Equity and the Informational Basis of Collective Choice*

Claude d'Aspremont Louis Gevers

October 1976

We consider the problem of a planner or ethical observer who wants to derive a collective preference ordering over a set of feasible alternatives from the knowledge of individual utility functions. By assumption, he is concerned with social welfare judgements, not with committee decisions.[1]

As a tool of analysis, we use the concept of social welfare functional (SWFL), which was developed by Sen [9] on foundations originally laid down by Arrow [1]. Rather than to compare SWFL's directly, we treat them somewhat like composite goods and we compare sets of axioms which characterize them. We select five such sets, which differ mainly with respect to the planner's informational basis. This term refers to an "invariance" axiom which defines in each case the measurability and comparability properties of individual utility functions.

Taking up a suggestion of Sen's [10], we focus our attention on the implications of each informational basis for the equity content of collective choice.

Our study does not treat all possible invariance axioms; it does not even exhaust all the most relevant ones. However, we think that it brings about some logical clarification. Among other things, we characterize utilitarianism and the leximin (or lexical maximin) principle by means of two sets of axioms which differ only in one respect, viz. the invariance axiom.

The paper is divided into three sections. In Section 1 we describe our problem formally, we discuss our invariance axioms, and we show that some of them are equivalent, in the light of the

* *Review of Economic Studies*, **44(2)**, 199-209, 1977.

[1] On this distinction, see Sen [11].

axiom of independence of irrelevant alternatives.

Section 2 proceeds from Arrow's celebrated theorem, which is based on the prohibition of interpersonal welfare comparisons. It then moves to two invariance axioms which respectively allow for interpersonal comparisons of welfare *gains* and for interpersonal comparisons of welfare *levels*. With help of the former axiom, we get a characterization of utilitarianism which is adapted from Milnor [7]. With the help of the latter, we characterize the leximin principle, making use of a theorem due to Hammond [3]. Our result is close to a theorem arrived at independently by Strasnick [13].

Proofs, technical lemmas and some heuristic comments are contained in Section 3.

1 General

We consider a set of individuals $N = \{1, 2, ..., i, j, ..., n\}$ which is finite, and a set of feasible social states, $X = \{x, y, z, ...\}$ which consists of s elements. This number may be infinite or finite. In any case, we assume $s \geq 3$.

We let $\Re$ be the set of all orderings[2] on X. For every $R \in \Re$, $\forall\, x, y \in X$, xRy means that x is at least as good as y from the collective standpoint, as the planner sees it. We denote strict preference and indifference by P and I, respectively.

We let U be the set of all numerical bounded functions which may be defined on $X \times N$. For every $u \in U$, $\forall i \in N$, $\forall x, y \in X$, $u(x, i) \geq u(y, i)$ means x is at least as good as y from the point of view of individual i, as the planner sees it. Thus, it is legitimate to interpret $u(\cdot, i)$ as agent i's utility function, seen through the planner's eyes.

To use Sen's [9] terminology, a social welfare functional (SWFL) is a function f from U to $\Re$. For every $u, u^1, u^2 \in U$ we shall write $R = f(u), R^1 = f(u^1)$, $R^2 = f(u^2)$, etc.

We proceed by describing some "reasonable" conditions or axioms one might wish to impose on SWFL's. Let us first specify the planner's informational basis. By this term, we mean his ability to discriminate more or less finely among the elements of U. It is embodied in an

[2] An ordering is defined as a total, reflexive and transitive binary relation on X (see Sen [9, p. 9]).

invariance axiom. We shall describe briefly six such axioms. For a fuller treatment, the reader is referred to Sen [12].

At one end of the spectrum of relevant possibilities, we may, as K. J. Arrow [1] did in his pioneering work, assume that individual utility functions are measurable up to a positive monotonic transformation and that interpersonal comparisons are ruled out.

In other words, individual utility functions are *ordinal* and *non-comparable*. Formally,

ON: *For every* u^1, $u^2 \in U$, $R^1 = R^2$, *if,* $\forall\, i \in N$, *there exists a strictly increasing numerical function* ϕ_i *such that,* $\forall\, x \in X$, $u^1(x,i) = \phi_i(u^2(x,i))$.

