Welfarism and Interpersonal Comparisons*

Claude d'Aspremont†

Abstract

The purpose of this presentation is to review the welfarist approach to the global evaluation of alternative economic situations (social states or institutions). More general than the utilitarian doctrine and centrally based on the Pareto principle, this approach leads to a plurality of evaluation criteria, requiring different kinds of interpersonal comparisons. From a pragmatic point of view it may be used to find useful inequality indices.

1 Introduction

Although Arrow's fundamental negative result on the consistency of Social Choice is generally connected to Condorcet's voting paradox, it is clear from the start in Arrow's original monograph (1951) that there are two different types of questions that he wants to address. One is to investigate the various methods of voting and is most relevant for political sciences; the other is the global evaluation of alternative economic situations (social states or institutions) and is most relevant for economics. We shall concentrate on this second question. In the first part we shall recall that this question received an early, but complete answer at the early stage of economics, in the eighteen-century moral philosophy: it is the "classical utilitarianism" doctrine of Hutcheson and Bentham. This doctrine was not really challenged before the beginning of our century, when Pareto collective optimality concept was developed and its "ordinal" character put forth. The Pareto criterion has lead to a new, but less demanding doctrine that has been

*Reprinted from *Investigaciones Economicas*, Volumen XVIII(1) - Enero 1994, pp. 3–17.
†Center for Operations Research and Econometrics, Louvain-la-Neuve

called “welfarism” (see Sen (1979)). However as we shall see welfarism is not sufficient to get an adequate social evaluation for all alternatives. In the second part, a reasonable attempt to complete the welfarist approach will bring us back to utilitarianism. Also, it will become clear that utilitarianism is just a special, although important, completion of welfarism. In the last part we shall see other instances and show how they can be applied to construct an important tool for the welfare economist, namely inequality indices.

2 From classical utilitarianism to welfarism

Classical utilitarianism attaches, to every individual member of society, a utility function measuring in the same common unit the level of satisfaction obtained in each social state. Then the global evaluation of a social state is given by the summation of utilities over all individuals. The main difficulty in this approach is the interpretation of the utility measures. In the XVIIIth century they were seen to measure, in the same common unit, the “happiness” of all the different individuals. But what is happiness? Is it, for instance, the satisfaction of desires according to tastes, or according to needs? In the sharing of a cake should the bigger part go to the one who likes the cake most or to the one who is more hungry? Also should the nature of the “social states” involved matter? For instance, should the goods to be distributed include, in addition to ordinary economic goods and services, “primary goods” in the sense of Rawls, involving higher-order interests of the individuals and including basic liberties, opportunities, powers and prerogatives? In that respect social choice could only fix the rules of the economic game and utilitarianism should be rule-utilitarianism not act-utilitarianism. In any case the application of utilitarianism involves difficult value judgements implying interpersonal comparisons.

This is precisely what the New Welfare Economists have tried to avoid. The ideal for them is to use the same utility information as the one needed in the standard consumer theory in microeconomics. There the informational requirements are, in principle, observable: they are supposedly given by the observed consumption choices of all individuals. If these choices are expressing individual preferences in a consistent (or rational) way, they might then be represented by an “ordinal” utility function, unique up to a monotone transformation. (All the required information is given by the set of individual indifference curves.) These utilities are supposed to provide the basis for the welfare evaluation of each social alternative, not by aggregating them into a single number, but by taking them as a point in an Euclidean space of dimension equal to the number of individuals. Although social choice is also reduced to utility comparisons, the comparisons do not involve *utility totals* but only *utility vectors*. No interpersonal comparisons are involved. This is the welfarist doctrine, weaker than the utilitarian doctrine.

Perhaps one essential difference between political social choice, usually based on voting methods, and economic social choice, based on some welfare evaluation, is given by the explicit possibility of paying compensations by transferring goods (or money) from one individual to the other. This is taken for granted in the utilitarian approach since the possibility of summing utilities indicates that transferring utility units from one individual to the other is meaningful. But “compensation principles” in terms of goods (or money) have been introduced in Welfare Economics in order to compare two social states without introducing interpersonal comparisons of utilities. Indeed, one formulation is to say that *society ought to strictly prefer one social state to another, if, at this other state, the losers (from the change) are effectively compensated by the gainers (by transfers of one or more goods) in such a way that everyone prefers the resulting state to the initial state (and at least some strictly prefers).* Of course this is nothing else than a Pareto-dominance principle (called the Strong Pareto condition) analogous to the unanimity principle used for voting rules. If we consider the set of all feasible social alternatives represented in the n-dimensional utility space (with n being the number of individuals), then using this principle we may obtain (under some regularity conditions) the set of all undominated states, that is the set of Pareto-optimal states (or the Pareto frontier). Two different points on this frontier cannot be discriminated socially as well as many other pairs of points in the utility feasibility set. The question is whether some more discrimination is possible without using more information than the one required by standard consumer theory or the application of the Pareto condition and, in particular, without introducing interpersonal comparisons of utilities? A new, extended compensation principle was proposed for that purpose (see Kaldor (1939) and Hicks

