Optimisation Combinatoire et Convexe.

Introduction, convexité, dualité.

Today

- Convex optimization: introduction
- Course organization and other gory details...
- Convex optimization: basic concepts

Convex Optimization

Convex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0, \ldots, f_m(x) \leq 0 \end{array}$$

$x \in \mathbb{R}^n$ is optimization variable; $f_i : \mathbb{R}^n \to \mathbb{R}$ are **convex**:

$$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f_i(y)$$

for all x, y, $0 \leq \lambda \leq 1$

- This template includes LS, LP, QP, and many others.
- **Good news:** convex problems (LP, QP, etc) are **fundamentally tractable**.
- **Bad news:** this is an exception, most nonconvex are **completely intractable**.

Convex optimization

A brief history. . .

- The field is about 50 years old.
- Starts with the work of Von Neumann, Kuhn and Tucker, etc.
- Explodes in the 60's with the advent of "relatively" cheap and efficient computers. . .
- Key to all this: fast linear algebra
- Some of the theory developed before computers even existed. . .

Convex optimization: history

- Historical view: nonlinear problems are hard, linear ones are easy.
- In reality: **Convexity** $\Longrightarrow$ low complexity

  "... *In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.*" **T. Rockafellar**.

- True: Nemirovskii and Yudin [1979].
- Very true: Karmarkar [1984].
- Seriously true: convex programming, Nesterov and Nemirovskii [1994].

Convexity, complexity

- All convex minimization problems with: a first order oracle (returning $f(x)$ and a subgradient) can be solved in polynomial time in size and number of precision digits.

- Proved using the **ellipsoid method** by Nemirovskii and Yudin [1979].

- Very slow convergence in practice.

Linear Programming

- Simplex algorithm by Dantzig (1949): exponential worst-case complexity, very efficient in most cases.

- Khachiyan [1979] then used the ellipsoid method to show the polynomial complexity of LP.

- Karmarkar [1984] describes the first efficient polynomial time algorithm for LP, using interior point methods.

From LP to structured convex programs

- Nesterov and Nemirovskii [1994] show that the interior point methods (IPM) used for LPs can be applied to a larger class of structured convex problems.

- The **self-concordance** analysis that they introduce extends the polynomial time complexity proof for LPs.

- Most operations that preserve convexity also preserve self-concordance.

Large-scale convex programs

Interior point methods.

- IPM essentially solved once and for all a broad range of medium-scale convex programs.
- For large-scale problems, computing a single Newton step is often too expensive

First order methods.

- Dependence on precision is polynomial $O(1/\epsilon^\alpha)$, not logarithmic $O(\log(1/\epsilon))$. This is OK in many applications (stats, etc).
- Run a much larger number of cheaper iterations. No Hessian means significantly lower memory and CPU costs per iteration.
- No unified analysis (self-concordance for IPM): large library of disparate methods.
- Algorithmic choices strictly constrained by **problem structure**.

Objective: classify these techniques, study their performance & complexity.

Symmetric cone programs

- An important particular case: linear programming on symmetric cones

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax - b \in \mathcal{K} \end{array}$$

- These include the LP, second-order (Lorentz) and semidefinite cone:

$$\begin{array}{ll} \text{LP:} & \{x \in \mathbb{R}^n : x \geq 0\} \\ \text{Second order:} & \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : \|x\| \leq y\} \\ \text{Semidefinite:} & \{X \in \mathbf{S}^n : X \succeq 0\} \end{array}$$

- Broad class of problems can be represented in this way.

- **Good news:** Fast, reliable, open-source solvers available (SDPT3, CVX, etc).

This course will describe some “exotic” applications of these programs.

A few “miracles”

Beyond convexity. . .

- **Hidden convexity.** Convex programs solving nonconvex problems ($\mathcal{S}$-lemma).
- **Approximation results.** Approximating combinatorial problems by convex programs.
  - Approximate $\mathcal{S}$-lemma.
  - Approximation ratio for MaxCut, etc.
- **Recovery results on ℓ_1 penalties.** Finding sparse solutions to optimization problems using convex penalties.
  - Sparse signal reconstruction.
  - Matrix completion (collaborative filtering, NETFLIX, etc.).

Course Organization

Course outline

- Fundamental definitions
  - A brief primer on convexity and duality theory
- Algorithmic complexity
  - Interior point methods, self-concordance.
  - First order algorithms: complexity and classification.
- Modern applications
  - Statistics
  - Geometrical problems, graphs.
  - $\cdots$
- Some "miracles": approximation, asymptotic and hidden convexity results
  - Measure concentration results.
  - $\mathcal{S}$-lemma, MaxCut, low rank SDP solutions, nonconvex QCQP, etc.
  - High dimensional geometry
  - ℓ_1 recovery, matrix completion, convex deconvolution, etc.

Info

- Course website with lecture notes, homework, etc.

  `http://www.di.ens.fr/~aspremon/`

- A final exam.

Short blurb

- Contact info on `http://www.di.ens.fr/~aspremon/`
- Email: `aspremon@ens.fr`
- Dual PhDs: Ecole Polytechnique & Stanford University
- Interests: Optimization, machine learning, statistics & finance.

References

All lecture notes will be posted online, none of the books below are required.

- Nesterov [2003], "Introductory Lectures on Convex Optimization", Springer.
- "Convex Optimization" by Lieven Vandenberghe and Stephen Boyd, available online at:

  http://www.stanford.edu/~boyd/cvxbook/

- See also Ben-Tal and Nemirovski [2001], "Lectures On Modern Convex Optimization: Analysis, Algorithms, And Engineering Applications", SIAM.

  http://www2.isye.gatech.edu/~nemirovs/

- Nesterov and Nemirovskii [1994], "Interior Point Polynomial Algorithms in Convex Programming", SIAM.

Convex Sets

Convex Sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Convex set

line segment between x_1 and x_2: all points

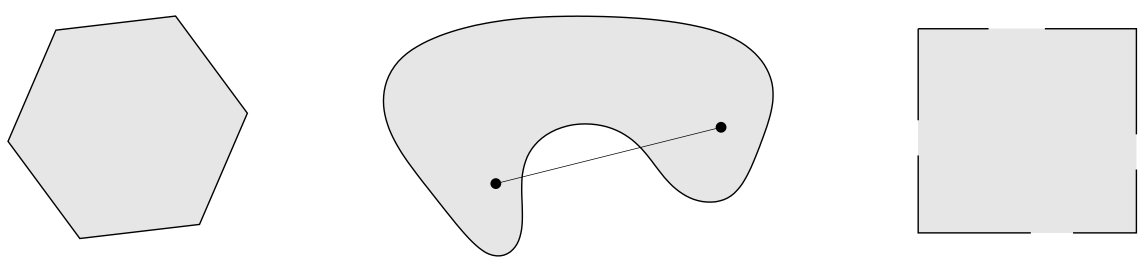
$$x = \theta x_1 + (1-\theta)x_2$$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

examples (one convex, two nonconvex sets)



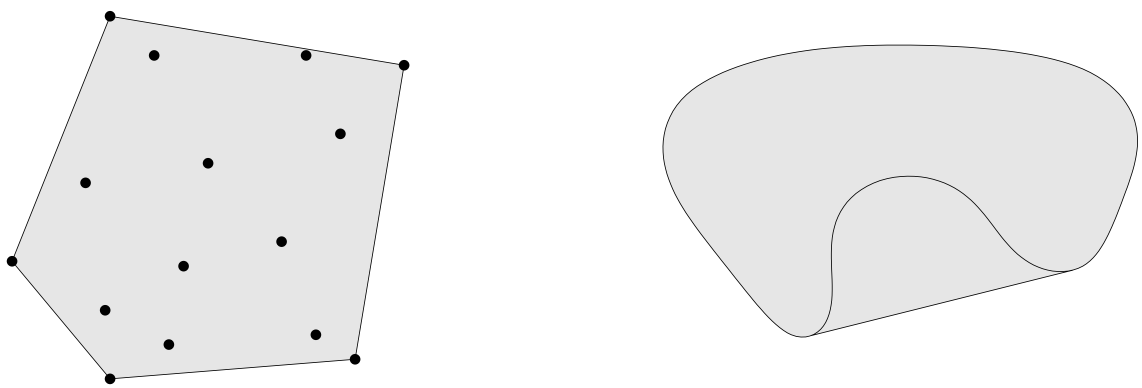
Convex combination and convex hull

convex combination of x_1,. . . , x_k: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

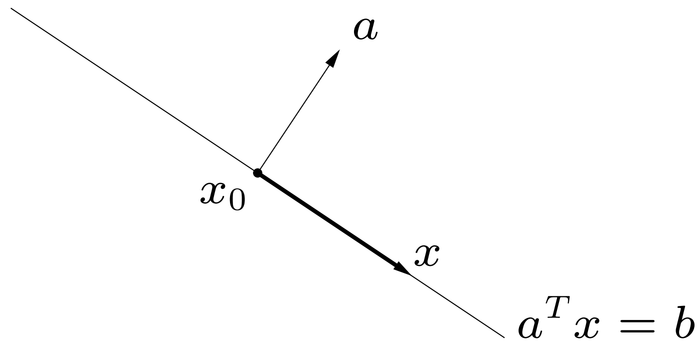
with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{Co}S$: set of all convex combinations of points in S

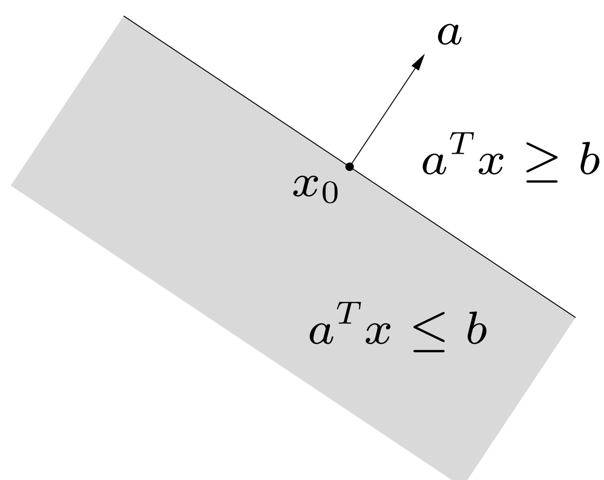


Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

Euclidean balls and ellipsoids

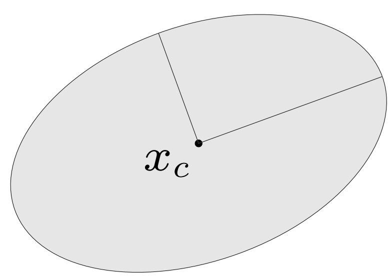
- **(Euclidean) ball** with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

- **Ellipsoid:** set of the form

$$\{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$$

  with $P \in \mathbf{S}^n_{++}$ ($i.e.$, P symmetric positive definite)



  other representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$, with A square and nonsingular.

