# A Market Test for the Positivity of Arrow-Debreu Prices

**Alexandre d'Aspremont**

ORFE, Princeton University.

# Introduction

- Classic Black & Scholes (1973) option pricing based on:
  - a *dynamic hedging* argument
  - a *model* for the asset dynamics (geometric BM)
- Sensitive to liquidity, transaction costs, model risk ...
- What can we say about derivative prices with much weaker assumptions?

# Static Arbitrage

Here, we rely on a *minimal set of assumptions*:

- no assumption on the asset distribution
- one period model

An arbitrage in this simple setting is a *buy and hold* strategy:

- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option's maturity

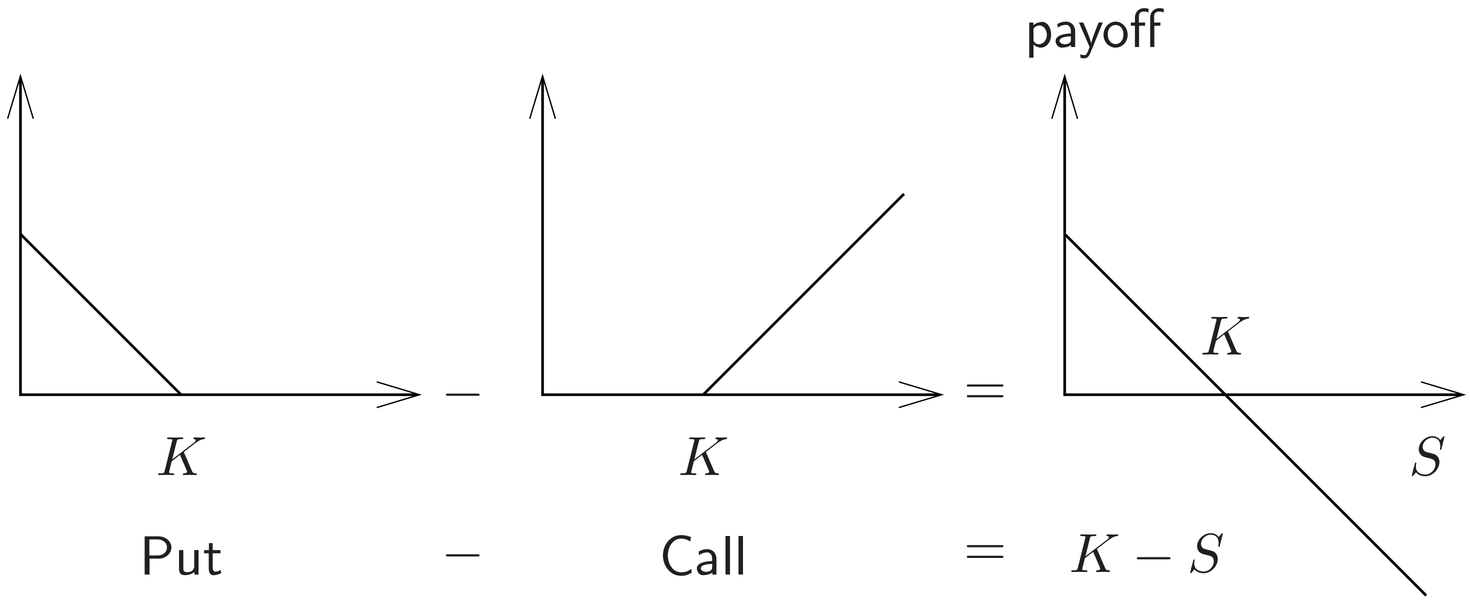
# What for?

