A Moment Approach to the Static Arbitrage Problem on Baskets

Alexandre d'Aspremont

EECS Dept., U.C. Berkeley

Introduction

- classic Black & Scholes (1973) option pricing based on:
  - a *dynamic hedging* argument
  - a *model* for the asset dynamics (geometric BM)
- sensitive to liquidity, transaction costs, model risk ...
- what can we say about option prices with much weaker assumptions?

Static Arbitrage

The *fundamental theorem of asset pricing* states that:

$$\text{Absence of Arbitrage} \quad \Leftrightarrow \quad \text{Price} = \mathbf{E}_\pi[\text{Payoff}]$$

Here, we rely on a *minimal set of assumptions*:

- no assumption on the asset distribution
- one period model

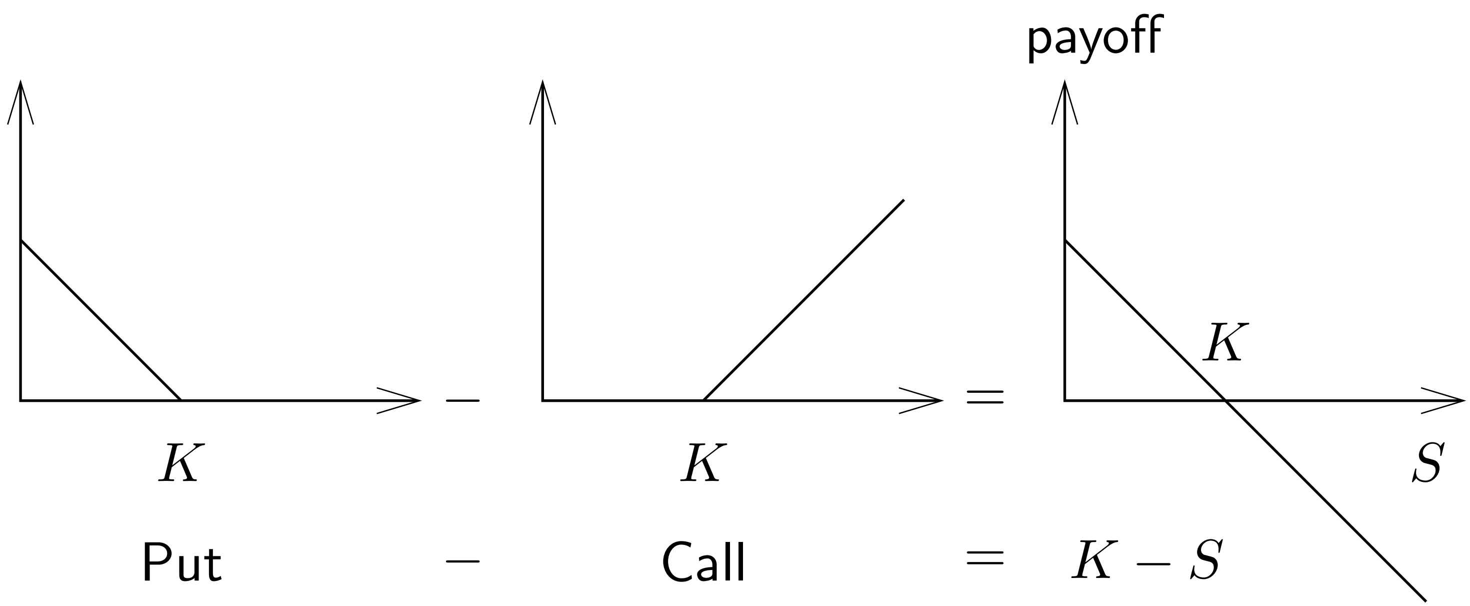
An arbitrage in this simple setting is a *buy and hold* strategy:

- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option's maturity

What for?

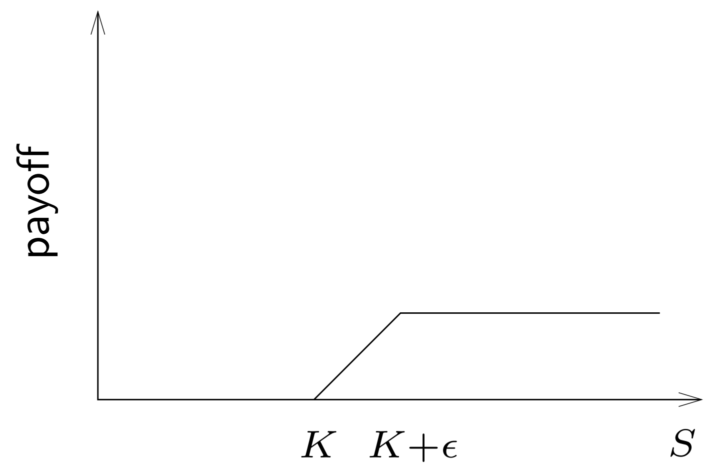
- arbitrage free data stripping before calibration
- test extrapolation formulas
- in illiquid markets, find optimal static hedge or bound risk at little cost

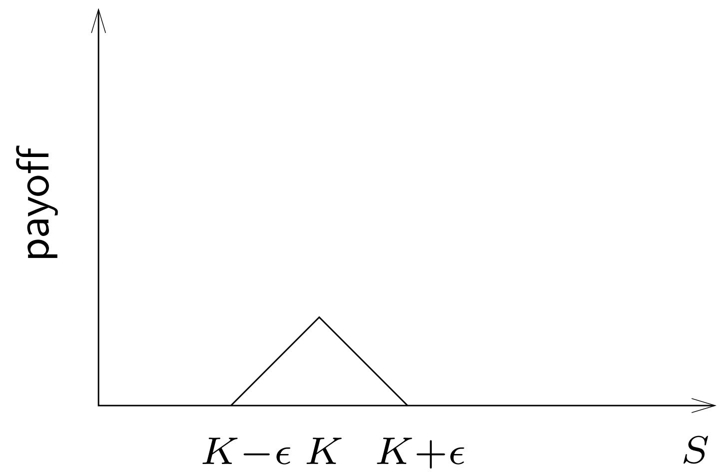
Simplest Example: Put Call Parity



We denote by $C(K)$ the price of the call with payoff $(S-K)^+$. If we know the forward prices, then we can deduce call prices from puts, ...