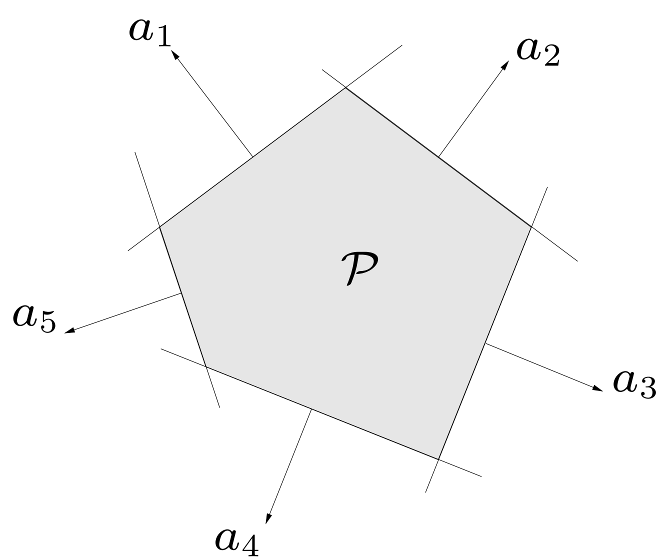
- Representation impacts problem formulation & complexity.
- Idem for polytopes, with polynomial number of vertices, exponential number of facets, and vice-versa.

Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

($A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times n}$, $\preceq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Positive semidefinite cone

notation:

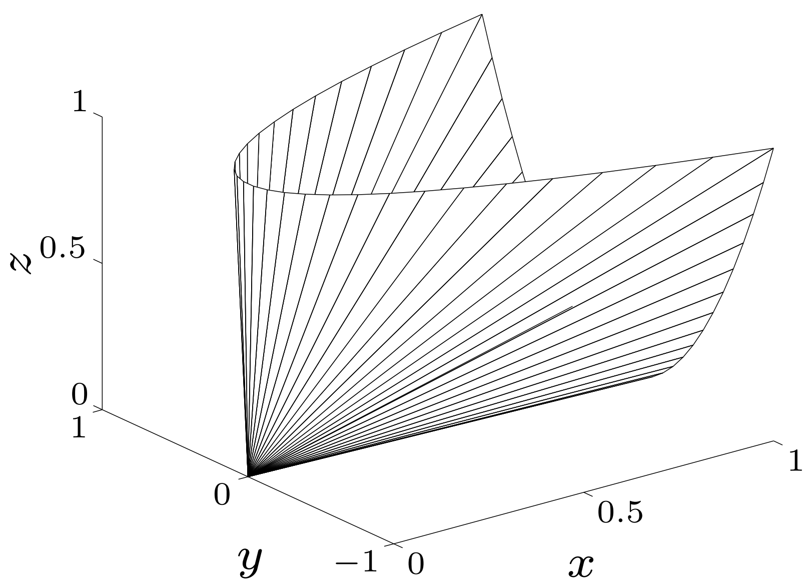
- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

  $\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$



Operations that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

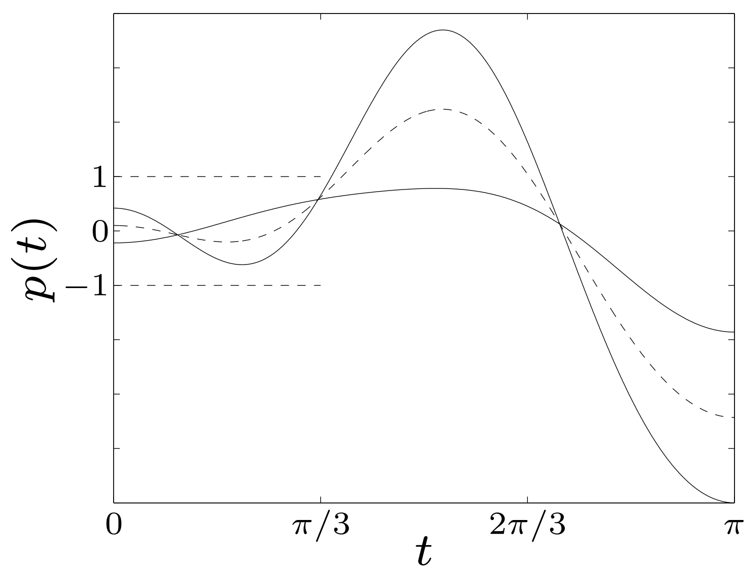
the intersection of (any number of) convex sets is convex

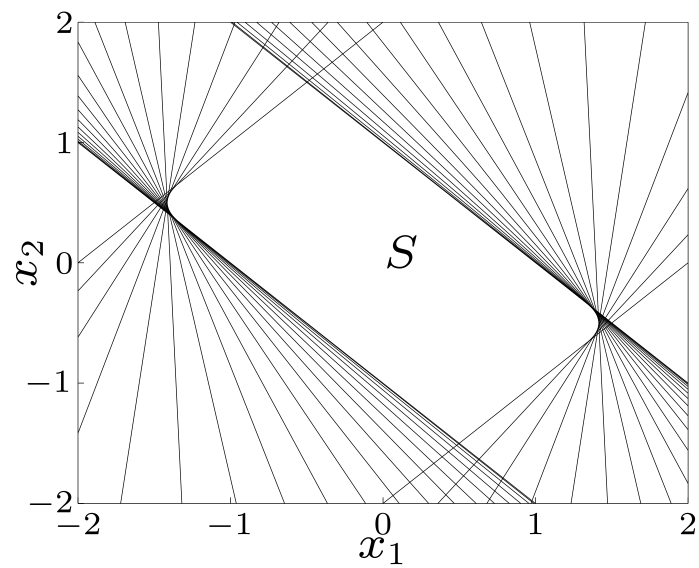
example:

$$S = \{x \in \mathbb{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:





Affine function

suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbb{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2,\ c^T x \geq 0\}$ (with $P \in \mathbf{S}^n_+$)

Perspective and linear-fractional function

perspective function $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

linear-fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

Generalized inequalities

a convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

examples

- nonnegative orthant $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbb{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

generalized inequality defined by a proper cone K:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K, \qquad\qquad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \mathbf{int}\, K$$

examples

- componentwise inequality $(K = \mathbb{R}^n_+)$

$$x \preceq_{\mathbf{R}^n_+} y \quad \Longleftrightarrow \quad x_i \leq y_i, \quad i = 1, \ldots, n$$

- matrix inequality $(K = \mathbf{S}^n_+)$

$$X \preceq_{\mathbf{S}^n_+} Y \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

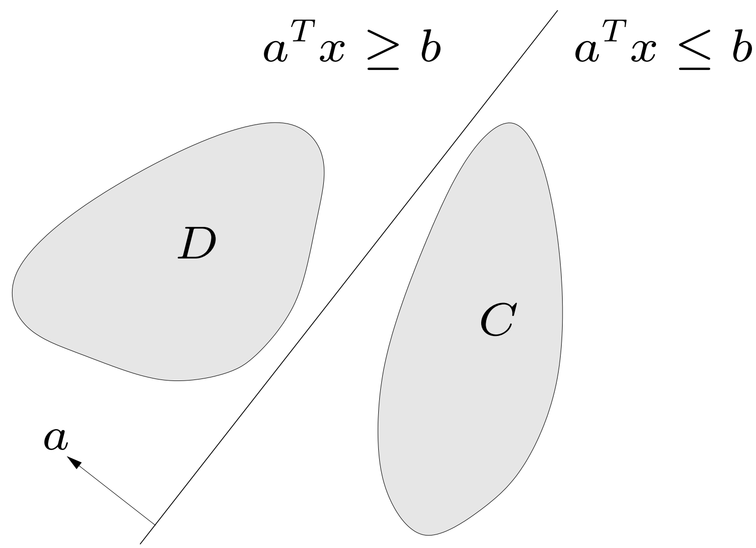
properties: many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$, $e.g.$,

$$x \preceq_K y, \quad u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$$

Separating hyperplane theorem

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

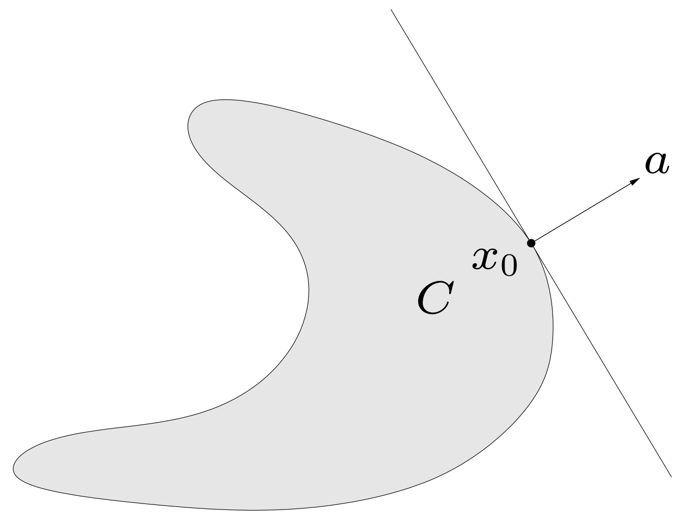
Classical result. Proof relies on minimizing distance between set, and using the argmin to explicitly produce separating hyperplane.

Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Dual cones and generalized inequalities

dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual** cones

dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

Convex Functions

Outline

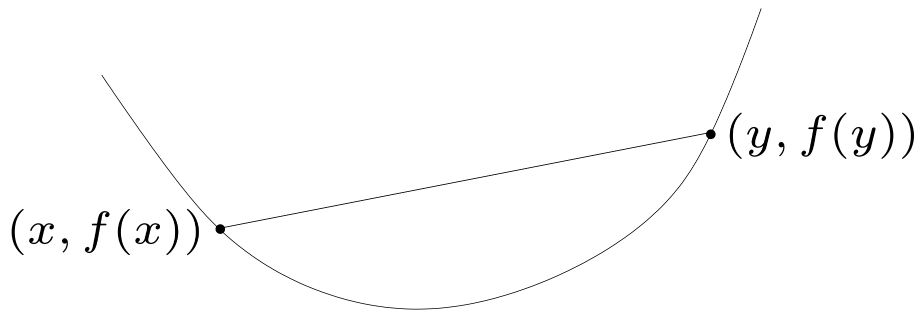
- basic properties and examples
- operations that preserve convexity
- the conjugate function
- quasiconvex functions
- log-concave and log-convex functions
- convexity with respect to generalized inequalities

Definition

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \mathbf{dom}\, f$, $x \neq y$, $0 < \theta < 1$

Examples on $\mathbb{R}$

convex:

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- exponential: e^{ax}, for any $a \in \mathbb{R}$
- powers: x^α on $\mathbb{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbb{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbb{R}_{++}$

concave:

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- powers: x^α on $\mathbb{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbb{R}_{++}$

