Convex Optimization
Homework 1

Exercise 1 Which of the following sets are convex ?

1) A rectangle, *i.e.*, a set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \dots, n\}$.
2) The hyperbolic set
$$\{x \in \mathbb{R}^2_+ \mid x_1 x_2 \ge 1\}$$
3) The set of points closer to a given point than a given set, *i.e.*,
$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$$
where $S \subseteq \mathbb{R}^n$.
4) The set of points closer to one set than another, *i.e.*,
$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\},$$
where $S, T \subseteq \mathbb{R}^n$, and $\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$.
5) The set
$$\{x \mid x + S_2 \subseteq S_1\},$$
where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

Exercise 2 For each of the following functions determine whether it is convex or concave or not. *Optional: Determine if they are quasiconvex or quasiconcave.*

1) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
2) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.
3) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.
4) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on $\mathbb{R}^2_{++}$.

Exercise 3 Show that following functions are convex

1) $f(X) = \mathbf{Tr}(X^{-1})$ on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.
2) $f(X, y) = y^T X^{-1} y$ on $\mathbf{dom}\, f = \mathbf{S}^n_{++} \times \mathbb{R}^n$ *Hint: express it as a supremum*
3) $f(X) = \sum_{i=1}^n \sigma_i(X)$ on $\mathbf{dom}\, f = \mathbf{S}^n$, where $\sigma_1(X), \dots, \sigma_n(X)$ are singular values of a matrix $X \in \mathbb{R}^{n \times n}$. *Hint: express it as a supremum*

Optional exercises

Exercise 4 We define the *monotone nonnegative cone* as
$$K_{\mathrm{m}+} = \{x \in \mathbb{R}^n \mid x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}.$$
i.e., all nonnegative vectors with components sorted in nonincreasing order.

1. Show that $K_{\mathrm{m}+}$ is a proper cone.
2. Find the dual cone $K^*_{\mathrm{m}+}$.

Exercise 5 Derive the conjugates of the following functions.

1) *Max function.* $f(x) = \max_{i=1,\dots,n} x_i$ on $\mathbb{R}^n$.

2) *Sum of largest elements.* $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$.

3) *Piecewise-linear function on* $\mathbb{R}$. $f(x) = \max_{i=1,\dots,m}(a_i x + b_i)$ on $\mathbb{R}$. You can assume that the a_i are sorted in increasing order, *i.e.*, $a_1 \leq \cdots \leq a_m$, and that none of the functions $a_i x + b_i$ is redundant, *i.e.*, for each k there is at least one x with $f(x) = a_k x + b_k$.