On the Versatility of the Nesterov Acceleration Scheme

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Optimization without Borders, Les Houches
Tribute to Y. Nesterov’s teaching and research

Book
Collaborators

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Julien Mairal

Publication

Focus of this work

Minimizing large finite sums

Consider the minimization of a large sum of convex functions

$$\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \psi(x) \right\},$$

where each $f_i$ is smooth and convex and $\psi$ is a convex but not necessarily differentiable penalty.

Goal of this work

- Design accelerated methods for minimizing large finite sums.
- Give a generic acceleration scheme which can apply to previously un-accelerated algorithms.
Why do large finite sums matter?

Empirical risk minimization

\[
\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \psi(x) \right\},
\]

- Typically, \( x \) represents **model parameters**.
- Each function \( f_i \) measures the **fidelity** of \( x \) to a data point.
- \( \psi \) is a **regularization function** to prevent overfitting.

For instance, given training data \((y_i, z_i)_{i=1, \ldots, n}\) with features \( z_i \) in \( \mathbb{R}^p \) and labels \( y_i \) in \( \{-1, +1\} \), we may want to predict \( y_i \) by \( \text{sign}(\langle z_i, x \rangle) \).

Functions \( f_i \) measures how far the prediction is from the true label.

This would be a **classification problem with a linear model**.
Why large finite sums matter?

A few examples

**Ridge regression:**

\[
\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} (y_i - \langle x, z_i \rangle)^2 + \frac{\lambda}{2} \| x \|_2^2.
\]

**Linear SVM:**

\[
\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \max(0, 1 - y_i \langle x, z_i \rangle) + \frac{\lambda}{2} \| x \|_2^2.
\]

**Logistic regression:**

\[
\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \log \left( 1 + e^{-y_i \langle x, z_i \rangle} \right) + \frac{\lambda}{2} \| x \|_2^2.
\]
Why does the composite problem matter?

A few examples

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**Linear SVM:**
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\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \max(0, 1 - y_i \langle x, z_i \rangle))^2 + \frac{\lambda}{2} \|x\|_2^2.
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\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \log \left( 1 + e^{-y_i \langle x, z_i \rangle} \right) + \frac{\lambda}{2} \|x\|_2^2.
\]

The **squared \( \ell_2 \)-norm** penalizes large entries in \( x \).
Why does the composite problem matter?

A few examples

**Ridge regression:** \[ \min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} (y_i - \langle x, z_i \rangle)^2 + \lambda \| x \|_1. \]

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When one knows in advance that \( x \) should be sparse, one should use a **sparsity-inducing** regularization such as the \( \ell_1 \)-norm.

[Chen et al., 1999, Tibshirani, 1996].
How to minimize a large sum composite problem?

Two major challenges

- **Non-differentiable regularization penalty.**
  Exclude existing solver such as MOSEK, CPLEX, etc.

- **Large-scale and high-dimensionality**
  Exclude higher-order (Newton) methods.

This leads us to first-order gradient-based methods.
Gradient descent methods

Let us consider the composite problem

$$\min_{x \in \mathbb{R}^p} f(x) + \psi(x),$$

where $f$ is convex, differentiable with $L$-Lipschitz continuous gradient and $\psi$ is convex, but not necessarily differentiable.

The classical forward-backward/ISTA algorithm

$$x_k \leftarrow \arg \min_{x \in \mathbb{R}^p} \frac{1}{2} \left\| x - \left( x_{k-1} - \frac{1}{L} \nabla f(x_{k-1}) \right) \right\|_2^2 + \frac{1}{L} \psi(x).$$

- $f(x_k) - f^* = O(1/k)$ for convex problems;
- $f(x_k) - f^* = O((1 - \mu/L)^k)$ for $\mu$-strongly convex problems;

Accelerated gradient descent methods

Nesterov introduced in 1983 an acceleration scheme for the gradient descent algorithm. It was generalized later to the composite setting [Nesterov, 1983, 2004, 2013].

FISTA [Beck and Teboulle, 2009]

\[ x_k \leftarrow \arg \min_{x \in \mathbb{R}^p} \frac{1}{2} \left\| x - \left( y_{k-1} - \frac{1}{L} \nabla f(y_{k-1}) \right) \right\|^2_2 + \frac{1}{L} \psi(x); \]

Find \( \alpha_k > 0 \) s.t. \( \alpha_k^2 = (1 - \alpha_k)\alpha_{k-1}^2 + \frac{\mu}{L} \alpha_k; \)

\[ y_k \leftarrow x_k + \beta_k (x_k - x_{k-1}) \quad \text{with} \quad \beta_k = \frac{\alpha_{k-1}(1 - \alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k}. \]

- \( f(x_k) - f^* = O(1/k^2) \) for convex problems;
- \( f(x_k) - f^* = O((1 - \sqrt{\mu/L})^k) \) for \( \mu \)-strongly convex problems;
- Acceleration works in many practical cases.

see also [Nesterov, 1983, 2004, 2013]
What do we mean by “acceleration”?