Weaker invariance axioms may be obtained by introducing cardinality or interpersonal comparability, or both. For instance, if we introduce interpersonal welfare comparisons while maintaining ordinality, we get, in Hammond's terms, an axiom of *coordinality*:

CO: *For every* $u^1, u^2 \in U, R^1 = R^2$ *if there exists a strictly increasing numerical function* ϕ *such that,* $\forall\, i \in N$, $\forall\, x \in X$,

$$u^1(x,i) = \phi(u^2(x,i)).$$

It should be clear that CO allows for interpersonal comparisons of welfare levels. As such, it rules out interpersonal comparisons of welfare gains.

Another way to relax ON is to assume that individual utility functions are *cardinal* and *non-comparable*. Formally, we get

CN: *For every* $u^1, u^2 \in U$, $R^1 = R^2$ *if there exist 2 n numbers* $\alpha_1, ..., \alpha_n$, $\beta_1 > 0, \cdots, \beta_n > 0$ *such that,* $\forall\, i \in N, \forall\, x \in X, u^1(x,i) = \alpha_i + \beta_i u^2(x,i)$.

If we require all α_i to be the same and all β_i to be the same, we impose in effect a common origin and a common scale to all individual utility functions. Thus, both comparisons of welfare levels and comparisons of welfare gains are allowed for among individuals. Utility functions are cardinal and fully comparable, or *co-cardinal.*

CC: *For every* $u^1, u^2 \in U, R^1 = R^2$ *if there exist 2 numbers* α *and* $\beta > 0$, *such that,* $\forall\, i \in N$, $\forall\, x \in X$,

$$u^1(x,i) = \alpha + \beta u^2(x,i).$$

Between CN and CC, we shall single out two interesting cases, although they may seem less natural than CC in the context of our problem. In the first case, the origin of individual utility functions is common while the scale factors may vary from individual to individual. We call this the axiom of invariance with respect to *individual units of measurement.*

IUM: *For every* $u^1, u^2 \in U, R^1 = R^2$ *if there exist* $1+n$ *numbers* α, $\beta_1 > 0, \cdots, \beta_n > 0$ *such that,* $\forall\, i \in N$, $\forall\, x \in X$,

$$u^1(x,i) = \alpha + \beta_i u^2(x,i).$$

In the other case of cardinality with imperfect interpersonal comparability, we assume that the origin of utility functions may vary, while their scale factor must be common. In this case, interpersonal comparisons of welfare *gains* are permitted, while interpersonal comparisons of welfare levels are prohibited. We call this the axiom of invariance with respect to *individual origins of utilities.*

IOU: *For every* $u^1, u^2 \in U, R^1 = R^2$ *if there exist* $n+1$ *numbers* $\alpha_1, \cdots, \alpha_n$, *and* $\beta > 0$ *such that,* $\forall\, i \in N$, $\forall\, x \in X$,

$$u^1(x,i) = \alpha_i + \beta u^2(x,i).$$

In Table I we compare our six invariance axioms with respect to both measurability and interpersonal comparability. An arrow between axioms indicates logical implication. The weaker an invariance axiom is, the more natural it becomes to view the aggregation process as based on subjective welfare evaluation. We take these subjective judgements as given, and we limit our discussion to the aggregation procedure proper.

Table I

	Comparability		
Measurability	Non-existent	Intermediate	Full
Ordinal	ON	$\cdots\cdots\cdots\cdots\cdots\rightarrow$	CO
	$\downarrow$		$\downarrow$
Cardinal	CN $\cdots\rightarrow$	$\left\{\begin{array}{c}\text{IOU}\\ \text{IUM}\end{array}\right\}\cdots\rightarrow$	CC

With our comprehensive definition of U, it turns out that IUM and CN are equivalent.[3] Formally, we can assert the following

Theorem 1 *A* SWFL *satisfies* IUM *if, and only if, it satisfies* CN*. This property remains valid when the range of the* SWFL *is widened.*

Another equivalence between invariance axioms shows up when f is required to satisfy a principle of independence of irrelevant alternatives. A short discussion of the latter is thus in order. When an individual is to rank the elements of a possibility set, his preferences with respect to elements which lie outside it should be considered as irrelevant. It seems reasonable to extend this idea to individual preferences aggregation. There are various ways of formalizing this principle which are more or less stringent. For every integer m such that $1 < m < s$, one can define what Blau [2] has called an *m-ary principle of independence of irrelevant alternatives:*

IR$_m$: *For every* $u^1, u^2 \in U$, $\forall\, B \subset X$, *such that* $|B| = m$, $R^1 = R^2$ *on* B *whenever* $u^1 = u^2$ *on* $B \times N$.

