A Market Test for the Positivity of Arrow-Debreu Prices

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Introduction

- Classic Black & Scholes (1973) option pricing based on:
  - a *dynamic hedging* argument
  - a *model* for the asset dynamics (geometric BM)

- Sensitive to liquidity, transaction costs, model risk ...

- What can we say about derivative prices with much weaker assumptions?
Static Arbitrage

Here, we rely on a *minimal set of assumptions*:

- no assumption on the asset distribution
- one period model

An arbitrage in this simple setting is a *buy and hold* strategy:

- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option’s maturity
What for?

• Data validation (e.g. before calibration), static arbitrage means market data is incompatible with *any* dynamic model.

• Test extrapolation formulas

• In illiquid markets, find optimal static hedge
Outline

• Static Arbitrage

• Harmonic Analysis on Semigroups

• No Arbitrage Conditions
Simplest Example: Put Call Parity

$$K - S = K - S$$
Static Arbitrage: Calls

Also, necessary and sufficient conditions on call prices:

Suppose we have a set of market prices for calls $C(K_i) = p_i$, then there is no arbitrage iff there is a function $C(K)$:

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex
- $C(K_i) = p_i$ and $C(0) = S$

This is very easy to test...
Dow Jones index call option prices on Mar. 17 2004, maturity Apr. 16 2004

Source: Reuters.

Why?

Data quality...

- All the prices are last quotes (not simultaneous)
- Low volume
- Some transaction costs

Problem: this data is used to calibrate models and price other derivatives...
Dimension n: Basket Options

- A basket call payoff is given by:

\[
\left( \sum_{i=1}^{k} w_i S_i - K \right)^+ 
\]

where \( w_1, \ldots, w_k \) are the basket’s weights and \( K \) is the option’s strike price.

- Examples include: Index options, spread options, swaptions...

- Basket option prices are used to gather information on correlation.

We denote by \( C(w, K) \) the price of such an option, can we get conditions to test basket price data?
Necessary Conditions

Similar to dimension one...

Suppose we have a set of market prices for calls $C(w_i, K_i) = p_i$, and there is no arbitrage, then the function $C(w, K)$ satisfies:

- $C(w, K)$ positive
- $C(w, K)$ decreasing in $K$, increasing in $w$
- $C(w, K)$ jointly convex in $(w, K)$
- $C(w_i, K_i) = p_i$ and $C(0) = S$

This is still \textit{tractable} in dimension $n$ as a \textit{linear program}. 
Sufficient?

A key difference with dimension one: Bertsimas & Popescu (2002) show that the exact problem is NP-Hard.

- These conditions are \textit{only necessary}...

- Numerical cost is minimal (small LP)

- We can show \textit{sufficiency} in some particular cases (see d’Aspremont & El Ghaoui (2005) and Davis & Hobson (2005) for details)

In practice: these conditions are far from being tight, how can we \textit{refine} them?
Arrow-Debreu prices

- **Arrow-Debreu**: There is no arbitrage in the static market iff there is a probability measure $\pi$ such that:

$$C(w, K) = E_\pi(w^T x - K)^+$$

- $\pi(x)$ represents Arrow-Debreu state prices.

- Discretize on a uniform grid: This turns this into a *linear program* with $m^n$ variables, where $n$ is the number of assets $x_i$ and $m$ is the number of bins.

- Numerically: hopeless.

- Explicit conditions derived by Henkin & Shananin (1990) (link with Radon transform), but intractable.
Tractable Conditions

- Bochner’s theorem on the Fourier transform of positive measures:
  \[ f(s) = \int e^{-i<s,x>} d\lambda(x) \quad \text{with } \lambda \text{ positive} \]
  \[ \uparrow \]
  \[ f(s) \text{ positive semidefinite} \]
  which means testing if the matrices \( f(s_i s_j) \) are positive semidefinite

- Can we generalize this result to other transforms? In particular:
  \[ \int_{\mathbb{R}_n^+} (w^T x - K)^+ d\pi(x) \]
Outline

- Static Arbitrage
- Harmonic Analysis on Semigroups
- No Arbitrage Conditions
Some quick definitions...