Complexity analysis for large finite sums

Since $f$ is a sum of $n$ functions, computing $\nabla f$ requires computing $n$ gradients $\nabla f_i$. The complexity to reach an $\varepsilon$-solution is given below

<table>
<thead>
<tr>
<th></th>
<th>$\mu &gt; 0$</th>
<th>$\mu = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ISTA</td>
<td>$O \left( \frac{nL}{\mu} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>$O \left( \frac{nL}{\varepsilon} \right)$</td>
</tr>
<tr>
<td>FISTA</td>
<td>$O \left( n \sqrt{\frac{L}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>$O \left( \frac{nL}{\sqrt{\varepsilon}} \right)$</td>
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</table>

Remarks

- $\varepsilon$-solution means here $f(x_k) - f^* \leq \varepsilon$.
- For $n = 1$, the rates of FISTA are optimal for a “first-order local black box” [Nesterov, 2004].
- For $n > 1$, the sum structure of $f$ is not exploited.
Can we do better for large finite sums?

Several *randomized* algorithms are designed with one $\nabla f_i$ computed per iteration, which yields a better *expected computational complexity*.

<table>
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<th>Complexity</th>
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<tr>
<td>FISTA</td>
<td>$O \left( n \sqrt{\frac{L}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
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<tr>
<td>SVRG, SAG, SAGA, SDCA, MISO, Finito</td>
<td>$O \left( \max \left( n, \frac{L}{\mu} \right) \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
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SVRG, SAG, SAGA, SDCA, MISO, Finito improve upon FISTA when

$$\max \left( n, \frac{L}{\mu} \right) \leq n \sqrt{\frac{L}{\mu}} \iff \sqrt{\frac{L}{\mu}} \leq n,$$

but they are not “accelerated” in the sense of Nesterov.

[Schmidt et al., 2013, Xiao and Zhang, 2014, Defazio et al., 2014a,b, Shalev-Shwartz and Zhang, 2012, Mairal, 2015, Zhang and Xiao, 2015]
Can we do even better for large finite sums?

Without vs with acceleration

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<tr>
<td>Acc-SDCA</td>
<td>$\tilde{O}\left(\max\left(n, \sqrt{n} \frac{L}{\mu}\right) \log \left(\frac{1}{\varepsilon}\right)\right)$</td>
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Acc-SDCA is due to Shalev-Shwartz and Zhang [2014].

- Acceleration occurs when $n \leq \frac{L}{\mu}$.
- see [Agarwal and Bottou, 2015] for discussions about optimality.

**Challenge:** can we accelerate these algorithms by a universal scheme for both convex and strongly convex objectives?
Catalyst is coming
Main idea

Catalyst, a meta-algorithm

Given an algorithm $\mathcal{M}$ that can solve a convex problem "appropriately".

- At iteration $k$, rather than minimizing $F$, we use $\mathcal{M}$ to minimize a function $G_k$, defined as follows,

$$G_k(x) \triangleq F(x) + \frac{\kappa}{2} \|x - y_{k-1}\|_2^2,$$

up to accuracy $\varepsilon_k$, i.e., such that $G_k(x_k) - G^*_k \leq \varepsilon_k$.

- Then compute the next prox-center $y_k$ using an extrapolation step

$$y_k = x_k + \beta_k (x_k - x_{k-1}).$$

The choices of $\beta_k, \varepsilon_k, \kappa$ are driven by the theoretical analysis.
Main idea

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The choices of $\beta_k, \varepsilon_k, \kappa$ are driven by the theoretical analysis.

Catalyst is a wrapper of $\mathcal{M}$ that yields an **accelerated** algorithm $\mathcal{A}$. 

Sources of inspiration

In addition to accelerated proximal algorithms [Beck and Teboulle, 2009, Nesterov, 2013], several works have inspired Catalyst.

The inexact accelerated proximal point algorithm of Güler [1992].
- Catalyst is a variant of inexact accelerated PPA.
- Complexity analysis for outer-loop only with non practical inexactness criterium.

Accelerated SDCA of Shalev-Shwartz and Zhang [2014].
- Accelerated SDCA is an instance of inexact accelerated PPA.
- Complexity analysis limited to $\mu$-strongly convex objectives.
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Other related work
This work

Contributions

- **Generic acceleration scheme**, which applies to previously unaccelerated algorithms such as SVRG, SAG, SAGA, SDCA, MISO, or Finito, and which is not tailored to finite sums.
- Provides explicit **support to non-strongly convex objectives**.
- Complexity analysis for $\mu$-strongly convex objectives.
- Complexity analysis for non-strongly convex objectives.

