RENEGAR’S CONDITION NUMBER AND COMPRESSED SENSING PERFORMANCE

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ABSTRACT. Renegar’s condition number is a data-driven computational complexity measure for convex programs, generalizing classical condition numbers in linear systems. We provide evidence that for a broad class of compressed sensing problems, the worst case value of this algorithmic complexity measure taken over all signals matches the restricted eigenvalue of the observation matrix, which controls compressed sensing performance. This means that, in these problems, a single parameter directly controls computational complexity and recovery performance.

1. INTRODUCTION

Several recent results have highlighted a clear tradeoff between computational complexity on one side, and statistical performance on the other (i.e., the number of samples required to recover the signal). We focus on recovery problems written

\[ \text{minimize} \quad \|x\| \]
\[ \text{subject to} \quad Ax = y \]  

in the variable \( x \in \mathbb{R}^p \), where \( A \in \mathbb{R}^{n \times p} \) is a sensing matrix and \( y \in \mathbb{R}^n \) is the vector of observations. Here, \( \| \cdot \| \) is a sparsity inducing norm (e.g., \( \ell_1 \)) whose properties will be specified below. Donoho and Tanner [Donoho and Tanner, 2005] and Candès and Tao [Candès and Tao, 2006] have shown that, for certain matrices \( A \), when the observations \( y \) are generated by a sparse signal, i.e., when \( y = Ax_0 \) and \( \text{Card}(x_0) = k \) so the signal is sparse, \( O(k \log p) \) observations suffice for stable recovery of \( x_0 \) by solving problem (1) with the \( \ell_1 \) norm, in which case (1) is a linear program. These results have been generalized to many other recovery problems with various assumptions on the signal structure (e.g., where \( x \) is a block-sparse vector, a low-rank matrix, etc.) and a library of corresponding convex relaxations has been developed to recover these more complex structures.

In sparse recovery problems, statistical performance is usually measured in terms of the number of samples required to guarantee stable recovery, while computational performance is usually measured in terms of classical bounds on the computational cost of the corresponding convex optimization problems or M-estimators. Early on, it was noticed, for example in [Donoho and Tsaig, 2008], that recovery problems which are easier to solve from a statistical point of view (i.e., where more samples are available), are also easier to solve numerically. The results in [Donoho and Tsaig, 2008] focused on homotopy methods and were essentially empirical. More recently, the authors of [Chandrasekaran and Jordan, 2013; Amelunxen et al., 2014] studied computational and statistical tradeoffs for increasingly tight convex relaxations of shrinkage estimators. They show that recovery performance is directly linked to the Gaussian squared-complexity of the tangent cone with respect to the constraint set and study the complexity of several convex relaxations. Their setting is slightly different from the one discussed here. In [Chandrasekaran and Jordan, 2013; Amelunxen et al., 2014] the structure of the convex relaxation is varying and affecting both complexity and recovery performance, while in [Donoho and Tsaig, 2008] and in what follows, the structure of the relaxation is fixed, but the data (i.e., the observation matrix \( A \)) varies.

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Here, we show that the cone restricted eigenvalues introduced in [Bickel et al., 2009] correspond to the worst-case value of Renegar’s condition number for problem (1) taken over a class of signals $x_0$. This means that a single quantity drives both the complexity of solving problem (1) and its recovery performance, i.e., the number of samples required for exact recovery, and the solution’s robustness when the observations $y$ are noisy. This same condition number also controls the impact of misspecification in $A$ on the optimal solution to (1). From a compressed sensing perspective, this confirms that obtaining more samples also makes the reconstructed solution more robust to experimental uncertainty in $A$.

2. COMPUTATIONAL COMPLEXITY

We begin by addressing computational complexity aspects of problem (1). Computational complexity for convex optimization problems is often described in terms of polynomial functions of the problem size. This produces a clear link between problem structure and computational complexity but fails to account for the nature of the data. If we use linear systems as a basic example, unstructured linear systems of dimension $n$ can be solved with complexity $O(n^3)$ regardless of the matrix values, but iterative solvers will converge much faster on systems that are better conditioned. The seminal work of [Renegar, 1995a, 2001] extends this notion of conditioning to optimization problems, producing data-driven bounds on the complexity of solving conic programs, and showing that the number of outer iterations of interior point algorithms increases as the distance to ill-posedness decreases.