(1939)). This is to say that *society ought to strictly prefer one social state to another if the losers could be (but are not) compensated in such a way that everyone prefers (and at least some strictly prefers) the resulting state.* In other words a social state x "Kaldor-dominates" a social state y, if there is a social state z_x that is obtained by compensating transfers from x and that "Pareto-dominates" y. A problem may arise though with the concept of Kaldor-domination. Indeed in general it is possible to have a social state x Kaldor-dominating a state y whereas the state y Kaldor-dominates the state x. In Figure 1, where we have represented for an economy of two individuals the set of feasible utility vectors $u(z) = (u_1(z), u_2(z))$, one can see that there are both a social state z_x obtained by compensating transfers from x to Pareto-dominate y and a social state z_y obtained by compensating transfers from y to Pareto-dominate x, hence the inconsistency of Kaldor-compensation principle.

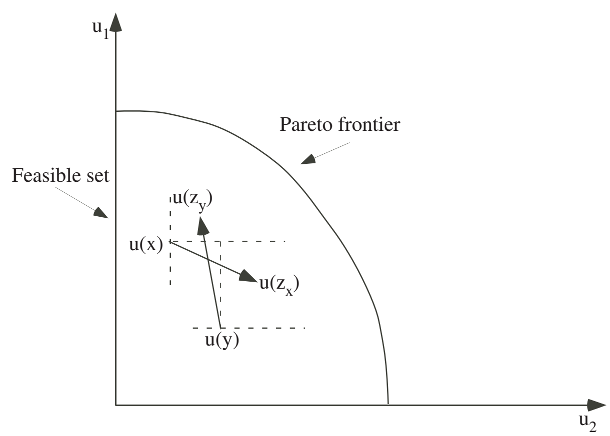

Figure 1:

There is a class of economic environments for which such inconsistencies cannot arise. In game-theoretic terms, this is called the class of "transferable utilities with side-payments": It is characterized by the existence of a perfectly divisible and transferable good playing the role of money, and such that, for some well-chosen calibration, the individual utility functions (without being interpersonally comparable) are additively separable in monetary transfers. Hence to any social alternative one can associate in utility space a "compensation curve" which is a straight line of slope -1. Then one can determine the best feasible (equivalence) class of social alternatives (given by the highest feasible compensation curve) without being forced to determine simultaneously the best redistribution (given by the choice of a particular point on the curve).

In general though the inconsistency of compensation principles such as Kaldor's domination criterion cannot be avoided. As shown by Arrow (1951), Scitovsky's modified principle, by which two social alternatives mutually Kaldor-dominating each other should be declared indifferent from a social viewpoint, leads also to intransitivities. The example is the following. Take an exchange economy consisting of two individuals ($i = 1, 2$) each consuming two kinds of goods, say good a and good b. The utility functions $u_i(q_a^i, q_b^i)$ for each individual i, are supposed to satisfy the inequalities:

$$u_1(21, 10) > u_1(10, 20) > u_1(24, 7) > u_1(17, 13) > u_1(20, 10),$$
$$u_2(14, 14) > u_2(10, 20) > u_2(16, 13) > u_2(18, 11) > u_2(20, 10).$$

Moreover, in order to get at least as much utility as with commodity bundle (10,20), individuals 1 and 2 should get a minimal amount of good b, respectively 9 and 12 units (see Figures 2 and 3).

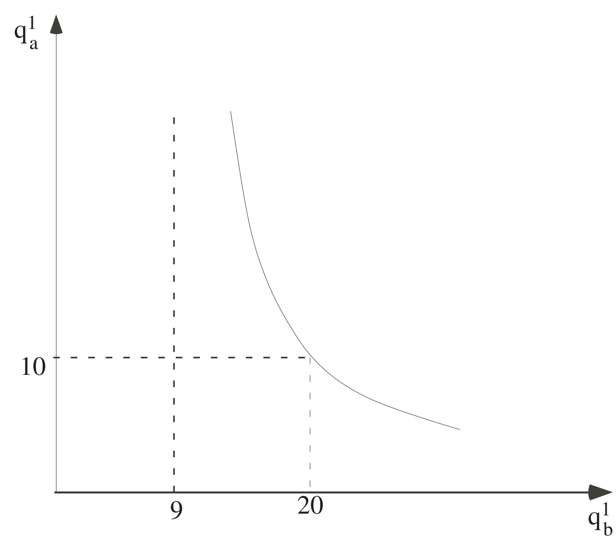

Figure 2:

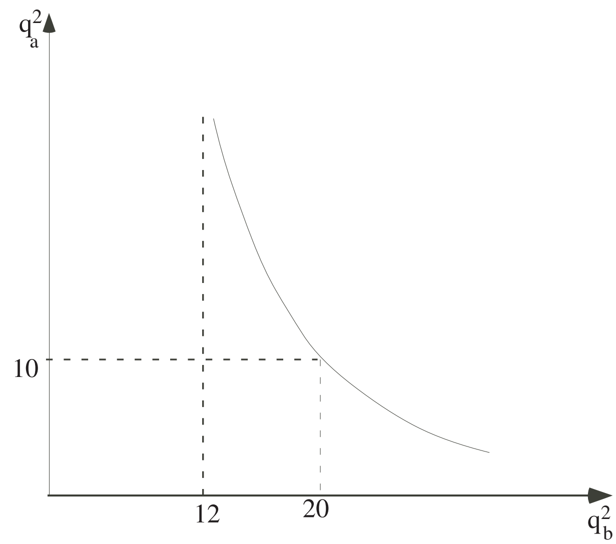

Figure 3:

A social alternative is given by a pair of commodity bundles $((q_a^1, q_b^1), (q_a^2, q_b^2))$ and the question is to rank the following three social alternatives:

$$x = ((20, 10), (20, 10), y = ((17, 13), (18, 11), z = ((10, 20), (10, 20)).$$

It is clear that y Pareto-dominates (and hence Kaldor-dominates) x and, moreover, by transfer of good a from 2 to 1 and of good b from 1 to 2, one may obtain a social alternative $z_x = ((24, 7), (16, 13))$ Pareto-dominating y. Hence x Kaldor-dominates y, so that x and y have to be declared Scitovsky-indifferent. Similarly z Pareto-dominates (and hence Kaldor-dominates) y and y Kaldor-dominates z since the social alternative $z_y = ((21, 10), (14, 14))$, obtained from y by transfers in both goods, Pareto-dominates z. Therefore z is Scitovsky-indifferent to y and, since y is Scitovsky-indifferent to x, transitivity would require that z be indifferent to x. But this is not possible because z Pareto-dominates x and x cannot Kaldor-dominate z: in order to Pareto-dominate z, at least 9 units of good b for individual 1 and 12 units for individual 2 would be required (see Figures 2 and 3) but only a total of 20 units are available in social alternative x, so that z dominates x for Scitovsky compensation principle. This principle is inconsistent. The well-known contribution of Arrow (1951) is to have turned out this counterexample into a "general impossibility theorem" about the whole New Welfare Economics project. This welfarist project was to go beyond the Pareto efficiency criterion by introducing some equity considerations but to keep the same informational basis, that is, individual utility numbers having only an ordinal meaning and prohibiting any kind of interpersonal comparisons.

3 Arrow's theorem and utilitarianism

The simplest way to characterize formally the welfarist project is to define a "social welfare ordering" R^* as a complete preference ordering of the utility space of dimension equal to the number n of individuals, say E^n, allowing to compare any two utility vectors $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ – in the sense that either $u\,R^*\,v$ or $v\,R^*\,u$ – and satisfying transitivity: for any u, v, w in E^n, $u\,R^*\,v$ and $v\,R^*\,w$ implies $u\,R^*\,w$. The social welfare ordering R^* should satisfy three basic properties: the Pareto-principle, an equity (or anonymity) principle and an ordinal, interpersonally noncomparable informational basis, namely.

SP^* (Stong Pareto Condition): for any u, v in E^n, $u\,R^*\,v$ whenever $u_i \geq v_i$ for $i = 1, 2, \cdots, n$, and moreover, $u\,P^*\,v$ (that is $u\,R^*\,v$ and not $v\,R^*\,u$) if also $u \neq v$ ($u_j > v_j$ for at least one j, $1 \leq j \leq n$).

A^* (Anonymity): If two vectors u and v in E^n have the same components but permuted, then $u\,I^*\,v$ (that is $u\,R^*\,v$ and $v\,R^*\,u$).

ON^* (Ordinality and Noncomparability): for any ordinal transformations $\varphi_i, \varphi_2, \varphi_n$ (strictly increasing numerical functions) and for any u, v in E^n, $u\,R^*\,v$ if and only if $(\varphi_1(u_1), \varphi_2(u_2), \cdots, \varphi_n(u_n))R^*(\varphi_1(v_1), \varphi_2(v_2), \cdots, \varphi_n(v_n))$.

To illustrate the interest of a social welfare ordering R^* we may come back to the two-person case and the representation of some set of feasible utility vectors as the hatched set in Figure 4. Then the social welfare ordering R^* is represented by "social indifference curves" and the "best" utility vector is the point u^* on the Pareto frontier.