Call Spread - Butterfly Spread





Here, the absence of arbitrage implies that the price of a call spread be positive, hence call prices must be *decreasing* with strike

$$C(K+\epsilon)-C(K)\leq 0$$

it also implies that the price of a butterfly spread be positive, and call prices must then be *convex* with strike

$$C(K+\epsilon)-2C(K)+C(K-\epsilon)\geq 0$$

Price Constraints

The absence of arbitrage implies that if $C(K)$ is a function giving the price of an option of strike K, then $C(K)$ must satisfy:

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex

With $C(0) = S$, we have a set of *necessary* conditions for the absence of arbitrage

Sufficient Conditions

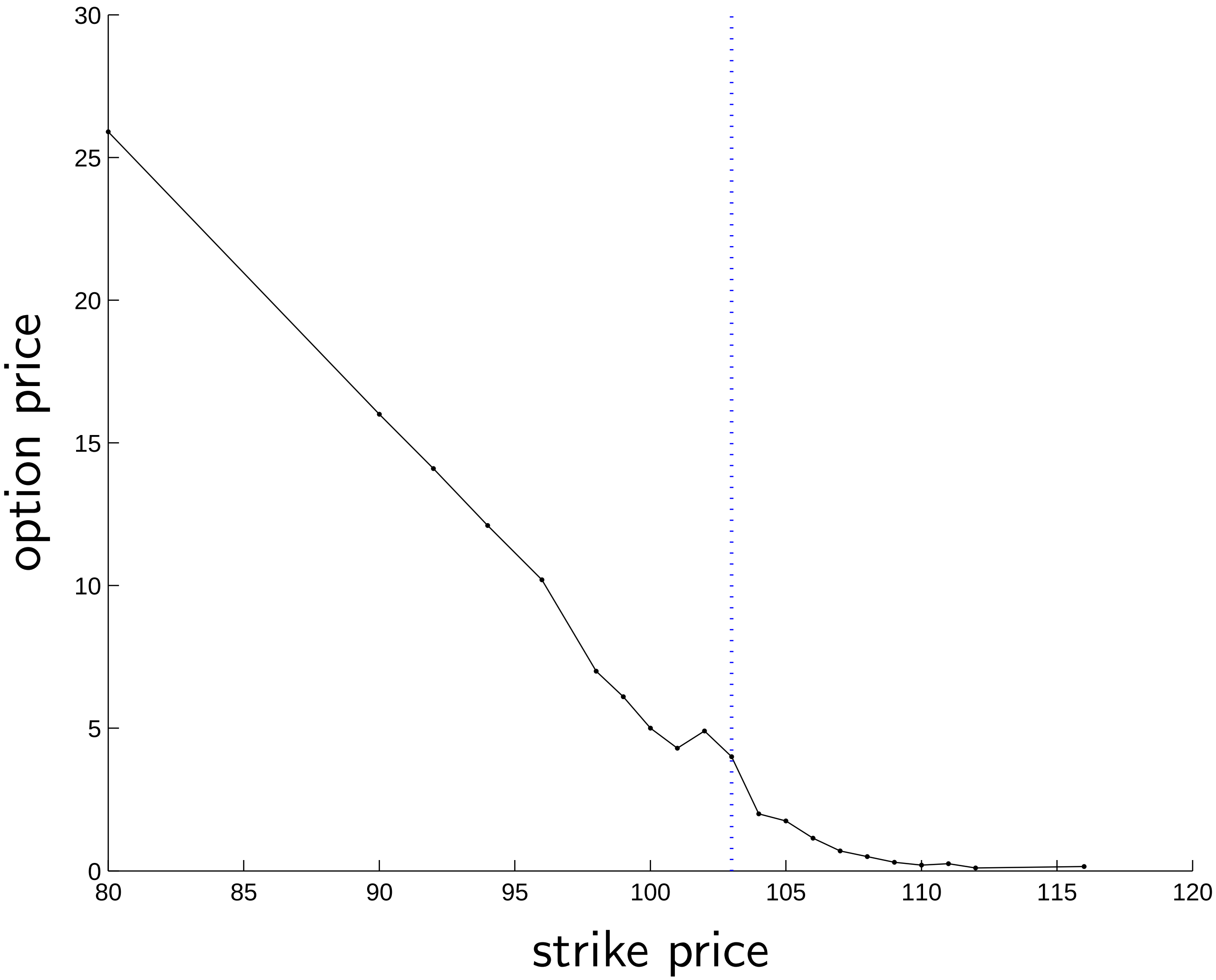
In fact, these conditions are also *sufficient*, see Laurent & Leisen (2000) and Bertsimas & Popescu (2002) among others. . .

Suppose we have a set of market prices for calls $C(K_i) = p_i$, then there is no arbitrage iff there is a function C(K):

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex
- $C(K_i) = p_i$ and $C(0) = S$

This is *very easy* to test. . .

Dow Jones index call option prices on Mar. $17\ 2004$, maturity Apr. $16\ 2004$



Source: Reuters.

Why?

data quality...

- all the prices are last quotes (not simultaneous)
- low volume
- some transaction costs

Problem: this data is used to calibrate models and price other derivatives...

Dimension n: Basket Options

- a basket call payoff is
$$\left(\sum_{i=1}^{k} w_i S_i - K\right)_+$$
where $w_1, \ldots, w_k$ are the basket's weights and K is the option's strike price

- examples include: Index options, spread options, swaptions...

- basket option prices are used to gather information on *correlation*

We denote by $C(w, K)$ the price of such an option, can we get conditions to test basket price data?

Necessary Conditions

Similar to dimension one...

Suppose we have a set of market prices for calls $C(w_i, K_i) = p_i$, and there is no arbitrage, then the function $C(w, K)$ satisfies:

- $C(w, K)$ positive
- $C(w, K)$ decreasing in K, increasing in w
- $C(w, K)$ jointly convex in (w, K)
- $C(w_i, K_i) = p_i$ and $C(0) = S$

Is this still *tractable* (in dimension n)?

Tractable?

