_{ii} + X_{jj} - 2X_{ij} \leq z d(x_i, x_j)^2, \quad i, j = 1, \ldots, n \\
& \quad X \succeq 0
\end{align*}
\]

in the variables $X \in S_n$ and $z \in \mathbb{R}$.

The result above shows $\sqrt{z^*}$ is $O(\log n)$ in the worst case.
Ellipsoidal approximations
Ellipsoidal approximations

Minimum volume ellipsoid $\mathcal{E}$ s.t. $C \subseteq \mathcal{E}$ (Löwner-John ellipsoid).

- parametrize $\mathcal{E}$ as $\mathcal{E} = \{ v \mid \|Av + b\|_2 \leq 1 \}$ with $A \succ 0$.
- $\text{vol} \mathcal{E}$ is proportional to $\det A^{-1}$; to compute minimum volume ellipsoid,

  $\begin{align*}
  \text{minimize (over } A, b) \quad & \log \det A^{-1} \\
  \text{subject to} \quad & \sup_{v \in C} \|Av + b\|_2 \leq 1
  \end{align*}$

  convex, but the constraint can be hard (for general sets $C$).

Finite set $C = \{x_1, \ldots, x_m\}$, or polytope with polynomial number of vertices:

$\begin{align*}
  \text{minimize (over } A, b) \quad & \log \det A^{-1} \\
  \text{subject to} \quad & \|Ax_i + b\|_2 \leq 1, \quad i = 1, \ldots, m
  \end{align*}$

also gives Löwner-John ellipsoid for polyhedron $\text{Co}\{x_1, \ldots, x_m\}$.

Similar result for the maximum volume inscribed ellipsoid when $C$ is a polyhedron given by its facets $\{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\}$.
**Ellipsoidal approximations**

**D-Optimal Experiment Design.**

Given experiment vectors $v_i \in \mathbb{R}^n$, we minimize the MLE estimation error in

$$y_j = v_j^T x + w_j$$

for Gaussian noise $w$. Assuming we run $\lambda_i$ times experiment $v_i$, the covariance matrix of the estimation error $x - \hat{x}$ is given by $\sum_{i=1}^{p} \lambda_i v_i v_i^T$ and we solve

$$\begin{align*}
\text{minimize} & \quad \log \det (\sum_{i=1}^{p} \lambda_i v_i v_i^T)^{-1} \\
\text{subject to} & \quad 1^T \lambda = 1, \quad \lambda \geq 0
\end{align*}$$

in the variable $\lambda \in \mathbb{R}^p$. The dual of this last problem is written

$$\begin{align*}
\text{minimize} & \quad \log \det(W)^{-1} \\
\text{subject to} & \quad v_i^T W v_i \leq 1
\end{align*}$$

which is a minimum volume ellipsoid problem in the variable $W \in S_n$. 
Ellipsoidal approximations

\( C' \subseteq \mathbb{R}^n \) convex, bounded, with nonempty interior

- Löwner-John ellipsoid, shrunk by a factor \( n \), lies inside \( C' \)
- maximum volume inscribed ellipsoid, expanded by a factor \( n \), covers \( C' \)

**example** (for two polyhedra in \( \mathbb{R}^2 \))

factor \( n \) can be improved to \( \sqrt{n} \) if \( C \) is symmetric. See [Boyd and Vandenberghe, 2004] for further examples.
Mixing rates for Markov chains & maximum variance unfolding
Mixing rates for Markov chains & unfolding

[Sun, Boyd, Xiao, and Diaconis, 2006]

- Let $G = (V, E)$ be an **undirected graph** with $n$ vertices and $m$ edges.
- We define a **Markov chain** on this graph, and let $w_{ij} \geq 0$ be the transition rate for edge $(i, j) \in V$.
- Let $\pi(t)$ be the state distribution at time $t$, its evolution is governed by the heat equation

$$d\pi(t) = -L\pi(t)dt$$

with

$$L_{ij} = \begin{cases} 
-w_{ij} & \text{if } i \neq j, (i, j) \in V \\
0 & \text{if } (i, j) \notin V \\
\sum_{(i,k) \in V} w_{ik} & \text{if } i = j
\end{cases}$$

the **graph Laplacian** matrix, which means

$$\pi(t) = e^{-Lt}\pi(0).$$

- The matrix $L \in S_n$ satisfies $L \succeq 0$ and its smallest eigenvalue is zero (associated with the uniform distribution).
Mixing rates for Markov chains & unfolding

- With

\[ \pi(t) = e^{-Lt}\pi(0) \]

the **mixing rate** is controlled by the second smallest eigenvalue \( \lambda_2(L) \).