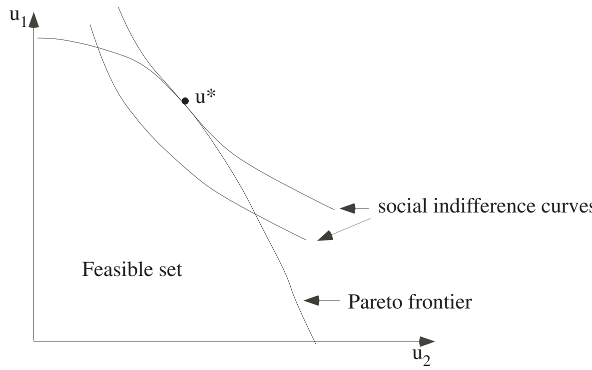

Figure 4:

One way to state Arrow's Impossibility Theorem is then the following:

Theorem 1 *If the social welfare ordering R^* satisfies SP^* and A^*, it cannot satisfy ON^*.*

Arrow's original theorem (1951-1963) is stated differently, in terms of individual preferences instead of individual utilities, and it is somewhat more general: it has a weaker Paretian condition and a weaker equity condition ("nondictatorship"). Also the proof given below for Theorem 1 is different: It will be based on a second theorem giving in analogous terms the characterization of classical utilitarianism, that is the social welfare ordering defined by the summation of individual utilities. The fact that the general impossibility applies to all rules including classical utilitarianism is well discussed in Arrow (1951) but, as he says, he has "not been able to construct a special proof of this fact for the sum of utilities which is essentially different from the proof of the general theorem" (p. 32). The proof below will answer the implicit question contained in this remark.

Of the above three conditions the one that should be incriminated from a utilitarian viewpoint is clearly the third. Indeed, to sum utilities across individuals requires that these utilities be cardinal and measured in the same units. Formally this amounts to replace ON^* by another condition, say:

CU^* (Cardinality and Unit Comparability): for any change of individual origins $a_1, a_2, \cdots, a_n$ (positive or nonpositive numbers), for any common change of unit b (a positive number) and for any u, v in E^n, $u\,R^*\,v$ if and only if

$$(a_1 + bu_1, a_2 + bu_2, \cdots, a_n + bu_n)\,R^*\,(a_1 + bv_1, a_2 + bv_2, \cdots, a_n + bv_n).$$

A result which is interesting to compare to Theorem 1 is the following (taken from d'Aspremont-Gevers (1977)).

Theorem 2 *The social welfare ordering R^* satisfies SP^*, A^* and CU^* if and only if it is the classical utilitarianism, i.e., for any u, v in $E^n, u\,R^*\,v$ if and only if $\sum_{i=1}^n u_i \geq \sum_{i=1}^n v_i$.*

The simultaneous considerations of the two theorems puts forward the main difference between the classical utilitarian approach and the New Welfare Economics approach. The first approach is characterized by the introduction of a particular kind of interpersonal comparisons, comparisons of differences in utilities (or marginal utilities), the other by the prohibition of any kind of interpersonal comparisons. This difference will become even clearer from the proofs.

Proof of Theorem 2. That classical utilitarianism satisfies SP^* and A^* is immediate. Also, for CU^*, one has that $\sum_{i=1}^n u_i \geq \sum_{i=1}^n v_i$ if and only if $\sum_{i=1}^n a_i + b\sum_{i=1}^n u_i \geq \sum_{i=1}^n a_i + b\sum_{i=1}^n v_i$, for any $a_1, a_2, \cdots, a_n$, and any $b > 0$. This proves the necessity of the three conditions. To prove their sufficiency one can adapt an argument due to Milnor (1954). First suppose u^0, v^0 in E^n satisfies $u^0 \neq v^0$ and $\sum_{i=1}^n u_i^0 = \sum_{i=1}^n v_i^0$. We want to show that $u^0\,I^*\,v^0$. This is done by a recursive argument implying two operations:

1. Re-order u^0 and v^0 as $u_{i,j}^0$ and $v_{i,j}^0$ in such a way that:

   $$u_{i(1)}^0 \leq u_{i(2)}^0 \leq \cdots \leq u_{i(n)}^0 \text{ and } v_{i(1)}^0 \leq v_{i(2)}^0 \leq \cdots \leq v_{i(n)}^0.$$

   Clearly by $A^*, u_{i,j}^0\,I^*\,u^0$ and $v_{i,j}^0\,I^*\,v^0$.