The problem can be formulated as:

$$\begin{array}{ll} \text{find} & z \\ \text{subject to} & Az \leq b, \; Cz = d \\ & z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T\right]^T \\ & g_i \text{ subgradient of } f \text{ at } x_i \quad i = 1, \ldots, k \\ & f \text{ monotone, convex} \end{array}$$

in the variables $f \in C(\mathbf{R}^n)$, $z \in \mathbf{R}^{(n+1)k}$, $g_1, \ldots, g_k \in \mathbf{R}^n$

- *discretize* and sample the convexity constraints to get a polynomial size LP feasibility problem

- enforce the convexity and subgradient constraints at the points $(x_i)_{i=1,\ldots,k}$ (monotonicity is a simple inequality on g) to get:

$$\begin{array}{ll} \text{find} & z \\ \text{subject to} & Cz = d,\ Az \leq b \\ & z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T\right]^T \\ & \langle g_i, x_j - x_i \rangle \leq f(x_j) - f(x_i) \quad i, j = 1, \ldots, k \end{array}$$

in the variables $f(x_i)_{i=1,\ldots,k}$ and g in $\mathbf{R}^k \times \mathbf{R}^{(n+1)\times k}$

- we note $z^{\mathrm{opt}} = \left[f^{\mathrm{opt}}(x_1), \ldots, f^{\mathrm{opt}}(x_k), (g_1^{\mathrm{opt}})^T, \ldots, (g_k^{\mathrm{opt}})^T\right]^T$ a solution to this problem

- from $z^{\rm opt}$, we define:

$$C(x) = \max_{i=1,\ldots,k} \left\{ f^{\rm opt}(x_i) + \left\langle g_i^{\rm opt}, x - x_i \right\rangle \right\}$$

- by construction, $C(x_i)$ solves the finite LP with:

$$C(x_i) = f^{\rm opt}(x_i), \quad i = 1, \ldots, k$$

- $C(x)$ is convex and monotone as the pointwise maximum of monotone affine functions

- so $C(x)$ is also a feasible point of the original problem

This means that $C(x)$ is a *solution* for the original (infinite dimensional) problem.

Relaxation

The previous result means that the price conditions remain tractable on basket options... They are equivalent to the following *feasibility problem*:

$$\begin{array}{ll}\text{find} & g_i \\ \text{subject to} & \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i \\ & g_{i,j} \geq 0, \quad j = 1, \ldots, n \\ & -1 \leq g_{i,n+1} \leq 0 \\ & \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 1, \ldots, m, \end{array}$$

where the variables $g_i \in \mathbf{R}^n$ are the subgradients of $C(w, K)$ at the points (w_i, K_i).

Sufficient?

A key difference with dimension one: Bertsimas & Popescu (2002) show that the exact problem is NP-Hard.

- the conditions are *only necessary*...
- however, numerical cost is minimal (small LP)
- we can show *sufficiency* in some particular cases
- how tight are these conditions in general?

Numerical Example

- two assets: x_1, x_2, we look for upper and lower bounds on the price of a particular basket $(x_1 + x_2 - K)^+$

- simple discrete model for the assets:

$$x = \{(0,0), (0,.8), (.8,.3), (.6,.6), (.1,.4), (1,1)\}$$

  with probability

$$\pi = (.2, .2, .2, .1, .1, .2)$$

- the forward prices are given, together with the following call prices:

$$(.2x_1 + x_2 - .1)^+, (.5x_1 + .8x_2 - .8)^+, (.5x_1 + .3x_2 - .4)^+,$$
$$(x_1 + .3x_2 - .5)^+, (x_1 + .5x_2 - .5)^+, (x_1 + .4x_2 - 1)^+, (x_1 + .6x_2 - 1.2)^+$$

- we compare the (outer) price bounds given by the previous relaxation with inner bounds computed by discretizing.

Numerical Example

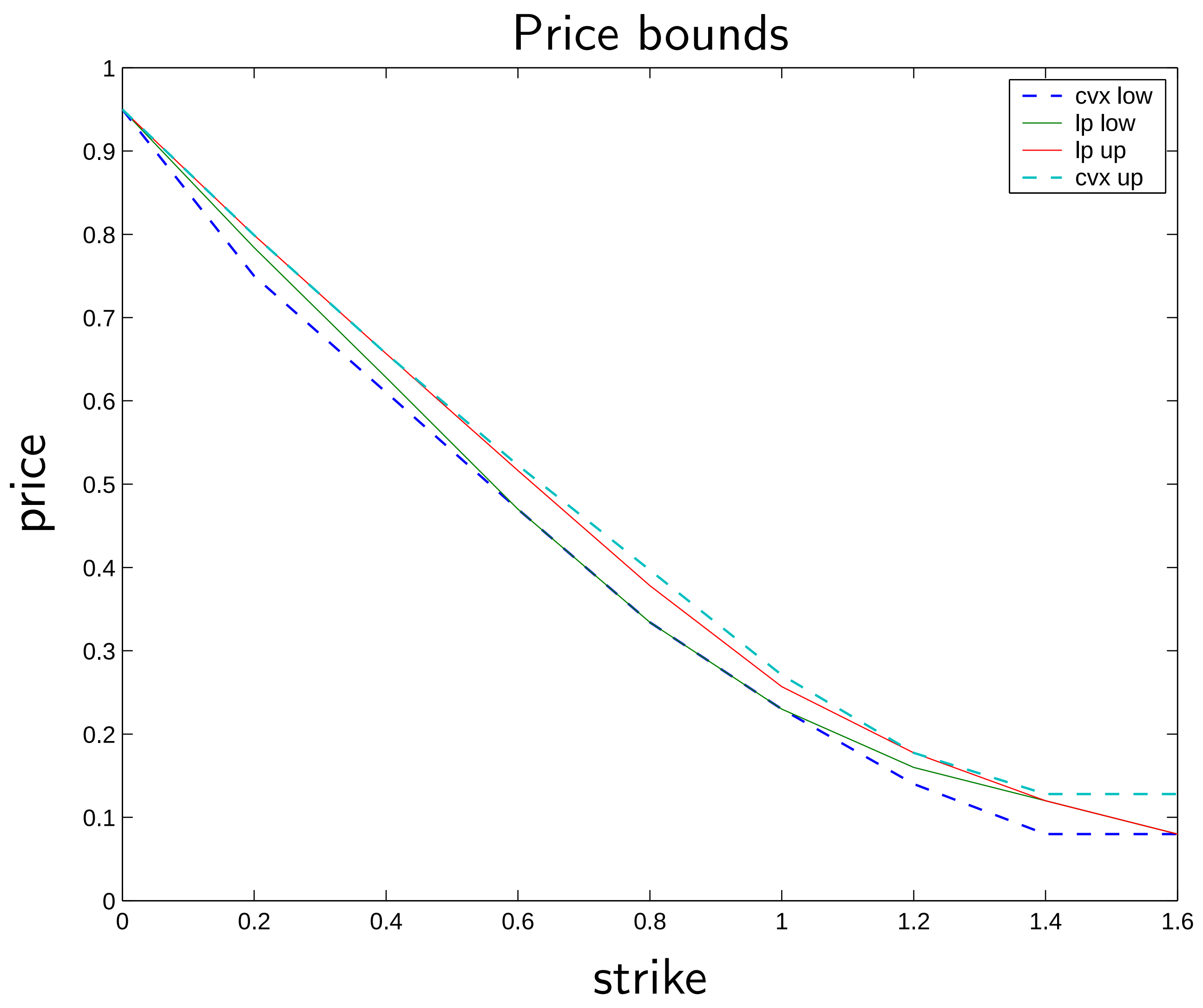
We compare the *outer bounds* on the price p_0 of the $(x_1 + x_2 - K)^+$ basket obtained by solving:

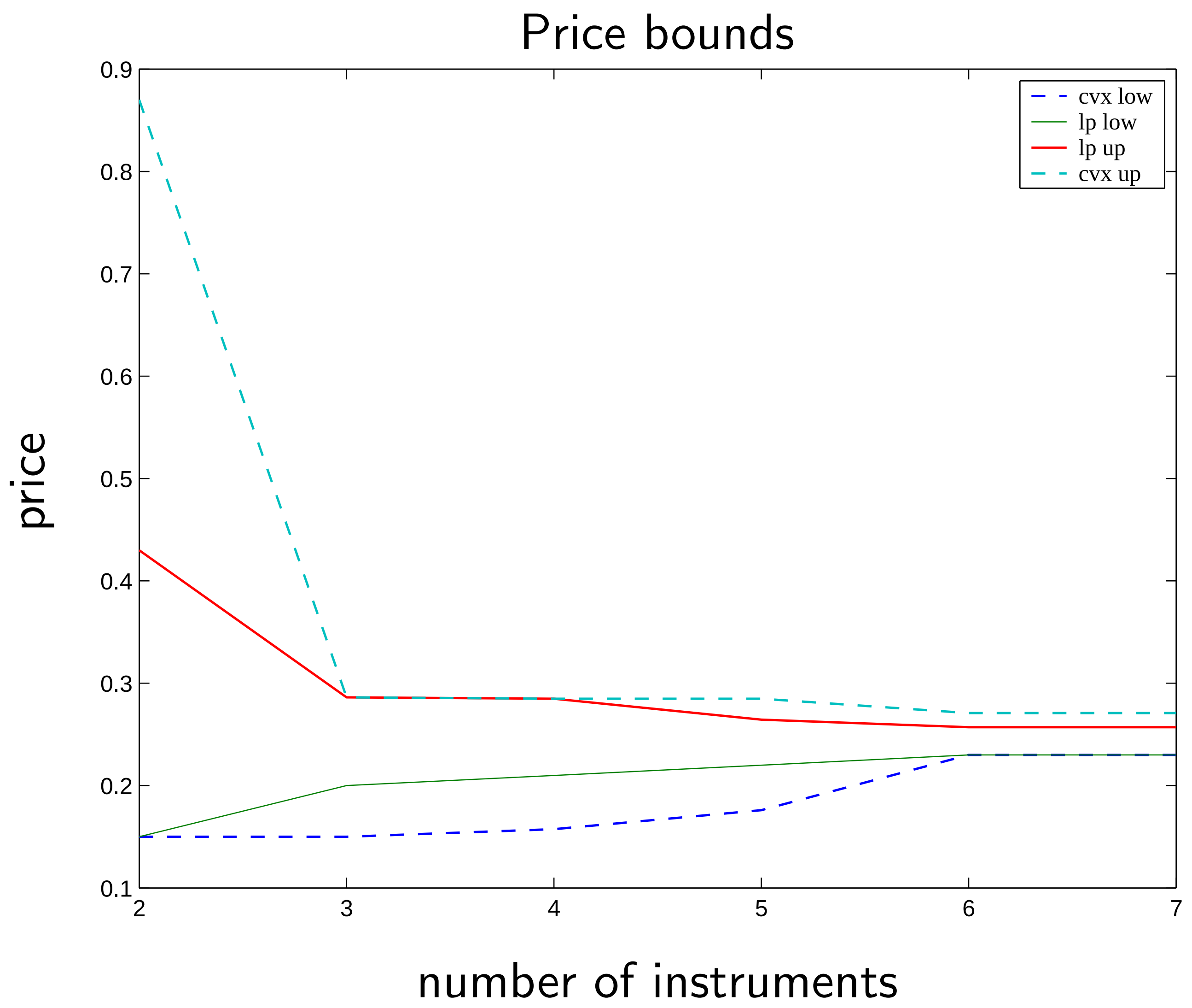
$$\begin{array}{ll} \text{max./min.} & p_0 \\ \text{subject to} & \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i \\ & g_{i,j} \geq 0, \quad j = 1, \ldots, n \\ & -1 \leq g_{i,n+1} \leq 0 \\ & \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 1, \ldots, m, \end{array}$$

with the *inner bounds* obtained by solving:

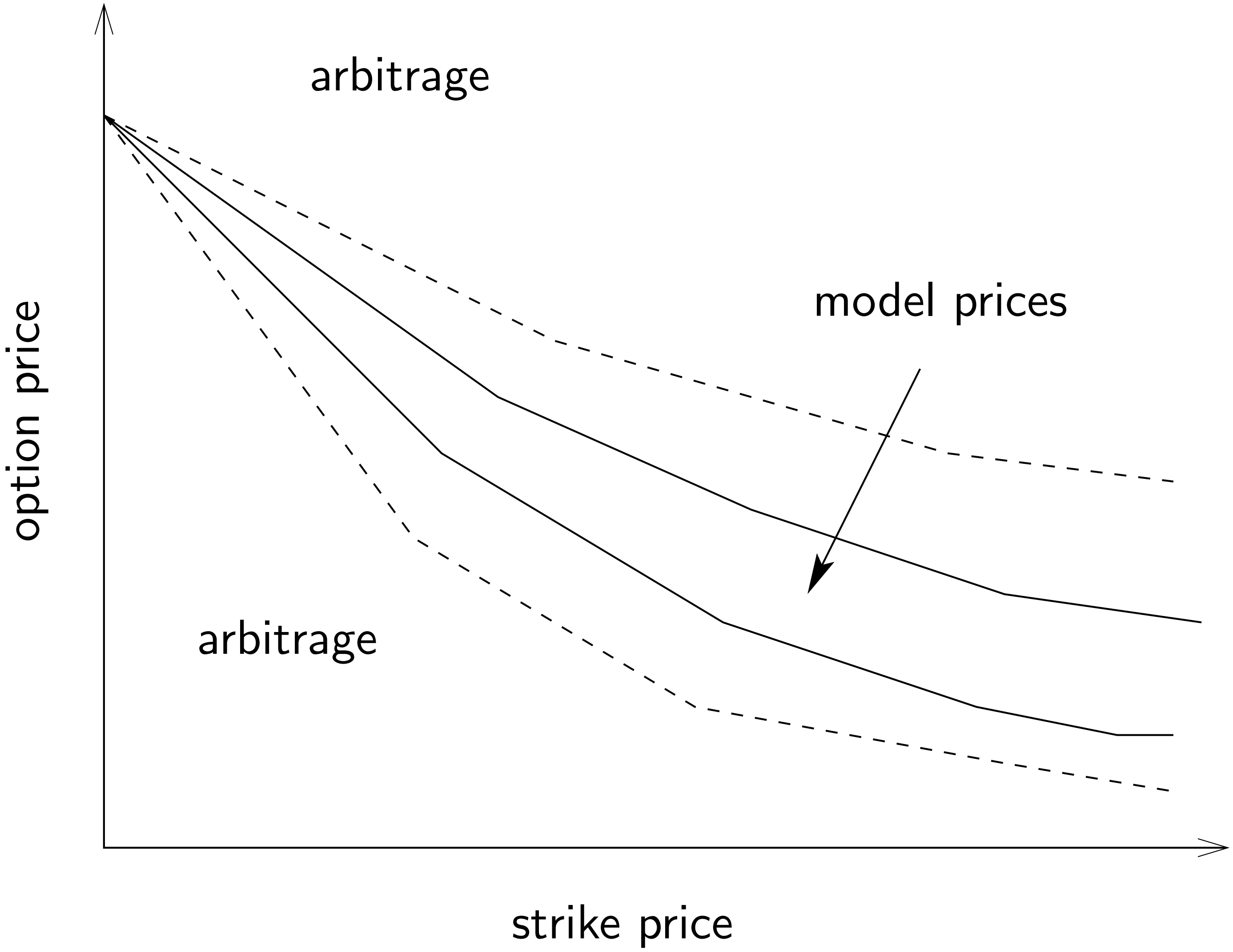
$$\begin{array}{ll} \text{max./min.} & \mathbf{E}_\pi(|w_0^T x - K_0|) \\ \text{subject to} & \mathbf{E}_\pi(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m, \end{array}$$

which becomes a simple (albeit large) linear program after we discretize π.


Price bounds
price
strike
cvx low
lp low
lp up
cvx up
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6


Price bounds
price
number of instruments
cvx low
lp low
lp up
cvx up
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
3
4
5
6
7

Price Bounds



Multivariate Black-Scholes Model

Here, we compare the *outer bounds* on the price p_0 of a basket obtained by solving the relaxation:

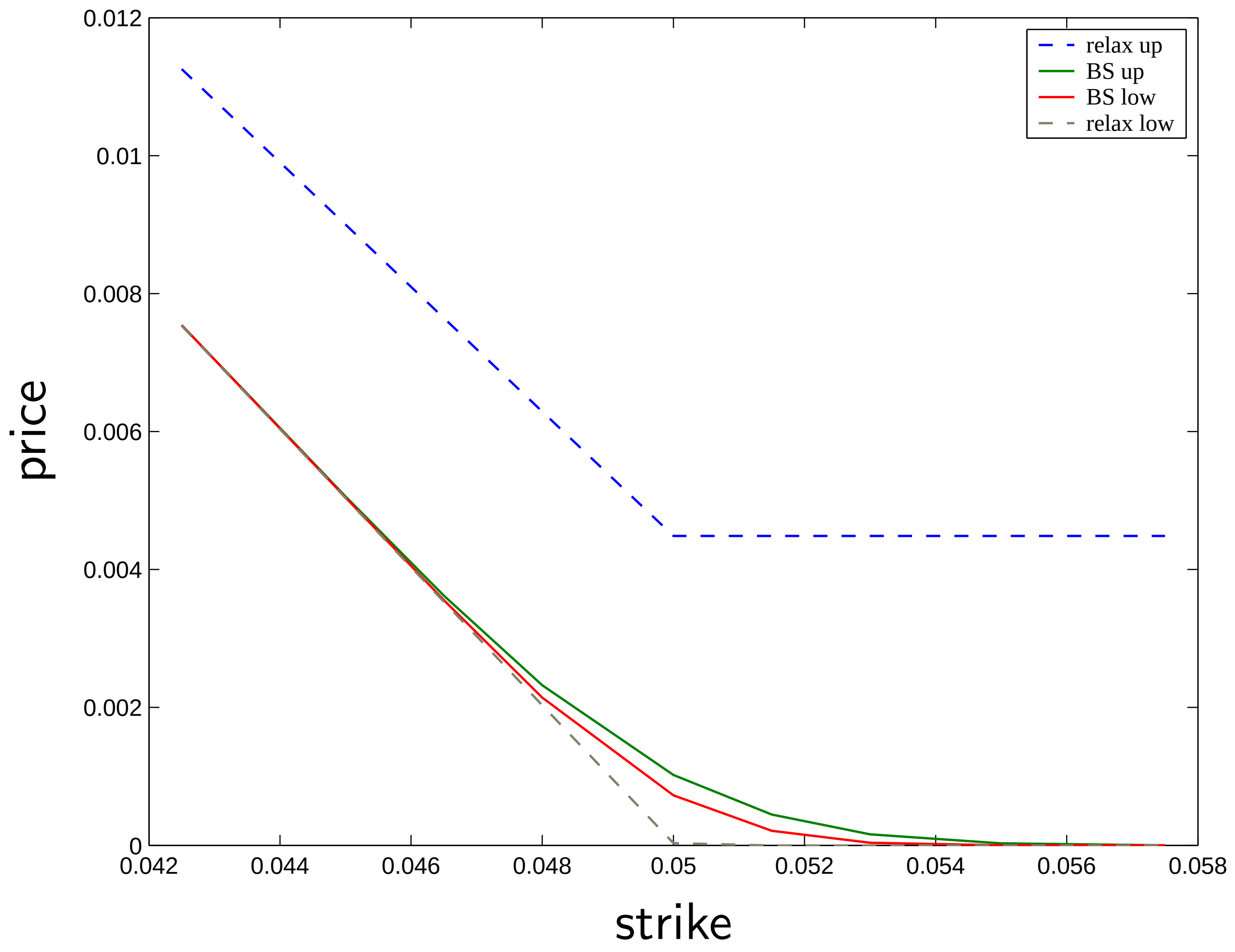
$$\begin{array}{ll} \text{max./min.} & p_0 \\ \text{subject to} & \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i \\ & g_{i,j} \geq 0, \quad j = 1, \ldots, n \\ & -1 \leq g_{i,n+1} \leq 0 \\ & \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 1, \ldots, m, \end{array}$$

with the *inner bounds* computed as:

$$\begin{array}{ll} \text{max./min.} & BS(T, w_0, V) \\ \text{subject to} & BS(T, w_i, V) = p_i, \quad i = 1, \ldots, m, \end{array}$$

in the variable $V \in \mathbf{S}^n$, corresponding to extreme prices on a basket option in a multivariate Black-Scholes model, given prices p_i of other basket options with weights w_i.