- Since the smallest eigenvalue of \( L \) is zero, with eigenvector \( \mathbf{1} \), we have

\[ \lambda_2(L) \geq t \iff L(w) \succeq t(I - (1/n)\mathbf{1}\mathbf{1}^T), \]

- Maximizing the mixing rate of the Markov chain means solving

\[
\begin{align*}
\text{maximize} & \quad t \\
\text{subject to} & \quad L(w) \succeq t(I - (1/n)\mathbf{1}\mathbf{1}^T) \\
& \quad \sum_{(i,j)\in V} d_{ij}^2 w_{ij} \leq 1 \\
& \quad w \geq 0
\end{align*}
\]

in the variable \( w \in \mathbb{R}^m \), with (normalization) parameters \( d_{ij}^2 \geq 0 \).

- Since \( L(w) \) is an affine function of the variable \( w \in \mathbb{R}^m \), this is a semidefinite program in \( w \in \mathbb{R}^m \).

- Numerical solution usually performs better than **Metropolis-Hastings**.
Mixing rates for Markov chains & unfolding

We can also form the dual of the maximum MC mixing rate problem.

The dual means solving

\[
\begin{align*}
\text{maximize} & \quad \text{Tr}(X(I - (1/n)11^T)) \\
\text{subject to} & \quad X_{ii} - 2X_{ij} + X_{jj} \leq d_{ij}^2, \quad (i, j) \in V \\
& \quad X \succeq 0,
\end{align*}
\]

in the variable \(X \in S_n\).

Here too, we can interpret \(X\) as the gram matrix of a set of \(n\) vectors \(v_i \in \mathbb{R}^d\). The program above maximizes the variance of the vectors \(v_i\)

\[
\text{Tr}(X(I - (1/n)11^T)) = \sum_i \|v_i\|^2 - \|\sum_i v_i\|^2
\]

while the constraints bound pairwise distances

\[
X_{ii} - 2X_{ij} + X_{jj} \leq d_{ij}^2 \iff \|v_i - v_j\|^2 \leq d_{ij}^2
\]

This is a maximum variance unfolding problem [Weinberger and Saul, 2006, Sun et al., 2006].
From [Sun et al., 2006]: we are given pairwise 3D distances for $k$-nearest neighbors in the point set on the right. We plot the maximum variance point set satisfying these pairwise distance bounds on the right.
Moment problems & positive polynomials
Moment problems & positive polynomials

[Nesterov, 2000]. Hilbert’s 17th problem has a positive answer for univariate polynomials: a polynomial is nonnegative iff it is a sum of squares

\[ p(x) = x^{2d} + \alpha_{2d-1}x^{2d-1} + \ldots + \alpha_0 \geq 0, \text{ for all } x \iff p(x) = \sum_{i=1}^{N} q_i(x)^2 \]

We can formulate this as a linear matrix inequality, let \( v(x) \) be the moment vector

\[ v(x) = (1, x, \ldots, x^d)^T \]

we have

\[ \sum_i \lambda_i u_i u_i^T = M \succeq 0 \iff p(x) = v(x)^T M v(x) = \sum_i \lambda_i (u_i^T v(x))^2 \]

where \((\lambda_i, u_i)\) are the eigenpairs of \(M\).
The dual to the cone of Sum-of-Squares polynomials is the cone of moment matrices

\[ E_\mu[x^i] = q_i, \ i = 0, \ldots, d \iff \begin{pmatrix} q_0 & q_1 & \cdots & q_d \\ q_1 & q_2 & \cdots & q_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ q_d & q_{d+1} & \cdots & q_{2d} \end{pmatrix} \succeq 0 \]