- Data validation (e.g. before calibration), static arbitrage means market data is incompatible with *any* dynamic model. . .
- Test extrapolation formulas
- In illiquid markets, find optimal static hedge

# Outline

- **Static Arbitrage**
- Harmonic Analysis on Semigroups
- No Arbitrage Conditions

# Simplest Example: Put Call Parity



# Static Arbitrage: Calls

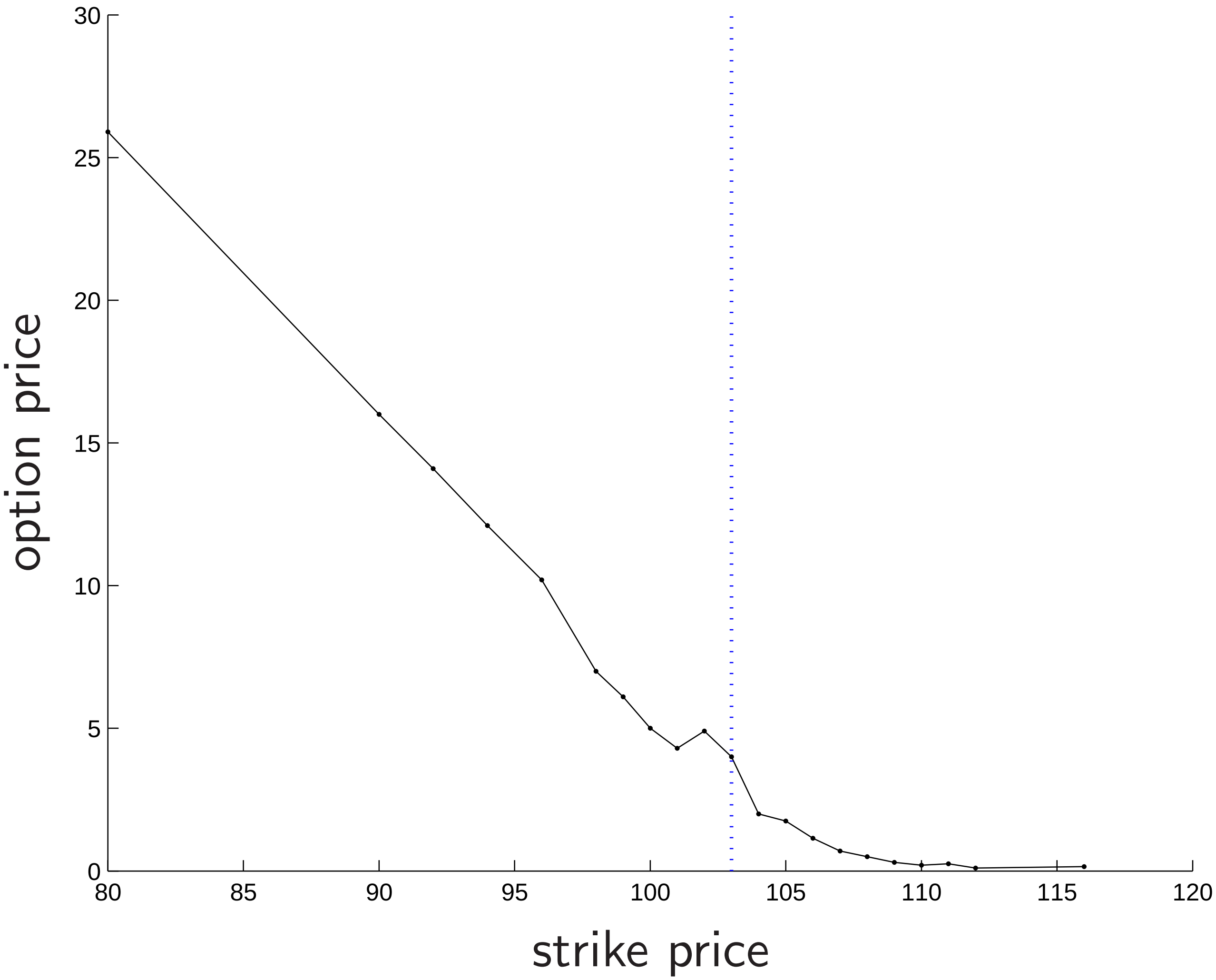
Also, necessary and sufficient conditions on call prices:

*Suppose we have a set of market prices for calls* $C(K_i) = p_i$*, then there is no arbitrage iff there is a function C(K):*

- $C(K)$ *positive*
- $C(K)$ *decreasing*
- $C(K)$ *convex*
- $C(K_i) = p_i$ *and* $C(0) = S$

This is *very easy* to test. . .