Multivariate Black-Scholes Model



Close the Gap

The gap is surprisingly large. . .

- ATM prices are not supposed to be very sensitive to the smile

- approx. lognormal model calibrate easily to swaption data in practice

How can we improve the static bounds (so we know when to blame the model)?

Integral Transform Solution

- we can write the set off call prices as:

$$\begin{aligned} C(w,K) &= \mathbf{E}_\pi (w^T x - K)_+ \\ &= \int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x), \end{aligned}$$

  and think of $C_\pi(w,K)$ as a particular integral transform of the measure π

- at least formally, we have:

$$\frac{\partial^2 C(w,K)}{\partial K^2} = \int_{\mathbf{R}_+^n} \delta(w^T x - K)\pi(x)dx$$

- this means that $\partial^2 C(w,K)/\partial K^2$ is the *Radon transform* (see Helgason (1999) or Ramm & Katsevich (1996)) of the measure π

A Range Characterization Problem...

- the general arbitrage problem can written as the following infinite dimensional problem:

$$\begin{array}{ll} \text{find} & C(w,K) \\ \text{subject to} & C(w_i,K_i)=p_i, \quad i=1,\ldots,m \\ & C(w,K)\in\mathcal{R}_C, \end{array}$$

- here, $\mathcal{R}_C$ is the range of the (linear) integral transform

$$\begin{aligned} C: \quad & \mathcal{K} \rightarrow \mathcal{R}_C \\ & \pi \rightarrow C(w,K)=\int_{\mathbf{R}^n_+}(w^Tx-K)_+ d\pi(x) \end{aligned}$$

Full Conditions

derived by Henkin & Shananin (1990). A function can be written

$$C(w, K) = \int_{\mathbf{R}_+^n} (w^T x - K)^+ d\pi(x)$$

with $w \in \mathbf{R}_+^n$ and $K > 0$, if and only if:

- $C(w, K)$ is *convex* and *homogenous* of degree one;
- $\lim_{K\to\infty} C(w, K) = 0$ and $\lim_{K\to 0^+} \frac{\partial C(w,K)}{\partial K} = -1$
- $F(w) = \int_0^\infty e^{-K} d\left(\frac{\partial C(w, K)}{\partial K}\right)$ belongs to $C_0^\infty(\mathbf{R}_+^n)$
- For some $\tilde{w} \in \mathbf{R}_+^n$ the inequalities: $(-1)^{k+1} D_{\xi_1}...D_{\xi_k} F(\lambda\tilde{w}) \geq 0$, for all positive integers k and $\lambda \in \mathbf{R}_{++}$ and all $\xi_1, \ldots, \xi_k$ in $\mathbf{R}_+^n$.

Finer Conditions

- the LP relaxations are sufficient in some particular cases
- can we improve their performance in the general case?
- how do we get super/subreplicating portfolio?
- the method in Bertsimas & Popescu (2002) only gives a relaxation for the case $x \in \mathbf{R}^n$
- the last two conditions (smoothness and total positivity) in the Radon range characterization are hard to implement, yet they suggest a *moment approach*. . .

Harmonic Analysis on Semigroups

some quick definitions...

- a pair $(\mathbb{S}, \cdot)$ is called a *semigroup* iff:
  - if $s, t \in \mathbb{S}$ then $s \cdot t$ is also in $\mathbb{S}$
  - there is a neutral element $e \in \mathbb{S}$ such that $e \cdot s = s$ for all $s \in \mathbb{S}$
- the *dual* $\mathbb{S}^*$ of $\mathbb{S}$ is the set of *semicharacters*, $i.e.$ applications $\chi : \mathbb{S} \rightarrow \mathbf{R}$ such that
  - $\chi(s)\chi(t) = \chi(s \cdot t)$ for all $s, t \in \mathbb{S}$
  - $\chi(e) = 1$, where e is the neutral element in $\mathbb{S}$
- a function α is called an *absolute value* on $\mathbb{S}$ iff
  - $\alpha(e) = 1$
  - $\alpha(s \cdot t) \leq \alpha(s)\alpha(t)$, for all $s, t \in \mathbb{S}$

Harmonic Analysis on Semigroups

last definitions (honest)...

- a function $f : \mathbb{S} \to \mathbf{R}$ is *positive semidefinite* iff for every family $\{s_i\} \subset \mathbb{S}$ the matrix with elements $f(s_i \cdot s_j)$ is positive semidefinite

- a function f is *bounded* with respect to the absolute value α iff there is a constant $C > 0$ such that

$$|f(s)| \leq C\alpha(s), \quad s \in \mathbb{S}$$

- f is *exponentially bounded* iff it is bounded with respect to an absolute value

Harmonic Analysis on Semigroups: Central Result

The central result, see Berg, Christensen & Ressel (1984) based on Choquet's theorem:

- the set of exponentially bounded *positive definite functions* is a *Bauer simplex* whose extreme points are the bounded semicharacters...

- this means that we have the following representation for positive definite functions on $\mathbb{S}$:

$$f(s) = \int_{\mathbb{S}^*} \chi(s) d\mu(\chi)$$

  where μ is a Radon measure on $\mathbb{S}^*$

Harmonic Analysis on Semigroups: Simple Examples

- *Berstein's theorem* for the Laplace transform

$$\mathbb{S} = (\mathbf{R}_+, +),\ \chi_x(t) = e^{-xt} \text{ and } f(t) = \int_{\mathbf{R}_+} e^{-xt} d\mu(x)$$

- with involution, *Bochner's theorem* for the Fourier transform

$$\mathbb{S} = (\mathbf{R}, +),\ \chi_x(t) = e^{2\pi ixt} \text{ and } f(t) = \int_{\mathbf{R}} e^{2\pi ixt} d\mu(x)$$

- *Hamburger's solution* to the unidimensional moment problem

$$\mathbb{S} = (\mathbf{N}, +),\ \chi_x(k) = x^k \text{ and } f(k) = \int_{\mathbf{R}} x^k d\mu(x)$$

