Shape Constrained Optimization
with Applications in Finance and Engineering

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Problem Statement

Shape Constrained Problem (SCP):

$$\begin{array}{ll} \text{minimize} & c^T z \\ \text{subject to} & Az \leq b, \; Cz = d \\ & z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T\right]^T \\ & g_i \text{ subgradient of } f \text{ at } x_i \quad i = 1, \ldots, k \\ & f \text{ bounded, convex \& monotone} \end{array}$$

in the variables $f \in C(\mathbf{R}^n)$, $z \in \mathbf{R}^{(n+1)k}$, $g_1, \ldots, g_k \in \mathbf{R}^n$

- particular case of Continuous Linear Program, which are intractable in general
- *reduces to a Linear Program* with a polynomial number of constraints
- extensions: replace "convex" by "positive" or "moment function"...

Outline

1. **Problem statement and motivation**

2. Convexity constraints

   (a) Main result
   (b) Applications:
      i. Consumer preference
      ii. Convex relaxations
      iii. Option pricing

3. Extension: moment problems

   (a) Harmonic analysis and positive semidefinite functions
   (b) The option pricing problem revisited

Examples...

Predicting Consumer Preference

- one model consumer, whose choices are repeatable

- his/her preferences are driven by an utility function u

- we get data on the consumer preferences among a set of goods baskets $a_1, \ldots, a_m$ with

$$u(a_i) \geq u(a_j), \quad i, j = 1, \ldots, m, \quad (i, j) \in \mathcal{P}$$

- strict appetite for goods and diversification mean u is *monotone* and *concave*

- the objective is to predict the consumer's preference on a new basket of goods versus the baskets $a_1, \ldots, a_m$

Convex Relaxations

$$\begin{array}{ll} \text{minimize} & \mathbf{Card}(x) \\ \text{subject to} & x \in \mathcal{C}, \end{array}$$

- most *concave minimization* problems are very hard...
- Fazel, Hindi & Boyd (2000): if $\mathcal{C}$ is convex, approximate solution replaces the objective by its convex envelope, $i.e.$ its largest convex lower bound

Basket Option Pricing

- given a set of market prices $p_1, \ldots, p_k$ corresponding to the payoffs $(w_i^T x - K_i)_+$ at maturity

- in a one period model, compute arbitrage bounds on the price p_0 of another basket, $i.e.$ solve

$$\begin{array}{ll} \text{max./min.} & \mathbf{E}_\pi (w_0^T x - K_0)_+ \\ \text{subject to} & \mathbf{E}_\pi (w_i^T x - K_i)_+ = p_i, \quad i = 1, \ldots, m, \end{array}$$

where the variable is $\pi \in \mathcal{K}$, a probability measure with support in $\mathbf{R}_+^n$

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Convexity Constraints: Main Result

consider the general Shape Constrained Problem (SCP):

$$\begin{array}{ll}\text{minimize} & c^T z \\ \text{subject to} & Az \leq b,\ Cz = d \\ & z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T\right]^T \\ & g_i \text{ subgradient of } f \text{ at } x_i \quad i = 1, \ldots, k \\ & f \text{ convex}\end{array}$$

in the variables $f \in C(\mathbf{R}^n)$, $z \in \mathbf{R}^{(n+1)k}$, $g_1, \ldots, g_k \in \mathbf{R}^n$

- we can discretize and sample the convexity constraints to get a polynomial size LP
- the solution will be a lower bound on the optimum of the original SCP

- we only enforce the convexity and subgradient constraints at the points $(x_i)_{i=1,\ldots,k}$ and get the following LP

$$
\begin{array}{ll}
\text{minimize} & c^T z \\
\text{subject to} & Cz = d, \;\; Az \leq b \\
 & z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T \right]^T \\
 & \langle g_i, x_j - x_i \rangle \leq f(x_j) - f(x_i) \quad i, j = 1, \ldots, k
\end{array}
$$

in the variables $f(x_i)_{i=1,\ldots,k}$ and g in $\mathbf{R}^n \times \mathbf{R}^{n \times k}$

- we note $z^{\mathrm{opt}} = \left[f^{\mathrm{opt}}(x_1), \ldots, f^{\mathrm{opt}}(x_k), (g_1^{\mathrm{opt}})^T, \ldots, (g_k^{\mathrm{opt}})^T \right]^T$ the optimal solution to this LP

- the optimal solution of the finite LP gives a lower bound on the optimal value of the SCP

- from z^{opt}, we construct a feasible point of the SCP and define:

$$s(x) = \max_{i=1,\ldots,k} \left\{ f^{\mathrm{opt}}(x_i) + \left\langle g_i^{\mathrm{opt}}, x - x_i \right\rangle \right\}$$

- by construction, $s(x_i)$ solves the finite LP with:

$$s(x_i) = f^{\mathrm{opt}}(x_i), \quad i = 1, \ldots, k$$

- $s(x)$ is convex and monotone as the pointwise maximum of monotone affine functions

- so $s(x)$ is also a feasible point of the SCP

this means that $s(x)$ is an *optimal solution* of the original SCP...

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Applications

- utility function assessment example adapted from Meyer & Pratt (1968), see also Pratt, Raiffa & Schlaifer (1964), Keeney & Raiffa (1993) and Keeney (1977), mostly parametric solutions...

- application on options pricing is based on Breeden & Litzenberger (1978), Buchen & Kelly (1996), Laurent & Leisen (2000) and Bertsimas & Popescu (2002), in dimension one...

- SCP also appear in nonlinear pricing and multidimensional screening problems, see Mirrlees (1971), Wilson (1993), Rochet & Chone (1998), Rochet & Stole (2000)

- applications on imaging and statistics (survival distributions) in Hansen & Lauritzen (1998) and Groeneboom, Jongbloed & Wellner (2001)

Predicting Consumer Preference

model assumptions:

- we have m baskets of goods $a_1, \ldots, a_m \in [0,1]^n$
- a consumer chooses among these goods based on a utility function u
- strict appetite for goods means u is monotone nondecreasing
- we also suppose u is concave: this models satiation, i.e. decreasing marginal utility as the amount of goods increases
- concavity also describes an appetite for diversification

Consumer Preference: Data

- the utility function is unknown...
- we are only given the consumer's preferences on the set of baskets

$$a_i \succsim a_j, \quad \text{with } (i,j) \text{ in a set } \mathcal{P} \subseteq \{1,\ldots,m\} \times \{1,...,m\}$$

- logically, $\mathcal{P}$ is transitive...
- this means that the preference information $\mathcal{P}$ gives us at most $m(m-1)/2$ inequalities on the utility function u at the points $a_1,\ldots,a_m \in [0,1]^n$:

$$u(a_i) \geq u(a_j), \quad i,j = 1,\ldots,m, \quad (i,j) \in \mathcal{P}$$

Consumer Preference: Objective

consider a new basket a_0, using monotonicity, concavity and the preference relations in $\mathcal{P}$, what can we infer on the consumer's preferences between a_0 and the other a_k?