- A pair \((S, \cdot)\) is called a **semigroup** iff:
  - if \(s, t \in S\) then \(s \cdot t\) is also in \(S\)
  - there is a neutral element \(e \in S\) such that \(e \cdot s = s\) for all \(s \in S\)

- The **dual** \(S^*\) of \(S\) is the set of **semicharacters**, i.e. applications \(\chi : S \rightarrow \mathbb{R}\) such that
  - \(\chi(s)\chi(t) = \chi(s \cdot t)\) for all \(s, t \in S\)
  - \(\chi(e) = 1\), where \(e\) is the neutral element in \(S\)

- A function \(f : S \rightarrow \mathbb{R}\) is **positive semidefinite** iff for every family \(\{s_i\} \subset S\), the matrix with elements \(f(s_i \cdot s_j)\) is positive semidefinite
Harmonic Analysis on Semigroups

Last definitions (honest)...

- A function $\alpha$ is called an *absolute value* on $\mathbb{S}$ iff
  - $\alpha(e) = 1$
  - $\alpha(s \cdot t) \leq \alpha(s) \alpha(t)$, for all $s, t \in \mathbb{S}$

- A function $f$ is *bounded* with respect to the absolute value $\alpha$ iff there is a constant $C > 0$ such that
  \[ |f(s)| \leq C \alpha(s), \quad s \in \mathbb{S} \]

- $f$ is *exponentially bounded* iff it is bounded with respect to an absolute value

Carleman type conditions on growth for moment determinacy, etc. . .
Harmonic Analysis on Semigroups: Central Result

The central result, see Berg, Christensen & Ressel (1984) based on Choquet's theorem:

- the set of exponentially bounded positive definite functions is a Bauer simplex whose extreme points are the bounded semicharacters...

- this means that we have the following representation for positive definite functions on $S$:

$$f(s) = \int_{S^*} \chi(s) d\mu(\chi)$$

where $\mu$ is a Radon measure on $S^*$
Harmonic Analysis on Semigroups: Simple Examples

- **Berstein’s theorem** for the Laplace transform

  \[ S = (\mathbb{R}_+ , +), \, \chi_x(t) = e^{-xt} \quad \text{and} \quad f(t) = \int_{\mathbb{R}_+} e^{-xt} \, d\mu(x) \]

- with involution, **Bochner’s theorem** for the Fourier transform

  \[ S = (\mathbb{R}, +), \, \chi_x(t) = e^{2\pi i xt} \quad \text{and} \quad f(t) = \int_{\mathbb{R}} e^{2\pi i xt} \, d\mu(x) \]

- **Hamburger’s solution** to the unidimensional moment problem

  \[ S = (\mathbb{N}, +), \, \chi_x(k) = x^k \quad \text{and} \quad f(k) = \int_{\mathbb{R}} x^k \, d\mu(x) \]
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The Option Pricing Problem Revisited

What is the appropriate semigroup here?

- Basket option payoffs \((w^T x - K)^+\) are not ideal in this setting.

- Solution: use straddles: \(|w^T x - K|\)

- Straddles are just the sum of a call and a put, their price can be computed from that of the corresponding call and forward by call-put parity.

- The fact that \(|w^T x - K|^2\) is a polynomial keeps the complexity low.
Payoff Semigroup

• The fundamental semigroup $\mathcal{S}$ here is the multiplicative payoff semigroup generated by the cash, the forwards and the straddles:

$$\mathcal{S} = \{1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots\}$$

• The semicharacters are the functions $\chi_x : \mathcal{S} \rightarrow \mathbb{R}$ which evaluate the payoffs at a certain point $x$

$$\chi_x(s) = s(x), \text{ for all } s \in \mathcal{S}$$
The original static arbitrage problem can be reformulated as

\[
\begin{align*}
\text{find} & \quad f \\
\text{subject to} & \quad f(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m \\
& \quad f(s) = \mathbb{E}_\pi[s], \quad s \in \mathbb{S} \quad (f \text{ moment function})
\end{align*}
\]

