Convex Optimization
Exam, November 2016.

You have three hours. You may use a single double-sided page of notes. Please keep your answers as concise as possible.

Exercise 1  (Duality) Derive the dual of the following LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad Dx = g \\
& \quad x \geq 0
\end{align*}
\]

in the variable \(x \in \mathbb{R}^n\). Start by writing the Lagrangian, then the dual function and finally the Lagrange dual problem.

Exercise 2  (QP) Derive a dual problem of the Support Vector Machine problem

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2}\|w\|^2 + C \sum_{i=1}^{m} v_i \\
\text{subject to} & \quad x_i^T w + v_i \geq 1, \quad i = 1, \ldots, m \\
& \quad v_i \geq 0, \quad i = 1, \ldots, m,
\end{align*}
\]

in the variables \(w \in \mathbb{R}^n, v \in \mathbb{R}^m\), where \(C > 0\) and \(x_1, \ldots, x_m \in \mathbb{R}^n\). Simplify it as much as you can.

Exercise 3  (Convexity) Show that the function

\[
\log \det(X)
\]

is concave in \(X \in \mathcal{S}_n\) (Hint: remember that we can always write \(X = X^{1/2}X^{1/2}\)).

Exercise 4  Let \(P\) be a polytope written

\[
P = \{x \in \mathbb{R}^n : a_i^T x \leq b_i, \ i = 1, \ldots, m\}
\]

Write the problem of finding the ball

\[
B = \{x_c + ru : \|u\|_2 \leq 1\}
\]

inscribed in \(P\) and with maximum radius, as an LP.
Exercise 5  (SDP) Here, we seek to approximate a given symmetric matrix $A \in \mathbb{S}_n$ such that $A \succeq 0$, by another one $X \in \mathbb{S}_n$ whose condition number

$$\kappa(X) = \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)}$$

is minimal. This number controls the stability of solutions to linear systems for example.

- Let $y, z \in \mathbb{R}^+$, show that the constraints $\lambda_{\max}(X) \leq y$ and $\lambda_{\min}(X) \geq z$ can both be written as linear matrix inequalities.

- Consider the optimization problem

$$\min_{X,y,z} \frac{y}{z} \quad s.t. \quad (X, y, z) \in C \quad y, z \geq 0; \quad X \succeq 0$$

where $C$ is the set formed by the inequalities

1. $\lambda_{\max}(X) \leq y$,
2. $\lambda_{\min}(X) \geq z$,
3. $\|X - A\|_F \leq \epsilon$ for some $\epsilon > 0$.

Show that $C$ is a convex set. Is the objective function convex?

- Rewrite the previous program as a convex minimization problem whose objective is affine. (*Hint:* with appropriate modifications, you can force $\lambda_{\min}(X) = 1$).