A Harmonic Analysis Solution to the Basket Arbitrage Problem

Alexandre d'Aspremont

ORFE, Princeton University.

Introduction

- Classic Black & Scholes (1973) option pricing based on:
  - a *dynamic hedging* argument
  - a *model* for the asset dynamics (geometric BM)
- Sensitive to liquidity, transaction costs, model risk ...
- What can we say about derivative prices with much weaker assumptions?

Static Arbitrage

Here, we rely on a *minimal set of assumptions*:

- no assumption on the asset distribution
- one period model

An arbitrage in this simple setting is a *buy and hold* strategy:

- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option's maturity

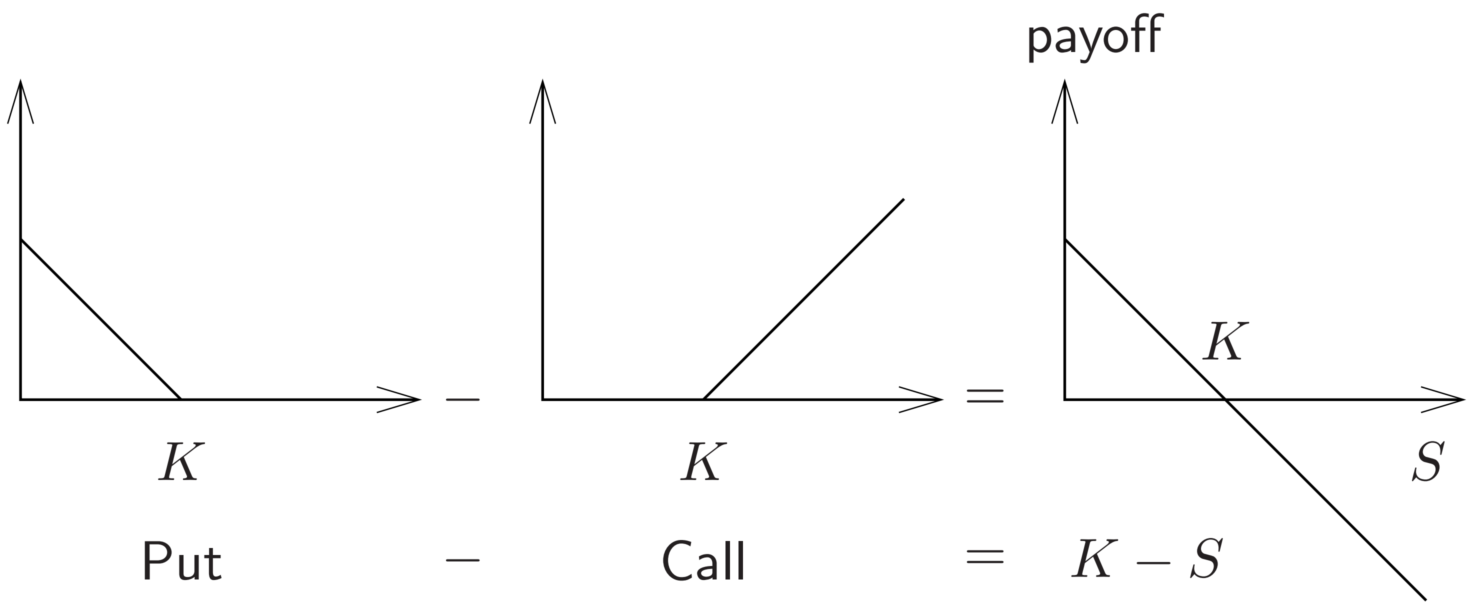
What for?

- Data validation (e.g. before calibration), static arbitrage means market data is incompatible with *any* dynamic model. . .
- Test extrapolation formulas
- In illiquid markets, find optimal static hedge

Outline

- **Static Arbitrage**
- Harmonic Analysis on Semigroups
- No Arbitrage Conditions