Examples on $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$

affine functions are convex and concave; all norms are convex

examples on $\mathbb{R}^n$

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

examples on $\mathbb{R}^{m\times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{Tr}(A^T X) + b = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

Restriction of a convex function to a line

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the function $g : \mathbb{R} \to \mathbb{R}$,

$$g(t) = f(x + tv), \qquad \mathbf{dom}\, g = \{t \mid x + tv \in \mathbf{dom}\, f\}$$

is convex (in t) for any $x \in \mathbf{dom}\, f$, $v \in \mathbb{R}^n$

can check convexity of f by checking convexity of functions of one variable

example. $f : \mathbf{S}^n \to \mathbb{R}$ with $f(X) = \log\det X$, $\mathbf{dom}\, X = \mathbf{S}^n_{++}$

$$\begin{aligned} g(t) = \log\det(X + tV) &= \log\det X + \log\det(I + tX^{-1/2}VX^{-1/2}) \\ &= \log\det X + \sum_{i=1}^n \log(1 + t\lambda_i) \end{aligned}$$

where λ_i are the eigenvalues of $X^{-1/2}VX^{-1/2}$

g is concave in t (for any choice of $X \succ 0$, V); hence f is concave

Extended-value extension

extended-value extension $\tilde{f}$ of f is

$$\tilde{f}(x) = f(x), \quad x \in \mathbf{dom}\, f, \qquad \tilde{f}(x) = \infty, \quad x \notin \mathbf{dom}\, f$$

often simplifies notation; for example, the condition

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad \tilde{f}(\theta x + (1-\theta)y) \leq \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

(as an inequality in $\mathbb{R} \cup \{\infty\}$), means the same as the two conditions

- $\mathbf{dom}\, f$ is convex
- for $x, y \in \mathbf{dom}\, f$,

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

First-order condition

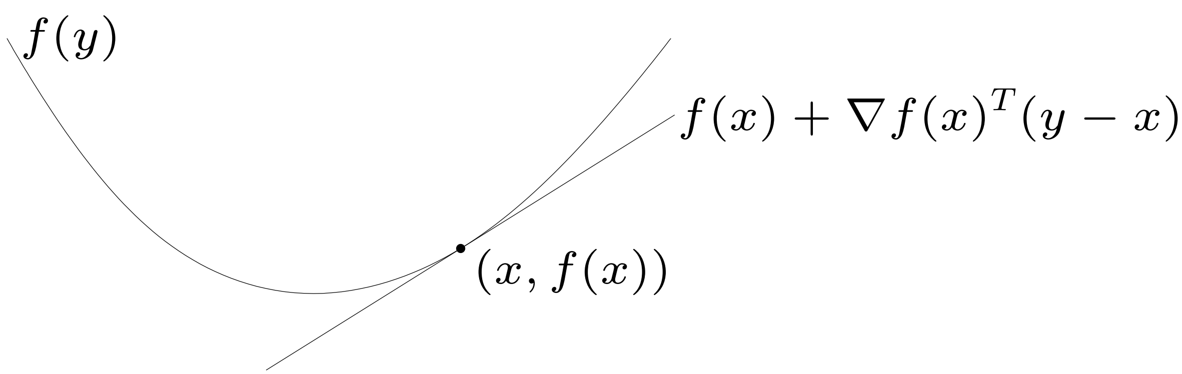
f is **differentiable** if $\mathbf{dom}\, f$ is open and the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in \mathbf{dom}\, f$

1st-order condition: differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$



first-order approximation of f is global underestimator

Second-order conditions

f is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in \mathbf{dom}\, f$

2nd-order conditions: for twice differentiable f with convex domain

- f is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

- if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then f is strictly convex

Examples

quadratic function: $f(x) = (1/2)x^T Px + q^T x + r$ (with $P \in \mathbf{S}^n$)

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

convex if $P \succeq 0$

least-squares objective: $f(x) = \|Ax - b\|_2^2$

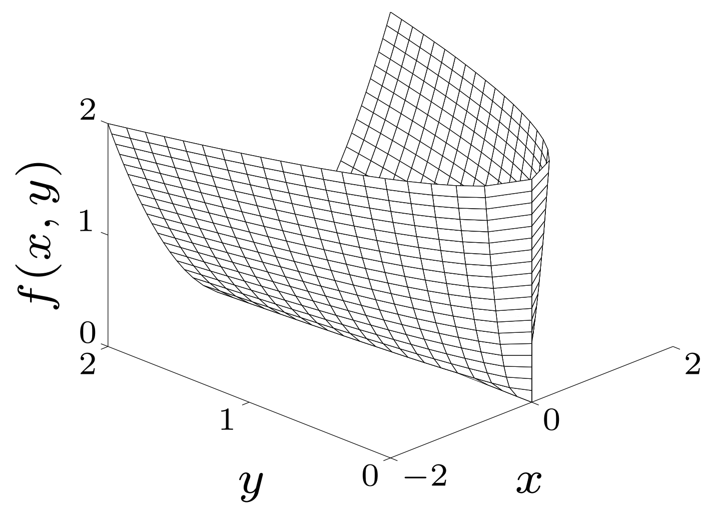
$$\nabla f(x) = 2A^T(Ax - b), \qquad \nabla^2 f(x) = 2A^T A$$

convex (for any A)

quadratic-over-linear: $f(x, y) = x^2/y$

$$\nabla^2 f(x,y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

convex for $y > 0$



log-sum-exp: $f(x) = \log \sum_{k=1}^{n} \exp x_k$ is convex

$$\nabla^2 f(x) = \frac{1}{\mathbf{1}^T z} \mathbf{diag}(z) - \frac{1}{(\mathbf{1}^T z)^2} z z^T \qquad (z_k = \exp x_k)$$

to show $\nabla^2 f(x) \succeq 0$, we must verify that $v^T \nabla^2 f(x) v \geq 0$ for all v:

$$v^T \nabla^2 f(x) v = \frac{(\sum_k z_k v_k^2)(\sum_k z_k) - (\sum_k v_k z_k)^2}{(\sum_k z_k)^2} \geq 0$$

since $(\sum_k v_k z_k)^2 \leq (\sum_k z_k v_k^2)(\sum_k z_k)$ (from Cauchy-Schwarz inequality)

geometric mean: $f(x) = (\prod_{k=1}^{n} x_k)^{1/n}$ on $\mathbb{R}_{++}^n$ is concave

(similar proof as for log-sum-exp)

Epigraph and sublevel set

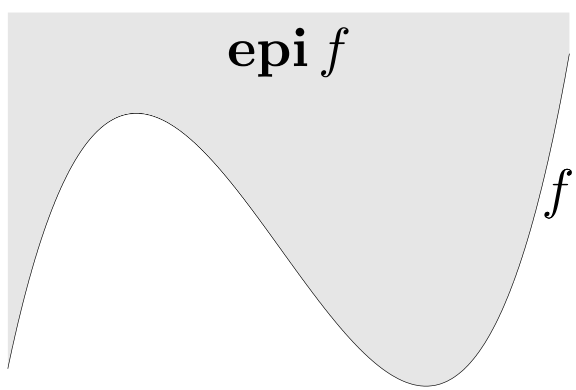
α-**sublevel set** of $f : \mathbb{R}^n \to \mathbb{R}$:

$$C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

epigraph of $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathbf{epi}\, f = \{(x,t) \in \mathbb{R}^{n+1} \mid x \in \mathbf{dom}\, f,\ f(x) \leq t\}$$



f is convex if and only if $\mathbf{epi}\, f$ is a convex set

Jensen's inequality

basic inequality: if f is convex, then for $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

extension: if f is convex, then

$$f(\mathbf{E}\, z) \leq \mathbf{E}\, f(z)$$

for any random variable z

basic inequality is special case with discrete distribution

$$\mathbf{Prob}(z = x) = \theta, \qquad \mathbf{Prob}(z = y) = 1 - \theta$$

Operations that preserve convexity

practical methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that f is obtained from simple convex functions by operations that preserve convexity

   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective

Positive weighted sum & composition with affine function

nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

composition with affine function: $f(Ax+b)$ is convex if f is convex

examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

Pointwise maximum

if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$ is convex
- sum of r largest components of $x \in \mathbb{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

  is convex ($x_{[i]}$ is ith largest component of x)

  proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

Pointwise supremum

if $f(x,y)$ is convex in x for each $y \in \mathcal{A}$, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x,y)$$

is convex

examples

- support function of a set C: $S_C(x) = \sup_{y \in C} y^T x$ is convex
- distance to farthest point in a set C:

$$f(x) = \sup_{y \in C} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$,

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$

Composition with scalar functions

composition of $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$:

$$f(x) = h(g(x))$$

f is convex if $\begin{array}{l} g \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing} \\ g \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing} \end{array}$

- proof (for $n = 1$, differentiable g, h)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

- note: monotonicity must hold for extended-value extension $\tilde{h}$

examples

- $\exp g(x)$ is convex if g is convex
- $1/g(x)$ is convex if g is concave and positive

Vector composition

composition of $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

f is convex if
- g_i convex, h convex, $\tilde{h}$ nondecreasing in each argument
- g_i concave, h convex, $\tilde{h}$ nonincreasing in each argument

proof (for $n = 1$, differentiable g, h)

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)$$

examples

- $\sum_{i=1}^m \log g_i(x)$ is concave if g_i are concave and positive
- $\log \sum_{i=1}^m \exp g_i(x)$ is convex if g_i are convex

Minimization

if $f(x, y)$ is convex in (x, y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex

examples

- $f(x, y) = x^T Ax + 2x^T By + y^T Cy$ with

$$\left[\begin{array}{cc} A & B \\ B^T & C \end{array}\right] \succeq 0, \qquad C \succ 0$$

  minimizing over y gives $g(x) = \inf_y f(x, y) = x^T (A - BC^{-1}B^T)x$

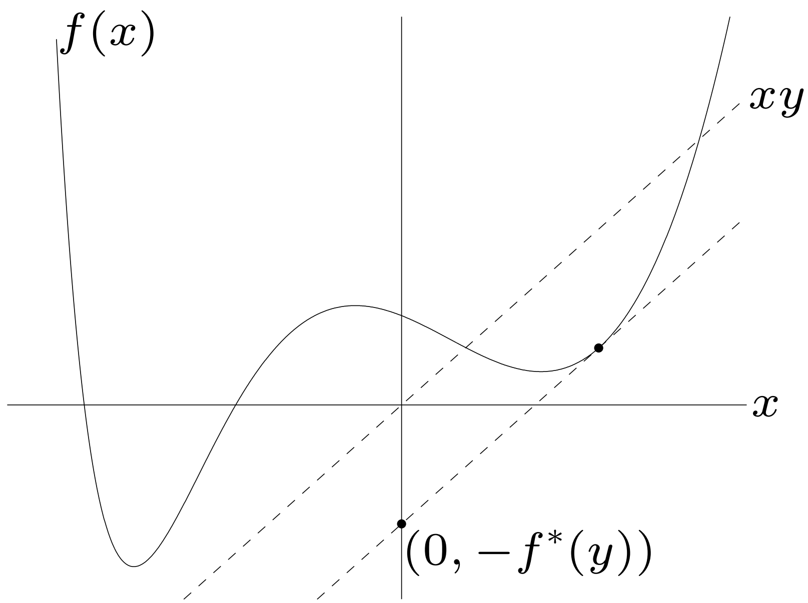
  g is convex, hence Schur complement $A - BC^{-1}B^T \succeq 0$