2. Construct u^1 and v^1 satisfying $\sum_{j=1}^n u_j^1 = \sum_{j=1}^n v_j^1$ and, for $j = 1, 2, \cdots, n$,

   $$u_j^1 = u_{i(j)}^0 - \min\{u_{i(j)}^0, v_{i(j)}^0\},$$
   $$v_j^1 = v_{i(j)}^0 - \min\{u_{i(j)}^0, v_{i(j)}^0\}.$$

Then u^1 and v^1 contains each (at least) one more zero component than u^0 and v^0 respectively and, by CU^*, $u^0\, I^*\, v^0$ if $u^1\, I^*\, v^1$. Repeating operations (1) and (2) n times (at most) we obtain a sequence of pairs of vectors: $(u^0, v^0), (u^1, v^1), (u^2, v^2), \cdots, (u^n, v^n)$ such that $u^0\, I^*\, v^0$ if $u^1\, I^*\, v^1$ if $\cdots u^n\, I^*\, v^n$. But in fact by construction $u^n\, I^*\, v^n$ (trivially) since $u^n = v^n = (0, 0, \cdots, 0)$. Here is an example to illustrate the procedure:

$$\begin{aligned}
u^0 &= (20, 7, 13), & u^0_{i(\cdot)} &= (7, 13, 20) \\
v^0 &= (9, 30, 1), & v^0_{i(\cdot)} &= (1, 9, 30), \\
u^1 &= (6, 4, 0), & u^1_{i(\cdot)} &= (0, 4, 6), \\
v^1 &= (0, 0, 10), & v^1_{i(\cdot)} &= (0, 0, 10), \\
u^2 &= (0, 4, 0), & u^2_{i(\cdot)} &= (0, 0, 4), \\
v^2 &= (0, 0, 4), & v^2_{i(\cdot)} &= (0, 0, 4), \\
u^3 &= (0, 0, 0), \\
v^3 &= (0, 0, 0).
\end{aligned}$$

Therefore $u^0\, I^*\, v^0$ whenever $\sum_{i=1}^n u_i^0 = \sum_{i=1}^n v_i^0$. What if instead $\sum_{i=1}^n u_i^0 > \sum_{i=1}^n v_i^0$? Then one may simply take w^0 such that $w_i^0 > v_i^0$ for all i and $\sum_{i=1}^n w_i^0 = \sum_{i=1}^n u_i^0$ (see Figure 5).

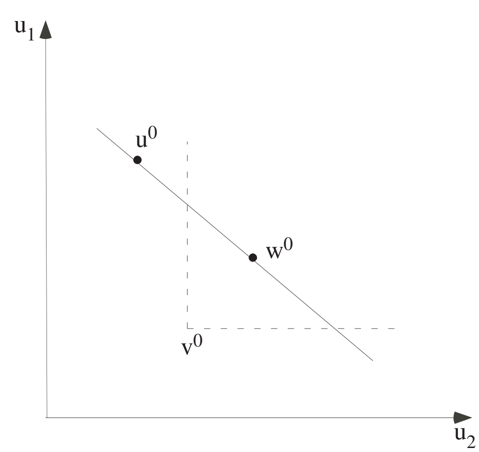

Figure 5:

Applying the same argument as before one gets $u^0\, I^* w^0$ and, applying SP^*, $w^0\, P^*\, v^0$ so that finally one gets $u^0\, P^*\, v^0$. Classical utilitarianism follows.

Proof of Theorem 1. Suppose R^* satisfies SP^*, A^* and ON^*. Then it satisfies SP^*, A^* and CU^* (ON^* obviously implies CU^*) so that, by Theorem 2, R^* is classical utilitarianism. Now, take any u, v in E^n such that $u_1 < v_1, u_j > v_j$ for all $j \neq 1$ and $\sum_{j=1}^n u_j \geq \sum_{j=1}^n v_j$. Then $u\, R^*\, v$ and, by ON^*,

$$(\lambda u_1, u_2, \cdots, u_n)\, R^*\, (\lambda v_1, v_2, \cdots, v_n),\ \text{for all}\, \lambda > 0.$$

But one cannot have

$$\lambda u_1 + \sum_{j \neq 1} u_j \geq \lambda v_1 + \sum_{j \neq 1} v_j,\ \text{for all}\, \lambda > 0\ (\text{since}\ v_1 > u_1)$$

in contradiction with R^* being classical utilitarianism. The inconsistency of SP^*, A^* and ON^* follows.

From this last argument one sees that Theorem 1 can be extended to weaker conditions than ON^* (but stronger than CU^*). In particular, ordinality plays no role. What really matters is that a monotone transformation (multiplying by a positive λ in this case) can be applied to the utility of one individual without being applied to any other. In conclusion Arrow's theorem tells us that in Welfare Economics it is not possible to go beyond the Pareto criterion by introducing equity considerations whenever interpersonal utility comparisons are meaningless. Classical utilitarianism is one way to make such comparisons. There are others.