The Option Pricing Problem Revisited

- the basket option payoffs $(w^T x - K)_+$ are not ideal in this setting
- solution, use *straddles*: $|w^T x - K|$
- as straddles are just the *sum of a call and a put*, their price can be computed from that of the corresponding call and forward by call-put parity
- the fact that $|w^T x - K|^2$ is a polynomial keeps the complexity low

Payoff Semigroup

- the fundamental semigroup $\mathbb{S}$ is here the multiplicative *payoff semigroup* generated by the cash, the forwards and the straddles:

$$\mathbb{S} = \{1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots\}$$

- the *semicharacters* are the functions $\chi_x : \mathbb{S} \to \mathbf{R}$ which evaluate the payoffs at a certain point x

$$\chi_x(s) = s(x), \quad \text{for all } s \in \mathbb{S}$$

The Option Pricing Problem Revisited

- the original static arbitrage problem can be reformulated as

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & f(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m \\ & f(s) = \mathbf{E}_\pi[s], \quad s \in \mathbb{S} \quad \text{(f moment function)} \end{array}$$

- the variable is now $f : \mathbb{S} \rightarrow \mathbf{R}$, a function that associates to each payoff s in $\mathbb{S}$, its price $f(s)$

- the representation result in Berg et al. (1984) shows when a (price) function $f : \mathbb{S} \rightarrow \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\pi[s]$$

Option Pricing: Main Theorem

If we assume that the asset distribution has a compact support included in $\mathbf{R}_+^n$, and note e_i for $i = 1, \ldots, n+m$ the forward and option payoff functions we get:

A function $f(s) : \mathbb{S} \to \mathbf{R}$ *can be represented as*

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \textit{for all } s \in \mathbb{S},$$

for some measure ν *with compact support, iff for some* $\beta > 0$*:*

(i) $f(s)$ *is positive semidefinite*

(ii) $f(e_i s)$ *is positive semidefinite for* $i = 1, \ldots, n+m$

(iii) $\left(\beta f(s) - \sum_{i=1}^{n+m} f(e_i s)\right)$ *is positive semidefinite*

this turns the basket arbitrage problem into a *semidefinite program*

Semidefinite Programming

A *semidefinite program* is written:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, CX \\ \text{subject to} & \mathbf{Tr}\, A_i X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0, \end{array}$$

in the variable $X \in \mathbf{S}^n$, with parameters $C, A_i \in \mathbf{S}^n$ and $b_i \in \mathbf{R}$ for $i = 1, \ldots, m$. Its *dual* is given by:

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & C - \sum_{i=1}^m \lambda_i A_i \succeq 0, \end{array}$$

in the variable $\lambda \in \mathbf{R}^m$.

A recent extension of interior point techniques for linear programming shows how to solve these convex programs *very efficiently* (see Nesterov & Nemirovskii (1994), Sturm (1999) and Boyd & Vandenberghe (2003)).

Feasibility Problems

Of course, the related feasibility problems:

$$\begin{array}{ll} \text{find} & X \\ \text{such that} & \mathbf{Tr}\, A_i X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0, \end{array}$$

and

$$\begin{array}{ll} \text{find} & \lambda \\ \text{such that} & C - \sum_{i=1}^{m} \lambda_i A_i \succeq 0, \end{array}$$

can be solved as efficiently (setting for example $C = I$ or $b = \mathbf{1}$ in the previous programs.

Also, because most solvers produce both primal and dual solution, we also get a Farkas type *certificate of infeasibility* or a *proof of optimality* in the duality gap.

Option Pricing: a Semidefinite Program

we get a relaxation by only sampling the elements of $\mathbb{S}$ up to a certain degree, the variable is then the vector $f(s)$ with

$$e = (1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots, |w_m^T x - K_m|^N)$$

testing for the absence of arbitrage is then a *semidefinite program*:

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\ & M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$

Price Bounds

We can also consider the related problem of finding *bounds* on the price of a straddle, given prices of other similar options:

$$\begin{array}{ll} \text{max./min.} & \mathbf{E}_\pi(|w_0^T x - K_0|) \\ \text{subject to} & \mathbf{E}_\pi(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m, \end{array}$$

which, using the previous result becomes the following semidefinite program:

$$\begin{array}{ll} \text{max./min.} & f(e_0) \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\ & M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$.

Duality

- the price maximization program is:

$$\begin{array}{ll} \text{maximize} & \int_{\mathbf{R}^n_+} (w_0^T x - K_0)^+ \pi(x) dx \\ \text{subject to} & \int_{\mathbf{R}^n_+} (w_i^T x - K_i)^+ \pi(x) dx = p_i, \quad i = 1, \ldots, m \\ & \int_{\mathbf{R}^n_+} \pi(x) dx = 1, \end{array}$$

in the variable $\pi \in \mathcal{K}$.

- the dual is a *portfolio problem*:

$$\begin{array}{ll} \text{minimize} & \lambda^T p + \lambda_0 \\ \text{subject to} & \sum_{i=1}^m \lambda_i (w_i^T x - K_i)^+ + \lambda_0 \geq \psi(x) \text{ for every } x \in \mathbf{R}^n_+ \end{array}$$

in the variable $\lambda \in \mathbf{R}^{m+1}$.

very intuitive, but completely intractable. . .

Conic Duality

let $\Sigma \subset \mathcal{A}(\mathbb{S})$ be the set of polynomials that are sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$, and $\mathcal{P}$ the set of positive semidefinite sequences on $\mathbb{S}$

- instead of the conic duality between probability measures and positive portfolios

$$p(x) \geq 0 \Leftrightarrow \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu$$

- we use the duality between positive semidefinite sequences $\mathcal{P}$ and sums of squares polynomials Σ

$$p \in \Sigma \Leftrightarrow \langle f, p \rangle \geq 0 \text{ for all } f \in \mathcal{P}$$

with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \rightarrow \mathbf{R}$, where $\langle f, p \rangle = \sum_i q_i f(s_i)$

Option Pricing: Dual

- the dual of the price maximization problem

$$
\begin{array}{ll}
\text{maximize} & f(e_0) \\
\text{subject to} & M_N(f(s)) \succeq 0 \\
& M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\
& M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\
& f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S}
\end{array}
$$

- now becomes...

$$
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{n+m} p_j \lambda_j + \lambda_{n+m+1} \\
\text{subject to} & \sum_{j=1}^{n+m} \lambda_j e_j(x) + \lambda_{n+m+1} - |w_0^T x - K_0| \\
& = q_0(x) + \sum_{j=1}^{n+m} q_j(x) e_j(x) + (\beta - \sum_{k=0}^{n+m} e_k(x)) q_{n+1}(x)
\end{array}
$$

in the variables $\lambda \in \mathbf{R}^{n+m+1}$ and $q_j \in \Sigma$ for $j = 0, \ldots, (n+1)$