This forms exponentially large, ill-conditioned semidefinite programs however.
Gordon-Slepian
and the maximum of Gaussian processes
Let $x \sim \mathcal{N}(0, X)$ and $y \sim \mathcal{N}(0, Y)$ be two Gaussian processes such that

$$
\begin{align*}
X_{ij} & \leq Y_{ij} \\
X_{ii} & = Y_{ii}, \quad i, j = 1, \ldots, n,
\end{align*}
$$

then

$$
\mathbb{E} \left[ \prod_{i=1}^{N} f(x_i) \right] \leq \mathbb{E} \left[ \prod_{i=1}^{N} f(y_i) \right]
$$

for every nonnegative and nonincreasing differentiable $f$ such that $f$ and $f'$ are bounded on $\mathbb{R}$. This is first order stochastic dominance.

This implies in particular that

$$
\mathbb{E} \left[ \sup_{i=1,\ldots,N} y_i \right] \leq \mathbb{E} \left[ \sup_{i=1,\ldots,N} x_i \right]
$$
Lemma

**Simple bound on Gaussian processes.** Let $y \sim \mathcal{N}(0, Y)$ be a Gaussian vector with covariance $Y \in \mathbb{S}_n$. Suppose $X \succeq 0$ satisfies

$$Y_{ii} - 2Y_{ij} + Y_{jj} \leq X_{ii} - 2X_{ij} + X_{jj}, \quad i,j = 1, \ldots, n$$

which are convex inequalities in $X \in \mathbb{S}_n$, then

$$\mathbb{E} \left[ \sup_{i=1,\ldots,n} y_i \right] \leq 2 \left( \text{Rank}(X) \max_{i=1,\ldots,n} X_{ii} \right)^{1/2}$$

**Proof.** Use Gordon-Slepian together with Cauchy inequality. Write $x = Vg$ with $V \in \mathbb{R}^{n \times k}$ such that $X = VV^T$ and $k = \text{Rank}(X)$, so

$$\sup_{i=1,\ldots,n} x_i = \sup_{i=1,\ldots,n} \sum_{j=1}^k V_{ij} g_j \leq \left( \sup_{i=1,\ldots,n} \| V_i \|_2 \right) \| g \|_2 \quad \text{where } g \text{ i.i.d. Gaussian, with } \mathbb{E}[\| g \|_2] \leq \sqrt{k} \text{ and } \| V_i \|_2 = X_{ii}.$$
**Proposition**

**SDP bounds on Gaussian processes.** Let \( y \sim \mathcal{N}(0, Y) \) be a Gaussian vector with covariance \( Y \in \mathbb{S}_n \). From a matrix \( X \in \mathbb{S}_n \) satisfying

\[
Y_{ii} - 2Y_{ij} + Y_{jj} \leq (X_{ii} - 2X_{ij} + X_{jj}), \quad i, j = 1, \ldots, n
\]

which are convex inequalities in \( X \in \mathbb{S}_n \), we can construct a low rank matrix \( X^r \) such that

\[
\mathbb{E} \left[ \sup_{i=1,\ldots,n} y_i \right] \leq c_2 \sqrt{\log n} \left( \max_{i=1,\ldots,n} \sqrt{X_{ii}} \right)
\]

where \( c_1, c_2 > 0 \) are absolute constants.

**Proof.** Combine previous lemma with approximate S-lemma. Explicit randomized procedure to find \( X^r \).
The previous result shows that the SDP bound always does as well as the classical bound

\[ E \left[ \sup_{i=1,...,n} y_i \right] \leq 2\sqrt{\log n} \left( \max_{i=1,...,n} \sqrt{Y_{ii}} \right), \]

up to a multiplicative constant.

We can exploit symmetries in the matrix \( Y \) to block diagonalize the SDP and reduce its complexity. [Gatermann and Parrilo, 2002, Vallentin, 2009]
Semidefinite programming formulation to some problems in geometry, probability, statistics.

Improvements over classical closed-form bounds on reasonably large problems.

Next: applications to compressed sensing...
References