- if for every concave, monotone function u that is consistent with the preferences $\mathcal{P}$ we have

$$u(a_0) \geq u(a_k), \quad \text{for some } k \in [1, m]$$

  then we know that the consumer will always prefer the basket a_0.

- idem if we always have $u(a_0) \leq u(a_k)$...

- if $u(a_0) \leq u(a_k)$ for some functions u and $u(a_0) \geq u(a_k)$ for others, then we can't conclude on the consumer's preference

Consumer Preference: Solution

to solve the preference problem we for the following SCP:

$$\begin{array}{ll}\text{minimize/maximize} & u(a_0) - u(a_k) \\ \text{subject to} & u \text{ concave and nondecreasing} \\ & u(a_i) \geq u(a_j), \quad i,\ j = 1, \ldots, m, \quad (i,j) \in \mathcal{P} \\ & u(0) = 0, \quad u(\mathbf{1}) = 1\end{array}$$

with (infinite-dimensional) variable $u : \mathbf{R}^n \rightarrow \mathbf{R}$, and we can find an optimal solution (utility function here) by solving the following finite LP:

$$\begin{array}{ll}\text{minimize/maximize} & \hat{u}_0 - \hat{u}_k \\ \text{subject to} & \hat{u}_i \geq \hat{u}_j, \quad i,\ j = 1, \ldots, m, \quad (i,j) \in \mathcal{P} \\ & \hat{u}_i \leq \hat{u}_j + g_j^T (a_i - a_j), \quad i,\ j = 0, \ldots, m+2 \\ & g_i \succeq 0, \quad i = 0, \ldots, m+2 \\ & \hat{u}_{m+1} = 0, \quad \hat{u}_{m+2} = 1\end{array}$$

with variables $\hat{u}_0, \ldots, \hat{u}_{m+2} \in \mathbf{R}$ and $g_0, \ldots, g_{m+2} \in \mathbf{R}^n$

Example

- for simplicity, we consider baskets of two goods...
- we compute 40 random points in $[0,1]^2$
- to generate the consumer preference data $\mathcal{P}$, we compare the baskets using the utility function

$$u(x_1, x_2) = (1.1x_1^{1/2} + 0.8x_2^{1/2})/1.9.$$

- we plot these goods baskets, and a few level curves of the utility function u, in figure 1

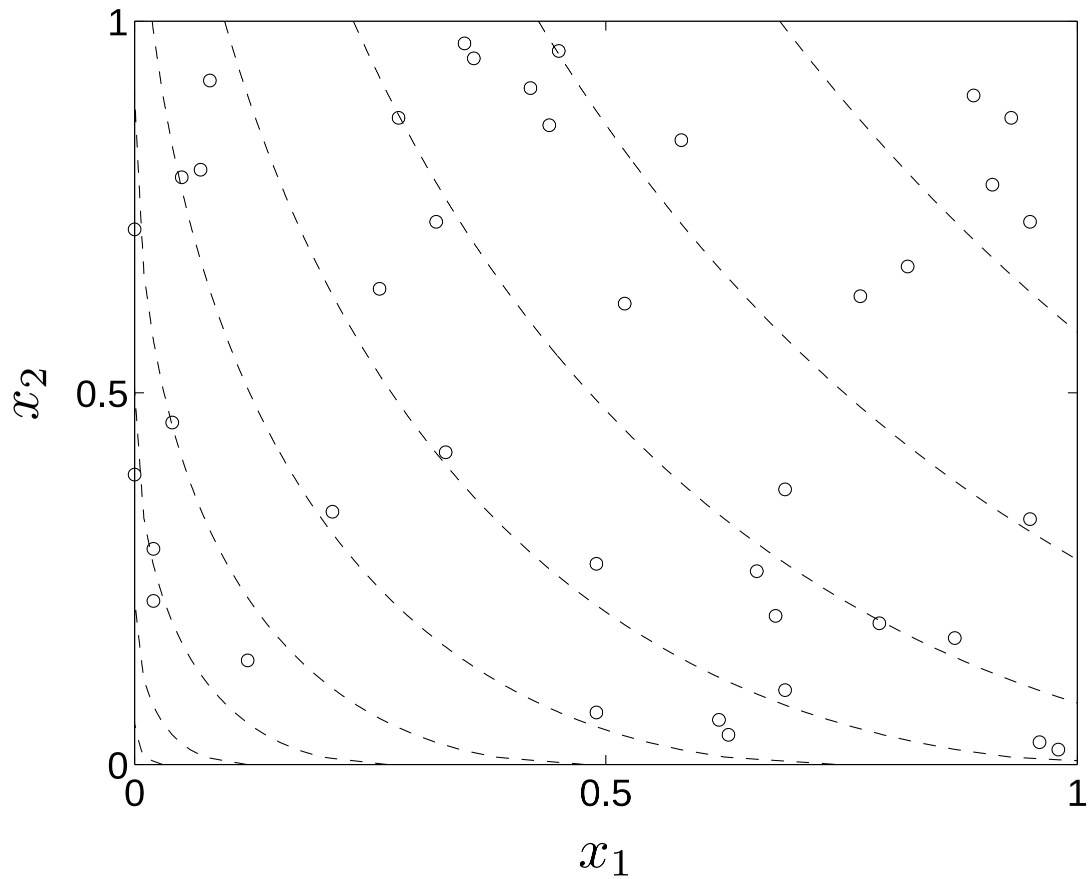


Figure 1: Forty goods baskets $a_1, \ldots, a_{40}$, shown as circles. The $0.1,\ 0.2, \ldots, 0.9$ level curves of the true utility function u are shown as dashed lines. This utility function is used to find the consumer preference data $\mathcal{P}$ among the 40 baskets.