Example of application

Garber and Hazan [2015] have used Catalyst to accelerate new principal component analysis algorithms based on convex optimization.
Appropriate $\mathcal{M} = \text{Linear convergence rate when } \mu > 0$

Linear convergence rate
Consider a strongly convex minimization problem

$$\min_{z \in \mathbb{R}^p} H(z).$$

We say that an algorithm $\mathcal{M}$ has a linear convergence rate if $\mathcal{M}$ generates a sequence of iterates $(z_t)_{t \in \mathbb{N}}$ such that there exists $\tau_{\mathcal{M},H}$ in $(0, 1)$ and a constant $C_{\mathcal{M},H}$ in $\mathbb{R}$ satisfying

$$H(z_t) - H^* \leq C_{\mathcal{M},H}(1 - \tau_{\mathcal{M},H})^t. \quad (1)$$

- $\tau_{\mathcal{M},H}$ depends usually on the condition number $L/\mu$, e.g., $\tau_{\mathcal{M},H} = \mu/L$ for ISTA and $\tau_{\mathcal{M},H} = \sqrt{\mu/L}$ for FISTA.
- $C_{\mathcal{M},H}$ depends usually on $H(z_0) - H^*$. 
Appropriate $\mathcal{M} = \text{Linear convergence rate when } \mu > 0$

**Linear convergence rate**

Consider a **strongly convex** minimization problem

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$$H(z_t) - H^* \leq C_{\mathcal{M},H}(1 - \tau_{\mathcal{M},H})^t. \quad (1)$$

**Important message:** we do not make any assumption for non strongly convex objectives. It is possible that $\mathcal{M}$ is not even defined for $\mu = 0.$
Catalyst action

\[ G_k(x) \triangleq F(x) + \frac{\kappa}{2} \|x - y_{k-1}\|_2^2, \]

- \( G_k \) is always strongly convex as long as \( F \) is convex.
- When \( F \) is strongly convex, the condition number of \( G_k \) is better than that of \( F \) since \( \frac{L+\kappa}{\mu+\kappa} < \frac{L}{\mu} \).
Catalyst action

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**Minimizing \( G_k \) is easier than minimizing \( F \) !**
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**Minimizing \( G_k \) is easier than minimizing \( F \) !**

- If \( \kappa \gg 1 \), then minimizing \( G_k \) is easy;
- If \( \kappa \approx 0 \), then \( G_k \) is a good approximation of \( F \).

We will choose \( \kappa \) to optimize the computational complexity.
Convergence analysis

An analysis in two stages

\[ G_k(x) \triangleq F(x) + \frac{\kappa}{2} \| x - y_{k-1} \|^2, \]

\( x_k \) is an approximate minimizer of \( G_k \) such that \( G_k(x_k) - G_k^* \leq \epsilon_k. \)

- Outer loop: once we obtain the sequence \( (x_k)_{k \in \mathbb{N}} \), what can we say about the convergence rate of \( F(x_k) - F^*? \)
  \( \Rightarrow \) Wisely choose \( (y_k) \) and control the accumulation of errors.

- Inner loop: how much effort do we need to obtain a \( x_k \) with accuracy \( \epsilon_k? \)
  \( \Rightarrow \) Wisely choose the starting point.
Choice of \((y_k)_{k \in \mathbb{N}}\)

Extrapolation

\[ y_k = x_k + \beta_k (x_k - x_{k-1}) \quad \text{with} \quad \beta_k = \frac{\alpha_{k-1}(1 - \alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k}. \]

- This update is identical to Nesterov’s accelerated gradient descent or FISTA.
- Unfortunately, the literature does not provide any simple geometric explanation why it yields an acceleration...
- The construction is purely theoretical by using a mechanism introduced by Nesterov, called "estimate sequence".
How does “acceleration” work?

If \( f \) is \( \mu \)-strongly convex and \( \nabla f \) is \( L \)-Lipschitz continuous

\[
\begin{align*}
  f(x) &\leq f(x_{k-1}) + \nabla f(x_{k-1})^T (x - x_{k-1}) + \frac{L}{2} \| x - x_{k-1} \|^2_2; \\
  f(x) &\geq f(x_{k-1}) + \nabla f(x_{k-1})^T (x - x_{k-1}) + \frac{\mu}{2} \| x - x_{k-1} \|^2_2;
\end{align*}
\]
How does “acceleration” work?

If \( \nabla f \) is \( L \)-Lipschitz continuous

\[
\begin{align*}
  f(x) & \leq f(x_{k-1}) + \nabla f(x_{k-1})^\top (x - x_{k-1}) + \frac{L}{2} \|x - x_{k-1}\|^2_2; \\
  x_k & = x_{k-1} - \frac{1}{L} \nabla f(x_{k-1}) \text{ (gradient descent step).}
\end{align*}
\]
How does “acceleration” work?

Definition of estimate sequence [Nesterov].