In what follows, we study the complexity of the oracle certifying optimality of a candidate solution $x^*$ to (1) as a proxy for the problem of computing an optimal solution to this problem. As we will see below, certifying optimality means solving a pair of alternative conic linear systems of the form

$$Ax = 0, \quad x \in C$$

and

$$-A^Ty \in C^*$$

for a given cone $C \subset \mathbb{R}^p$. Several references have connected Renegar’s condition number $C(A)$ (which will be defined more precisely below) and the complexity of solving conic linear systems using various algorithms [Renegar, 1995a; Freund and Vera, 1999b; Epelman and Freund, 2000; Renegar, 2001; Vera et al., 2007; Belloni et al., 2009]. In particular, Vera et al. [Vera et al., 2007] link $C(A)$ to the complexity of solving the primal dual pair (2)–(3) using a barrier method. They show that the number of outer barrier method iterations grows as

$$O(\sqrt{\nu C} \log (\nu C C(A)))$$

where $\nu C$ is the barrier parameter, while the conditioning (hence the complexity) of the linear systems arising at each interior point iteration is controlled by $C(A)^2$. This link was also tested empirically on linear programs using the NETLIB library of problems in [Ordóñez and Freund, 2003], where computing times and number of iterations were regressed against estimates of the condition number computed using the approximations for $C(A)$ detailed in [Freund and Vera, 2003].

Studying the complexity of computing an optimality certificate in (2) gives insights on the performance of oracle based optimization techniques such as the ellipsoid method. Of course, these abstract methods are very different from those used to solve problem (1) in practice. However, we will observe in the numerical experiments of Section 5 that the condition number is strongly correlated with the empirical performance of efficient recovery algorithms such as LARS [Efron et al., 2004] and Homotopy [Donoho and Tsaig, 2008; Asif and Romberg, 2014].

We now briefly recall optimality conditions for problem (1) and two equivalent constructions for the condition number of a conic linear system. Define the tangent cone at point $x$ with respect to the norm $\|\cdot\|$, that is, the set of descent directions for the norm $\|\cdot\|$ at $x$, as

$$\mathcal{T}(x) = \text{cone}\{z : \|x + z\| \leq \|x\|\}.$$  \hspace{1cm} (4)

The simple lemma below characterizes unique optimal solutions to problem (1) in terms of $\mathcal{T}(x)$. 

Lemma 2.1. The point $x^*$ is the unique minimizer of (1) if and only if $\text{Null}(A) \cap \mathcal{T}(x^*) = \{0\}$.

Proof. This follows from standard KKT conditions (see for example [Chandrasekaran et al., 2012, Prop 2.1]).

In other words, $x^*$ is the unique optimizer if and only if the following feasibility problem is infeasible

$$\begin{align*}
\text{find} & \quad z \\
\text{subject to} & \quad Az = 0 \\
& \quad z \in \mathcal{T}(x^*), \; z \neq 0,
\end{align*}$$

in the variable $z \in \mathbb{R}^p$. To certify feasibility, it is sufficient to exhibit a solution. One way of certifying infeasibility of (P) is to solve the dual problem

$$\begin{align*}
\text{find} & \quad u \\
\text{subject to} & \quad -A^T u \in \mathcal{T}(x^*)^\circ, \; u \neq 0,
\end{align*}$$

in the variable $u \in \mathbb{R}^n$, where $\mathcal{T}(x^*)^\circ$ is the polar cone of $\mathcal{T}(x^*)$. Renegar’s condition number [Renegar, 1995a,b; Peña, 2000] provides a data-driven measure of the complexity of this task. It is rooted in the sensible idea that certifying infeasibility is easiest if the problem is far from being feasible. Formally, the distance to feasibility $\rho_{x^*}(A)$ is defined as follows. Let $\mathcal{M}_{x^*} = \{A \in \mathbb{R}^{n \times p} : (P) \text{ is infeasible}\}$. Then, using the spectral norm as matrix norm,

$$\rho_{x^*}(A) \triangleq \inf_{\Delta A} \{\|\Delta A\|_2 : A + \Delta A / \in \mathcal{M}_{x^*}\}. \quad (5)$$

Renegar’s condition number for problem (P) with respect to $x^*$ is then defined as the scale-invariant reciprocal of this distance

$$C_{x^*}(A) \triangleq \frac{\|A\|_2}{\rho_{x^*}(A)}. \quad (6)$$

Interestingly, the distance to feasibility is also given by the following formula, reminiscent of a conically restricted minimal singular value of $A$

$$\mu_{x^*}(A) = \inf_{z \in \mathcal{T}(x^*)} \frac{\|Az\|_2}{\|z\|_2}. \quad (7)$$

We have the following result.