Simplest Example: Put Call Parity



Put $-$ Call $=$ $K - S$

Static Arbitrage: Calls

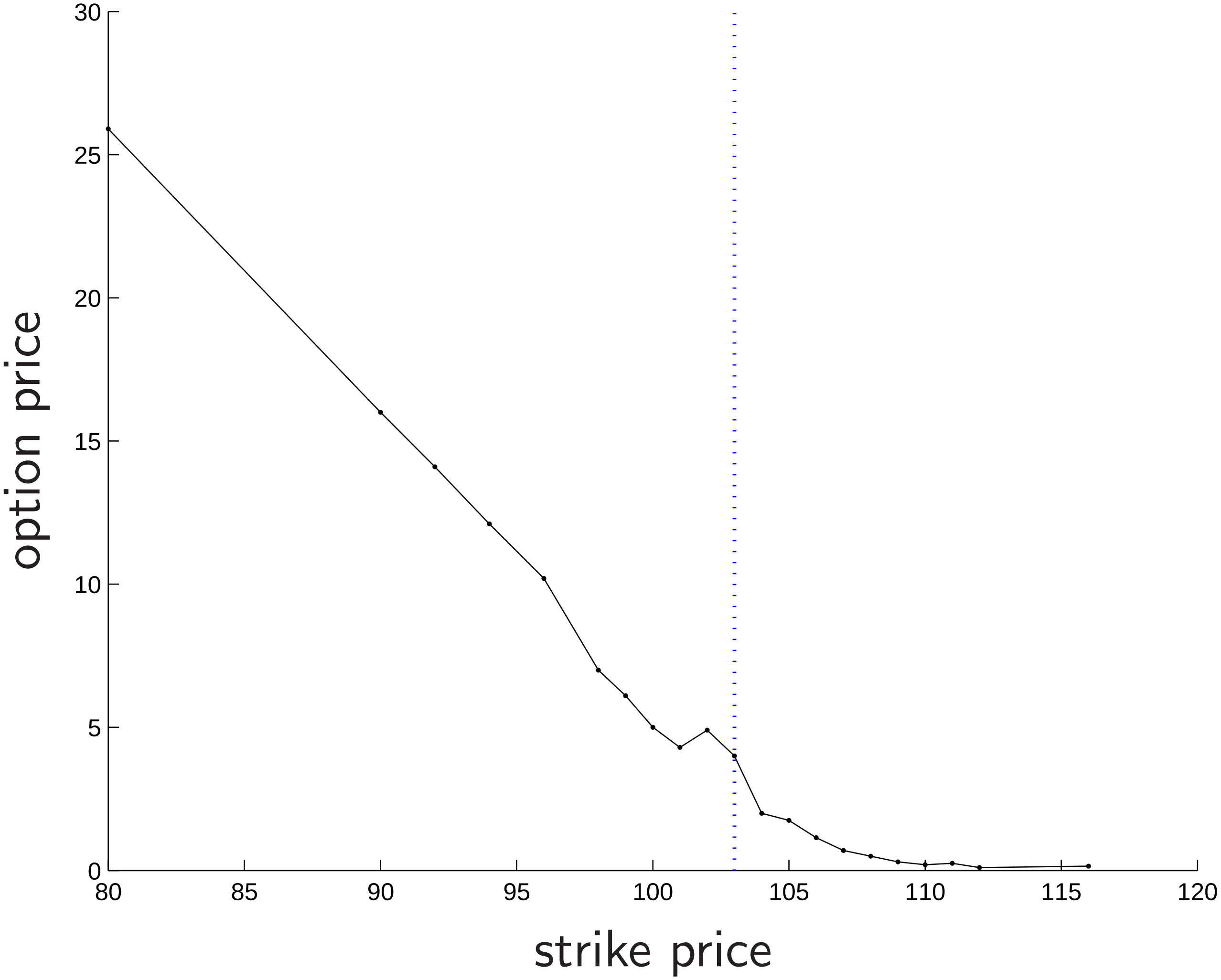
Also, necessary and sufficient conditions on call prices:

Suppose we have a set of market prices for calls $C(K_i) = p_i$, then there is no arbitrage iff there is a function C(K):

- $C(K)$ *positive*
- $C(K)$ *decreasing*
- $C(K)$ *convex*
- $C(K_i) = p_i$ *and* $C(0) = S$

This is *very easy* to test. . .

Dow Jones index call option prices on Mar. $17\ 2004$, maturity Apr. $16\ 2004$



Source: Reuters.

Why?

Data quality...

- All the prices are last quotes (not simultaneous)
- Low volume
- Some transaction costs

Problem: this data is used to calibrate models and price other derivatives...

Dimension n: Basket Options

- A basket call payoff is given by:

$$\left(\sum_{i=1}^{k} w_i S_i - K\right)^+$$

where $w_1, \ldots, w_k$ are the basket's weights and K is the option's strike price

- Examples include: Index options, spread options, swaptions...

- Basket option prices are used to gather information on *correlation*

We denote by $C(w, K)$ the price of such an option, can we get conditions to test basket price data?

Necessary Conditions

Similar to dimension one...

Suppose we have a set of market prices for calls $C(w_i, K_i) = p_i$, and there is no arbitrage, then the function $C(w, K)$ satisfies:

- $C(w, K)$ positive
- $C(w, K)$ decreasing in K, increasing in w
- $C(w, K)$ jointly convex in (w, K)
- $C(w_i, K_i) = p_i$ and $C(0) = S$

This is still *tractable* in dimension n as a *linear program*.