The variable is now \( f : \mathbb{S} \to \mathbb{R} \), a function that associates to each payoff \( s \) in \( \mathbb{S} \), its price \( f(s) \)

The \textit{representation result} in Berg et al. (1984) shows when a (price) function \( f : \mathbb{S} \to \mathbb{R} \) can be represented as

\[
f(s) = \mathbb{E}_\pi[s]
\]
Option Pricing: Main Theorem

If we assume that the asset distribution has a compact support included in $\mathbb{R}^n_+$, and note $e_i$ for $i = 1, \ldots, n + m$ the forward and option payoff functions we get:

A function $f(s) : S \rightarrow \mathbb{R}$ can be represented as

$$f(s) = \mathbb{E}_\nu[s(x)], \text{ for all } s \in S,$$

for some measure $\nu$ with compact support, iff for some $\beta > 0$:

(i) $f(s)$ is positive semidefinite

(ii) $f(e_i s)$ is positive semidefinite for $i = 1, \ldots, n + m$

(iii) $\left( \beta f(s) - \sum_{i=1}^{n+m} f(e_i s) \right)$ is positive semidefinite

this turns the basket arbitrage problem into a semidefinite program
A **semidefinite program** is written:

\[
\begin{align*}
\text{minimize} & \quad \text{Tr } CX \\
\text{subject to} & \quad \text{Tr } A_i X = b_i, \quad i = 1, \ldots, m \\
& \quad X \succeq 0,
\end{align*}
\]

in the variable \( X \in \mathbb{S}^n \), with parameters \( C, A_i \in \mathbb{S}^n \) and \( b_i \in \mathbb{R} \) for \( i = 1, \ldots, m \). Its **dual** is given by:

\[
\begin{align*}
\text{maximize} & \quad b^T \lambda \\
\text{subject to} & \quad C - \sum_{i=1}^{m} \lambda_i A_i \succeq 0,
\end{align*}
\]

in the variable \( \lambda \in \mathbb{R}^m \).

Extension of interior point techniques for linear programming show how to solve these convex programs **efficiently** (see Nesterov & Nemirovskii (1994), Sturm (1999) and Boyd & Vandenberghe (2004)).
Option Pricing: a Semidefinite Program

We get a relaxation by only sampling the elements of $S$ up to a certain degree, the variable is then the vector $f(s)$ with

$$e = (1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots, |w_m^T x - K_m|^N)$$

testing for the absence of arbitrage is then a semidefinite program:

$$\begin{align*}
\text{find} & \quad f \\
\text{subject to} & \quad M_N(f(s)) \succeq 0 \\
& \quad M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\
& \quad M_N \left( f(\left( \beta - \sum_{k=1}^{n+m} e_k \right) s) \right) \succeq 0 \\
& \quad f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n + m \text{ and } s \in S
\end{align*}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$

Conic Duality

Let $\Sigma \subset A(S)$ be the set of polynomials that are sums of squares of polynomials in $A(S)$, and $\mathcal{P}$ the set of positive semidefinite sequences on $S$

- instead of the conic duality between probability measures and positive portfolios

$$p(x) \geq 0 \iff \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu$$

- we use the duality between positive semidefinite sequences $\mathcal{P}$ and sums of squares polynomials $\Sigma$

$$p \in \Sigma \iff \langle f, p \rangle \geq 0 \text{ for all } f \in \mathcal{P}$$

with $p = \sum_i q_i \chi_{s_i}$ and $f : S \to \mathbb{R}$, where $\langle f, p \rangle = \sum_i q_i f(s_i)$
Option Pricing: Caveats

- **Size**: grows exponentially with the number of assets: no free lunch...

- In dimension 2, for spread options, this is:

\[
\binom{2 + d}{2}(k + 1)
\]

where \(d\) is the degree of the relaxation and \(k\) the number of assets.

- Conditioning issues...
Conclusion

• Testing for static arbitrage in option price data is easy in dimension one.

• The extension on basket options (swaptions, etc) is NP-hard but good relaxations can be found.

• We get a computationally friendly set of conditions for the absence of arbitrage.

• Small scale problems are tractable in practice as semidefinite programs.
References