4 Other social welfare orderings and inequality indices

An alternative way of introducing interpersonal comparisons has been defended more recently by Rawls (1972). In reaction to the hedonistic philosophy underlying the utilitarian approach, he proposes a conception of distributive justice based on a fair allocation of the primary goods supposed to provide every moral person with the means to achieve his own conception of the good. Although Rawls goes beyond welfarism by applying different principles (with an order of priority) to different kinds of primary goods, proposing pure egalitarianism for the first two kinds, namely the principles of equal liberty and of equality of chances, we may concentrate on the third principle “the difference principle” applying to all other kinds of primary goods. This last principle is an ordinal principle requiring interpersonal comparisons of an aggregate index (to avoid the term utility) of those primary goods: it says that for each social state individual levels should be ranked in increasing order according to this index and the Pareto efficient state chosen should be the one favouring the least advantaged individuals according to that order. In our notation, this lexicographic application of the maximin criterion (called “Leximin”) can be assimilated to the social welfare ordering R^* defined by the property that $u\, P^*\, v$ whenever the respective ranked vectors $u_{i(\cdot)}$ and $v_{i(\cdot)}$ (as in the proof of Theorem 2) satisfy:

$$\begin{aligned} &u_{i(j)} > v_{i(j)} \text{ for some } j, 1 \leq j \leq n, \\ &u_{i(k)} = v_{i(k)} \text{ for } k = 1, 2, \cdots, j-1. \end{aligned}$$

This criterion is the most egalitarian criterion compatible with the Pareto principle. It can be contrasted to another extreme, the most inequitable “Leximax” for which, in the definition, one starts instead from the most advantaged individual ($u_{i(k)} = v_{i(k)}$ for $k = n, n-1, \cdots, j+1$). Leximin (or Leximax, but we exclude it) shares many properties with Classical Utilitarianism. This is clear for SP^* and A^*. But this is also true for the fact that, in comparing two utility vectors, they only take into account unequal corresponding components, namely the property

SE^* (Separability): For any u, v and any u', v' in E^n, if $u_i = u'_i$ and $v_i = v'_i$ for all i in some set $S \subset \{1, 2, \cdots, n\}$ but $u_i = v_i$ and $u'_i = v'_i$ for all others (i not in S), then

$$u\, R^*\, v \textit{ if and only if } u'\, R^*\, v'.$$

The set S in this definition can be interpreted as the set of “concerned” (or non-indifferent) individuals: The condition says that only the utility levels of the concerned individuals should matter for social choice. It is easy to verify that Leximin as well as Classical Utilitarianism both satisfy SE^* (in addition to SP^* and A^*). Moreoveoro a theorem can be proved (see the proofs in d’Aspremont (1985) or d’Aspremont-Gevers (1977) using other characterizations of Leximin given by Hammond (1976) and Strasnick (1976)), showing that, ultimately, the only distinguishing property of Leximin with respect to Classical Utilitarianism is the sort of information and the type of interpersonal comparisons it relies on, namely the replacing of ON^* by

OC^* (Co-ordinality): For any common monotone transformation φ (an increasing numerical function), and any u, v in E^n, $u\, R^*\, v$ if and only if

$$\varphi(u_1), \varphi(u_2), \cdots, \varphi(u_n))\, R^*\, (\varphi(v_1), \varphi(v_2), \cdots, \varphi(v_n)).$$

The fact that, excluding Lexima, the Leximin criterion is characterized by SP^*, A^*, SE^* and OC^* and Classical Utilitarianism by SP^*, A^*, SE^* and CU^* stresses again the fact that the choice of the informational basis is crucial for social choice. Posing SP^*, A^* and SE^* as granted, if we take the informational basis corresponding to ON^* one gets Arrow's impossibility; if instead we allow interpersonal comparisons by CU^* or by OC^* then we can only get Classical Utilitarianism on one hand, Leximin or Leximax on the other.

To avoid such a drastic conclusion one can introduce still more discriminating interpersonal comparisons, allowing for a larger class of social welfare orderings. Actually this is standard practice in the literature on inequality indices. This literature is mainly concerned by the comparison in terms of equity of different income distributions. As such this implies specific kinds of informational invariance conditions. The main distinction is between a relative index or an absolute index of inequality. A relative index does not discriminate between two income distributions which differ only in their scale (or units) but it discriminates if the origin (say fixed at zero income) is changed. An absolute index does not discriminate between two income distributions, if they differ only by a common change of origin (say by adding the same amount of income to everyone) but discriminates if the unit is changed.

As shown by Kolm (1969), Atkinson (1970) and Sen (1973), in the tradition of Dalton (1920), it is possible to associate an inequality index to any social welfare ordering, thus giving a normative foundation to the choice of such index. Reciprocally, a social welfare ordering may be, in some way, associated to a chosen index of inequality (see Blackorby and Donaldson (1978, 1980)), thus allowing to introduce more judgements about inequality in the choice of a social welfare ordering. Let us illustrate this fact by looking at some relative indices of inequality.