The most demanding member of this family of axioms would seem to be IR$_2$, as it clearly implies the other members, while the converse is not so evident. If IR$_2$ is adopted, all the relevant information pertaining to collective choice within any pair $\{x, y\}$ is contained in the two lists of utility levels $u(x,i), u(y,i), i = 1, ..., n$. As Sen [9] observed, CN must then be equivalent to ON. Indeed, one needs at least three distinct measurements to get a nontrivial comparison of

[3]This property was suggested to us by A. K. Sen.

gains. However, Sen's argument is not enough to get this equivalence when IR_m with $m > 2$ is adopted.

Now Blau [2] has shown that for an Arrow social welfare function, an m-ary independence principle is equivalent to a binary principle. His arguments apply also to SWFL's, whatever the invariance axiom may be, even when a wider definition of their range is adopted. Hence, whatever m-ary principle of independence is chosen, CN and ON are equivalent. Let us record these facts formally.

Lemma 1 *For every integer* $m, 1 < m < s-1$, f *satisfies* IR_m *if, and only if, it satisfies* IR_{m+1}.

In view of the above lemma, we let the reader choose the m-ary independence principle he pleases, provided $1 < m < s$, and we denote it by IR, for brevity's sake. Finally, using again Sen's argument, Lemma 1 trivially implies:

Theorem 2 *A* SWFL *satisfying* IR *satisfies* CN *if, and only if, it satisfies* ON.

By Theorems 1 and 2 our six initial invariance axioms may be reduced to four, viz. ON, IOU, CO and CC.

We turn now to the strong version of the Pareto principle, which we shall use repeatedly.

SP: *For every* $x, y \in X$, $\forall\, u \in U$, xRy *if,* $\forall\, i \in N, u(x,i) \geq u(y,i)$*; and if, moreover, for some* $j \in N$, $u(x,j) > u(y,j)$*, then* xPy.

Taken together, IR and SP imply a property which we call *extended neutrality* (XNE). It says that all the information which matters for collective choice is captured by the relevant utility levels. In particular, state labels do not matter. Formally, f satisfies XNE if, and only if, $\forall\, x, y \in X, \forall\, u^1 \in U$, xR^1y (respectively xP^1y), whenever there exist $w, z \in X$, and $u^0 \in U$, such that $\forall\, i \in N, u^1(x,i) = u^0(w,i), u^1(y,i) = u^0(z,i)$ and wR^0z (respectively wP^0z). It is worth observing that XNE implies IR_2, so that if may be considered as more demanding than May's [6] neutrality axiom. From another viewpoint, the latter axiom is much stronger than XNE, as it implies ON, while XNE does not imply any invariance axiom.

Lemma 2 *If f satisfies* IR *and* SP*, it satisfies* XNE*.*

To conclude this section we want to mention that if $f$satisfies XNE, one may associate with it a binary relation R^* on the utility space, which is an ordering if R is an ordering. This ordering is an essential link between the SWFL and the familiar concept of social welfare function as it was formulated by Bergson.

Let E^N be the n-dimensional Euclidean space, where each coordinate bears the name of a distinct individual. We define on E^N a binary relation R^* as follows: $\forall\, a, b \in E^N$, aR^*b iff there exist $x, y \in X$, and $u \in U$, such that $\forall\, i \in N$, $u(x, i) = a_i, u(y, i) = b_i$ and xRy.

Lemma 3 *If f satisfies XNE, R^* is an ordering on E^N.*

2 Utilitarianism and the leximin principle

As a starting-point, we would like to remind the reader that ruling out interpersonal welfare comparisons is potentially quite dangerous from the point of view of equity. Indeed, we may interpret Arrow's classical theorem as follows: if we require our SWFL to satisfy both IR and SP, then we know that the only f which satisfies ON (or, equivalently, CN or IUM) is dictatorial.