A pair of sequences \((\varphi_k)_{k \geq 0}\) and \((\lambda_k)_{k \geq 0}\), with \(\varphi_k : \mathbb{R}^p \to \mathbb{R}\) and \(\lambda_k \geq 0\), is called an \textbf{estimate sequence} of function \(F\) if

- \(\lambda_k \to 0\);
- \(\varphi_k(x) \leq (1 - \lambda_k)F(x) + \lambda_k \varphi_0(x)\), for any \(k, x\);
- There exists a sequence \((x_k)_{k \geq 0}\) such that

\[
F(x_k) \leq \varphi_k^* \triangleq \min_{x \in \mathbb{R}^p} \varphi_k(x).
\]

Remarks

- \(\varphi_k\) is neither an upper-bound, nor a lower-bound;
- Finding the right estimate sequence is often nontrivial.
Convergence of outer-loop algorithm

Analysis for \( \mu \)-strongly convex objective functions

Choose \( \alpha_0 = \sqrt{q} \) with \( q = \frac{\mu}{(\mu + \kappa)} \) and

\[
\epsilon_k = \frac{2}{9}(F(x_0) - F^*)(1 - \rho)^k \quad \text{with} \quad \rho < \sqrt{q}.
\]

Then, the algorithm generates iterates \((x_k)_{k \geq 0}\) such that

\[
F(x_k) - F^* \leq C(1 - \rho)^{k+1}(F(x_0) - F^*) \quad \text{with} \quad C = \frac{8}{(\sqrt{q} - \rho)^2}.
\]

In practice

- Choice of \( \rho \) can safely be set to \( \rho = 0.9\sqrt{q} \).
- Choice of \((\epsilon_k)_{k \geq 0}\) typically follows from a duality gap at \( x_0 \). When \( F \) is non-negative, we can set \( \epsilon_k = (2/9)F(x_0)(1 - \rho)^k \).
Convergence of outer-loop algorithm

Analysis for non-strongly convex objective functions, $\mu = 0$

Choose $\alpha_0 = (\sqrt{5} - 1)/2$ and

$$\epsilon_k = \frac{2(F(x_0) - F^*)}{9(k + 2)^{4+\eta}} \quad \text{with } \eta > 0.$$ 

Then, the meta-algorithm generates iterates $(x_k)_{k \geq 0}$ such that

$$F(x_k) - F^* \leq \frac{8}{(k + 2)^2} \left( \left(1 + \frac{2}{\eta}\right)^2 (F(x_0) - F^*) + \frac{\kappa}{2} \|x_0 - x^*\|^2 \right).$$

(2)

In practice

- Choice of $\eta$ can be set to $\eta = 0.1$. 
How many iterates of $\mathcal{M}$ do we need to obtain $x_k$?

Control of inner-loop complexity

For minimizing $G_k$, consider a method $\mathcal{M}$ generating iterates $(z_t)_{t \geq 0}$ with linear convergence rate

$$G_k(z_t) - G_k^* \leq A(1 - \tau_\mathcal{M})^t(G_k(z_0) - G_k^*).$$

Then by choosing $z_0 = x_{k-1}$, the precision $\varepsilon_k$ is reached with at most

- A constant number of iterations $T_\mathcal{M}$ when $\mu > 0$;
- A logarithmic increasing number of iterations $T_\mathcal{M} \log(k + 2)$ when $\mu = 0$.

where $T_\mathcal{M} = \tilde{O}(1/\tau_\mathcal{M})$ is independent of $k$. 

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Catalyst
Global computational complexity

Analysis for $\mu$-strongly convex objective functions

The global convergence rate of the accelerated algorithm $\mathcal{A}$ is

$$F_s - F^* \leq C \left(1 - \frac{\rho}{T_M}\right)^s (F(x_0) - F^*). \quad (3)$$

where $F_s$ is the objective function value obtained after performing $s = kT_M$ iterations of the method $\mathcal{M}$. As a result,

$$\tau_{\mathcal{A},F} = \frac{\rho}{T_M} = \tilde{O}(\tau_M \sqrt{\mu} / \sqrt{\mu + \kappa}),$$

where $\tau_M$ typically depends on $\kappa$ (the greater, the faster).

$\kappa$ will be chosen to maximize the ratio $\tau_M / \sqrt{\mu + \kappa}$. 
Global computational complexity

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$$\tau_{\mathcal{A},F} = \frac{\rho}{T_M} = \tilde{O}(\tau_{\mathcal{M}} \sqrt{\mu} / \sqrt{\mu + \kappa}),$$

where $\tau_{\mathcal{M}}$ typically depends on $\kappa$ (the greater, the faster).

e.g., $\kappa = L - 2\mu$ when $\tau_{\mathcal{M}} = \frac{\mu + \kappa}{L + \kappa} \Rightarrow \tau_{\mathcal{A}} = \tilde{O} \left( \sqrt{\frac{\mu}{L}} \right)$. 
Global computational complexity

Analysis for non-strongly convex objective functions

The global convergence rate of the accelerated algorithm $\mathcal{A}$ is

$$F_s - F^* \leq \frac{8 T_{\mathcal{M}}^2 \log^2(s)}{s^2} \left( \left(1 + \frac{2}{\eta}\right)^2 (F(x_0) - F^*) + \frac{\kappa}{2} \|x_0 - x^*\|^2 \right).$$

If $\mathcal{M}$ is a first-order method, this rate is near-optimal, up to a logarithmic factor, when compared to the optimal rate $O(1/s^2)$, which may be the price to pay for using a generic acceleration scheme.