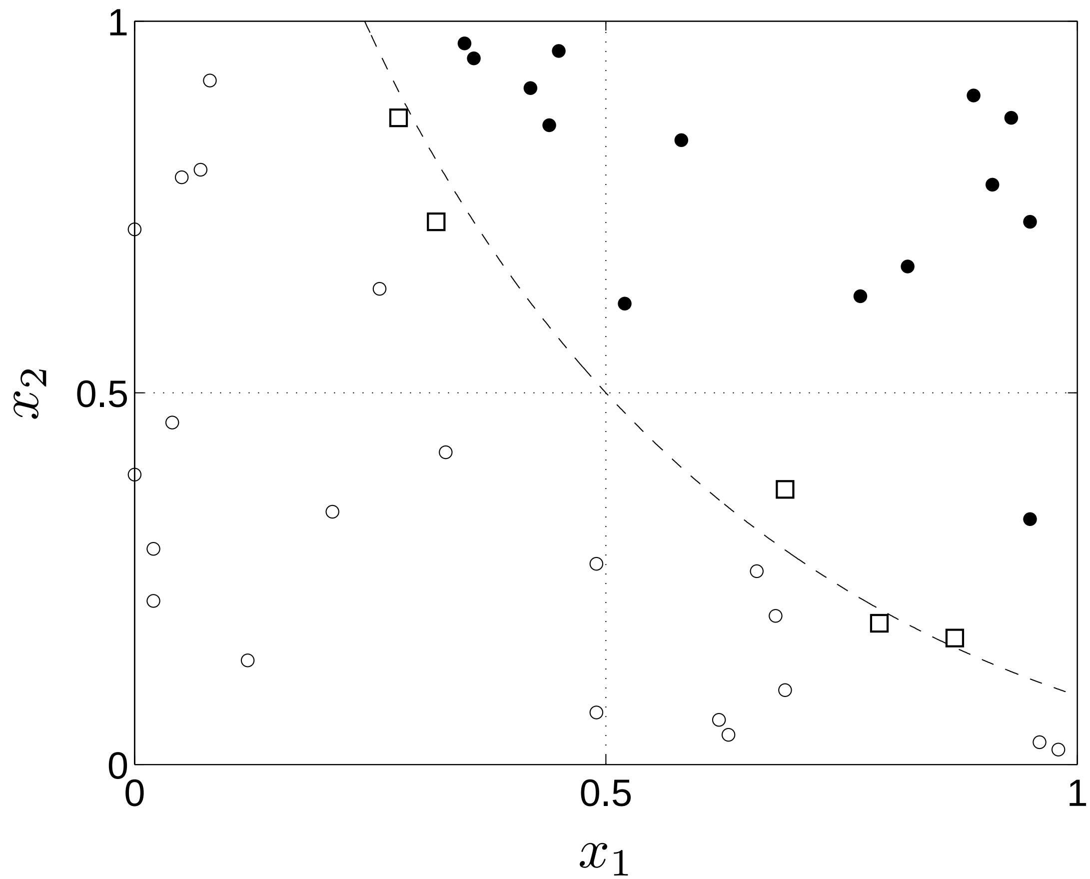
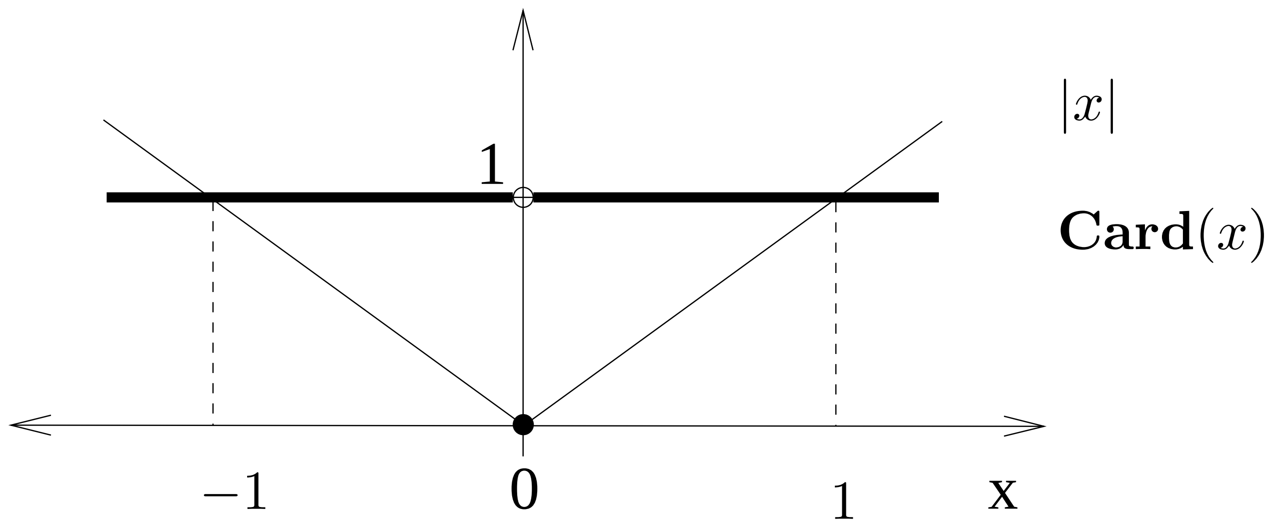


Figure 2: for a new goods basket $(0.5, 0.5)$. The original baskets are displayed as open circles if they are definitely rejected, as solid black circles if they are definitely preferred, and as squares when no conclusion can be made. The level curve of the underlying utility function, that passes through $(0.5, 0.5)$, is shown as a dashed curve

Convex Relaxations



simple result on convex lower bounds of concave functions on polyhedral convex sets (see Veinott (2003) for example):

$f : \mathcal{C} \rightarrow \mathbf{R}$ be a concave function on $\mathcal{C} \subset \mathbf{R}^n$, a (bounded) polyhedral convex set. Then the convex envelope of f is equal to the convex polyhedral function h with vertices defined by the set

$$S = \{(x, f(x)) : x \text{ vertex of } \mathcal{C}\}.$$

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Option Pricing: Basket Options

- we note the asset prices $x_1, \ldots, x_k$, the payoff at maturity of a basket call is given by:

$$\left(\sum_{i=1}^{k} w_i S_i - K \right)_+$$

where $w_1, \ldots, w_k$ are the basket's weights and K is the option's strike price

- examples include: Index options, spread options, swaptions, ...

- we note $C(w, K)$ the price of such an option

History

- we are given basket option prices and we are interested in computing arbitrage bounds on the price of another option
- static arbitrage bounds for options on a single asset are well-known (see for example Breeden & Litzenberger (1978), Bertsimas & Popescu (2002) or Laurent & Leisen (2000))...
- bounds for some continuous time models are known too (simplest example: the bounds obtained by varying the volatility in the unidimensional Black & Scholes (1973) model match the static bounds)
- what happens for baskets, in dimension n?