Under dictatorship, there is an individual which is given all the weight in the aggregation process. Nothing accounts for this privilege, but his name. Equity would seem to require that collective choice be unchanged when individuals exchange position: what really matters in each state is the list of individual utility levels, not the names attached to them. This idea is captured by the well known *anonymity* axiom.

A: *Let σ be any permutation of N; $\forall\, u^0, u^1 \in U, R^0 = R^1$ if u^0, u^1 are such that, $\forall\, i \in N$, $\forall\, x \in X$,*

$$u^0(x, i) = u^1(x, \sigma(i)).$$

By allowing the planner to compare welfare gains interpersonally, we get in position to move away from dictatorship to utilitarianism. Our theorem is a straightforward adaptation of a result presented by Milnor [7] in the context of the theory of decision under uncertainty.

In the narrow sense of the term, we define *utilitarianism* as the SWFL which is such that, $\forall\, u \in U, \forall\, x, y \in X, xRy$ if, and only if,

$$\sum_i u(x,i) \geq \sum_i u(y,i).$$

Theorem 3 *Utilitarianism is characterized by* IR*,* SP*,* A *and* IOU*.*

In the pure income distribution problem, diminishing marginal utility is a natural assumption which guarantees equal incomes for any two individuals who may be distinguished only by their names. This is an attractive feature from the point of view of equity.

On the other hand, utilitarianism implies a complete lack of concern for the distribution of utility levels. Many authors criticize it on this ground. See for instance Rawls [8] and Sen [10]. This feature is of course built in the invariance axiom IOU.

Our next step is to substitute coordinality for IOU. We require again our SWFL to satisfy IR, SP and A. Another axiom proves very useful in the present context. It deals with the effect on collective choice of individuals, who are indifferent between all states in X. Sen [9] aptly calls them unconcerned individuals. It would seem unreasonable to allow the welfare level of unconcerned individuals to influence collective choice. Thus we introduce an axiom of *separability* with respect to unconcerned individuals.

SE: *For every u^1, $u^2 \in U$, $R^1 = R^2$ if there exists $M \subset N$, for which, $\forall\, i \in M, \forall\, x \in X$, $u^1(x,i) = u^2(x,i)$, while, $\forall\, h \in N \setminus M$, $\forall\, x, y \in X$, $u^1(x,h) = u^1(y,h)$ and*

$$u^2(x,h) = u^2(y,h).$$

It is easy to see that all invariance axioms which prohibit interpersonal comparisons of welfare *levels* imply SE. Combining SE with CO, IR, SP and A has drastic consequences from the point of view of equity. This is for instance brought forth in cases where only two individuals are in conflict about two states. The other individuals being indifferent, we further assume that one of the conflicting individuals is worse off than the other in both states. *Extremist equity* considerations would let the worst off individual win in all cases.

XE: *For every* $u \in U$, $\forall\, x, y \in X, \forall\, i, j \in N$, xPy *whenever,* $\forall\, g \in (N \setminus \{i, j\}), u(x, g) = u(y, g)$, *and* $u(y, i) < u(x, i) < u(x, j) < u(y, j)$.

In total contrast with XE, the planner might decide that the better off individual wins in all cases. We would have thus the following *inequity* principle.

IN: *For every* $u \in U$, $\forall\, x, y \in X, \forall\, i, j \in N$, yPx *whenever,* $\forall\, g \in (N \setminus \{i, j\}), u(x, g) = u(y, g)$, *and* $u(y, i) < u(x, i) < u(x, j) < u(y, j)$.

We are now ready for

Theorem 4 *If* f *satisfies* IR, SP, A, CO *and* SE, *either it satisfies* XE *or it satisfies* IN.

Despite some cogent examples due to Harsanyi [4], some people may find XE appealing. However, the full force of its extremism becomes clear when the worst off individual is in conflict with all the more favoured individuals. Indeed, under our assumptions, his strict preference is always endorsed by the collective ordering. The notion of dictatorship which applies to individuals in Arrow's framework extends naturally to ranks in the present context.

Formally, let $\underline{N} = \{1, 2, \cdots, n\}$ be the set of ranks. Given any rank $h \in \underline{N}$, we need identify whoever occupies it in any particular state. For this purpose, we define for every $u \in U$, $\forall\, x \in X$, a one-to-one function i_x from $\underline{N}$ to N, which satisfies the following condition: $\forall\, h, k \in \underline{N}, u(x, i_x(h)) < u(x, i_x(k))$ implies $h < k$. If there are ties, several i_x functions satisfy this requirement. Whichever function is selected has, as it will become clear, no bearing on the final outcome.