Clearly if we want to associate social welfare orderings and relative indices of inequality such as the one used for income distributions, then the class of social welfare orderings considered must, taking "incomes" as proxies for utilities, be invariant to common unit changes. This is the condition:

RS (Ratio-Scale comparability). For any* u, v *in* E^n*, for any* $b > 0, u\,R^*\,v$ *if and only if* $(bu_1, bu_2, \cdots, bu_n)\,R^*\,(bv_1, bv_2, \cdots, bv_n)$*.*

Also it seems natural then to consider zero as the (fixed) common origin: We shall assume that R^* is defined only for positive utility vectors (denoting by E^n_{++} the positive orthant of E^n) and that it can be "represented" by a "social welfare function", i.e. for any u, v in E^n_{++}, $u\,R^*\,v$ if and only if $W(u) \geq W(v)$. Notice that by A^* the social welfare function W is symmetric and by SP^* it is increasing in all its arguments. We may also restrict our considerations to the case where W is a continuous function. More importantly we may introduce a condition which is traditionally imposed to inequality measurement, the so-called *Pigou-Dalton principle*, whereby a transfer of utility from one individual to another reducing the difference in their utility levels should increase (or at least not reduce) social welfare. There are in fact two other equivalent ways to state the Pigou-Dalton principle (see Dasgupta, Sen and Starrett (1973) and Moulin (1988)).

1. Lorenz Criterion. For any u and v in E^n_{++} $u\,P^*\,v$ (or at least $u\,R^*\,v$) whenever

$$\sum_{j=1}^{k} u_{i(j)} \geq \sum_{j=1}^{k} v_{i(j)} \text{ for } k = 1, 2, \cdots, n$$

and at least one inequality is strict.

This is nothing else then saying that the "Lorenz curve associated to w", that is the vector $(u_{i(1)}, u_{i(1)} + u_{i(2)}, \cdots, u_{i(1)} + u_{i(2)} + \cdots + u_{i(n)})$, dominates in the usual Pareto sense the "Lorenz curve associated to v", that is the vector $(v_{i(1)}, v_{i(1)} + v_{i(2)}, \cdots, v_{i(1)} + v_{i(2)} + \cdots + v_{i(n)})$. It can be shown that this is equivalent to the existence of a sequence of Pigou-Dalton transfers.

2. S-concavity of W. For any doubly-stochastic $n \times n$-matrix Q (i.e. the sum of the elements of each line = the sum of the elements of each column = 1, and each element is nonnegative) which is not a permutation (with only zeros and ones), for any u in E^n_{++},

$$W(Qu) > W(u) \text{ [or at least } W(Qu) \geq W(u)].$$

Of course when Q is a permutation, then by A^*, $W(Qu) = W(u)$. It can be shown that 2. is equivalent to 1.

Now if a social welfare function W, representing a social welfare ordering satisfying A^* and SP^*, is continuous and S-concave it is possible to construct an associated index of inequality (following Atkinson, Kolm and Sen) by defining first, for any u in E^n_{++}, the "equally distributed equivalent utility level" w_u, i.e. w_u is the positive level of utility such that

$$W(w_u, w_u, \cdots, w_u) = W(u).$$

Then inequality might be measured by the relative difference between the equally distributed equivalent utility level w_u and the mean utility level $\overline{u} = \frac{1}{n}\sum_{i=1}^{n} u_i$. In Figure 6 this difference is given by the ratio AB/OB.

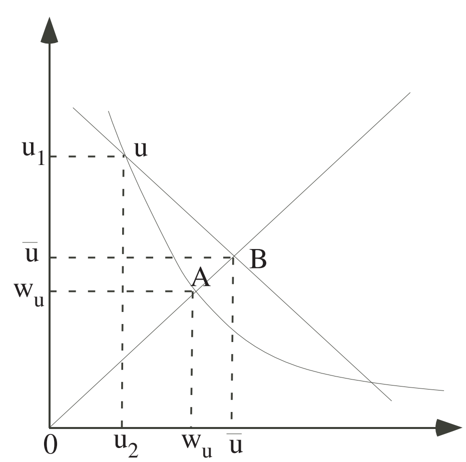

Figure 6:

Formally if R^* satisfies RS^* then W is "homothetic": for any u in E^n_{++}, $W(u) = \varphi[\tilde{W}(u)]$, with φ is a monotone transformation and $\tilde{W}$ a positive linear homogeneous function defined on E^n_{++}, i.e. $\tilde{W}(bu) = b\tilde{W}(u)$, for any $b > 0$. So, for any u, we can write

$$\tilde{W}(u) = \tilde{W}(w_u, w_u, \cdots, w_u) = w_u \tilde{W}(1, 1, \cdots, 1).$$

That is

$$w_u = \frac{\tilde{W}(w_u, w_u, \cdots, w_u)}{\tilde{W}(1, 1, \cdots, 1)}.$$

The relative index of inequality obtained (continuous and homogeneous of degree zero) is defined by:

$$I(u) = 1 - \frac{w_u}{\overline{u}} = 1 - \frac{\tilde{W}(w_u, w_u, \cdots, w_u)}{\tilde{W}(\overline{u}, \overline{u}, \cdots, \overline{u})}.$$

Clearly, $0 \leq I(u) \leq 1$ and $I(u) = 0$ if and only if $w_u = \overline{u}$.