Dow Jones index call option prices on Mar. $17\ 2004$, maturity Apr. $16\ 2004$




Source: Reuters.

# Why?

Data quality...

- All the prices are last quotes (not simultaneous)
- Low volume
- Some transaction costs

Problem: this data is used to calibrate models and price other derivatives...

# Dimension n: Basket Options

- A basket call payoff is given by:

$$\left(\sum_{i=1}^{k} w_i S_i - K\right)^+$$

  where $w_1, \ldots, w_k$ are the basket's weights and $K$ is the option's strike price

- Examples include: Index options, spread options, swaptions...

- Basket option prices are used to gather information on *correlation*

We denote by $C(w, K)$ the price of such an option, can we get conditions to test basket price data?

# Necessary Conditions

Similar to dimension one...

Suppose we have a set of market prices for calls $C(w_i, K_i) = p_i$, and there is no arbitrage, then the function $C(w, K)$ satisfies:

- $C(w, K)$ positive
- $C(w, K)$ decreasing in $K$, increasing in $w$
- $C(w, K)$ jointly convex in $(w, K)$
- $C(w_i, K_i) = p_i$ and $C(0) = S$

This is still *tractable* in dimension $n$ as a *linear program*.

# Sufficient?

A key difference with dimension one: Bertsimas & Popescu (2002) show that the exact problem is NP-Hard.

- These conditions are *only necessary*...
- Numerical cost is minimal (small LP)
- We can show *sufficiency* in some particular cases (see d'Aspremont & El Ghaoui (2005) and Davis & Hobson (2005) for details)

In practice: these conditions are far from being tight, how can we *refine* them?

# Arrow-Debreu prices

- *Arrow-Debreu*: There is no arbitrage in the static market iff there is a probability measure $\pi$ such that:

$$C(w, K) = \mathbf{E}_\pi (w^T x - K)^+$$

- $\pi(x)$ represents Arrow-Debreu state prices.
- Discretize on a uniform grid: This turns this into a *linear program* with $m^n$ variables, where $n$ is the number of assets $x_i$ and $m$ is the number of bins.
- Numerically: hopeless. . .
- Explicit conditions derived by Henkin & Shananin (1990) (link with Radon transform), but intractable. . .

# Tractable Conditions

- Bochner's theorem on the Fourier transform of positive measures:

$$f(s) = \int e^{-i<s,x>} d\lambda(x) \quad \text{with } \lambda \text{ positive}$$

$$\Updownarrow$$

$$f(s) \text{ positive semidefinite}$$

which means testing if the *matrices* $f(s_i s_j)$ are positive semidefinite

- Can we generalize this result to other transforms? In particular:

$$\int_{\mathbf{R}_+^n} (w^T x - K)^+ d\pi(x)$$

# Outline

- Static Arbitrage
- **Harmonic Analysis on Semigroups**
- No Arbitrage Conditions

# Harmonic Analysis on Semigroups

Some quick definitions...

- A pair $(\mathbb{S}, \cdot)$ is called a *semigroup* iff:
  - if $s, t \in \mathbb{S}$ then $s \cdot t$ is also in $\mathbb{S}$
  - there is a neutral element $e \in \mathbb{S}$ such that $e \cdot s = s$ for all $s \in \mathbb{S}$

- The *dual* $\mathbb{S}^*$ of $\mathbb{S}$ is the set of *semicharacters*, $i.e.$ applications $\chi : \mathbb{S} \to \mathbf{R}$ such that
  - $\chi(s)\chi(t) = \chi(s \cdot t)$ for all $s, t \in \mathbb{S}$
  - $\chi(e) = 1$, where $e$ is the neutral element in $\mathbb{S}$

- A function $f : \mathbb{S} \to \mathbf{R}$ is *positive semidefinite* iff for every family $\{s_i\} \subset \mathbb{S}$ the matrix with elements $f(s_i \cdot s_j)$ is positive semidefinite

# Harmonic Analysis on Semigroups

Last definitions (honest)...