The leximin (or lexical maximim) principle is the SWFL selecting as a dictator the least favoured non-indifferent rank. It was defined by Sen [9] in relation to Rawls' [8] work. It was further studied by Kolm [5], among others.

The *leximin principle* is defined as follows: $\forall\, u \in U, \forall\, x, y \in X$, xPy if, and only if, $\exists$, $m \in \underline{N}$ such that, $\forall\, h \in \underline{N}$, $h < m, u(y, i_y(h)) = u(x, i_x(h))$, and $u(y, i_y(m)) < u(x, i_x(m))$. The strength of XE is manifested by the fact that the following theorem does not rely on any invariance axiom. The separability axiom is also conspicuously absent.

Theorem 5 *The leximin principle is characterized by* IR, SP, A *and* XE.

The argument used in proving this theorem is due to Hammond [3], who points out in turn his debt to Strasnick [13]. Going back to the definition of leximin, and writing $m < h$ instead of $h < m$, we get the formal definition of what may be called the *leximax principle*, a SWFL based on the dictatorship of the most favoured non-indifferent rank. The following theorem is symmetric to Theorem 5.

Theorem 6 *The leximax principle is characterized by* IR, SP, A *and* IN.

We next introduce a *minimal equity* axiom which requires no comments.

ME: *The* SWFL *is not the leximax principle.*

Finally, as an easy consequence of the last three theorems, we get a new characterization of the leximin principle.

Theorem 7 *The leximin principle is characterized by* IR, SP, A, CO, SE *and* ME.

We may now usefully compare utilitarianism with the leximin principle by collecting in Table II the most important results of this paper.

Axioms, which taken together, characterize a SWFL, are marked with a star in the table. Axioms which are consistent (respectively inconsistent) with a SWFL get a + (respectively −) mark. We observe that the consistency of utilitarianism with ME does not mean that for all members of U the leximax principle and utilitarianism give different collective orderings. We also observe that the only source of discrepancy lies in the informational basis of each SWFL. If the cost of gathering and processing information is high, one may be tempted to adopt the leximin principle on the ground that CO is less demanding in this respect than IOU. We feel, however, that, once interpersonal comparisons are allowed for, informational cost differentials are illusory.

TABLE II

	Utilitarianism	Leximin principle
IR	$*$	$*$
SP	$*$	$*$
A	$*$	$*$
SE	$+$	$*$
ME	$+$	$*$
IOU	$*$	$-$
CO	$-$	$*$

The only natural form of invariance would then be expressed by the axiom of co-cardinality. We conclude that it would be a worthwhile task to find out which SWFL's, besides utilitarianism and the leximin principle, satisfy CC, IR, SP, A, SE and ME.

3 Technical

Proof of Theorem 1. We need only prove that IUM implies CN. Choose any $u^1 \in U$, any $(\beta_1, \cdots, \beta_n) \in E^n_+$ and any $(\alpha_1, \cdots, \alpha_n) \in E^n$.

Select $u^0 \in U$, such that $\forall\, z \in X$, $\forall\, i \in N$, $u^0(z,i) = \alpha_i + \beta_i u^1(z,i)$. We must show that if f satisfies IUM, then $R^1 = R^0$.

Select $u^2 \in U$, such that $\forall\, z \in X$, $i \in N$, $u^2(z,i) = 1 + \beta^1_i u^1(z,i)$ where, for some $\theta < \min\limits_{j \in N}\{\alpha_j\}$,

$$\frac{1}{\beta^1_i} = \frac{1}{\beta_i}(\alpha_i - \theta).$$

By construction, we have $\beta^1_i > 0$, $\forall\, i \in N$; hence, by IUM, $R^2 = R^1$. Also we can write, $\forall\, i \in N$,

$\forall\, z \in X$,

$$\begin{aligned} u^1(z,i) &= \frac{1}{\beta_i^1}(u^2(z,i)-1) \\ u^0(z,i) &= \alpha_i + \beta_i u^1(z,i) \\ &= \alpha_i + \frac{\beta_i}{\beta_i^1}(u^2(z,i)-1) \\ &= \alpha_i + (\alpha_i - \theta)(u^2(z,i)-1) \\ &= \theta + (\alpha_i - \theta)u^2(z,i). \end{aligned}$$