Lemma 2.2. Distance to feasibility and cone restricted eigenvalues match, i.e. $\rho_{x^*}(A) = \mu_{x^*}(A)$.

Proof. When (P) is feasible, both vanish. Otherwise, see [Freund and Vera, 1999a, Th. 2], or simplified versions in [Belloni and Freund, 2009, Lem. 3.2] and [Amelunxen and Lotz, 2014].

Notice that, if $\mathcal{T}(x^*)$ were the whole space $\mathbb{R}^p$, and if $A^T A$ were full-rank (never the case if $n < p$), then $\mu(A)$ would be the smallest singular value of $A$. As a result, $C(A)$ would reduce to the classical condition number of $A$ (and to $\infty$ when $A^T A$ is rank-deficient). Renegar’s condition number is necessarily smaller (better) than the latter, as it further incorporates the notion that $A$ need only be well-conditioned along those directions that matter with respect to the norm $\|\cdot\|$ at $x^*$. Later, we will remove the dependence on $x^*$ by considering a worst-case condition number over classes of “simple” signals.

Naturally, the condition number also controls the sensitivity of the solution to changes in the matrix $A$, with [Renegar, 1994, 1995b] bounding changes in the solution to (2) in terms of $C(A)$ and changes $\Delta A$ in the system matrix. This means that $C(A)$ also measures the robustness of the solution to the recovery with respect to misspecification of the observation matrix $A$, a point rarely addressed by classical recovery results.
3. Statistical Performance

We now focus on the link between condition number and the statistical performance of the solution of problem (1). To this end, assume now that the observations $y$ are affected by noise and that we solve a robust version of problem (1), written

$$\begin{align*}
\text{minimize} & \quad \|x\| \\
\text{subject to} & \quad \|Ax - y\|_2 \leq \delta\|A\|_2,
\end{align*}$$

in the variable $x \in \mathbb{R}^p$, with the same design matrix $A \in \mathbb{R}^{n \times p}$, observations $y \in \mathbb{R}^n$ and noise level $\delta > 0$.

3.1. Recovery bounds using $C(A)$. The following classical result bounds the reconstruction error in terms of $C(A)$.

Lemma 3.1. Suppose we observe $y = Ax_0 + w$ where $\|w\|_2 \leq \delta\|A\|_2$ and let $x^*$ be an optimal solution of problem (8). We get the following error bound:

$$\|x^* - x_0\|_2 \leq 2\frac{\delta\|A\|_2}{\mu_{x_0}(A)} = 2\delta \cdot C_{x_0}(A).$$

Proof. We recall the short proof of [Chandrasekaran et al., 2012, Prop. 2.2]. Both $x^*$ and $x_0$ are feasible for (8) and $x^*$ is optimal, so that $\|x^*\| \leq \|x_0\|$. Thus, the error vector $x^* - x_0$ is in the tangent cone $T(x_0)$ (4). By the triangle inequality,

$$\|A(x^* - x_0)\|_2 \leq \|Ax^* - y\|_2 + \|Ax_0 - y\|_2 \leq 2\delta\|A\|_2.$$

Furthermore, by definition of $\mu_{x_0}$ (7),

$$\|A(x^* - x_0)\|_2 \geq \mu_{x_0}(A) \|x^* - x_0\|_2.$$

Combining the two concludes the proof. $\blacksquare$

Notice that, in the above lemma, the condition number is evaluated at $x_0$ (the true signal) rather than at $x^*$ (the estimator). This will be convenient when considering worst cases over classes of target signals.

This means that Renegar’s condition number defined in (6) also controls the statistical performance of estimators built on solving the approximate recovery problem (8). This at least partially explains the common empirical observation (see, e.g., [Donoho and Tsaig, 2008]) that problem instances where statistical estimators built on solving the approximate recovery problem (8) and $\delta$ small. In our experiments, we did observe sometimes significant differences in behavior between the noisy and noiseless case.