$\kappa$ will be chosen to maximize the ratio $\tau_{\mathcal{M}}/\sqrt{L + \kappa}$. 
Applications

Expected computational complexity in the regime $n \leq L/\mu$ when $\mu > 0$,

<table>
<thead>
<tr>
<th>Algorithm</th>
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<th>Catalyst $\mu &gt; 0$</th>
<th>Cat. $\mu = 0$</th>
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<tr>
<td>FG</td>
<td>$O \left( n \left( \frac{L}{\mu} \right) \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>$O \left( n \frac{L}{\varepsilon} \right)$</td>
<td>$\tilde{O} \left( n \sqrt{\frac{L}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>$\tilde{O} \left( n \frac{L}{\sqrt{\varepsilon}} \right)$</td>
</tr>
<tr>
<td>SAG</td>
<td>$O \left( \frac{L}{\mu} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>$O \left( n \frac{L}{\varepsilon} \right)$</td>
<td>$\tilde{O} \left( \sqrt{\frac{nL}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>$\tilde{O} \left( n \frac{L}{\sqrt{\varepsilon}} \right)$</td>
</tr>
<tr>
<td>SAGA</td>
<td>$O \left( \frac{L}{\mu} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>NA</td>
<td>$\tilde{O} \left( \sqrt{\frac{nL}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td></td>
</tr>
<tr>
<td>Finito/MISO</td>
<td>$O \left( \frac{L'}{\mu} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>NA</td>
<td>$\tilde{O} \left( \sqrt{\frac{nL'}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td></td>
</tr>
<tr>
<td>SDCA</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>SVRG</td>
<td>$O \left( \frac{L'}{\mu} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>NA</td>
<td>$\tilde{O} \left( \sqrt{\frac{nL'}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td></td>
</tr>
<tr>
<td>Acc-FG</td>
<td>$O \left( n \sqrt{\frac{L}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>$O \left( n \frac{L}{\sqrt{\varepsilon}} \right)$</td>
<td>no acceleration</td>
<td></td>
</tr>
<tr>
<td>Acc-SDCA</td>
<td>$\tilde{O} \left( \sqrt{\frac{nL}{\mu}} \log \left( \frac{1}{\varepsilon} \right) \right)$</td>
<td>NA</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Experiments with MISO/SAG/SAGA

\( \ell_2 \)-logistic regression formulation

Given some data \((y_i, z_i)\), with \(y_i\) in \((-1, +1)\) and \(z_i\) in \(\mathbb{R}^p\), minimize

\[
\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \log(1 + e^{-y_i x^\top z_i}) + \frac{\mu}{2} \|x\|_2^2,
\]

\(\mu\) is the regularization parameter and the strong convexity modulus.

Datasets

<table>
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<tr>
<th>name</th>
<th>rcv1</th>
<th>real-sim</th>
<th>covtype</th>
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<th>alpha</th>
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</thead>
<tbody>
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<td>781 265</td>
<td>72 309</td>
<td>581 012</td>
<td>2 500 000</td>
<td>250 000</td>
</tr>
<tr>
<td>(p)</td>
<td>47 152</td>
<td>20 958</td>
<td>54</td>
<td>1 155</td>
<td>500</td>
</tr>
</tbody>
</table>
Experiments with MISO/SAG/SAGA

The complexity analysis is not just a theoretical exercise since it provides the values of $\kappa, \varepsilon_k, \beta_k$, which are required in concrete implementations.

Here, theoretical values match practical ones.

Restarting

The theory tells us to restart $\mathcal{M}$ with $x_{k-1}$. For SDCA/Finito/MISO, the theory tells us to use instead $x_{k-1} + \frac{\kappa}{\mu + \kappa} (y_{k-1} - y_{k-2})$. We also tried this as a heuristic for SAG and SAGA.

One-pass heuristic

constrain $\mathcal{M}$ to always perform at most $n$ iterations in inner loop; we call this variant AMISO2 for MISO, whereas AMISO1 refers to the regular “vanilla” accelerated variant; idem to accelerate SAG and SAGA.
Experiments without strong convexity, $\mu = 0$

![Graphs showing objective function values for different number of passes on various datasets.](image)

**Figure:** Objective function value for different number of passes on data.

**Conclusions**

- SAG, SAGA are accelerated when they do not perform well already;
- $\text{AMISO2} \geq \text{AMISO1} \ (\text{vanilla}), \ MISO \ does \ not \ apply$. 
Experiments without strong convexity, $\mu = 10^{-1}/n$

**Figure:** Relative duality gap (log-scale) for different number of passes on data.

**Conclusions**
- SAG, SAGA are not always accelerated, but often.
- AMISO2, AMISO1 $\gg$ MISO.
Experiments without strong convexity, $\mu = 10^{-3}/n$

![Graphs showing relative duality gap for different datasets](image)

**Figure:** Relative duality gap (log-scale) for different number of passes on data.

**Conclusions**

- same conclusions as $\mu = 10^{-1}/n$;
- $\mu$ is so small that (unaccelerated) MISO becomes numerically unstable.
General conclusions about Catalyst

Summary: lots of nice features

- Simple acceleration scheme with broad application range.
- Recover near-optimal rates for known algorithms.
- Effortless implementation.