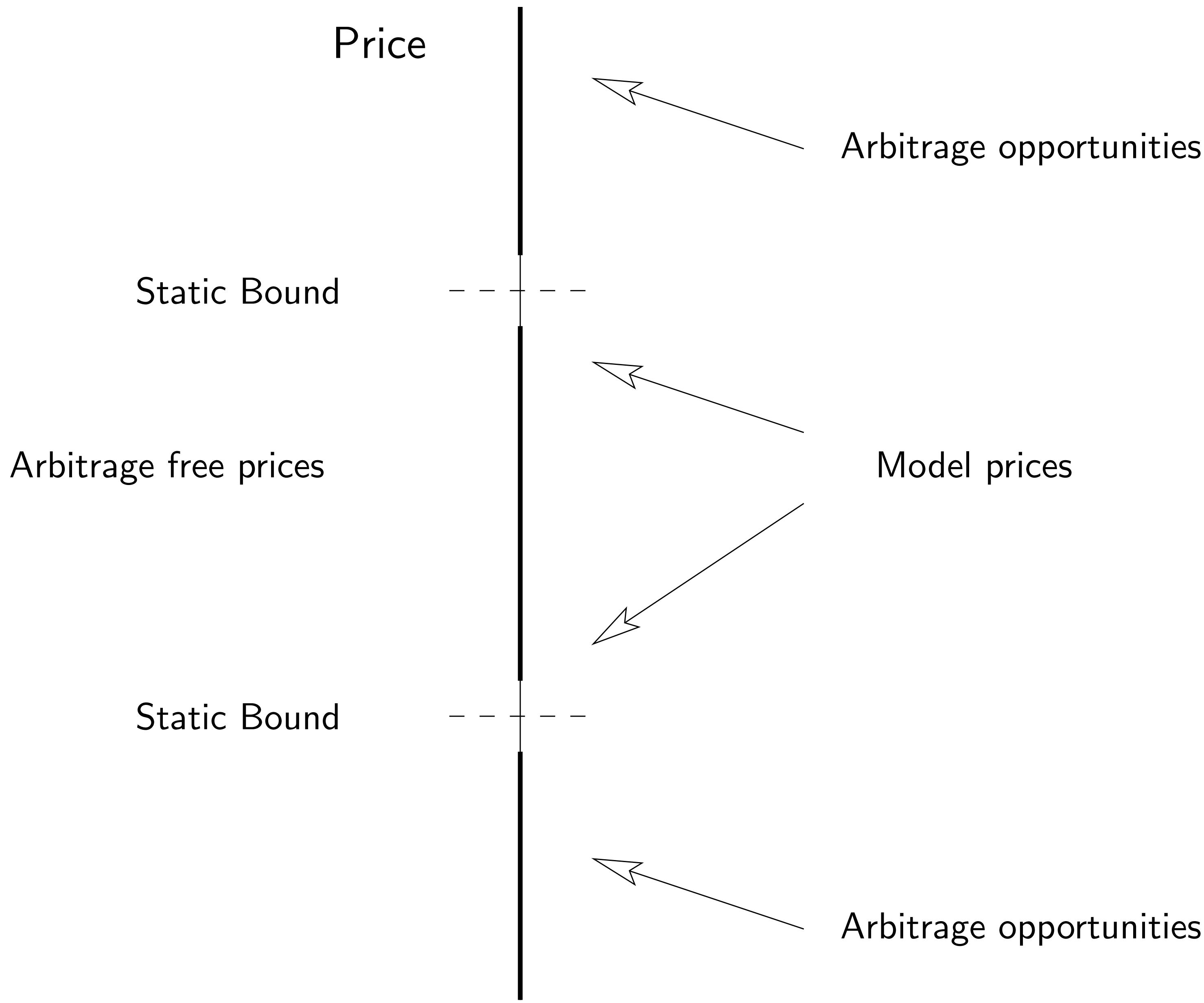
Problem Statement

solve the following program:

$$\begin{array}{ll} \text{max./min.} & \mathbf{E}_\pi (w_0^T x - K_0)_+ \\ \text{subject to} & \mathbf{E}_\pi (w_i^T x - K_i)_+ = p_i, \quad i = 1, \ldots, m, \end{array}$$

in the variable $\pi \in \mathcal{K}$, where $\mathcal{K}$ is the set of probability measures with support included in $\mathbf{R}_+^n$

- *objective*: compute upper and lower bounds on the price of an European basket call option with strike K_0 and weight vector w_0
- *inputs*: $p \in \mathbf{R}_+^m$, $K \in \mathbf{R}^m$, $w \in \mathbf{R}^n$, $w_i \in \mathbf{R}^n$, for $i = 1, \ldots, m$ and $K_0 \geq 0$
- *assumptions*: one period model, no transaction costs, perfect liquidity, but no particular assumption on π


Price
Arbitrage opportunities
Static Bound
Arbitrage free prices
Model prices
Static Bound
Arbitrage opportunities

Option Pricing: Motivation

- diagnostic: what happens when model calibration fails?
- low quality data: difference in time, missing quotes, illiquidity etc...
- sparse data: arbitrage free interpolation of basket option prices
- synthesize an option from liquid ones: spark/crack spreads on the NYMEX, real options ...
- market idea of correlation?

Dimension One

- the problem directly reduces to a SCP...
- Breeden & Litzenberger (1978), Buchen & Kelly (1996) simply impose convexity in K:

$$\pi(K) = \frac{\partial^2 C(K)}{\partial K^2} \geq 0 \quad \text{where } C(K) = \mathbf{E}_\pi(x - K)_+$$

  and use this to recover the distribution
- further developed by Laurent & Leisen (2000) who detail necessary and sufficient conditions for the absence of arbitrage and calibrate a discrete mulitperiod model
- Bertsimas & Popescu (2002) mix this with moments constraints, and show that the multivariate problem is NP-Hard

Multidimensional Problem

$$
\begin{array}{ll}
\text{max./min.} & \int_{\mathbf{R}^n_+} (w_0^T x - K_0)_+ \pi(x) dx \\
\text{subject to} & \int_{\mathbf{R}^n_+} (w_i^T x - K_i)_+ \pi(x) dx = p_i, \quad i = 1, \ldots, m \\
& \int_{\mathbf{R}^n_+} \pi(x) dx = 1
\end{array}
$$

three possible approaches:

- infinite LP or *semi-infinite program* (see Hettich & Kortanek (1993)
- *integral transform* inversion problem (tomography, ... see Henkin & Shananin (1990))
- generalized *moment problem* (see Bertsimas & Popescu (2002))

LP Solution

special case: we examine the simpler problem of computing bounds on:

$$\mathbf{E}_\pi (w^T x - K_0)_+$$

given the $2n$ constraints

$$\mathbf{E}_\pi (x_i - K_i)_+ = p_i, \quad \mathbf{E}_\pi x_i = q_i, \quad i = 1, \ldots, n,$$

on n forwards and n options on each individual asset.