3.2. Generalized restricted eigenvalues. We now further specify the sparsity inducing norms in order to study Renegar’s condition number on a class of signals that share the same sparsity properties. We use the framework of sparsity structure introduced by Juditsky et al. [Juditsky et al., 2014] that allows a common treatment of popular norms such as the $\ell_1$ norm, group-$\ell_1$ norms and the nuclear norm.

Sparsity systems suggest defining simplicity of signals through projectors. We begin by briefly recalling the setting in [Juditsky et al., 2014]. Consider $\mathcal{X}$ and $\mathcal{E}$, two Euclidean spaces, and a map $B: \mathcal{X} \to \mathcal{E}$. In most cases, notably including the $\ell_1$ and nuclear norm cases, $\mathcal{X} = \mathcal{E}$ and we may think of $B$ as the identity map, but it is useful to consider this more general case to model group norms as well. In this setting, the problem under consideration is that of finding a sparse representation $Bx$ of a signal, given noisy observations $y = Ax$.

A sparsity structure on $\mathcal{E}$ is defined as a norm $\|\cdot\|$ on $\mathcal{E}$, together with a family $\mathcal{P}$ of linear maps of $\mathcal{E}$ into itself satisfying three assumptions:
Then, given a sparsity structure \( \nu(P) \geq 0 \) and a linear map \( \bar{P} \) on \( E \) such that \( P \bar{P} = 0 \), one has
\[
\| P^* f + P^* g \|_* \leq \max(\| f \|_*, \| g \|_*),
\]
where \( \| \cdot \|_* \) is the dual norm of \( \| \cdot \| \) and \( P^* \) is the conjugate mapping of the linear map \( P \). Arguably, the last condition is the least intuitive. Note that the polar of assumption (iii) implies, for \( u \in E \),
\[
\| u \| \geq \| P u \| + \| \bar{P} u \|.
\]
Hence, \( \| u \| = \| P u \| + \| \bar{P} u \| \) when \( \bar{P} = I - P \), in which case these systems match the decomposable norms setting in [Negahban et al., 2009]. This includes the \( \ell_1 \) norm and group sparsity, but not the nuclear norm.

For \( k \geq 0 \), let \( \mathcal{P}_k = \{ P \in \mathcal{P} : \nu(P) \leq k \} \). The notion of simplicity is defined as follows: a vector \( w \) is said to be \( k \)-sparse if there exists \( P \in \mathcal{P}_k \) such that \( P w = w \). A signal \( x \) is said to be \( k \)-sparse if its representation \( Bx \) is \( k \)-sparse.

**\( \ell_1 \) norm.** In the the \( \ell_1 \) norm case, the sparsity structure is defined over \( E = \mathcal{X} = \mathbb{R}^p \), the map \( B \) reduces to the identity, and \( \mathcal{P} \) is the set of projectors on coordinate subspaces of \( \mathbb{R}^p \), that is, \( \mathcal{P} \) contains all projectors which zero out all coordinates of a vector except for a subset of them, which are left unaffected. The companion maps are the complementary projectors: \( \bar{P} = I - P \). Naturally, the complexity level corresponds to the number of coordinates preserved by \( P \), i.e., \( \nu(P) = \text{Rank}(P) \). These definitions recover the usual notion of sparsity.

**Nuclear norm.** The nuclear norm is defined for matrices \( X \in \mathbb{R}^{p \times q} \) with singular values \( \sigma_i(X) \) as \( \| X \| = \sum_{k=1}^{\min(p,q)} \sigma_k(X) \). It can be cast as a sparsity system by setting \( \mathcal{X} = \mathcal{E} = \mathbb{R}^{p \times q}, B = I \). Its associated family of linear maps is
\[
P : X \mapsto P_{\text{left}}XP_{\text{right}}, \quad \text{and} \quad \bar{P} : X \mapsto (I - P_{\text{left}})X(I - P_{\text{right}}),
\]
where \( P_{\text{left}} \in \mathbb{R}^{p \times q} \) and \( P_{\text{right}} \in \mathbb{R}^{p \times q} \) are orthogonal projectors. Their weights are defined as \( \nu(P) = \max(\text{Rank}(P_{\text{left}}), \text{Rank}(P_{\text{right}})) \) defining therefore \( k \)-sparse matrices as matrices of rank at most \( k \).