... but also lots of unsolved problems

- Acceleration occurs when \( n \leq L/\mu \); otherwise, the “unaccelerated” complexity \( O(n \log(1/\varepsilon)) \) seems unbeatable.
- \( \mu \) is an estimate of the true strong convexity parameter \( \mu' \geq \mu \).
- \( \mu \) is the global strong convexity parameter, not a local one \( \mu^* \geq \mu \).
- When \( n \leq L/\mu \), but \( n \geq L/(\mu' \text{ or } \mu^*) \), a method \( \mathcal{M} \) that adapts to the unknown strong convexity may be impossible to accelerate.
- The optimal restart for \( \mathcal{M} \) is not yet fully understood.
Happy birthday!
Catalyst, the algorithm

**Algorithm 1 Catalyst**

**input** initial estimate $x_0 \in \mathbb{R}^p$, parameters $\kappa$ and $\alpha_0$, sequence $(\varepsilon_k)_{k \geq 0}$, optimization method $\mathcal{M}$; initialize $q = \mu / (\mu + \kappa)$ and $y_0 = x_0$;

1: **while** the desired stopping criterion is not satisfied **do**

2: Find an approx. solution $x_k$ using $\mathcal{M}$ s.t. $G_k(x_k) - G_k^* \leq \varepsilon_k$

3: \[ x_k \approx \arg \min_{x \in \mathbb{R}^p} \left\{ G_t(x) \overset{\Delta}{=} F(x) + \frac{\kappa}{2} \| x - y_{k-1} \|^2 \right\} \]

3: Compute $\alpha_k \in (0, 1)$ from equation $\alpha_k^2 = (1 - \alpha_k) \alpha_{k-1}^2 + q \alpha_k$;

4: Compute \[ y_k = x_k + \beta_k (x_k - x_{k-1}) \] with \[ \beta_k = \frac{\alpha_{k-1} (1 - \alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k} \]

5: **end while**

**output** $x_k$ (final estimate).
Ideas of the proofs

Main theorem

Let us denote

$$\lambda_k = \prod_{i=0}^{k-1} (1 - \alpha_i),$$

where the $\alpha_i$'s are defined in Catalyst. Then, the sequence $(x_k)_{k \geq 0}$ satisfies

$$F(x_k) - F^* \leq \lambda_k \left( \sqrt{S_k} + 2 \sum_{i=1}^{k} \sqrt{\frac{\epsilon_i}{\lambda_i}} \right)^2,$$

where $F^*$ is the minimum value of $F$ and

$$S_k = F(x_0) - F^* + \frac{\gamma_0}{2} \|x_0 - x^*\|^2 + \sum_{i=1}^{k} \frac{\epsilon_i}{\lambda_i} \quad \text{where} \quad \gamma_0 = \frac{\alpha_0 ((\kappa + \mu)\alpha_0 - \mu)}{1 - \alpha_0},$$

where $x^*$ is a minimizer of $F$. 
Ideas of the proofs

Then, the theorem will be used with the following lemma to control the convergence rate of the sequence \((\lambda_k)_{k \geq 0}\), whose definition follows the classical use of estimate sequences. This will provide us convergence rates both for the strongly convex and non-strongly convex cases.

Lemma 2.2.4 from Nesterov [2004]

If in the quantity \(\gamma_0\) defined in (6) satisfies \(\gamma_0 \geq \mu\), then the sequence \((\lambda_k)_{k \geq 0}\) from (4) satisfies

\[
\lambda_k \leq \min \left\{ \left(1 - \sqrt{q}\right)^k, \frac{4}{\left(2 + k \sqrt{\frac{\gamma_0}{\kappa + \mu}}\right)^2} \right\},
\]

(7)

where \(q \triangleq \mu / (\mu + \kappa)\).
Ideas of proofs

Step 1: build an approximate estimate sequence

- Remember that in general, we build $\varphi_k$ from $\varphi_{k-1}$ as follows

$$
\varphi_k(x) \triangleq (1 - \alpha_k)\varphi_{k-1}(x) + \alpha_k d_k(x),
$$

where $d_k$ is a lower bound.

- Here, a natural lower bound would be

$$
F(x) \geq d_k(x) \triangleq F(x_k^*) + \langle \kappa (y_{k-1} - x_k^*), x - x_k^* \rangle + \frac{\mu}{2} \| x - x_k^* \|^2,
$$

where $x_k^* \triangleq \arg\min_{x \in \mathbb{R}^p} \left\{ G_k(x) \triangleq F(x) + \frac{\kappa}{2} \| x - y_{k-1} \|^2 \right\}$.