Option Pricing: Numerical Example

- two assets: x_1, x_2, we look for bounds on the price of $|x_1 + x_2 - K|$
- simple discrete model for the assets:

$$x = \{(0,0), (0,3), (3,0), (1,2), (5,4)\}$$

with probability

$$p = (.2, .2, .2, .3, .1)$$

- the forward prices are given, together with the following straddles:

$$|x_1 - .9|, |x_1 - 1|, |x_2 - 1.9|, |x_2 - 2|, |x_2 - 2.1|$$

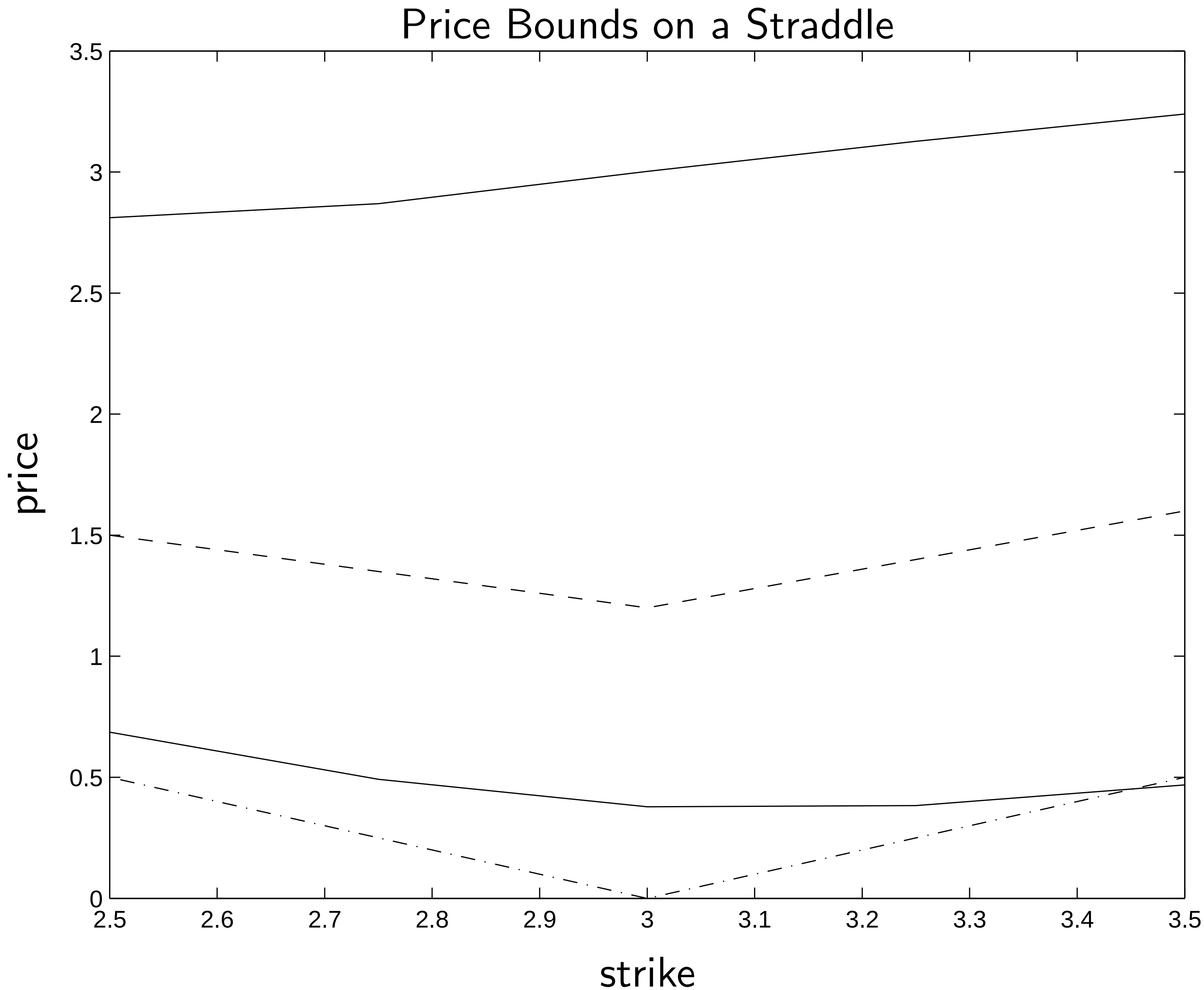


Figure 1: Upper and lower price bounds on a straddle.

Option Pricing: Caveats

- *size*: grows exponentially with the number of assets: no free lunch, even in numerical complexity. . .

- some numerical difficulties

Conclusion

- testing for static arbitrage in option price data is easy in dimension one
- the extension on basket options (swaptions, etc) is NP-hard but good relaxations can be found
- we get a computationally friendly set of conditions for the absence of arbitrage
- small scale problems are tractable in practice as semidefinite programs

References

Berg, C., Christensen, J. P. R. & Ressel, P. (1984), *Harmonic analysis on semigroups : theory of positive definite and related functions*, Vol. 100 of *Graduate texts in mathematics*, Springer-Verlag, New York.

Bertsimas, D. & Popescu, I. (2002), 'On the relation between option and stock prices: a convex optimization approach', *Operations Research* **50**(2), 358–374.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 637–659.

Boyd, S. & Vandenberghe, L. (2003), *Convex Optimization*, Cambridge University Press.

Helgason, S. (1999), *The Radon transform*, 2nd edn, Birkhauser, Boston. Progress in mathematics (Boston, Mass.) ; vol. 5.

Henkin, G. & Shananin, A. (1990), 'Bernstein theorems and Radon transform, application to the theory of production functions', *American Mathematical Society: Translation of mathematical monographs* **81**, 189–223.

Laurent, J. & Leisen, D. (2000), Building a consistent pricing model from observed option prices, *in* M. Avellaneda, ed., 'Quantitative Analysis in Financial Markets', World Scientific Publishing.

Nesterov, I. & Nemirovskii, A. (1994), *Interior-point polynomial algorithms in convex programming*, Society for Industrial and Applied Mathematics, Philadelphia.

Ramm, A. G. & Katsevich, A. I. (1996), *The radon transform and local tomography*, CRC Press, Boca Raton.

Sturm, J. F. (1999), 'Using sedumi 1.0x, a matlab toolbox for optimization over symmetric cones.', *Working paper, Department of Quantitative Economics, Maastricht University, The Netherlands.* .