- A function $\alpha$ is called an *absolute value* on $\mathbb{S}$ iff
  - $\alpha(e) = 1$
  - $\alpha(s \cdot t) \leq \alpha(s)\alpha(t)$, for all $s, t \in \mathbb{S}$

- A function $f$ is *bounded* with respect to the absolute value $\alpha$ iff there is a constant $C > 0$ such that

$$|f(s)| \leq C\alpha(s), \quad s \in \mathbb{S}$$

- $f$ is *exponentially bounded* iff it is bounded with respect to an absolute value

Carleman type conditions on growth for moment determinacy, etc. . .

# Harmonic Analysis on Semigroups: Central Result

The central result, see Berg, Christensen & Ressel (1984) based on Choquet's theorem:

- the set of exponentially bounded *positive definite functions* is a *Bauer simplex* whose extreme points are the bounded semicharacters...
- this means that we have the following representation for positive definite functions on $\mathbb{S}$:
$$f(s) = \int_{\mathbb{S}^*} \chi(s) d\mu(\chi)$$
where $\mu$ is a Radon measure on $\mathbb{S}^*$

# Harmonic Analysis on Semigroups: Simple Examples

- *Berstein's theorem* for the Laplace transform

$$\mathbb{S} = (\mathbf{R}_+, +),\ \chi_x(t) = e^{-xt} \text{ and } f(t) = \int_{\mathbf{R}_+} e^{-xt} d\mu(x)$$

- with involution, *Bochner's theorem* for the Fourier transform

$$\mathbb{S} = (\mathbf{R}, +),\ \chi_x(t) = e^{2\pi ixt} \text{ and } f(t) = \int_{\mathbf{R}} e^{2\pi ixt} d\mu(x)$$

- *Hamburger's solution* to the unidimensional moment problem

$$\mathbb{S} = (\mathbf{N}, +),\ \chi_x(k) = x^k \text{ and } f(k) = \int_{\mathbf{R}} x^k d\mu(x)$$

# Outline

- Static Arbitrage
- Harmonic Analysis on Semigroups
- **No Arbitrage Conditions**

# The Option Pricing Problem Revisited

What is the appropriate semigroup here?

- Basket option payoffs $(w^T x - K)^+$ are not ideal in this setting.
- Solution: use *straddles*: $|w^T x - K|$
- Straddles are just the *sum of a call and a put*, their price can be computed from that of the corresponding call and forward by call-put parity.
- The fact that $|w^T x - K|^2$ is a polynomial keeps the complexity low.

# Payoff Semigroup

- The fundamental semigroup $\mathbb{S}$ here is the multiplicative *payoff semigroup* generated by the cash, the forwards and the straddles:

$$\mathbb{S} = \{1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots\}$$

- The *semicharacters* are the functions $\chi_x : \mathbb{S} \to \mathbf{R}$ which evaluate the payoffs at a certain point $x$

$$\chi_x(s) = s(x), \quad \text{for all } s \in \mathbb{S}$$

# The Option Pricing Problem Revisited

- The original static arbitrage problem can be reformulated as

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & f(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m \\ & f(s) = \mathbf{E}_\pi[s], \quad s \in \mathbb{S} \quad \text{(f moment function)} \end{array}$$

- The variable is now $f : \mathbb{S} \to \mathbf{R}$, a function that associates to each payoff $s$ in $\mathbb{S}$, its price $f(s)$

- The *representation result* in Berg et al. (1984) shows when a (price) function $f : \mathbb{S} \to \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\pi[s]$$

# Option Pricing: Main Theorem

If we assume that the asset distribution has a compact support included in $\mathbf{R}^n_+$, and note $e_i$ for $i = 1, \ldots, n+m$ the forward and option payoff functions we get:

*A function $f(s) : \mathbb{S} \to \mathbf{R}$ can be represented as*

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \textit{for all } s \in \mathbb{S},$$