- But $x_k^*$ is unknown! Then, use instead $d'_k(x)$ defined as

$$
d'_k(x) \triangleq F(x_k) + \langle \kappa (y_{k-1} - x_k), x - x_k \rangle + \frac{\mu}{2} \| x - x_k \|^2.
$$
Ideas of proofs

Step 2: Relax the condition $F(x_k) \leq \varphi_k^*$. 

- We can show that Catalyst generates iterates $(x_k)_{k \geq 0}$ such that 

$$F(x_k) \leq \phi_k^* + \xi_k,$$

where the sequence $(\xi_k)_{k \geq 0}$ is defined by $\xi_0 = 0$ and 

$$\xi_k = (1 - \alpha_{k-1})(\xi_{k-1} + \varepsilon_k - (\kappa + \mu)\langle x_k - x_k^*, x_k - x_{k-1} \rangle).$$

- The sequences $(\alpha_k)_{k \geq 0}$ and $(y_k)_{k \geq 0}$ are chosen in such a way that all the terms involving $y_{k-1} - x_k$ are cancelled.

- We will control later the quantity $x_k - x_k^*$ by strong convexity of $G_k$: 

$$\frac{\kappa + \mu}{2} \|x_k - x_k^*\|_2^2 \leq G_k(x_k) - G_k^* \leq \varepsilon_k.$$
Ideas of proofs

Step 3: Control how this errors sum up together.

- Do cumbersome calculus.
Catalyst in practice

General strategy and application to randomized algorithms

Calculating the iteration-complexity decomposes into three steps:

1. When $F$ is $\mu$-strongly convex, find $\kappa$ that maximizes the ratio $\frac{\tau_{\mathcal{M}, G_k}}{\sqrt{\mu + \kappa}}$ for algorithm $\mathcal{M}$. When $F$ is non-strongly convex, maximize instead the ratio $\frac{\tau_{\mathcal{M}, G_k}}{\sqrt{L + \kappa}}$.

2. Compute the upper-bound of the number of outer iterations $k_{\text{out}}$ using the theorems.

3. Compute the upper-bound of the expected number of inner iterations

$$\max_{k=1, \ldots, k_{\text{out}}} \mathbb{E}[T_{\mathcal{M}, G_k}(\varepsilon_k)] \leq k_{\text{in}},$$

Then, the expected iteration-complexity denoted $\text{Comp}$ is given by

$$\text{Comp} \leq k_{\text{in}} \times k_{\text{out}}.$$
Applications

Deterministic and Randomized Incremental Gradient methods

- Stochastic Average Gradient (SAG and SAGA) [Schmidt et al., 2013, Defazio et al., 2014a];
- Finito and MISO [Mairal, 2015, Defazio et al., 2014b];
- Semi-Stochastic/Mixed Gradient [Konečný et al., 2014, Zhang et al., 2013];
- Stochastic Dual coordinate Ascent [Shalev-Shwartz and Zhang, 2012];
- Stochastic Variance Reduced Gradient [Xiao and Zhang, 2014].

But also, randomized coordinate descent methods, and more generally first-order methods with linear convergence rates.
Appendix on proximal MISO
Original motivation

Given some data, learn some model parameters $x$ in $\mathbb{R}^p$ by minimizing

$$\min_{x \in \mathbb{R}^p} \left\{ F(x) \equiv \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right\},$$

where each $f_i$ may be nonsmooth and nonconvex.

The original MISO algorithm is an incremental extension of the majorization-minimization principle [Lange et al., 2000].

Paper

Majorization-minimization principle

Iteratively minimize locally tight upper bounds of the objective.

The objective monotonically decreases.

Under some assumptions, we get similar convergence rates as gradient-based approaches for convex problems.
Algorithm 2 Incremental scheme MISO

**Input** $x_0 \in \mathbb{R}^p$; $T$ (number of iterations).

1. Choose surrogates $g_i^0$ of $f_i$ near $x_0$ for all $i$;
2. for $k = 1, \ldots, K$ do
3. Randomly pick up one index $\hat{i}_k$ and choose a surrogate $g_{\hat{i}_k}^k$ of $f_{\hat{i}_k}$ near $x_{k-1}$. Set $g_i^k \triangleq g_i^{k-1}$ for $i \neq \hat{i}_k$;
4. Update the solution:

$$x_k \in \arg \min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} g_i^k(x).$$

5. end for

**Output** $x_K$ (final estimate);
Incremental Optimization: MISO

Update rule with basic upper bounds

We want to minimize $\frac{1}{n} \sum_{i=1}^{n} f_i(x)$, where the $f_i$’s are smooth.

$$x_k \leftarrow \arg\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f_i(y_i^k) + \nabla f_i(y_i^k)^\top (x - y_i^k) + \frac{L}{2} \|x - y_i^k\|_2^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} y_i^k - \frac{1}{Ln} \sum_{i=1}^{n} \nabla f_i(y_i^k).$$

At iteration $k$, randomly draw one index $\hat{i}_k$, and update $y_{\hat{i}_k}^k \leftarrow x_k$.