Given a sparsity structure and \( k \geq 0 \), Lemma 2.2 shows that the worst-case distance to infeasibility on the class of \( k \)-sparse signals can be written as
\[
\mu_k(A) \triangleq \inf_{P \in \mathcal{P}, \nu(P) = k} \inf_{x \in \mathcal{X}, PBx = Bx} \frac{\| Az \|_2}{\| z \|_2},
\]
(the first infimum covers all signals \( x \) of sparsity \( k \)), where the tangent cone is defined here as
\[
\mathcal{T}(x) = \text{cone}\{ z \in \mathcal{X} : \| Bx + Bz \| \leq \| Bx \| \}.
\]

In the following lemma, we show that this worst-case distance to infeasibility \( \mu_k \) is directly related to generalized restricted eigenvalues.

**Lemma 3.2.** Given a sparsity structure \( (\| \cdot \|, \mathcal{P}) \), for \( P \in \mathcal{P} \), let
\[
\mathcal{C}_P = \bigcup_{\{ x \in \mathcal{X} : PBx = Bx \}} \mathcal{T}(x), \quad \text{and} \quad \mathcal{D}_P = \{ z \in \mathcal{X} : \| \bar{P} Bz \| \leq \| PBz \| \}.
\]
Then, \( \mathcal{C}_P \subseteq \mathcal{D}_P \). Hence, for any \( k \geq 0 \),
\[
\mu_k(A) \geq \sigma_k(A) = \inf_{P \in \mathcal{P}_k} \frac{\| Az \|_2}{\| z \|_2}.
\]
Proof. Let $P \in \mathcal{P}$ and $z \in \mathcal{T}(x)$ for $x \in \mathcal{X}$ such that $PBx = Bx$. Using [Juditsky et al., 2014, Lem. 3.1], we have
\[
\|Bx\| + \|PBz\| - \|PBz\| \leq \|Bx + Bz\| \leq \|Bx\|.
\]
So $\|PBz\| \leq \|PBz\|$ and $z \in \mathcal{D}_P$. ■

The inverse of the generalized restricted eigenvalue therefore also bounds the worst case computational complexity through the condition number. We remark that, in general, one does not have $\mu_k(A) = \sigma_k(A)$. A simple counterexample can be derived for the nuclear norm. Indeed, let $\mathcal{E} = \mathcal{X} = \mathbb{R}^{2 \times 2}$, $B$ be the identity and $(\| \cdot \|, \mathcal{P})$ be the nuclear norm and its associated family of linear maps. Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}$ with $u \neq 0$. Setting $P : \mathcal{X} \rightarrow QXQ$, so that $P \in \mathcal{P}$, we have $\|P\| = \|P\| = 0$, hence $U \in \mathcal{D}_P$. Now let
\[
\mathcal{X}_P = \{X \in \mathcal{X} : PX = X\} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\}.
\]
For any $X \in \mathcal{X}_P$, $\|X+U\| = \sqrt{x^2 + 4u^2} > |x| = \|X\|$, hence $U \notin \cup_{X \in \mathcal{X}_P} \mathcal{T}(X)$, showing that $\mathcal{D}_P \notin \mathcal{C}_P$. As shown below however, with the additional assumption that the norm is strictly decomposable, that is, $\bar{P} = I - P$ and that $B$ is bijective (non-overlapping groups) the bound in Lemma 3.2 is tight and $\mu_k(A) = \sigma_k(A)$.

Lemma 3.3. Given a sparsity structure $(\| \cdot \|, \mathcal{P})$, assume that for any $P \in \mathcal{P}$, $\bar{P} = I - P$. Then $\mathcal{C}_P = \mathcal{D}_P$ and, for any $k \geq 0$, we have
\[
\mu_k(A) = \sigma_k(A).
\]

Proof. Let $P \in \mathcal{P}$ and $z \in \mathcal{X}$, $\|\bar{PBz}\| \leq \|PBz\|$. Let $w = -PBz$. We have $Pw = w$ and
\[
\|w + Bz\| = \|(I - P)Bz\| = \|PBz\| \leq \|PBz\| = \|w\|.
\]
Thus, $z \in \mathcal{T}(x)$ for $x$ such that $w = Bx$, and $PBx = Bx$ implies $z \in \mathcal{C}_P$. ■

In the $\ell_1$ case, our definition for $\mu_k(A) = \sigma_k(A)$ matches the definition of restricted eigenvalue in [Bickel et al., 2009], with
\[
\sigma_k(A) = \inf_{S \subset \{1, \ldots, p\}, \text{Card}(S) = k} \frac{\|Az\|_2}{\|z\|_2}.
\]
This, we believe, makes for an interesting link between the statistical notion of restricted eigenvalue, and the computational notion of Renegar condition number.