- distance to a set: $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$ is convex if S is convex

The conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$



- f^* is convex (even if f is not)
- Used in regularization, duality results, . . .

examples

- negative logarithm $f(x) = -\log x$

$$\begin{aligned} f^*(y) &= \sup_{x>0}(xy + \log x) \\ &= \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

- strictly convex quadratic $f(x) = (1/2)x^TQx$ with $Q \in \mathbf{S}^n_{++}$

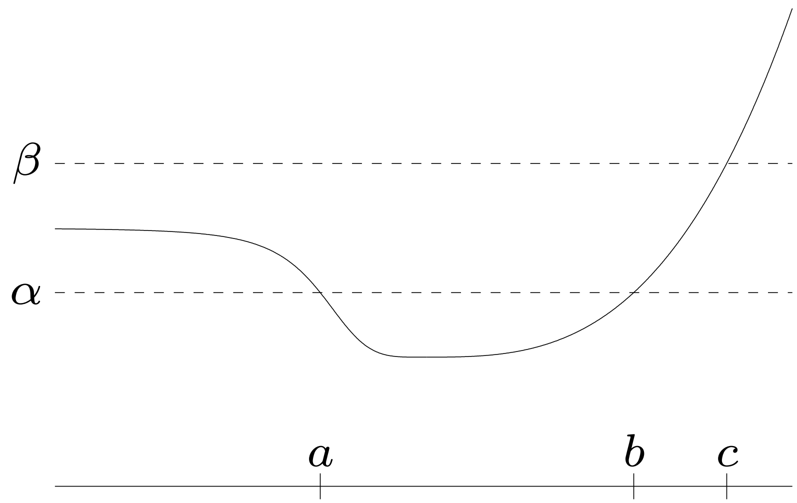
$$\begin{aligned} f^*(y) &= \sup_x (y^Tx - (1/2)x^TQx) \\ &= \frac{1}{2}y^TQ^{-1}y \end{aligned}$$

Quasiconvex functions

$f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if $\mathbf{dom}\, f$ is convex and the sublevel sets

$$S_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$$

are convex for all α



- f is quasiconcave if $-f$ is quasiconvex
- f is quasilinear if it is quasiconvex and quasiconcave

Examples

- $\sqrt{|x|}$ is quasiconvex on $\mathbb{R}$
- $\operatorname{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$ is quasilinear
- $\log x$ is quasilinear on $\mathbb{R}_{++}$
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbb{R}^2_{++}$
- linear-fractional function

$$f(x) = \frac{a^T x + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

  is quasilinear
- distance ratio

$$f(x) = \frac{\|x - a\|_2}{\|x - b\|_2}, \qquad \mathbf{dom}\, f = \{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$$

  is quasiconvex

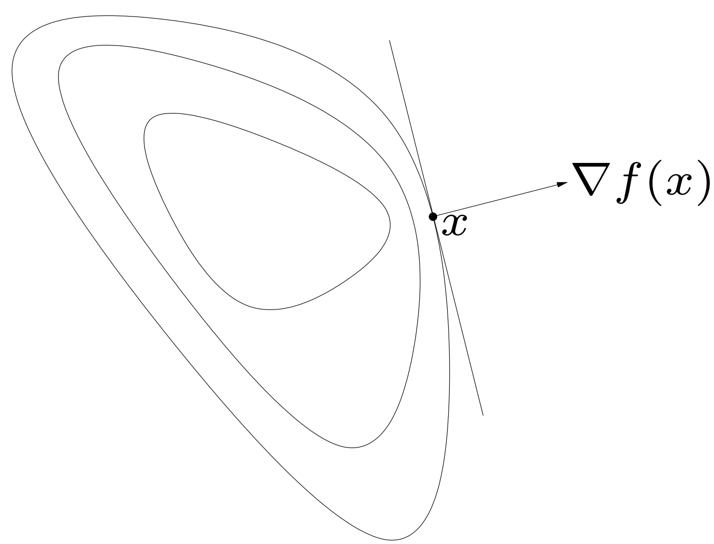
Properties

modified Jensen inequality: for quasiconvex f

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$$

first-order condition: differentiable f with cvx domain is quasiconvex iff

$$f(y) \leq f(x) \quad \Longrightarrow \quad \nabla f(x)^T (y - x) \leq 0$$



sums of quasiconvex functions are not necessarily quasiconvex

Log-concave and log-convex functions

a positive function f is log-concave if $\log f$ is concave:

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta} \quad \text{for } 0 \leq \theta \leq 1$$

f is log-convex if $\log f$ is convex

- powers: x^a on $\mathbb{R}_{++}$ is log-convex for $a \leq 0$, log-concave for $a \geq 0$
- many common probability densities are log-concave, $e.g.$, normal:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})}$$

- cumulative Gaussian distribution function Φ is log-concave

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$$

Properties of log-concave functions

- twice differentiable f with convex domain is log-concave if and only if

$$f(x)\nabla^2 f(x) \preceq \nabla f(x)\nabla f(x)^T$$

  for all $x \in \mathbf{dom}\, f$

- product of log-concave functions is log-concave
- sum of log-concave functions is not always log-concave
- integration: if $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is log-concave, then

$$g(x) = \int f(x, y)\, dy$$

  is log-concave (not easy to show)

consequences of integration property

- convolution $f * g$ of log-concave functions f, g is log-concave

$$(f * g)(x) = \int f(x-y)g(y)dy$$

- if $C \subseteq \mathbb{R}^n$ convex and y is a random variable with log-concave pdf then

$$f(x) = \mathbf{Prob}(x + y \in C)$$

  is log-concave

  proof: write $f(x)$ as integral of product of log-concave functions

$$f(x) = \int g(x+y)p(y)\,dy, \qquad g(u) = \begin{cases} 1 & u \in C \\ 0 & u \notin C, \end{cases}$$

  p is pdf of y

Convex Optimization Problems

Outline

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- geometric programming
- generalized inequality constraints
- semidefinite programming
- vector optimization

Optimization problem in standard form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- $x \in \mathbb{R}^n$ is the optimization variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the objective or cost function
- $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbb{R}^n \to \mathbb{R}$ are the equality constraint functions

optimal value:

$$
p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}
$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Optimal and locally optimal points

x is **feasible** if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints

a feasible x is **optimal** if $f_0(x) = p^\star$; X_{opt} is the set of optimal points

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize (over } z\text{)} & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \leq R \end{array}$$

examples (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbb{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbb{R}_{++}$: $p^\star = -\infty$
- $f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbb{R}_{++}$: $p^\star = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $p^\star = -\infty$, local optimum at $x = 1$

Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- we call $\mathcal{D}$ the **domain** of the problem
- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints $(m = p = 0)$

example:

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Convex optimization problem

standard form convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array}$$

- $f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine
- problem is *quasiconvex* if f_0 is quasiconvex (and $f_1, \ldots, f_m$ convex)

often written as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

important property: feasible set of a convex optimization problem is convex

example

$$\begin{array}{ll}\text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1+x_2)^2 = 0\end{array}$$

- f_0 is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine
- equivalent (but not identical) to the convex problem

$$\begin{array}{ll}\text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0\end{array}$$

Local and global optima

any locally optimal point of a convex problem is (globally) optimal

Proof: suppose x is locally optimal and y is optimal with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Longrightarrow \quad f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and

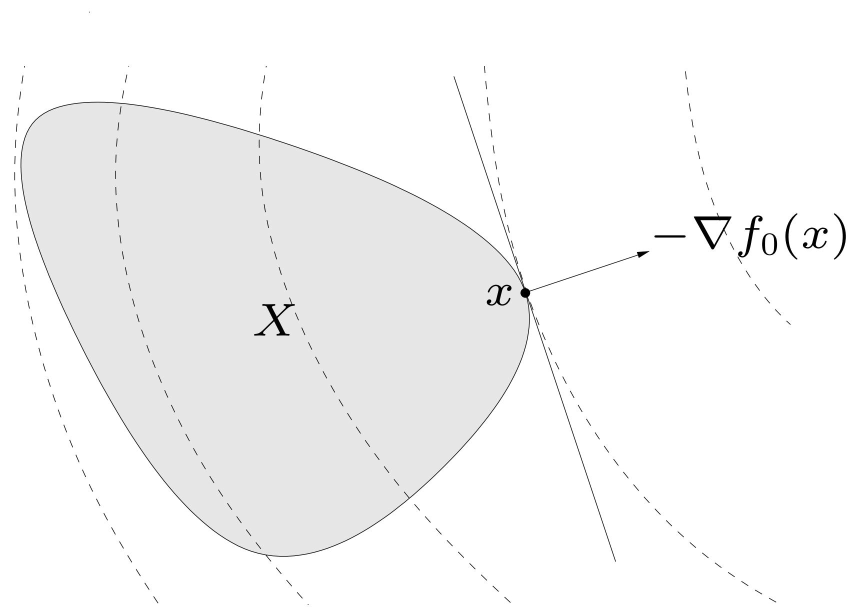
$$f_0(z) \leq \theta f_0(x) + (1 - \theta) f_0(y) < f_0(x)$$

which contradicts our assumption that x is locally optimal

Optimality criterion for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$



if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

Equivalent convex problems

two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- **eliminating equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  is equivalent to

$$\begin{array}{ll} \text{minimize (over } z) & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m \end{array}$$

  where F and x_0 are such that

$$Ax = b \quad \Longleftrightarrow \quad x = Fz + x_0 \text{ for some } z$$

- **introducing equality constraints**

$$\begin{array}{ll}\text{minimize} & f_0(A_0x+b_0)\\ \text{subject to} & f_i(A_ix+b_i)\leq 0, \quad i=1,\ldots,m\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{minimize (over } x,\ y_i) & f_0(y_0)\\ \text{subject to} & f_i(y_i)\leq 0, \quad i=1,\ldots,m\\ & y_i=A_ix+b_i, \quad i=0,1,\ldots,m\end{array}$$

- **introducing slack variables for linear inequalities**

$$\begin{array}{ll}\text{minimize} & f_0(x)\\ \text{subject to} & a_i^Tx\leq b_i, \quad i=1,\ldots,m\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{minimize (over } x,\ s) & f_0(x)\\ \text{subject to} & a_i^Tx+s_i=b_i, \quad i=1,\ldots,m\\ & s_i\geq 0, \quad i=1,\ldots m\end{array}$$