Remarks

- replace $(1/n) \sum_{i=1}^{n} y_i^k$ by $x_{k-1}$ yields SAG [Schmidt et al., 2013].
- replace $(1/L)$ by $(1/\mu)$ for strongly convex problems is close to a variant of SDCA [Shalev-Shwartz and Zhang, 2012].
Incremental Optimization: MISO$\mu$.

**Update rule with lower bounds?**

We want to minimize $\frac{1}{n} \sum_{i=1}^{n} f_i(x)$, where the $f_i$'s are smooth.

$$x_k = \arg \min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f_i(y_i^k) + \nabla f_i(y_i^k) \top (x - y_i^k) + \frac{\mu}{2} \|x - y_i^k\|_2^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} y_i^k - \frac{1}{\mu n} \sum_{i=1}^{n} \nabla f_i(y_i^k).$$

**Remarks**

- Requires strong convexity.
- Use a counter-intuitive minorization-minimization principle.
- Close to a variant of SDCA [Shalev-Shwartz and Zhang, 2012].
- Much faster than the basic MISO (faster rate).
Incremental Optimization: MISOμ.

In the first part of this presentation, what we have called MISO is the algorithm that uses $1/(\mu n)$ step-sizes (sorry for the confusion). To minimize $F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, MISOμ has the following guarantees

Proposition [Mairal, 2015]

When the functions $f_i$ are $\mu$-strongly convex, differentiable with $L$-Lipschitz gradient, and non-negative, MISOμ satisfies

$$
\mathbb{E}[F(x_k) - F^*] \leq \left(1 - \frac{1}{3n}\right)^k \cdot nf^*,
$$

under the condition $n \geq 2L/\mu$.

Remarks

- When $n \leq 2L/\mu$, the algorithm may diverge;
- When $\mu$ is very small, numerical stability is an issue.
- The condition $f_i \geq 0$ does not really matter.
Proximal MISO [Lin, Mairal, and Harchaoui, 2015]

Main goals

- Remove the condition \( n \leq 2L/\mu \);
- Allow a composite term \( \psi \):

\[
\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \psi(x) \right\},
\]

Starting points

MISO\( \mu \) is iteratively updating/minimizing a lower-bound of \( F \)

\[
x_k \leftarrow \arg \min_{x \in \mathbb{R}^p} \left\{ D_k(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} d_i^k(x) \right\},
\]

[Lin, Mairal, and Harchaoui, 2015].
Proximal MISO

Adding the proximal term

\[ x_t \leftarrow \arg \min_{x \in \mathbb{R}^p} \left\{ D_k(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} d_i^k(x) + \psi(x) \right\} , \]

Remove the condition \( n \geq 2L/\mu \)

For \( i = \hat{i}_k \),

\[ d_i^k(x) = (1-\delta)d_i^{k-1}(x) + \delta \left( f_i(x_{k-1}) + \langle \nabla f_i(x_{k-1}), x - x_{k-1} \rangle + \frac{\mu}{2} \| x - x_{k-1} \|^2 \right) \]

Remarks

- the original MISO\( \mu \) uses \( \delta = 1 \). To get rid of the condition \( n \geq 2L/\mu \), proximal MISO uses instead \( \delta = \min \left( 1, \frac{\mu n}{2(L-\mu)} \right) \).
- variant “5” of SDCA [Shalev-Shwartz and Zhang, 2012] is identical with another value \( \delta = \frac{\mu n}{L+\mu n} \) in \( (0, 1) \).
Proximal MISO

Convergence of MISO-Prox

Let \((x_k)_{k \geq 0}\) be obtained by MISO-Prox, then

\[
\mathbb{E}[F(x_k)] - F^* \leq \frac{1}{\tau} (1 - \tau)^{k+1} (F(x_0) - D_0(x_0)) \quad \text{with} \quad \tau \geq \min \left\{ \frac{\mu}{4L}, \frac{1}{2n} \right\}.
\]

Furthermore, we also have fast convergence of the certificate

\[
\mathbb{E}[F(x_k) - D_k(x_k)] \leq \frac{1}{\tau} (1 - \tau)^k (F^* - D_0(x_0)).
\]

Differences with SDCA

- The construction is \textbf{primal}. The proof of convergence and the algorithm do not use duality, while SDCA is a dual ascent technique.
- \(D_k(x_k)\) is a lower-bound of \(F^*\); it plays the same role as the dual in SDCA, but is \textbf{easier to evaluate}. 
Conclusions

- Relatively simple algorithm, with simple convergence proof, and simple optimality certificate.
- Catalyst not only accelerates it, but also stabilizes it numerically, with the parameter $\delta = 1$.
- Close to SDCA, but without duality.
References I


References II


References III