4. Computing $\mathcal{C}_{x_0}(A)$

The condition number $\mathcal{C}_{x_0}(A)$ appears here in upper bounds on computational complexities and statistical performances. In order to test numerically whether this quantity truly explains those features (as opposed to merely appearing in a wildly pessimistic bound), we explicitly compute it in numerical experiments.

We focus on the $\ell_1$ norm. To compute $\mathcal{C}_{x_0}(A)$, we propose a heuristic which computes $\mu_{x_0}(A)$ in (7), the value of a nonconvex minimization problem over the cone of descent directions $\mathcal{T}(x_0)$. The closure of the latter is the polar of the cone generated by the subdifferential to the $\ell_1$-norm ball at $x_0$ [Chandrasekaran et al., 2012, §2.3]. Let $S \subset \{1, \ldots, p\}$ denote the support of $x_0$, $\bar{S}$ denote its complement, and $|\bar{S}|$ denote the cardinality of $\bar{S}$. Then, with $s = \text{sign}(x_0)$,
\[
\mathcal{T}(x_0) = \text{cone}\left\{ z \in \mathbb{R}^p : z_S = s_S, z_{\bar{S}} \in [-1, 1]^{|\bar{S}|} \right\} = \{ z \in \mathbb{R}^p : \|z_S\|_1 \leq -s_{\bar{S}}^T z_S = -s^T z \}. 
\]
Thus, \( \mu_{x_0}(A) \) is the square root of

\[
\min_{z \in \mathbb{R}^p} z^T A^T A z \quad \text{s.t.} \quad \|z\|_2 = 1 \quad \text{and} \quad \|z_S\|_1 \leq -s^T z. \tag{11}
\]

Let \( \lambda \) denote the largest eigenvalue of \( A^T A \). If it were not for the cone constraint, solutions of this problem would be the dominant eigenvectors of \( \lambda I - A^T A \), which suggests a projected power method [Deshpande et al., 2014] as follows. Given an initial guess \( z(0) \in \mathbb{R}^p, \|z(0)\|_2 = 1 \), iterate

\[
\hat{z}_{(k+1)} = \text{Proj}_{x_0} \left( (\lambda I - A^T A)z(k) \right), \quad z_{(k+1)} = \hat{z}_{(k+1)}/\|\hat{z}_{(k+1)}\|_2; \tag{12}
\]

where we used the orthogonal projector to \( T(x_0) \),

\[
\text{Proj}_{x_0}(\hat{z}) = \arg\min_{z \in \mathbb{R}^p} \|z - \hat{z}\|_2^2 \quad \text{s.t.} \quad \|z_S\|_1 \leq -s^T z. \tag{13}
\]

This convex, linearly constrained quadratic program is easily solved with CVX [Grant et al., 2001]. As can be seen from KKT conditions, this iteration is a generalized power iteration Journé et al. [2010] 100 times for each value of \( \|\delta\|_2 \). Furthermore, for a fixed value of \( \|\delta\|_2 \), and sparsity \( k \), we modify (11) by smoothly penalizing the inequality constraint in the cost function, which results in a smooth optimization problem on the \( \ell_2 \) sphere. Specifically, for small \( \varepsilon_1, \varepsilon_2 > 0 \), we use smooth proxies

\[
h(x) = \sqrt{x^2 + \varepsilon_1^2} - \varepsilon_1 \approx |x| \quad \text{and} \quad q(x) = \varepsilon_2 \log(1 + \exp(x/\varepsilon_2)) \approx \max(0, x). \]

Then, with \( \gamma > 0 \) as Lagrange multiplier, we consider

\[
\min_{\|z\|_2 = 1} \|Az\|_2^2 + \gamma \cdot q \left( s^T z + \sum_{i \in S} h(z_i) \right). \]

We solve the latter locally with Manopt [Boumal et al., 2014], itself with a uniformly random initial guess on the sphere, to obtain \( z(0) \). Then, we iterate the projected power method. The value \( \|Az\|_2 \) is an upper bound on \( \mu_{x_0}(A) \), so that we obtain a lower bound on \( C_{x_0}(A) \). Empirically, this procedure, which is random only through the initial guess on the sphere, consistently returns the same value, up to five digits of accuracy, which suggests the proposed heuristic computes a good approximation of the condition number. Similarly positive results have been reported on other cones in [Deshpande et al., 2014], where the special structure of the cone even made it possible to certify that this procedure indeed attains a global optimum in proposed experiments. This gives us confidence in the estimate produced here.