Sufficient?

A key difference with dimension one: Bertsimas & Popescu (2002) show that the exact problem is NP-Hard.

- These conditions are *only necessary*...
- Numerical cost is minimal (small LP)
- We can show *sufficiency* in some particular cases

In practice: these conditions are far from being tight, how can we *refine* them?

Arrow-Debreu prices

- *Arrow-Debreu*: There is no arbitrage in the static market iff there is a probability measure π such that:

$$C(w, K) = \mathbf{E}_\pi (w^T x - K)^+$$

- $\pi(x)$ represents Arrow-Debreu state prices.

- Discretize on a uniform grid: This turns this into a *linear program* with m^n variables, where n is the number of assets x_i and m is the number of bins.

- Numerically: hopeless. . .

- Explicit conditions derived by Henkin & Shananin (1990) (link with Radon transform), but intractable. . .

Tractable Conditions

- Bochner's theorem on the Fourier transform of positive measures:

$$f(s) = \int e^{-i<s,x>} g(x) dx \quad \text{with } g(x) \geq 0$$

$$\Updownarrow$$

$$f(s) \text{ positive semidefinite}$$

  which means testing if the *matrices* $f(s_i s_j)$ are positive semidefinite

- Can we generalize this result to other transforms? In particular:

$$\int_{\mathbf{R}^n_+} (w^T x - K)^+ d\pi(x)$$

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Harmonic Analysis on Semigroups

Some quick definitions...

- A pair $(\mathbb{S}, \cdot)$ is called a *semigroup* iff:
  - if $s, t \in \mathbb{S}$ then $s \cdot t$ is also in $\mathbb{S}$
  - there is a neutral element $e \in \mathbb{S}$ such that $e \cdot s = s$ for all $s \in \mathbb{S}$
- The *dual* $\mathbb{S}^*$ of $\mathbb{S}$ is the set of *semicharacters*, $i.e.$ applications $\chi : \mathbb{S} \rightarrow \mathbf{R}$ such that
  - $\chi(s)\chi(t) = \chi(s \cdot t)$ for all $s, t \in \mathbb{S}$
  - $\chi(e) = 1$, where e is the neutral element in $\mathbb{S}$
- A function $f : \mathbb{S} \rightarrow \mathbf{R}$ is *positive semidefinite* iff for every family $\{s_i\} \subset \mathbb{S}$ the matrix with elements $f(s_i \cdot s_j)$ is positive semidefinite

Harmonic Analysis on Semigroups

Last definitions (honest)...

- A function α is called an *absolute value* on $\mathbb{S}$ iff
  - $\alpha(e) = 1$
  - $\alpha(s \cdot t) \leq \alpha(s)\alpha(t)$, for all $s, t \in \mathbb{S}$

- A function f is *bounded* with respect to the absolute value α iff there is a constant $C > 0$ such that

$$|f(s)| \leq C\alpha(s), \quad s \in \mathbb{S}$$

- f is *exponentially bounded* iff it is bounded with respect to an absolute value

Carleman type conditions on growth for moment determinacy, etc. . .

Harmonic Analysis on Semigroups: Central Result

The central result, see Berg, Christensen & Ressel (1984) based on Choquet's theorem:

- the set of exponentially bounded *positive definite functions* is a *Bauer simplex* whose extreme points are the bounded semicharacters...

- this means that we have the following representation for positive definite functions on $\mathbb{S}$:
$$f(s) = \int_{\mathbb{S}^*} \chi(s) d\mu(\chi)$$
where μ is a Radon measure on $\mathbb{S}^*$

Harmonic Analysis on Semigroups: Simple Examples

- *Berstein's theorem* for the Laplace transform

$$\mathbb{S} = (\mathbf{R}_+, +),\ \chi_x(t) = e^{-xt} \ \text{ and } \ f(t) = \int_{\mathbf{R}_+} e^{-xt} d\mu(x)$$

- with involution, *Bochner's theorem* for the Fourier transform

$$\mathbb{S} = (\mathbf{R}, +),\ \chi_x(t) = e^{2\pi i x t} \ \text{ and } \ f(t) = \int_{\mathbf{R}} e^{2\pi i x t} d\mu(x)$$

- *Hamburger's solution* to the unidimensional moment problem

$$\mathbb{S} = (\mathbf{N}, +),\ \chi_x(k) = x^k \ \text{ and } \ f(k) = \int_{\mathbf{R}} x^k d\mu(x)$$

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The Option Pricing Problem Revisited

What is the appropriate semigroup here?