Also, this formula allows to go the other way: From a relative inequality index to a social welfare function (or a social welfare ordering). To conclude on a more pragmatic note, we illustrate this possibility for two classes of inequality indices.

Example 1. *The Gini index (generalized).* For a sequence of n numbers $a_1 > a_2 \geq \cdots\cdots a_n$, define the index:

$$I_G(u) = 1 - \frac{\sum_{j=1}^n a_j u_{i(j)}}{\overline{u} \sum_{j=1}^n a_j}, \text{ for all } u \text{ in } E^n_{++}.$$

Then we can let, for all u in E^n_{++}:

$$W_G(u) = \tilde{W}_G(u) = \sum_{j=1}^n a_j u_{i(j)}.$$

For the original Gini Index the sequence of numbers is:

$$a_j = 2(n - j) + 1; \quad j = 1, 2, \cdots, n.$$

The index then corresponds to the area between the Lorenz curve of the equal distribution and the Lorenz curve of the given distribution u in E^n_+, both normalized so that the sum of individual utilities equals 1.

EXAMPLE 2. *Atkinson's indices.* This is

$$I_A(u) = 1 - \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{u_i}{\overline{u}}\right)^A\right]^{1/A} \quad \text{for } 0 < A < 1 \text{ or } A \neq 0.$$

We may let

$$\tilde{W}_A(u) = \left[\frac{1}{n}\sum_{i=1}^{m} u_i^A\right]^{1/A} = w_u \text{ (since } \tilde{W}_A(1, \cdots, 1) = 1)$$

and

$$\begin{aligned} W_A(u) &= \textstyle\sum_{i=1}^{n} u_i^A && \text{for } 0 < A < 1 \\ W_A(u) &= -\textstyle\sum_{i=1}^{n} u_i^A && \text{for } A < 0. \end{aligned}$$

Notice that for $A = 1$ we would have utilitarianism and that for $A \to -\infty$ we get close to the Leximin.

Hence these two examples show how the two theories, the theory of social welfare orderings and the theory of inequality indices, can reinforce each other.

References

Arrow, K.J. *Social Choice and Individual Values, 2nd ed.*. New Haven: Yale University Press.

Atkinson, A.B. On the measurement of inequality. *Journal of Economic Theory*, 2, 244–263, 1970.

Bentham, J. *An Introduction to the Principle of Morals and Legislation.* Payne, 1789; Oxford: Clarendon Press, 1907.

Blackorby, C. and D. Donaldson. Measures of relative equality and their meaning in terms of social welfare. *Journal of Economic Theory*, 18, 59–80, 1978.

Blackorby, C. and D. Donaldson. A theoretical treatment of indices of absolute inequality. *International Economic Review*, 21, 107–136, 1980.

Dalton, H. The measurement of the inequality of incomes. *Economic Journal*, 30, 1920.

Dasgupta, P., Sen, A. and D. Starrett. Notes on the measurement of inequality. *Journal of Economic Theory*, 180–187, 1973.

d'Aspremont, C. Axioms for social welfare orderings. In L. Hurwicz, D. Schmeidler and H. Sonnenschein (eds.), *Social Goals and Social Organization, Essays in Memory of Elisha Pazner*. Cambridge, Cambridge University Press, 1985.

d'Aspremont, C. and L. Gevers. Equity and the informational basis of collective choice. *Review of Economic Studies*, 44, 199-209, 1977.

Hammond, P.J. (1976). Equity, Arrow's conditions and Rawl's difference principle. *Econometrica*, 44, 793–804, 1976.

Hicks, J.R. Foundations of welfare economics. *Economic Journal*, 49, 696–712, 1939.

Hutcheson, F. An inquiry concerning moral good and evil, 1725.

Kaldor, N. Welfare proposition in economics. *Economic Journal*, 49, 1939.

Kolm, S.C. The optimum of social justice. In J. Margolis and H. Guitton (eds.), *Public Economics*. London: Macmillan, 1969.

Milnor, J. Games against nature. In R.M. Thrall, C.H. Coombs and R.L. Davis (eds.), *Decision Processes*. New York: John Wiley, 1954.

Moulin, H. *Axioms of Cooperative Decision*. Cambridge: Cambridge University Press, 1988.

Rawls, J. *A Theory of Justice*. Oxford: Clarendon Press.

Scitovsky, T. A note on welfare propositions in economics. *Review of Economic Studies*, 9, 77–88, 1941.

Scitovsky, T. A reconsideration of the theory of tariffs. *Review of Economic Studies*, 9, 89–110, 1942.

Sen, A.K. *On Economic Inequality.* Oxford, Clarendon Press; and New York: Norton, 1973.

Sen, A.K. Utilitariansim and welfarism. *The Journal of Philosophy*, 76, 463–489, 1979.

Strasnick, S.L. Social choice and the derivation of Rawls's difference principle. *The Journal of Philosophy*, 73, 85–99, 1976.