5. NUMERICAL EXPERIMENTS

We conduct numerical experiments in the \( \ell_1 \) case to illustrate the connection between the condition number \( C_{x_0}(A) \), the computational complexity of solving (1), and the statistical efficiency of the estimator (8). Importantly, throughout the experiments, the classical condition number of \( A \) will remain essentially constant, so that the main variations cannot be attributed to the latter.

We follow a standard setup, similar to some of the experiments in [Donoho and Tsaig, 2008]. Fixing the ambient dimension \( p = 300 \) and sparsity \( k = \text{Card}(x_0) = 15 \), we let the number of linear measurements \( n \) vary from 1 to 150. For each value of \( n \), we generate a random signal \( x_0 \in \mathbb{R}^p \) (uniformly random support, i.i.d. Gaussian entries, unit \( \ell_2 \)-norm) and a random sensing matrix \( A \in \mathbb{R}^{n \times p} \) with i.i.d. standard Gaussian entries. Furthermore, for a fixed value \( \delta = 10^{-2} \), we generate a random noise vector \( w \in \mathbb{R}^n \) with i.i.d. standard Gaussian entries, normalized such that \( \|w\|_2 = \delta \|A\|_2 \), and we let \( y = Ax_0 + w \). This is repeated 100 times for each value of \( n \).
For each triplet \((A,x_0,y)\), we first solve the noisy problem (8) with the L1-Homotopy algorithm \((\tau = 10^{-7})\) [Asif and Romberg, 2014], and report the estimation error \(\|x^* - x_0\|_2\). Then, we solve the noiseless problem (1) with L1-Homotopy and the TFOCS routine for basis pursuit \((\mu = 1)\) Becker et al. [2011]. Exact recovery is declared when the error is less than \(10^{-5}\), and we report the empirical probability of exact recovery, together with the number of iterations required by each of the solvers. The number of iterations of LARS [Efron et al., 2004] is also reported, for comparison. For L1-Homotopy, we report the computation time, normalized by the computation time required for one least-squares solve in \(A\), as in [Donoho and Tsaig, 2008, Fig. 3], which accounts for the growth in \(n\). Finally, we compute the classical condition number of \(A\), \(\kappa(A)\), as well as (a lower bound on) the cone restricted condition number \(C_{x_0}(A)\), as per the previous section. As it is the computational bottleneck of the experiment, it is only computed for 20 of the 100 repetitions.

The results of Figure 1 show that the cone-restricted condition number explains both the computational complexity of (1) and the statistical complexity of (8): fewer samples mean bad conditioning which in turn implies high computational complexity. We caution that our estimate of \(C_{x_0}(A)\) is only a lower bound. Indeed, for small \(n\), the third plot on the left shows that, even in the absence of noise, recovery of \(x_0\) is not achieved by (8). Lemma 3.1 then requires \(C_{x_0}(A)\) to be infinite. But the computational complexity of solving (1) is visibly favorable for small \(n\), where far from the phase transition, problem (P) is far from infeasibility, which is just as easy to verify as it is to certify that (P) is infeasible when \(n\) is comfortably larger than needed. This phenomenon is best explained using a symmetric version of the condition number [Amelunxen and Lotz, 2014] (omitted here to simplify computations).

We also solved problem (1) with interior point methods via CVX. The number of iterations appeared mostly constant throughout the experiments, suggesting that the practical implementation of such solvers renders their complexity mostly data agnostic in the present setting. Likewise, the computation time required by L1-Homotopy on the noisy problem (8), normalized by the time of a least-squares solve, is mostly constant (at about 150). This hints that the link between computational complexity of (1) and (8) remains to be fully explained.

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Figure 1. We plot the cone-restricted condition number of $A$ (upper left), explaining both the computational complexity of problem (1) (right column) and the statistical complexity of problem (8) (second on the left). Central curves represent the mean (geometric mean in log-scale plots), red curves correspond to 10th and 90th percentile.

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