*for some measure $\nu$ with compact support, iff for some $\beta > 0$:*

*(i)* $f(s)$ *is positive semidefinite*

*(ii)* $f(e_i s)$ *is positive semidefinite for* $i = 1, \ldots, n+m$

*(iii)* $\left(\beta f(s) - \sum_{i=1}^{n+m} f(e_i s)\right)$ *is positive semidefinite*

this turns the basket arbitrage problem into a *semidefinite program*

# Semidefinite Programming

A *semidefinite program* is written:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, CX \\ \text{subject to} & \mathbf{Tr}\, A_i X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0, \end{array}$$

in the variable $X \in \mathbf{S}^n$, with parameters $C, A_i \in \mathbf{S}^n$ and $b_i \in \mathbf{R}$ for $i = 1, \ldots, m$. Its *dual* is given by:

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & C - \sum_{i=1}^m \lambda_i A_i \succeq 0, \end{array}$$

in the variable $\lambda \in \mathbf{R}^m$.

Extension of interior point techniques for linear programming show how to solve these convex programs *efficiently* (see Nesterov & Nemirovskii (1994), Sturm (1999) and Boyd & Vandenberghe (2004)).

# Option Pricing: a Semidefinite Program

We get a relaxation by only sampling the elements of $\mathbb{S}$ up to a certain degree, the variable is then the vector $f(s)$ with

$$e = (1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots, |w_m^T x - K_m|^N)$$

testing for the absence of arbitrage is then a *semidefinite program*:

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\ & M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$

# Conic Duality

Let $\Sigma \subset \mathcal{A}(\mathbb{S})$ be the set of polynomials that are sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$, and $\mathcal{P}$ the set of positive semidefinite sequences on $\mathbb{S}$

- instead of the conic duality between probability measures and positive portfolios

$$p(x) \geq 0 \Leftrightarrow \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu$$

- we use the duality between positive semidefinite sequences $\mathcal{P}$ and sums of squares polynomials $\Sigma$

$$p \in \Sigma \Leftrightarrow \langle f, p \rangle \geq 0 \text{ for all } f \in \mathcal{P}$$

with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \to \mathbf{R}$, where $\langle f, p \rangle = \sum_i q_i f(s_i)$

# Option Pricing: Caveats

- *Size*: grows exponentially with the number of assets: no free lunch. . .

- In dimension 2, for spread options, this is:

$$\binom{2+d}{2}(k+1)$$

  where $d$ is the degree of the relaxation and $k$ the number of assets.

- Conditioning issues. . .

# Conclusion

- Testing for static arbitrage in option price data is easy in dimension one
- The extension on basket options (swaptions, etc) is NP-hard but good relaxations can be found
- We get a computationally friendly set of conditions for the absence of arbitrage
- Small scale problems are tractable in practice as semidefinite programs

# References

Berg, C., Christensen, J. P. R. & Ressel, P. (1984), *Harmonic analysis on semigroups : theory of positive definite and related functions*, Vol. 100 of *Graduate texts in mathematics*, Springer-Verlag, New York.

Bertsimas, D. & Popescu, I. (2002), 'On the relation between option and stock prices: a convex optimization approach', *Operations Research* **50**(2), 358–374.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 637–659.

Boyd, S. & Vandenberghe, L. (2004), *Convex Optimization*, Cambridge University Press.

d'Aspremont, A. & El Ghaoui, L. (2005), 'Static arbitrage bounds on basket option prices', *Mathematical Programming* .

Davis, M. H. & Hobson, D. G. (2005), 'The range of traded option prices', *Working Paper* .

Henkin, G. & Shananin, A. (1990), 'Bernstein theorems and Radon transform, application to the theory of production functions', *American Mathematical Society: Translation of mathematical monographs* **81**, 189–223.

Nesterov, Y. & Nemirovskii, A. (1994), *Interior-point polynomial algorithms in convex programming*, Society for Industrial and Applied Mathematics, Philadelphia.

Sturm, J. F. (1999), 'Using SEDUMI 1.0x, a MATLAB toolbox for optimization over symmetric cones', *Optimization Methods and Software* **11**, 625–653.