- **epigraph form**: standard form convex problem is equivalent to

$$\begin{array}{ll} \mbox{minimize (over } x,\ t) & t \\ \mbox{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- **minimizing over some variables**

$$\begin{array}{ll} \mbox{minimize} & f_0(x_1, x_2) \\ \mbox{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \mbox{minimize} & \tilde{f}_0(x_1) \\ \mbox{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

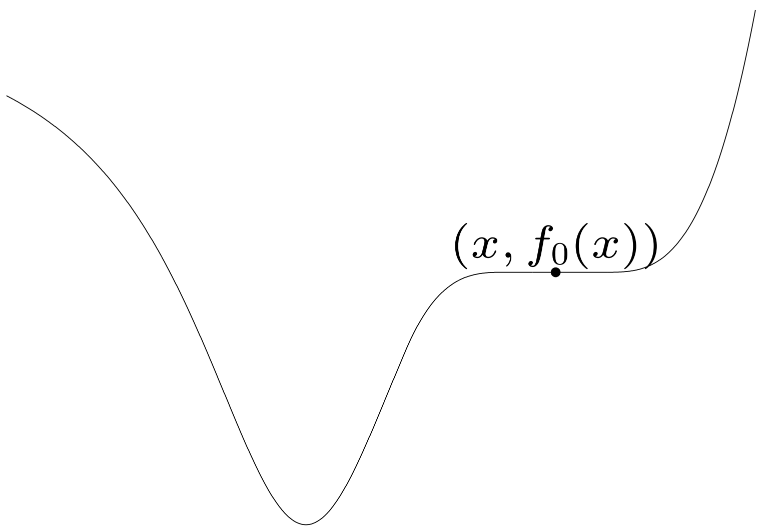
where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

Quasiconvex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

with $f_0 : \mathbb{R}^n \to \mathbb{R}$ quasiconvex, $f_1, \ldots, f_m$ convex

can have locally optimal points that are not (globally) optimal



quasiconvex optimization via convex feasibility problems

$$f_0(x) \leq t, \qquad f_i(x) \leq 0, \quad i = 1, \ldots, m, \qquad Ax = b \tag{1}$$

- for fixed t, a convex feasibility problem in x
- if feasible, we can conclude that $t \geq p^\star$; if infeasible, $t \leq p^\star$

Bisection method for quasiconvex optimization

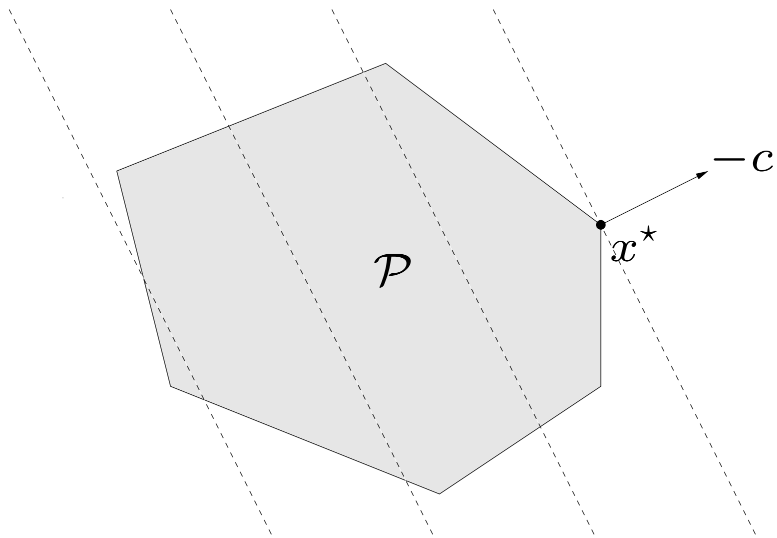
given $l \leq p^\star$, $u \geq p^\star$, tolerance $\epsilon > 0$.
repeat
  1. $t := (l + u)/2$.
  2. Solve the convex feasibility problem (1).
  3. **if** (1) is feasible, $u := t$; **else** $l := t$.
until $u - l \leq \epsilon$.

requires exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations (where u, l are initial values)

Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



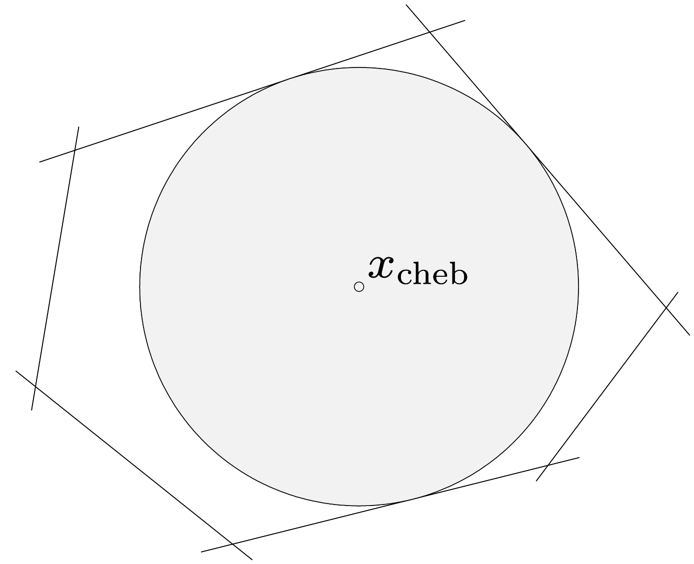
Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \le r\}$$



- $a_i^T x \le b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) \mid \|u\|_2 \le r\} = a_i^T x_c + r\|a_i\|_2 \le b_i$$

- hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \le b_i, \quad i = 1, \ldots, m \end{array}$$

(Generalized) linear-fractional program

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

linear-fractional program

$$f_0(x) = \frac{c^T x + d}{e^T x + f}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e^T x + f > 0\}$$

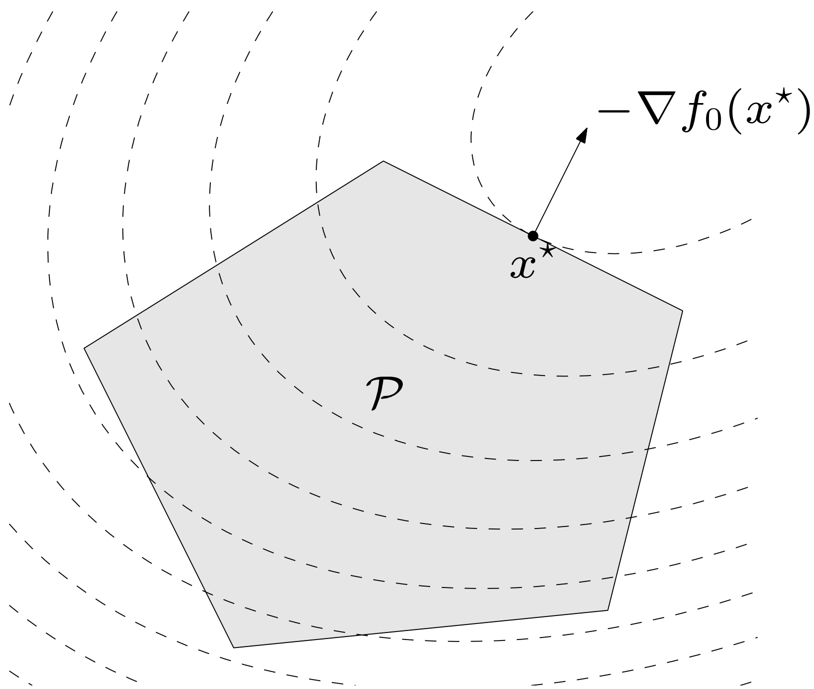
- a quasiconvex optimization problem; can be solved by bisection
- also equivalent to the LP (variables y, z)

$$\begin{array}{ll} \text{minimize} & c^T y + dz \\ \text{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0 \end{array}$$

Quadratic program (QP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P x + q^T x + r \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Examples

least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution $x^\star = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)
- can add linear constraints, $e.g.$, $l \preceq x \preceq u$

linear program with random cost

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}\, c^T x + \gamma\, \mathbf{var}(c^T x) \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

- c is random vector with mean $\bar{c}$ and covariance Σ
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

Quadratically constrained quadratic program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, feasible region is intersection of m ellipsoids and an affine set

Second-order cone programming

$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\
& Fx = g
\end{array}
$$

($A_i \in \mathbb{R}^{n_i \times n}$, $F \in \mathbb{R}^{p \times n}$)

- inequalities are called second-order cone (SOC) constraints:

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbb{R}^{n_i+1}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP
- more general than QCQP and LP

Robust linear programming

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

there can be uncertainty in c, a_i, b_i

two common approaches to handling uncertainty (in a_i, for simplicity)

- **deterministic model:** constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m, \end{array}$$

- **stochastic model:** a_i is random variable; constraints must hold with probability η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

deterministic approach via SOCP

- choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\} \qquad (\bar{a}_i \in \mathbb{R}^n, \quad P_i \in \mathbb{R}^{n \times n})$$

center is $\bar{a}_i$, semi-axes determined by singular values/vectors of P_i

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

(follows from $\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

stochastic approach via SOCP

- assume a_i is Gaussian with mean $\bar{a}_i$, covariance Σ_i ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$)
- $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \Sigma_i x$; hence

$$\mathbf{Prob}(a_i^T x \leq b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2}\right)$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2}\, dt$ is CDF of $\mathcal{N}(0,1)$

- robust LP

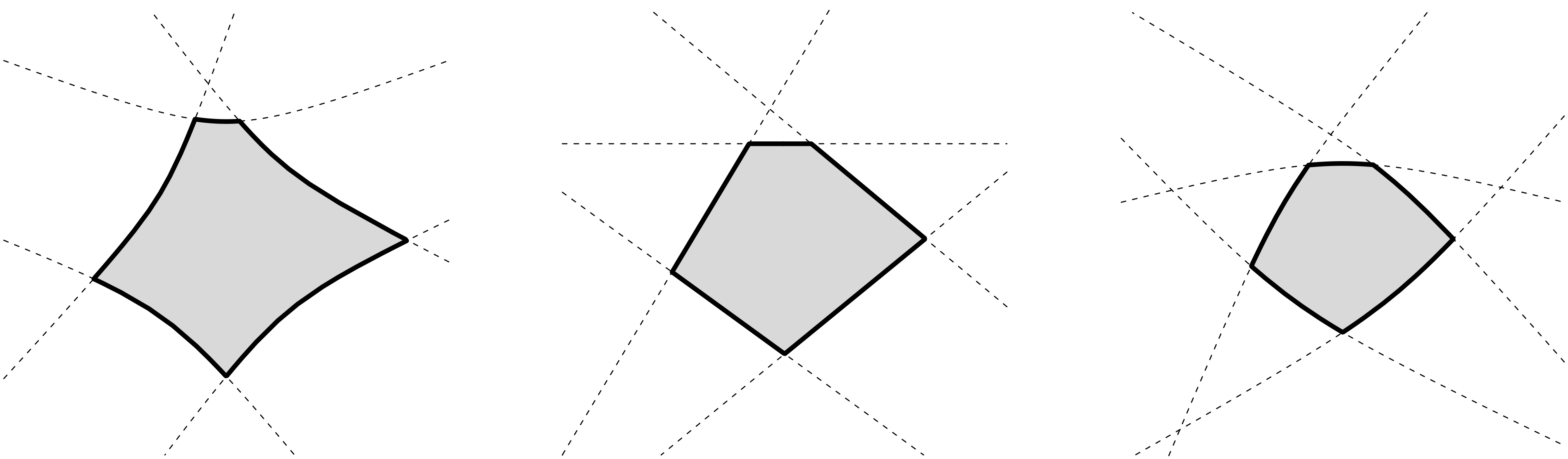
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m, \end{array}$$

with $\eta \geq 1/2$, is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Impact of reliability

$$\{x \mid \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta, \ i = 1, \ldots, m\}$$