LP Solution: Upper Bound

- the dual of the upper bound problem is

$$d^{\sup} \quad = \quad \inf_{\lambda+\mu \geq w} \sup_{x \geq 0} \lambda^T p + \mu^T q + (w^T x - K_0)_+ - \lambda^T (x - K)_+ - \mu^T x,$$

- decompose domain into

$$D_I = \{x \ : \ x_i > K_i, \quad i \in I, \quad 0 \leq x_i \leq K_i, \quad i \in J\},$$

- write $\left(w^T x - K\right)_+ = \max_{t \in [0,1]} t \left(w^T x - K\right)$

LP Solution: Upper Bound

- the dual of the resulting problem can be solved explicitly
- finally...

$$d^{\sup} = \max_{0 \leq j \leq n+1} w^T p + \sum_i w_i \min(q_i - p_i, \beta_j K_i) - \beta_j K_0,$$

with the convention $\beta_0 = 0$, $\beta_{n+1} = 1$

- We can check that the above bound satisfies some basic properties: it is convex in w and concave in p, q. Also, when $w = e_i$ (the i-th unit vector), and $K_0 = K_i$, we obtain $d^{\sup} = p_i$, while for $K_i = 0$, we obtain $d^{\sup} = q_i$.

LP Solution: Lower Bound

- the lower bound is computed from the dual (portfolio) problem:

$$d^{\text{inf}} = \sup_{\lambda+\mu\leq w} \inf_{x\geq 0} \lambda^T p + \mu^T q + (w^T x - K_0)_+ - \lambda^T (x-K)_+ - \mu^T x,$$

- similar techniques show that the solution can be computed from the following LP:

$$\begin{array}{ll} \sup\limits_{\lambda,\mu,\alpha_0,\ldots,\alpha_n} & \lambda^T p + \mu^T (q-K) + h \\ \text{subject to} & \lambda + \mu \leq w \\ & h \leq \alpha_0 (w^T K - K_0) - (\alpha_0 w - \mu)_+^T K, \quad 0 \leq \alpha_0 \leq 1 \\ & h \leq \alpha_i (w^T K - K_0) - \sum_{j\neq i} (\alpha_i w_j - \mu_j)_+ K_j \\ & (\lambda_i + \mu_i)_+ / w_i \leq \alpha_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

- perfect duality not guaranteed here: lower bound on lower bound

Integral Transform Solution

- we can write the set off call prices as:

$$
\begin{aligned}
C(w,K) &= \mathbf{E}_\pi (w^T x - K)_+ \\
&= \int_{\mathbf{R}^n_+} (w^T x - K)_+ d\pi(x),
\end{aligned}
$$

and think of $C_\pi(w,K)$ as a particular integral transform of the measure π

- at least formally, we have:

$$
\frac{\partial^2 C(w,K)}{\partial K^2} = \int_{\mathbf{R}^n_+} \delta(w^T x - K)\pi(x)dx
$$

- this means that $\partial^2 C(w,K)/\partial K^2$ is the *Radon transform* (see Helgason (1999) or Ramm & Katsevich (1996)) of the measure π

Integral Transform: a Range Characterization Problem...

- the general pricing problem can written as the following infinite dimensional problem:

$$\begin{array}{ll} \text{min./max.} & C(w_0, K_0) \\ \text{subject to} & C(w_i, K_i) = p_i, \quad i = 1, \ldots, m \\ & C(w, K) \in \mathcal{R}_C, \end{array}$$

- here, $\mathcal{R}_C$ is the range of the (linear) integral transform

$$\begin{aligned} C: \quad & \mathcal{K} \to \mathcal{R}_C \\ & \pi \to C(w, K) = \int_{\mathbf{R}^n_+} (w^T x - K)_+ d\pi(x) \end{aligned}$$

Integral Transform: Range Characterization

Range characterized by Henkin & Shananin (1990). A function can be written

$$C(w,K)=\int_{\mathbf{R}_+^n}(w^Tx-K)_+d\pi(x)$$

with $w\in\mathbf{R}_+^n$ and $K>0$, if and only if:

- $C(w,K)$ is *convex* and *homogenous* of degree one;
- $\lim_{K\to\infty}C(w,K)=0$ and $\lim_{K\to 0^+}\frac{\partial C(w,K)}{\partial K}=-1$
- $F(w)=\displaystyle\int_0^\infty e^{-K}d\left(\frac{\partial C(w,K)}{\partial K}\right)$ belongs to $C_0^\infty(\mathbf{R}_+^n)$
- For some $\tilde{w}\in\mathbf{R}_+^n$ the inequalities: $(-1)^{k+1}D_{\xi_1}...D_{\xi_k}F(\lambda\tilde{w})\geq 0$, for all positive integers k and $\lambda\in\mathbf{R}_{++}$ and all $\xi_1,\ldots,\xi_k$ in $\mathbf{R}_+^n$.

Integral Transform: Relaxation

Simply drop the last two constraints: If a function $C(w, K)$, with $w \in \mathbf{R}^n_+$ and $K > 0$ belongs to $\mathcal{R}_C$ and can be represented as

$$C(w, K) = \int_{\mathbf{R}^n_+} (w^T x - K)_+ d\pi(x),$$

then necessarily

- $C(w, K)$ is *convex* and *homogenous* of degree one;
- for every $w \in \mathbf{R}^n_{++}$, we have

$$\lim_{K \to \infty} C(w, K) = 0 \text{ and } \lim_{K \to 0^+} \frac{\partial C(w, K)}{\partial K} = -1$$

Integral Transform: Relaxation

- the constraint $C(w,K) \in \mathcal{R}_C$ becomes $C(w,K)$ convex, homogeneous... this turns the problem into a *shape constrained problem*

- this relaxation is equivalent to the following *linear program*:

$$
\begin{array}{ll}
\text{max./min.} & p_0 \\
\text{subject to} & \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i, \quad i,j = 0, \ldots, m+n+1 \\
& g_{i,j} \geq 0 \\
& -1 \leq g_{i,n+1} \leq 0 \\
& \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 0, ..., m+n+1, \quad j = 1, ..., n
\end{array}
$$

where the variables g_i are subgradients

Integral Transform: Tightness

- in the case where *only options* are given, the relaxation is *tight*
- when *forwards and options* are given, the *upper bound is tight* while the lower bound is not
- in general, only upper bound on upper bound, lower bound on lower bound

Numerical Example

- the x_i are the simulated Black & Scholes (1973) lognormal asset prices at maturity, with S the initial stock values

