Convex Optimization
Exam, November 2016.

You have three hours. You may use a single double-sided page of notes. Please keep your answers as concise as possible.

Exercise 1 (Duality) Derive the dual of the following LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Dx = g \\ & x \geq 0 \end{array}$$

in the variable $x \in \mathbf{R}^n$. Start by writing the Lagrangian, then the dual function and finally the Lagrange dual problem.

Exercise 2 (QP) Derive a dual problem of the Support Vector Machine problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|w\|_2^2 + C\sum_{i=1}^m v_i \\ \text{subject to} & x_i^T w + v_i \geq 1, \quad i = 1, \ldots, m \\ & v_i \geq 0, \quad i = 1, \ldots, m, \end{array}$$

in the variables $w \in \mathbf{R}^n$, $v \in \mathbf{R}^m$, where $C > 0$ and $x_1, \ldots, x_m \in \mathbf{R}^n$. Simplify it as much as you can.

Exercise 3 (Convexity) Show that the function

$$\log\det(X)$$

is concave in $X \in \mathbf{S}_n$ (*Hint: remember that we can always write* $X = X^{\frac{1}{2}} X^{\frac{1}{2}}$).

Exercise 4 Let P be a polytope written

$$P = \{x \in \mathbf{R}^n : a_i^T x \leq b_i,\ i = 1, \ldots, m\}$$

Write the problem of finding the ball

$$B = \{x_c + ru : \|u\|_2 \leq 1\}$$

inscribed in P and with maximum radius, as an LP.

Exercise 5 (SDP) Here, we seek to approximate a given symmetric matrix $A \in \mathbf{S}_n$ such that $A \succeq 0$, by another one $X \in \mathbf{S}_n$ whose condition number

$$\kappa(X) = \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)}$$

is minimal. This number controls the stability of solutions to linear systems for example.

- Let $y, z \in \mathbf{R}^+$, show that the constraints $\lambda_{\max}(X) \leq y$ and $\lambda_{\min}(X) \geq z$ can both be written as linear matrix inequalities.

- Consider the optimization problem

$$\begin{array}{ll} \min\limits_{X,y,z} & \frac{y}{z} \\ s.t. & (X, y, z) \in C \\ & y, z \geq 0\,; \quad X \succeq 0 \end{array}$$

  where C is the set formed by the inequalities

  1. $\lambda_{\max}(X) \leq y$,
  2. $\lambda_{\min}(X) \geq z$,
  3. $\|X - A\|_F \leq \epsilon$ for some $\epsilon > 0$.

  Show that C is a convex set. Is the objective function convex?

- Rewrite the previous program as a convex minimization problem whose objective is affine. (*Hint*: with appropriate modifications, you can force $\lambda_{\min}(X) = 1$).