$\eta = 10\%$ $\eta = 50\%$ $\eta = 90\%$

Generalized inequality constraints

convex problem with generalized inequality constraints

$$\begin{array}{ll}\text{minimize} & f_0(x)\\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i=1,\ldots,m\\ & Ax=b\end{array}$$

- $f_0 : \mathbb{R}^n \to \mathbb{R}$ convex; $f_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$ K_i-convex w.r.t. proper cone K_i
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

conic form problem: special case with affine objective and constraints

$$\begin{array}{ll}\text{minimize} & c^T x\\ \text{subject to} & Fx+g \preceq_K 0\\ & Ax=b\end{array}$$

extends linear programming ($K = \mathbb{R}^m_+$) to nonpolyhedral cones

Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

with F_i, $G \in \mathbf{S}^k$

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

$$x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \qquad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

  is equivalent to single LMI

$$x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0$$

LP and SOCP as SDP

LP and equivalent SDP

$$\begin{array}{llll} \text{LP:} & \text{minimize} & c^T x \\ & \text{subject to} & Ax \preceq b \end{array} \qquad \begin{array}{lll} \text{SDP:} & \text{minimize} & c^T x \\ & \text{subject to} & \mathbf{diag}(Ax - b) \preceq 0 \end{array}$$

(note different interpretation of generalized inequality $\preceq$)

SOCP and equivalent SDP

$$\begin{array}{lll} \text{SOCP:} & \text{minimize} & f^T x \\ & \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

$$\begin{array}{lll} \text{SDP:} & \text{minimize} & f^T x \\ & \text{subject to} & \begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m \end{array}$$

Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{S}^k$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

- variables $x \in \mathbb{R}^n$, $t \in \mathbb{R}$
- follows from

$$\lambda_{\max}(A) \leq t \quad \Longleftrightarrow \quad A \preceq tI$$

Matrix norm minimization

$$\text{minimize} \quad \|A(x)\|_2 = \left(\lambda_{\max}(A(x)^T A(x))\right)^{1/2}$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{S}^{p \times q}$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} tI & A(x) \\ A(x)^T & tI \end{array}\right] \succeq 0 \end{array}$$

- variables $x \in \mathbb{R}^n$, $t \in \mathbb{R}$
- constraint follows from

$$\begin{aligned} \|A\|_2 \leq t \quad &\Longleftrightarrow \quad A^T A \preceq t^2 I, \quad t \geq 0 \\ &\Longleftrightarrow \quad \left[\begin{array}{cc} tI & A \\ A^T & tI \end{array}\right] \succeq 0 \end{aligned}$$

Duality

Outline

- Lagrange dual problem
- weak and strong duality
- optimality conditions
- perturbation and sensitivity analysis
- examples
- generalized inequalities

Lagrangian

standard form problem (not necessarily convex)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

variable $x \in \mathbb{R}^n$, domain $\mathcal{D}$, optimal value $p^\star$

Lagrangian: $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$, with $\mathbf{dom}\, L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$

- weighted sum of objective and constraint functions
- λ_i is Lagrange multiplier associated with $f_i(x) \leq 0$
- ν_i is Lagrange multiplier associated with $h_i(x) = 0$

Lagrange dual function

Lagrange dual function: $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$,

$$\begin{aligned} g(\lambda,\nu) &= \inf_{x\in\mathcal{D}} L(x,\lambda,\nu) \\ &= \inf_{x\in\mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) \end{aligned}$$

g is concave, can be $-\infty$ for some λ, ν

lower bound property: if $\lambda \succeq 0$, then $g(\lambda,\nu) \leq p^\star$

proof: if $\tilde{x}$ is feasible and $\lambda \succeq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x},\lambda,\nu) \geq \inf_{x\in\mathcal{D}} L(x,\lambda,\nu) = g(\lambda,\nu)$$

minimizing over all feasible $\tilde{x}$ gives $p^\star \geq g(\lambda,\nu)$

Least-norm solution of linear equations

$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax = b \end{array}$$

dual function

- Lagrangian is $L(x, \nu) = x^T x + \nu^T (Ax - b)$
- to minimize L over x, set gradient equal to zero:

$$\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \quad \Longrightarrow \quad x = -(1/2) A^T \nu$$

- plug in in L to obtain g:

$$g(\nu) = L((-1/2) A^T \nu, \nu) = -\frac{1}{4} \nu^T A A^T \nu - b^T \nu$$

a concave function of ν

lower bound property: $p^\star \geq -(1/4) \nu^T A A^T \nu - b^T \nu$ for all ν

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$

dual function

- Lagrangian is

$$\begin{array}{rcl} L(x, \lambda, \nu) & = & c^T x + \nu^T (Ax - b) - \lambda^T x \\ & = & -b^T \nu + (c + A^T \nu - \lambda)^T x \end{array}$$

- L is linear in x, hence

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

  g is linear on affine domain $\{(\lambda, \nu) \mid A^T \nu - \lambda + c = 0\}$, hence concave

lower bound property: $p^\star \geq -b^T \nu$ if $A^T \nu + c \succeq 0$

Equality constrained norm minimization

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = b \end{array}$$

dual function

$$g(\nu) = \inf_x (\|x\| - \nu^T Ax + b^T \nu) = \begin{cases} b^T \nu & \|A^T \nu\|_* \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

where $\|v\|_* = \sup_{\|u\| \leq 1} u^T v$ is dual norm of $\|\cdot\|$

proof: follows from $\inf_x (\|x\| - y^T x) = 0$ if $\|y\|_* \leq 1$, $-\infty$ otherwise

- if $\|y\|_* \leq 1$, then $\|x\| - y^T x \geq 0$ for all x, with equality if $x = 0$
- if $\|y\|_* > 1$, choose $x = tu$ where $\|u\| \leq 1$, $u^T y = \|y\|_* > 1$:

$$\|x\| - y^T x = t(\|u\| - \|y\|_*) \to -\infty \quad \text{as } t \to \infty$$

lower bound property: $p^\star \geq b^T \nu$ if $\|A^T \nu\|_* \leq 1$

Two-way partitioning

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

- a nonconvex problem; feasible set contains 2^n discrete points
- interpretation: partition $\{1, \ldots, n\}$ in two sets; W_{ij} is cost of assigning i, j to the same set; $-W_{ij}$ is cost of assigning to different sets

dual function

$$\begin{aligned} g(\nu) = \inf_x (x^T W x + \sum_i \nu_i (x_i^2 - 1)) &= \inf_x x^T (W + \mathbf{diag}(\nu)) x - \mathbf{1}^T \nu \\ &= \begin{cases} -\mathbf{1}^T \nu & W + \mathbf{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

lower bound property: $p^\star \geq -\mathbf{1}^T \nu$ if $W + \mathbf{diag}(\nu) \succeq 0$

example: $\nu = -\lambda_{\min}(W)\mathbf{1}$ gives bound $p^\star \geq n\lambda_{\min}(W)$

The dual problem

Lagrange dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- finds best lower bound on $p^\star$, obtained from Lagrange dual function
- a convex optimization problem; optimal value denoted $d^\star$
- λ, ν are dual feasible if $\lambda \succeq 0$, $(\lambda, \nu) \in \mathbf{dom}\, g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom}\, g$ explicit

example: standard form LP and its dual (page 99)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & -b^T \nu \\ \text{subject to} & A^T \nu + c \succeq 0 \end{array}$$

Weak and strong duality

weak duality: $d^\star \leq p^\star$

- always holds (for convex and nonconvex problems)
- can be used to find nontrivial lower bounds for difficult problems

  for example, solving the SDP

$$\begin{array}{ll} \text{maximize} & -\mathbf{1}^T \nu \\ \text{subject to} & W + \mathbf{diag}(\nu) \succeq 0 \end{array}$$

  gives a lower bound for the two-way partitioning problem on page 101

strong duality: $d^\star = p^\star$

- does not hold in general
- (usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called **constraint qualifications**

Slater's constraint qualification

strong duality holds for a convex problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

if it is **strictly feasible**, *i.e.*,

$$\exists x \in \mathbf{int}\, \mathcal{D}: \qquad f_i(x) < 0, \quad i = 1, \ldots, m, \qquad Ax = b$$

- also guarantees that the dual optimum is attained (if $p^\star > -\infty$)
- can be sharpened: *e.g.*, can replace $\mathbf{int}\, \mathcal{D}$ with $\mathbf{relint}\, \mathcal{D}$ (interior relative to affine hull); linear inequalities do not need to hold with strict inequality, . . .
- there exist many other types of constraint qualifications

Inequality form LP

primal problem

$$\begin{array}{ll}\text{minimize} & c^T x\\ \text{subject to} & Ax \preceq b\end{array}$$

dual function

$$g(\lambda) = \inf_x \left((c + A^T\lambda)^T x - b^T\lambda\right) = \begin{cases} -b^T\lambda & A^T\lambda + c = 0\\ -\infty & \text{otherwise}\end{cases}$$

dual problem

$$\begin{array}{ll}\text{maximize} & -b^T\lambda\\ \text{subject to} & A^T\lambda + c = 0, \quad \lambda \succeq 0\end{array}$$

- from Slater's condition: $p^\star = d^\star$ if $A\tilde{x} \prec b$ for some $\tilde{x}$
- in fact, $p^\star = d^\star$ except when primal and dual are infeasible

Quadratic program

primal problem (assume $P \in \mathbf{S}^n_{++}$)

$$\begin{array}{ll} \mbox{minimize} & x^T P x \\ \mbox{subject to} & Ax \preceq b \end{array}$$

dual function

$$g(\lambda) = \inf_x \left(x^T P x + \lambda^T (Ax - b) \right) = -\frac{1}{4} \lambda^T A P^{-1} A^T \lambda - b^T \lambda$$

dual problem

$$\begin{array}{ll} \mbox{maximize} & -(1/4) \lambda^T A P^{-1} A^T \lambda - b^T \lambda \\ \mbox{subject to} & \lambda \succeq 0 \end{array}$$