- the numerical values used here are $S = \{0.7, 0.5, 0.4, 0.4, 0.4\}$, $w_0 = \{0.2, 0.2, 0.2, 0.2, 0.2\}$, $T = 5$ years and the covariance matrix is given by:

$$V = \frac{11}{100}\begin{pmatrix} 0.64 & 0.59 & 0.32 & 0.12 & 0.06 \\ 0.59 & 1 & 0.67 & 0.28 & 0.13 \\ 0.32 & 0.67 & 0.64 & 0.29 & 0.14 \\ 0.12 & 0.28 & 0.29 & 0.36 & 0.11 \\ 0.06 & 0.13 & 0.14 & 0.11 & 0.16 \end{pmatrix}$$

- all individual options are ATM, hence $K = \{0.7, 0.5, 0.4, 0.4, 0.4\}$

- we get $p = \{0.0161, 0.0143, 0.0093, 0.0070, 0.0047\}$

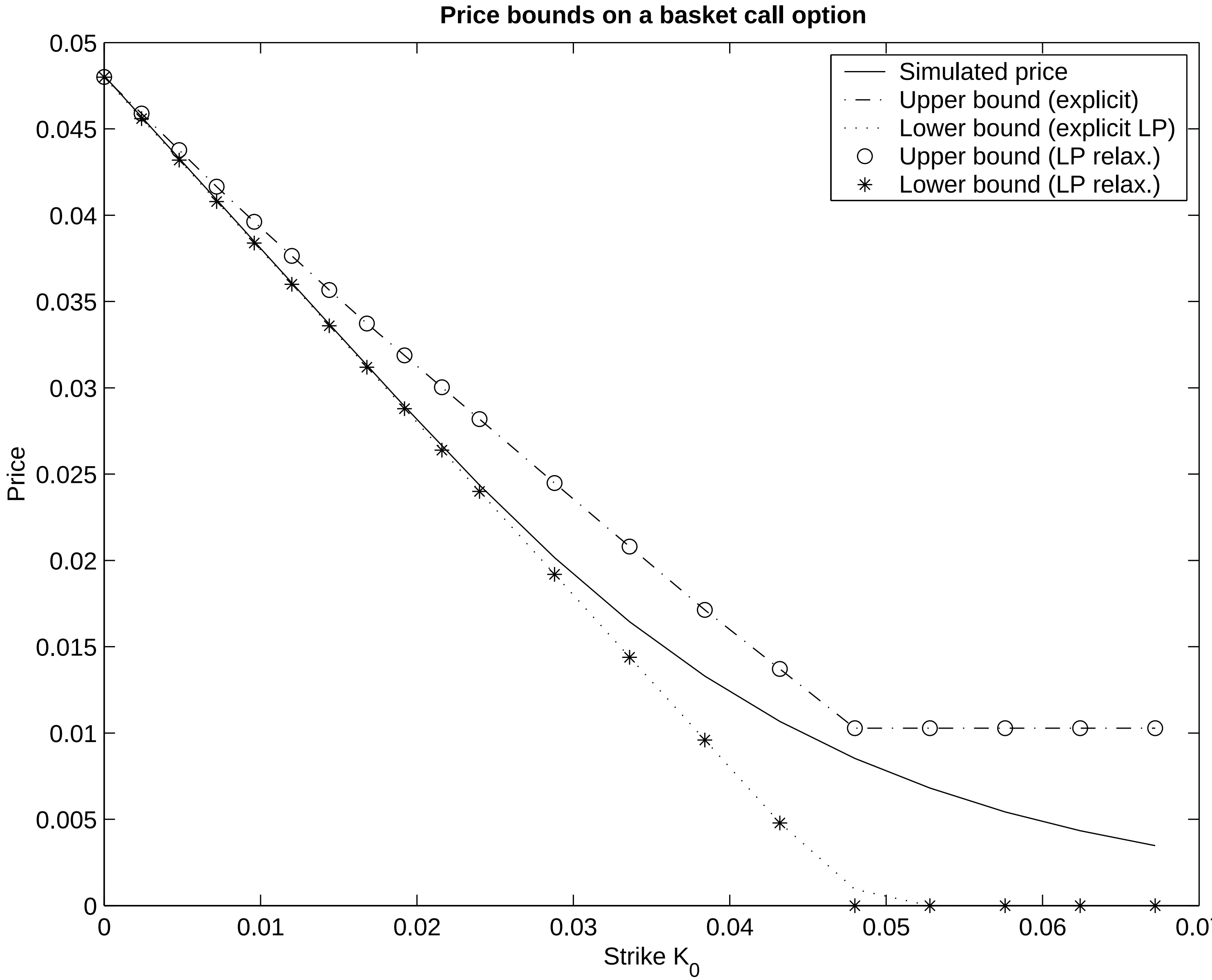


Figure 3: Upper and lower price bounds.

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Extension...

- the bounds computed using the previous LP relaxation are tight in some particular cases

- can we improve their performance in the general case?

- how do we get the super/subreplicating portfolio?

- the method in Bertsimas & Popescu (2002) only gives relaxations for the case $x \in \mathbf{R}^n$

- the last two conditions (smoothness and total positivity) in the Radon range characterization could be implemented by interpolation, but this cannot guarantee a lower bound...

Extension: a Moment Problem?

- Berstein-Bochner results answer the question of when a function $f(t)$ is a *characteristic function*

$$f(t) = \int_{\mathbf{R}} e^{2\pi i x t} d\pi(x)$$

- can we obtain the same kind of result for the call payoff?

$$C(w, K) = \int_{\mathbf{R}^n_+} (w^T x - K)_+ d\pi(x)$$

- the solution is *harmonic analysis on semigroups*, more tractable than total positivity in Henkin & Shananin (1990) (but same origin)...

Harmonic Analysis on Semigroups

some quick definitions...

- a pair $(\mathbb{S}, \cdot)$ is called a *semigroup* iff:
  - if $s, t \in \mathbb{S}$ then $s \cdot t$ is also in $\mathbb{S}$
  - there is a neutral element $e \in \mathbb{S}$ such that $e \cdot s = s$ for all $s \in \mathbb{S}$

- the *dual* $\mathbb{S}^*$ of $\mathbb{S}$ is the set of *semicharacters*, $i.e.$ applications $\chi : \mathbb{S} \to \mathbf{R}$ such that
  - $\chi(s)\chi(t) = \chi(s \cdot t)$ for all $s, t \in \mathbb{S}$
  - $\chi(e) = 1$, where e is the neutral element in $\mathbb{S}$

- a function α is called an *absolute value* on $\mathbb{S}$ iff
  - $\alpha(e) = 1$
  - $\alpha(s \cdot t) \leq \alpha(s)\alpha(t)$, for all $s, t \in \mathbb{S}$

Harmonic Analysis on Semigroups

last definitions (honest)...

