Risk management methods for the market model of interest rates using semidefinite programming

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Semidefinite programming

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1. Introduction

• Option prices are a function of the underlying asset prices today and the market volatility (variance).

• Derivative pricing and hedging requires daily model calibration of that variance to option prices quoted by the market.

• Multivariate option models (on interest-rate derivatives) have a co-variance matrix as their fundamental parameter.
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- Current methods heavily parameterize this covariance and use Monte-Carlo estimates of option prices to calibrate the model.
- In practice, however, the calibration problem can be approximated by an SDP with excellent precision.
- Both primal and dual problems have direct, intuitive interpretations.
- Robustness, smoothness, Bid-Ask spread constraints, can be included.

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at a certain fixed maturity $T$.

$$\text{Call}_T^+ = \max(K - S_T, 0)$$

which pay

The most heavily traded derivative products are European Call options.

Given by $dS_t = \sigma S_t \, dW_t$ where $W_t$ is a B.M., i.e. log $S_T$ is Gaussian.

In the Black & Scholes (1973) model, the stock price dynamics $S_t$ are

2.1 Option pricing in dimension one

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The central "no arbitrage" argument in Black & Scholes (1973) and Merton (1973) shows that Calls are redundant. 

There is a self-financing dynamic portfolio strategy in stock and cash.

The option is perfectly hedged by holding \( \text{Call}(S^T, K, \sigma^2 T) \) in stock \( S_t \) and the rest in cash.

\[ \left[ + (K - S_T) \right] \mathbb{Q} = \text{Call}(S_0, K, \sigma^2 T) \]

The option price is given by:

\[ \text{Call}(S_0, K, \sigma^2 T) = \mathbb{Q} \left[ + (K - S_T) \right] \]

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- In fact, the market quotes option prices using their BS variance $\sigma^2$.

- Call prices and BS volatility.

- Because $S_0$ is quoted by the market today and $\text{Call}(S_0, K, \sigma^2 T)$ is strictly increasing in $\sigma^2$, there is a one-to-one relationship between $S_0$ and $\text{Call}(S_0, K, \sigma^2 T)$.

- In fact, the market quotes option prices using their BS variance $\sigma^2$.

- The expectation $E^Q\left[ (S_T - K)^+ \right]$ can be computed explicitly:

\[
BS(S_0, K, \sigma^2 T) = S_0 N\left( \frac{\ln(S_0/K) + \sigma^2 T}{\sigma \sqrt{T}} \right) - KN\left( \frac{\ln(S_0/K) - \sigma^2 T}{\sigma \sqrt{T}} \right)
\]

where $N$ is the CDF of the Gaussian density.

\[
\left( \frac{\ln(S_0/K) + \sigma^2 T}{\sigma \sqrt{T}} \right) N X - \left( \frac{\ln(S_0/K) - \sigma^2 T}{\sigma \sqrt{T}} \right) N 0 S = (\ln(S_T/K), \sigma^2 T) S 0 S
\]

- The expectation can be computed explicitly:

$E^Q\left[ (S_T - K)^+ \right] = \left[ 0 \right]^{+} (X - \mathcal{L} S)$
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2.2 Multivariate option pricing

The stocks today and by the covariance matrix

\[ u^i, \ldots, u^n \in \mathbb{R} \]

The model is entirely parametrized by \( u^i, \ldots, u^n \) for \( i = 1, \ldots, n \)

The model is entirely parametrized by \( S_t \) for \( i = 1, \ldots, n \)

\[ S_t = u^i \sigma_i \cdot \sqrt{t} \]

\[ X = \sigma_1 \sigma_2 \ldots \sigma_n \]

\[ \sigma_i \in \mathbb{R}_+ \]

\[ W_t \] is a \( n \)-dimensional Brownian motion.

\[ S_t = u^i \sigma_i \cdot \sqrt{t} \]

\[ \mathbb{M} \]

\[ \mathbb{P} \]

\[ \mathbb{S} \]

\[ \mathbb{P}_i \sigma_i S_t = u^i S_t \]

\[ W_t \]

\[ u^i \]

\[ u^i, \ldots, u^n \]

\[ X \]

\[ \sigma_i \]

\[ \mathbb{M}_i \]

\[ \mathbb{P}_i \sigma_i S_t = u^i S_t \]

\[ \mathbb{S}_i \]

\[ \mathbb{P}_i \sigma_i S_t = u^i S_t \]

\[ \mathbb{S}_i \]

\[ \mathbb{P}_i \sigma_i S_t = u^i S_t \]
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• The simplest derivative products are European Basket Call options (Swaptions) which pay:

\[ \forall \sum_{i=1}^{n} w_i S_i(T) - K \]

\[ \mathbb{E}^Q \left[ + \left( K - \sum_{i=1}^{n} w_i S_i(T) \right) \right] \]

• No closed form solution is available to compute the price.

\[ + \left( K - \sum_{i=1}^{n} w_i S_i(T) \right) \]

\[= \text{Call} \]

(\text{Swaptions}) which pay:

The simplest derivative products are European Basket Call options.
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- The one-to-one relationship between variance and price is lost.
- The calibration is performed with a heavily parametrized (often non-convex) set of covariances.
- Monte-Carlo pricing introduces additional instability.
- Because a calibration is performed every day, the "numerical noise" hedging can become very costly.
- Derivative desks stay perfectly hedged ($\partial \text{Call}(S_t, K, \sigma^2 T) / \partial S = 0$).
In practice, we can approximate the price of a basket option by:

\[ L \mathbf{m} = \mathcal{U} \quad \text{and} \quad \left( \mathbf{w} \cdots \mathbf{w} \right)^T \mathbf{1} = \mathbf{1}^{\mathbf{m}} \]

where

\[ (\mathcal{U} \mathbf{w})^T \mathbf{1} = \frac{m \cdot \rho}{n} \]

which can be rewritten:

\[ \frac{0 \cdots 0 \cdots 0 \cdot \mathbf{w}^{\mathbf{m}}}{0 \cdots 0 \cdot \mathbf{w}^{\mathbf{m}}} = \mathbf{m} \quad \text{with} \quad \left\| \mathbf{w}^{\mathbf{m}} \right\| = \frac{m \cdot \rho}{2} \]

where:

\[ \left( L \mathbf{m}, \mathcal{U}^{\mathbf{m}} \right) \mathbf{S} \]

3.1 Semidefinite Programming formulation

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Figure 1: Order zero (dashed) and order one (plain) absolute approximation error versus the multidimensional Black-Scholes basket prices obtained by simulation for various strikes.

Moneyness in Delta

Error

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The approximation of the basket price as a Black-Scholes price with variance given by $\sigma^2_w = \text{Tr}(\Omega X)$ defines the calibration set as the feasible set of an SDP.

Given market prices $\sigma^2_k$ for $k = 1, \ldots, m$ on a set of options $(\Omega_k, T_k)$, the calibration problem becomes:

\[
\begin{align*}
\text{Find } X & \text{ s.t. } \text{Tr}(\Omega_k X) = \sigma^2_k \text{ for } k = 1, \ldots, m \text{ on a set of options } (\Omega_k, T_k) \\
& \text{with feasible set of an SDP.}
\end{align*}
\]

The approximation of the basket price as a Black-Scholes price with variance given by $\sigma^2_w = \text{Tr}(\Omega X)$ defines the calibration set as the feasible set of an SDP.
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Figure 2: The semidefinite cone in dimension 3.
We can minimize the surface of the solution matrix with:

\[ \sum_{i,j} \in [2,n] \| \Delta_{i,j}X \|_2 \]

where

\[ \Delta_{i,j}X = (X_{i,j} - X_{i,j-1}, X_{i,j} - X_{i-1,j}) \]

The calibration program becomes:

\[
\min t \\
\text{subject to} \\
\sum_{i,j} \in [2,n] \| \Delta_{i,j}X \|_2 \leq t \sigma^2 \]

\[ \text{Bid},kT(k) \leq \text{Tr}(\Omega_kX) \leq \text{Ask},kT(k) \]

\[ X \succeq 0 \]

\[ A \]

**3.2 Smoothness**

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Figure 3: Solution to the calibration problem with smoothness constraints.
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We can look at the eigenvectors of this purely market implied matrix to compare them with classical PCA results.

Figure 4: First eigenvector "level", second eigenvector "spread".
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3.3 Robustness

We can make the solution (uniformly) robust to a change in market conditions by solving:

\[
\begin{align*}
\text{maximize} & \quad t \\
\text{s.t.} & \quad \sigma^2_{\text{Bid},k}T_k + t \leq (X^{\gamma L})_{\gamma \gamma} \preceq t + \sigma^2_{\text{Ask},k}T_k - t \\
& \quad \forall k = 1, \ldots, m
\end{align*}
\]
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3.4 The dual problem

Let \( \Sigma_0 \) be the matrix associated with a particular target option. The dual problem:

\[
\begin{align*}
\text{maximize} & \quad \text{Tr}(\Sigma_0 XX) \\
\text{subject to} & \quad \text{Tr}(\Sigma_k XX) = \sigma^2_k \quad \text{for} \quad k = 1, \ldots, m \\
& \quad X \succeq 0
\end{align*}
\]

will compute an upper arbitrage bound on the price of \( \Sigma_0 \). The dual, in this case:

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=1}^{m} y_k \sigma^2_k T_k \\
\text{subject to} & \quad \Sigma_0 \preceq \sum_{k=1}^{m} y_k \Sigma_k \\
& \quad y_k \geq 0 \quad \text{for} \quad k = 1, \ldots, m
\end{align*}
\]

will give the coefficients of the associated hedging portfolio:

\[
\lambda_k = -y_k \frac{\partial B_S(\Sigma_0 \text{Tr}(\Sigma_0 XX))}{\partial \sigma^2} + \frac{\partial B_S(\Sigma_k \text{Tr}(\Sigma_k XX))}{\partial \sigma^2}
\]
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Figure 5: Calibration result and price bounds.
Monte-Carlo pricing of American options is making progress.

3.5 The rank issue

- A low rank solution is desirable.
- Very good heuristics methods exist.
- A low rank solution is desirable.

American option pricing is usually done by dynamic programming and
3.6 Conclusion

• Increased flexibility and stability should significantly improve the pricing and hedging performance.

• Multivariate derivative models calibration is a very intuitive, direct application of semidefinite programming.
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References