- from Slater's condition: $p^\star = d^\star$ if $A\tilde{x} \prec b$ for some $\tilde{x}$
- in fact, $p^\star = d^\star$ always

A nonconvex problem with strong duality

$$\begin{array}{ll} \text{minimize} & x^T A x + 2b^T x \\ \text{subject to} & x^T x \leq 1 \end{array}$$

nonconvex if $A \not\succeq 0$

dual function: $g(\lambda) = \inf_x (x^T(A + \lambda I)x + 2b^T x - \lambda)$

- unbounded below if $A + \lambda I \not\succeq 0$ or if $A + \lambda I \succeq 0$ and $b \notin \mathcal{R}(A + \lambda I)$
- minimized by $x = -(A + \lambda I)^\dagger b$ otherwise: $g(\lambda) = -b^T(A + \lambda I)^\dagger b - \lambda$

dual problem and equivalent SDP:

$$\begin{array}{ll} \text{maximize} & -b^T(A + \lambda I)^\dagger b - \lambda \\ \text{subject to} & A + \lambda I \succeq 0 \\ & b \in \mathcal{R}(A + \lambda I) \end{array} \qquad \begin{array}{ll} \text{maximize} & -t - \lambda \\ \text{subject to} & \begin{bmatrix} A + \lambda I & b \\ b^T & t \end{bmatrix} \succeq 0 \end{array}$$

strong duality although primal problem is not convex (more later)

Complementary slackness

Assume strong duality holds, $x^\star$ is primal optimal, $(\lambda^\star, \nu^\star)$ is dual optimal

$$\begin{aligned} f_0(x^\star) = g(\lambda^\star, \nu^\star) &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^\star f_i(x) + \sum_{i=1}^p \nu_i^\star h_i(x) \right) \\ &\leq f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star f_i(x^\star) + \sum_{i=1}^p \nu_i^\star h_i(x^\star) \\ &\leq f_0(x^\star) \end{aligned}$$

hence, the two inequalities hold with equality

- $x^\star$ minimizes $L(x, \lambda^\star, \nu^\star)$
- $\lambda_i^\star f_i(x^\star) = 0$ for $i = 1, \ldots, m$ (known as **complementary slackness**):

$$\lambda_i^\star > 0 \Longrightarrow f_i(x^\star) = 0, \qquad f_i(x^\star) < 0 \Longrightarrow \lambda_i^\star = 0$$

Karush-Kuhn-Tucker (KKT) conditions

the following four conditions are called KKT conditions (for a problem with differentiable f_i, h_i):

1. **Primal feasibility**: $f_i(x) \leq 0$, $i = 1, \ldots, m$, $h_i(x) = 0$, $i = 1, \ldots, p$
2. **Dual feasibility**: $\lambda \succeq 0$
3. **Complementary slackness**: $\lambda_i f_i(x) = 0$, $i = 1, \ldots, m$
4. Gradient of Lagrangian with respect to x vanishes (**first order condition**):

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

If strong duality holds and x, λ, ν are optimal, then they must satisfy the KKT conditions

KKT conditions for convex problem

If $\tilde{x}$, $\tilde{\lambda}$, $\tilde{\nu}$ satisfy KKT for a **convex problem**, then they are optimal:

- from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

If **Slater's condition** is satisfied, x is optimal if and only if there exist λ, ν that satisfy KKT conditions

- recall that Slater implies strong duality, and dual optimum is attained
- generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

Summary:

- When strong duality holds, the KKT conditions are necessary conditions for optimality
- If the problem is **convex**, they are also sufficient

example: water-filling (assume $\alpha_i > 0$)

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^n \log(x_i + \alpha_i) \\ \text{subject to} & x \succeq 0, \quad \mathbf{1}^T x = 1 \end{array}$$

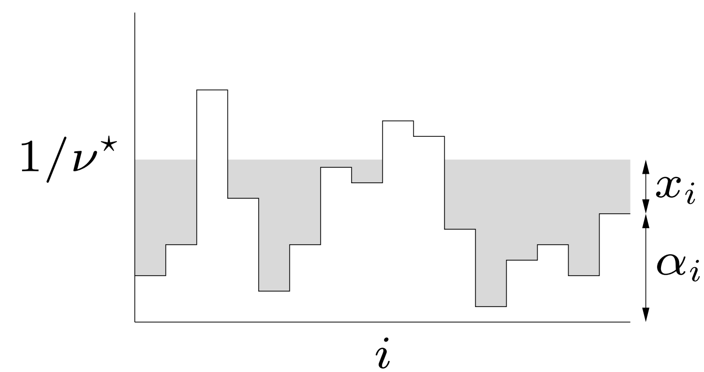
x is optimal iff $x \succeq 0$, $\mathbf{1}^T x = 1$, and there exist $\lambda \in \mathbb{R}^n$, $\nu \in \mathbb{R}$ such that

$$\lambda \succeq 0, \qquad \lambda_i x_i = 0, \qquad \frac{1}{x_i + \alpha_i} + \lambda_i = \nu$$

- if $\nu < 1/\alpha_i$: $\lambda_i = 0$ and $x_i = 1/\nu - \alpha_i$
- if $\nu \geq 1/\alpha_i$: $\lambda_i = \nu - 1/\alpha_i$ and $x_i = 0$
- determine ν from $\mathbf{1}^T x = \sum_{i=1}^n \max\{0, 1/\nu - \alpha_i\} = 1$

interpretation

- n patches; level of patch i is at height α_i
- flood area with unit amount of water
- resulting level is $1/\nu^\star$



Perturbation and sensitivity analysis

(unperturbed) optimization problem and its dual

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array} \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

perturbed problem and its dual

$$\begin{array}{lll} \text{min.} & f_0(x) & \\ \text{s.t.} & f_i(x) \leq u_i, & i = 1, \ldots, m \\ & h_i(x) = v_i, & i = 1, \ldots, p \end{array} \qquad \begin{array}{ll} \text{max.} & g(\lambda, \nu) - u^T\lambda - v^T\nu \\ \text{s.t.} & \lambda \succeq 0 \end{array}$$

- x is primal variable; u, v are parameters
- $p^\star(u, v)$ is optimal value as a function of u, v
- we are interested in information about $p^\star(u, v)$ that we can obtain from the solution of the unperturbed problem and its dual

Perturbation and sensitivity analysis

global sensitivity result Strong duality holds for unperturbed problem and $\lambda^\star$, $\nu^\star$ are dual optimal for unperturbed problem. Apply **weak duality** to perturbed problem:

$$\begin{array}{rcl} p^\star(u,v) & \geq & g(\lambda^\star,\nu^\star) - u^T\lambda^\star - v^T\nu^\star \\ & = & p^\star(0,0) - u^T\lambda^\star - v^T\nu^\star \end{array}$$

local sensitivity: if (in addition) $p^\star(u,v)$ is differentiable at $(0,0)$, then

$$\lambda_i^\star = -\frac{\partial p^\star(0,0)}{\partial u_i}, \qquad \nu_i^\star = -\frac{\partial p^\star(0,0)}{\partial v_i}$$

Duality and problem reformulations

- equivalent formulations of a problem can lead to very different duals
- reformulating the primal problem can be useful when the dual is difficult to derive, or uninteresting

common reformulations

- introduce new variables and equality constraints
- make explicit constraints implicit or vice-versa
- transform objective or constraint functions

  $e.g.$, replace $f_0(x)$ by $\phi(f_0(x))$ with ϕ convex, increasing

Introducing new variables and equality constraints

$$\text{minimize} \quad f_0(Ax+b)$$

- dual function is constant: $g = \inf_x L(x) = \inf_x f_0(Ax+b) = p^\star$
- we have strong duality, but dual is quite useless

reformulated problem and its dual

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & Ax+b-y=0 \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & b^T\nu - f_0^*(\nu) \\ \text{subject to} & A^T\nu = 0 \end{array}$$

dual function follows from

$$\begin{aligned} g(\nu) &= \inf_{x,y}(f_0(y) - \nu^T y + \nu^T Ax + b^T\nu) \\ &= \begin{cases} -f_0^*(\nu) + b^T\nu & A^T\nu = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

norm approximation problem: minimize $\|Ax - b\|$

$$\begin{array}{ll} \text{minimize} & \|y\| \\ \text{subject to} & y = Ax - b \end{array}$$

can look up conjugate of $\|\cdot\|$, or derive dual directly

$$\begin{aligned} g(\nu) &= \inf_{x,y}(\|y\| + \nu^T y - \nu^T Ax + b^T\nu) \\ &= \begin{cases} b^T\nu + \inf_y(\|y\| + \nu^T y) & A^T\nu = 0 \\ -\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} b^T\nu & A^T\nu = 0, \quad \|\nu\|_* \leq 1 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

(see page 98)

dual of norm approximation problem

$$\begin{array}{ll} \text{maximize} & b^T\nu \\ \text{subject to} & A^T\nu = 0, \quad \|\nu\|_* \leq 1 \end{array}$$

Implicit constraints

LP with box constraints: primal and dual problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & -\mathbf{1} \preceq x \preceq \mathbf{1} \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T \nu - \mathbf{1}^T \lambda_1 - \mathbf{1}^T \lambda_2 \\ \text{subject to} & c + A^T \nu + \lambda_1 - \lambda_2 = 0 \\ & \lambda_1 \succeq 0, \quad \lambda_2 \succeq 0 \end{array}$$

reformulation with box constraints made implicit

$$\begin{array}{ll} \text{minimize} & f_0(x) = \begin{cases} c^T x & -\mathbf{1} \preceq x \preceq \mathbf{1} \\ \infty & \text{otherwise} \end{cases} \\ \text{subject to} & Ax = b \end{array}$$

dual function

$$\begin{aligned} g(\nu) &= \inf_{-\mathbf{1} \preceq x \preceq \mathbf{1}} (c^T x + \nu^T (Ax - b)) \\ &= -b^T \nu - \|A^T \nu + c\|_1 \end{aligned}$$

dual problem: maximize $-b^T \nu - \|A^T \nu + c\|_1$

Problems with generalized inequalities

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

$\preceq_{K_i}$ is generalized inequality on $\mathbb{R}^{k_i}$

definitions are parallel to scalar case:

- Lagrange multiplier for $f_i(x) \preceq_{K_i} 0$ is vector $\lambda_i \in \mathbb{R}^{k_i}$
- Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m} \times \mathbb{R}^p \to \mathbb{R}$, is defined as

$$L(x, \lambda_1, \cdots, \lambda_m, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- dual function $g : \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m} \times \mathbb{R}^p \to \mathbb{R}$, is defined as

$$g(\lambda_1, \ldots, \lambda_m, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda_1, \cdots, \lambda_m, \nu)$$

lower bound property: if $\lambda_i \succeq_{K_i^*} 0$, then $g(\lambda_1, \ldots, \lambda_m, \nu) \leq p^\star$

proof: if $\tilde{x}$ is feasible and $\lambda \succeq_{K_i^*} 0$, then

$$\begin{aligned} f_0(\tilde{x}) &\geq f_0(\tilde{x}) + \sum_{i=1}^m \lambda_i^T f_i(\tilde{x}) + \sum_{i=1}^p \nu_i h_i(\tilde{x}) \\ &\geq \inf_{x \in \mathcal{D}} L(x, \lambda_1, \ldots, \lambda_m, \nu) \\ &= g(\lambda_1, \ldots, \lambda_m, \nu) \end{aligned}$$

minimizing over all feasible $\tilde{x}$ gives $p^\star \geq g(\lambda_1, \ldots, \lambda_m, \nu)$

dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda_1, \ldots, \lambda_m, \nu) \\ \text{subject to} & \lambda_i \succeq_{K_i^*} 0, \quad i = 1, \ldots, m \end{array}$$

- weak duality: $p^\star \geq d^\star$ always
- strong duality: $p^\star = d^\star$ for convex problem with constraint qualification (for example, Slater's: primal problem is strictly feasible)

Semidefinite program

primal SDP $(F_i, G \in \mathbf{S}^k)$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n \preceq G \end{array}$$

- Lagrange multiplier is matrix $Z \in \mathbf{S}^k$
- Lagrangian $L(x, Z) = c^T x + \mathbf{Tr}\,(Z(x_1 F_1 + \cdots + x_n F_n - G))$
- dual function

$$g(Z) = \inf_x L(x, Z) = \begin{cases} -\mathbf{Tr}(GZ) & \mathbf{Tr}(F_i Z) + c_i = 0, \quad i = 1, \ldots, n \\ -\infty & \text{otherwise} \end{cases}$$

dual SDP

$$\begin{array}{ll} \text{maximize} & -\mathbf{Tr}(GZ) \\ \text{subject to} & Z \succeq 0, \quad \mathbf{Tr}(F_i Z) + c_i = 0, \quad i = 1, \ldots, n \end{array}$$

$p^\star = d^\star$ if primal SDP is strictly feasible ($\exists x$ with $x_1 F_1 + \cdots + x_n F_n \prec G$)

Proof

Convex problem & constraint qualification

$$\Downarrow$$

Strong duality

Slater's constraint qualification

Convex problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

The problem satisfies Slater's condition if it is **strictly feasible**, $i.e.$,

$$\exists x \in \mathbf{int}\, \mathcal{D} : \qquad f_i(x) < 0, \quad i = 1, \ldots, m, \qquad Ax = b$$

- also guarantees that the dual optimum is attained (if $p^\star > -\infty$)
- there exist many other types of constraint qualifications

KKT conditions for convex problem

If $\tilde{x}$, $\tilde{\lambda}$, $\tilde{\nu}$ satisfy KKT for a **convex problem**, then they are optimal:

- from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$ with $(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$ feasible.

If **Slater's condition** is satisfied, x is optimal if and only if there exist λ, ν that satisfy KKT conditions

- Slater implies strong duality (more on this now), and dual optimum is attained
- generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

Summary

- For a convex problem satisfying constraint qualification, the KKT conditions are **necessary & sufficient** conditions for optimality.

Proof

To simplify the analysis. We make two additional technical assumptions:

- The domain $\mathcal{D}$ has nonempty interior (hence, $\mathbf{relint}\,\mathcal{D} = \mathbf{int}\,\mathcal{D}$)
- We also assume that A has full rank, i.e. $\mathbf{Rank}\,A = p$.

Proof

- We define the set $\mathcal{A}$ as

$$\mathcal{A} = \{(u, v, t) \mid \exists x \in \mathcal{D},\ f_i(x) \leq u_i,\ i = 1, \ldots, m,\\ h_i(x) = v_i,\ i = 1, \ldots, p,\ f_0(x) \leq t\},$$

  which is the set of values taken by the constraint and objective functions.

- If the problem is convex, $\mathcal{A}$ is defined by a list of convex constraints hence is **convex**.

- We define a second convex set $\mathcal{B}$ as

$$\mathcal{B} = \{(0, 0, s) \in \mathbf{R}^m \times \mathbf{R}^p \times \mathbf{R} \mid s < p^\star\}.$$

- The sets $\mathcal{A}$ and $\mathcal{B}$ do not intersect (otherwise $p^\star$ could not be optimal value of the problem).

First step: The hyperplane separating $\mathcal{A}$ and $\mathcal{B}$ defines a supporting hyperplane to $\mathcal{A}$ at $(0, p^\star)$.

Geometric proof

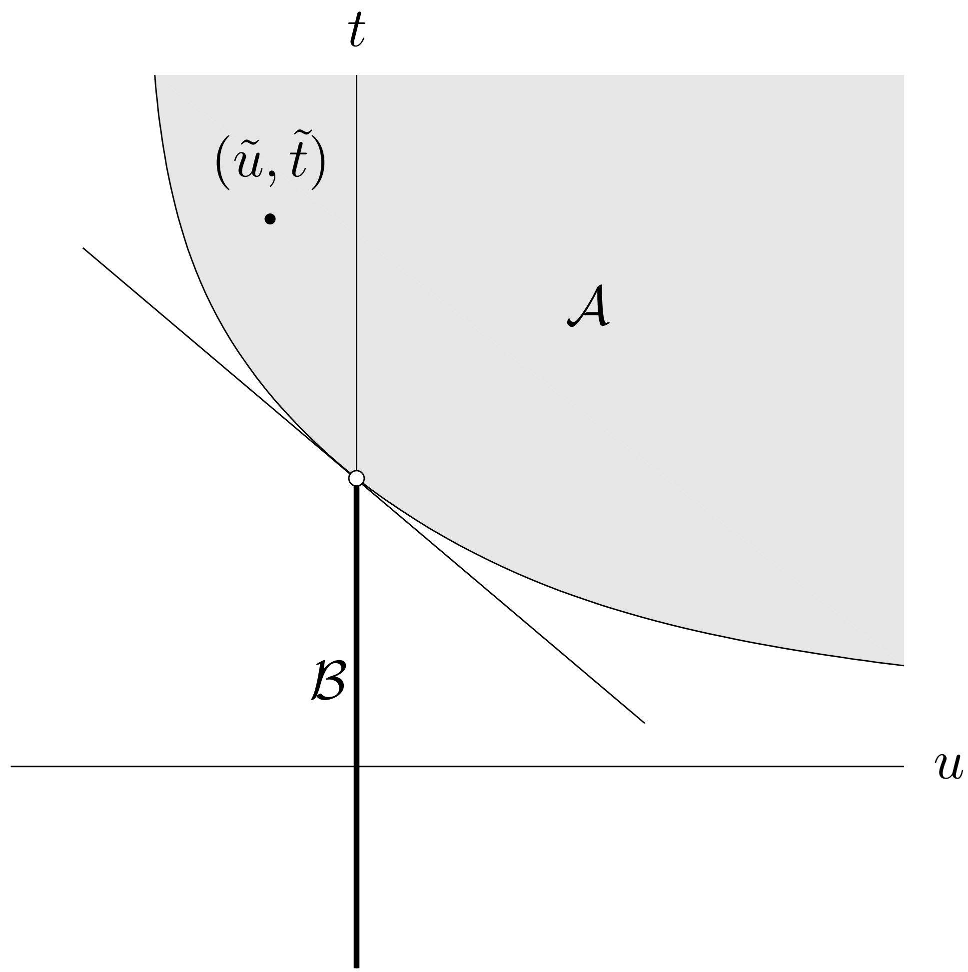


Illustration of strong duality proof, for a convex problem that satisfies Slater's constraint qualification. The two sets $\mathcal{A}$ and $\mathcal{B}$ are convex and do not intersect, so they can be separated by a hyperplane. Slater's constraint qualification guarantees that any separating hyperplane must be nonvertical.

Proof

- By the separating hyperplane theorem there exists $(\tilde{\lambda}, \tilde{\nu}, \mu) \neq 0$ and α such that
$$(u, v, t) \in \mathcal{A} \implies \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \geq \alpha, \tag{2}$$
and
$$(u, v, t) \in \mathcal{B} \implies \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \leq \alpha. \tag{3}$$
- From (2) we conclude that $\tilde{\lambda} \succeq 0$ and $\mu \geq 0$. (Otherwise $\tilde{\lambda}^T u + \mu t$ is unbounded below over $\mathcal{A}$, contradicting (2).)
- The condition (3) simply means that $\mu t \leq \alpha$ for all $t < p^\star$, and hence, $\mu p^\star \leq \alpha$.

Together with (2) we conclude that for any $x \in \mathcal{D}$,

$$\mu p^\star \leq \alpha \leq \mu f_0(x) + \sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T (Ax - b) \tag{4}$$

Proof

Let us assume that $\mu > 0$ (separating hyperplane is nonvertical)

- We can divide the previous equation by μ to get

$$L(x, \tilde{\lambda}/\mu, \tilde{\nu}/\mu) \geq p^\star$$

  for all $x \in \mathcal{D}$

- Minimizing this inequality over x produces $p^\star \leq g(\lambda, \nu)$, where

$$\lambda = \tilde{\lambda}/\mu, \qquad \nu = \tilde{\nu}/\mu.$$

- By weak duality we have $g(\lambda, \nu) \leq p^\star$, so in fact $g(\lambda, \nu) = p^\star$.

This shows that strong duality holds, and that the dual optimum is attained, whenever $\mu > 0$. The normal vector has the form $(\lambda^\star, 1)$ and produces the Lagrange multipliers.

Proof

Second step: Slater's constraint qualification is used to establish that the hyperplane must be **nonvertical**, i.e. $\mu > 0$.

By contradiction, assume that $\mu = 0$. From (4), we conclude that for all $x \in \mathcal{D}$,

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T (Ax - b) \geq 0. \tag{5}$$

- Applying this to the point $\tilde{x}$ that satisfies the Slater condition, we have

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(\tilde{x}) \geq 0.$$

- Since $f_i(\tilde{x}) < 0$ and $\tilde{\lambda}_i \geq 0$, we conclude that $\tilde{\lambda} = 0$.

Proof

This is where we use the two technical assumptions.

- Then (5) implies that for all $x \in \mathcal{D}$, $\tilde{\nu}^T(Ax - b) \geq 0$.
- But $\tilde{x}$ satisfies $\tilde{\nu}^T(A\tilde{x} - b) = 0$, and since $\tilde{x} \in \mathbf{int}\,\mathcal{D}$, there are points in $\mathcal{D}$ with $\tilde{\nu}^T(Ax - b) < 0$ unless $A^T\tilde{\nu} = 0$.
- This contradicts our assumption that $\mathbf{Rank}\,A = p$.

This means that we cannot have $\mu = 0$ and ends the proof.

*

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