- a function $f : \mathbb{S} \to \mathbf{R}$ is *positive semidefinite* iff for every family $\{s_i\} \subset \mathbb{S}$ the matrix with elements $f(s_i \cdot s_j)$ is positive semidefinite

- a function f is *bounded* with respect to the absolute value α iff there is a constant $C > 0$ such that

$$|f(s)| \leq C\alpha(s), \quad s \in \mathbb{S}$$

- f is *exponentially bounded* iff it is bounded with respect to an absolute value

Harmonic Analysis on Semigroups: Central Result

central result, see Berg, Christensen & Ressel (1984) based on Choquet's theorem:

- the set of exponentially bounded *positive definite functions* is a *Bauer simplex* whose extreme points are the bounded semicharacters...
- this means that we have the following representation:

$$f(s) = \int_{\mathbb{S}^*} \chi(s) d\mu(\chi), \quad \text{for all } s \in \mathbb{S}$$

  where μ is a Radon measure on $\mathbb{S}^*$

Harmonic Analysis on Semigroups: Simple Examples

- *Berstein's theorem* for the Laplace transform

$$\mathbb{S} = (\mathbf{R}_+, +),\ \chi_x(t) = e^{-xt} \text{ and } f(t) = \int_{\mathbf{R}_+} e^{-xt} d\mu(x)$$

- with involution, *Bochner's theorem* for the Fourier transform

$$\mathbb{S} = (\mathbf{R}, +),\ \chi_x(t) = e^{2\pi ixt} \text{ and } f(t) = \int_{\mathbf{R}} e^{2\pi ixt} d\mu(x)$$

- *Hamburger's solution* to the unidimensional moment problem

$$\mathbb{S} = (\mathbf{N}, +),\ \chi_x(k) = x^k \text{ and } f(k) = \int_{\mathbf{R}} x^k d\mu(x)$$

Outline

1. Problem statement and motivation

2. Convexity constraints

   (a) Main result
   (b) Applications:
      i. Consumer preference
      ii. Convex relaxations
      iii. Option pricing

3. Extension: moment problems

   (a) Harmonic analysis and positive semidefinite functions
   (b) **The option pricing problem revisited**

The Option Pricing Problem Revisited

- the basket option payoffs $(w^T x - K)_+$ are not ideal in this setting
- solution, use *straddles*: $|w^T x - K|$
- as straddles are just the *sum of a call and a put*, their price can be computed from that of the corresponding call and forward by call-put parity
- the fact that $|w^T x - K|^2$ is a polynomial keeps the complexity low

Payoff Semigroup

- the fundamental semigroup $\mathbb{S}$ is here the multiplicative *payoff semigroup* generated by the *cash*, the *forwards*, the *straddles*

$$1 \quad x_i \quad |w_j^T x - K_j|$$

- the *semicharacters* are the functions $\chi_x : \mathbb{S} \to \mathbf{R}$ which evaluate the payoffs at a certain point x

$$\chi_x(s) = s(x), \quad \text{for all } s \in \mathbb{S}$$

The Option Pricing Problem Revisited

- the original static arbitrage problem can be reformulated as an SCP

$$\begin{array}{ll} \text{max./min.} & f(|w_0^T x - K_0|) \\ \text{subject to} & f(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m \\ & f(s) = \mathbf{E}_\pi[s], \quad s \in \mathbb{S} \quad \text{(f moment function)} \end{array}$$

- the variable is now $f : \mathbb{S} \rightarrow \mathbf{R}$, a function that associates to each payoff s in $\mathbb{S}$, its price $f(s)$

- the representation result in Berg et al. (1984) shows when a (price) function $f : \mathbb{S} \rightarrow \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\pi[s]$$

Option Pricing: Main Theorem

If we assume that the asset distribution has a compact support included in $\mathbf{R}^n_+$, and note e_i for $i = 0, \ldots, n+m$ the forward and option payoff functions we get:

A function $f(s) : \mathbb{S} \to \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \textit{for all } s \in \mathbb{S},$$

for some measure ν with compact support, if and only if:

(i) $f(s)$ *is positive semidefinite*

(ii) $f(e_i s)$ *is positive semidefinite for* $i = 0, \ldots, n+m$

(iii) $\left(\beta f(s) - \sum_{i=0}^{n+m} f(e_i s)\right) f$ *is positive semidefinite*

this turns the basket arbitrage problem into a *semidefinite program*

Option Pricing: a Semidefinite Program

we get a relaxation by only sampling the elements of $\mathbb{S}$ up to a certain degree, the variable is then the vector $f(s)$ with

$$s = (1, x_1, \ldots, x_n, |w_0^T x - K_0|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots, |w_m^T x - K_m|^N)$$

this is a *semidefinite program*

$$\begin{array}{ll}
\text{minimize} & f(|w_0^T x - K_0|) \\
\text{subject to} & M_N(f(s)) \succeq 0 \\
& M_N(f(s_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\
& M_N\left(f((\beta - \sum_{k=0}^{n+m} s_k)s)\right) \succeq 0 \\
& f(s_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S}
\end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(s_k s))_{ij} = f(s_k s_i s_j)$

Classical Duality

- the general program is:

$$\sup_{\pi \in \mathcal{K}} \int_{\mathbf{R}_+^n} \psi(x)\pi(x)dx \quad \text{subject to} \int_{\mathbf{R}_+^n} \phi(x)\pi(x)dx = p, \quad \int_{\mathbf{R}_+^n} \pi(x)dx = 1$$

  where $\psi(x) = (w_0^T x - K_0)_+$ and $\phi(x)_i = (w_i^T x - K_i)_+$

- Lagrangian:

$$L(\pi, \lambda, \lambda_0) = \int_{\mathbf{R}_+^n} \left(\psi(x) - \lambda^T \phi(x) - \lambda_0\right) \pi(x)dx + \lambda^T p + \lambda_0,$$

- the dual is a *portfolio problem*:

$$\inf_{\lambda_0, \lambda} \lambda^T p + \lambda_0 \;:\; \lambda^T \phi(x) + \lambda_0 \geq \psi(x) \text{ for every } x \in \mathbf{R}_+^n$$

Conic Duality

let $\Sigma \subset \mathcal{A}(\mathbb{S})$ be the set of polynomials that are sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$, and $\mathcal{P}$ the set of positive semidefinite sequences on $\mathbb{S}$

- instead of the conic duality between probability measures and positive portfolios

$$p(x) \geq 0 \Leftrightarrow \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu$$

- we use the duality between positive semidefinite sequences $\mathcal{P}$ and sums of squares polynomials Σ

$$p \in \Sigma \Leftrightarrow \langle f, p \rangle \geq 0 \text{ for all } f \in \mathcal{P}$$

with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \rightarrow \mathbf{R}$, where $\langle f, p \rangle = \sum_i q_i f(s_i)$

Option Pricing: Dual

- the classic dual is a *hedging problem*

$$\begin{array}{ll} \text{maximize} & \lambda_{n+m+1} + \sum_{i=1}^{n+m} \lambda_i p_i \\ \text{subject to} & |w_0^T x - K_0| - \sum_{i=1}^{n+m} \lambda_i s_i(x) - \lambda_{n+m+1} \geq 0 \end{array}$$

- it becomes...