- Basket option payoffs $(w^T x - K)^+$ are not ideal in this setting.
- Solution: use *straddles*: $|w^T x - K|$
- Straddles are just the *sum of a call and a put*, their price can be computed from that of the corresponding call and forward by call-put parity.
- The fact that $|w^T x - K|^2$ is a polynomial keeps the complexity low.

Payoff Semigroup

- The fundamental semigroup $\mathbb{S}$ here is the multiplicative *payoff semigroup* generated by the cash, the forwards and the straddles:

$$\mathbb{S} = \{1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots\}$$

- The *semicharacters* are the functions $\chi_x : \mathbb{S} \to \mathbf{R}$ which evaluate the payoffs at a certain point x

$$\chi_x(s) = s(x), \quad \text{for all } s \in \mathbb{S}$$

The Option Pricing Problem Revisited

- The original static arbitrage problem can be reformulated as

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & f(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m \\ & f(s) = \mathbf{E}_\pi[s], \quad s \in \mathbb{S} \quad \text{(f moment function)} \end{array}$$

- The variable is now $f : \mathbb{S} \to \mathbf{R}$, a function that associates to each payoff s in $\mathbb{S}$, its price $f(s)$

- The *representation result* in Berg et al. (1984) shows when a (price) function $f : \mathbb{S} \to \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\pi[s]$$

Option Pricing: Main Theorem

If we assume that the asset distribution has a compact support included in $\mathbf{R}^n_+$, and note e_i for $i = 1, \ldots, n+m$ the forward and option payoff functions we get:

A function $f(s) : \mathbb{S} \to \mathbf{R}$ *can be represented as*

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \textit{for all } s \in \mathbb{S},$$

for some measure ν *with compact support, iff for some* $\beta > 0$*:*

(i) $f(s)$ *is positive semidefinite*

(ii) $f(e_i s)$ *is positive semidefinite for* $i = 1, \ldots, n+m$

(iii) $\left(\beta f(s) - \sum_{i=1}^{n+m} f(e_i s)\right)$ *is positive semidefinite*

this turns the basket arbitrage problem into a *semidefinite program*

Semidefinite Programming

A *semidefinite program* is written:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, CX \\ \text{subject to} & \mathbf{Tr}\, A_i X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0, \end{array}$$

in the variable $X \in \mathbf{S}^n$, with parameters $C, A_i \in \mathbf{S}^n$ and $b_i \in \mathbf{R}$ for $i = 1, \ldots, m$. Its *dual* is given by:

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & C - \sum_{i=1}^m \lambda_i A_i \succeq 0, \end{array}$$

in the variable $\lambda \in \mathbf{R}^m$.

Extension of interior point techniques for linear programming show how to solve these convex programs *efficiently* (see Nesterov & Nemirovskii (1994), Sturm (1999) and Boyd & Vandenberghe (2004)).

Option Pricing: a Semidefinite Program

We get a relaxation by only sampling the elements of $\mathbb{S}$ up to a certain degree, the variable is then the vector $f(s)$ with

$$e = (1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots, |w_m^T x - K_m|^N)$$

testing for the absence of arbitrage is then a *semidefinite program*:

$$\begin{array}{ll}
\text{find} & f \\
\text{subject to} & M_N(f(s)) \succeq 0 \\
& M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\
& M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k) s)\right) \succeq 0 \\
& f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S}
\end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$

Conic Duality

Let $\Sigma \subset \mathcal{A}(\mathbb{S})$ be the set of polynomials that are sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$, and $\mathcal{P}$ the set of positive semidefinite sequences on $\mathbb{S}$

- instead of the conic duality between probability measures and positive portfolios

$$p(x) \geq 0 \Leftrightarrow \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu$$

- we use the duality between positive semidefinite sequences $\mathcal{P}$ and sums of squares polynomials Σ

$$p \in \Sigma \Leftrightarrow \langle f, p \rangle \geq 0 \text{ for all } f \in \mathcal{P}$$

with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \to \mathbf{R}$, where $\langle f, p \rangle = \sum_i q_i f(s_i)$

Option Pricing: Caveats

- *Size*: grows exponentially with the number of assets: no free lunch. . .
- In dimension 2, for spread options, this is:

$$\binom{2+d}{2}(k+1)$$

where d is the degree of the relaxation and k the number of assets.

- Conditioning issues. . .

Conclusion

- Testing for static arbitrage in option price data is easy in dimension one
- The extension on basket options (swaptions, etc) is NP-hard but good relaxations can be found
- We get a computationally friendly set of conditions for the absence of arbitrage
- Small scale problems are tractable in practice as semidefinite programs

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