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n+m} p_j \lambda_j + \lambda_{n+m+1} \\ \text{subject to} & |w_0^T x - K_0| - \sum_{j=1}^{n+m} \lambda_j s_j(x) - \lambda_{n+m+1} \\ & = q_0(x) + \sum_{j=1}^{n+m} q_j(x) s_j(x) + (\beta - \sum_{k=0}^{n+m} s_k(x)) q_{n+1}(x) \end{array}$$

in the variables $\lambda \in \mathbf{R}^{n+m+1}$ and $q_j \in \Sigma$ for $j = 0, \ldots, (n+1)$

Option Pricing: Numerical Example

- two assets: x_1, x_2, we look for bounds on the price of $|x_1 + x_2 - K|$
- simple discrete model for the assets:

$$x = \{(0,0), (0,3), (3,0), (1,2), (5,4)\}$$

  with probability

$$p = (.2, .2, .2, .3, .1)$$

- the forward prices are given, together with the following straddles:

$$|x_1 - .9|, |x_1 - 1|, |x_2 - 1.9|, |x_2 - 2|, |x_2 - 2.1|$$

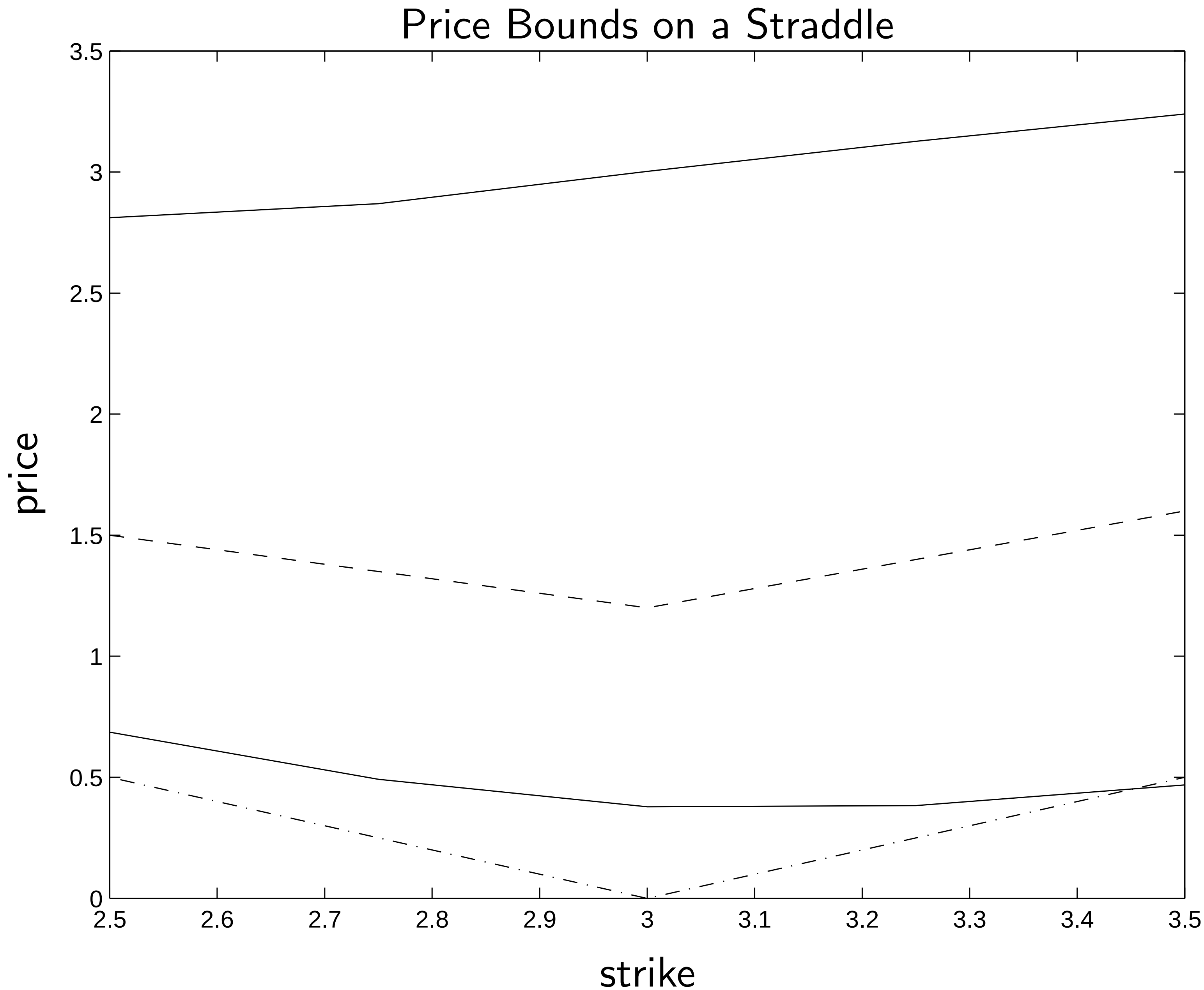

Figure 4: Upper and lower price bounds on a straddle.

Option Pricing: Caveats

- *size*: grows exponentially with the number of assets...
- *bad conditioning*: as inverse problems, the programs naturally tend to be ill-conditioned
- *transaction costs*: proportional transaction costs are usually OK, fixed are much harder

Option Pricing: Possible Extensions

- minimum entropy prices
- triangular FOREX arbitrage relationship: if x_1 is USD/EUR and x_2 is EUR/GBP, then an option on the USD/GBP is written as $(x_1x_2 - K)_+$
- swaptions are baskets, size limit?
- exploit sparsity, only 1% nonzero entries
- what happens when the payoff is not algebraic?

Conclusion

- some infinite dimensional LPs reduce to finite ones
- easy, constructive proof...
- applications on the basket arbitrage problem
- extension to moment problem gives a set of arbitrarily precise relaxations

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