In view of IUM, we have $R^0 = R^2$. We conclude that $R^0 = R^2 = R^1$. ∎

Proof of Lemma 1. We only prove sufficiency. Consider $u^1, u^2 \in U$, such that $u^1 = u^2$ on $A \times N$, where $A \subset X$ and $|A| = m$. Since $m < s - 1$, it is possible to find $c, d \in X \setminus A$, $c \neq d$, and to find $u \in U$, such that

$$u = u^1 \text{ on } (A \cup \{c\}) \times N,$$

and

$$u = u^2 \text{ on } (A \cup \{d\}) \times N.$$

By IR_{m+1},

$$f(u) = f(u^1) \text{ on } A \cup \{c\}, \text{ and hence on } A,$$

and

$$f(u) = f(u^2) \text{ on } A \cup \{d\}, \text{ and hence on } A.$$

Therefore, $\forall\, x, y \in A$, xR^1y iff xRy iff xR^2y. ∎

Proof of Lemma 2. Three cases must e distinguished, according as $\{x,y\} \cap \{w,z\}$ is empty, has one element or two. Consider the last case; suppose, $\forall\, i \in N$,

$$u^1(x,i) = u^0(y,i) = a_i \text{ and } u^1(y,i) = u^0(x,i) = b_i.$$

By assumption, $s > 2$. In view of our comprehensive definition of U, we may choose $v \in X$, $x \neq v \neq y$, and $u^2, u^3, u^4 \in U$, as described in Table III.

TABLE III

	u^0	u^1	u^2	u^3	u^4
x	b	a	a	b	b
y	a	b	b	b	a
v	$*$	$*$	a	a	a

Now by IR we can write:

$$\begin{aligned} xR^1y &\quad \text{iff} \quad xR^2y \\ vR^2y &\quad \text{iff} \quad vR^3y \\ vR^3y &\quad \text{iff} \quad vR^4x \\ yR^4x &\quad \text{iff} \quad yR^0x. \end{aligned}$$

By SP we can also write:

$$\begin{aligned} xR^2y &\quad \text{iff} \quad vR^2y \\ vR^3y &\quad \text{iff} \quad vR^3x \\ vR^4x &\quad \text{iff} \quad yR^4x. \end{aligned}$$

Combining these equivalences, we get:

$$xR^1y \text{ iff } yR^0x.$$

In this set of equivalent assertions, strict collective preference may obviously be substituted for weak collective preference. When $\{x, y\}$ and $\{w, z\}$ have fewer common elements, analogous, but shorter, arguments lead to the desired conclusion. The strict preference case is proved along the same lines. ∎

Proof of Lemma 3. In view of our definition of U, and because R is total and reflexive, the relation R^* is total and reflexive. To establish transitivity, take any $a, b, c \in E^N$ such that aR^*b and bR^*c. By our definition of U, there exist $x, y, z \in X$ and $u \in U$, such that, $\forall\, i \in N$, $a_i = u(x, i), b_i = u(y, i)$, and $c_i = u(z, i)$. By XNE, xRy and yRz. As R is an ordering, we get xRz, which in turn implies aR^*c. Hence R^* is an ordering. ∎

We turn next to a useful implication of IR, SP and A.

Lemma 4 *Suppose f satisfies* IR, SP *and* A. *Then,* $\forall\, u \in U$, $\forall\, x, y \in X$, xIy *if there exists a permutation σ of N such that,* $\forall\, i \in N$, $u(x,i) = u(y, \sigma(i))$.

Proof. Consider first the simple case in which only two agents exchange welfare levels. Suppose for some $i, j \in N, u(x,i) = u(y,j) = \alpha$ and $u(x,j) = u(y,i) = \beta$, while

$$\forall\, g \in N \setminus (i, j\}, u(x,g) = u(y,g) = \gamma_g.$$

Select $u^0, u^1 \in U$ such that, for every $g, i \neq g \neq j$, the utility level is γ_g in all cases and all states, and for i and j, we assume

$$\forall\, z \in X \setminus \{x, y\}, u^0(z,i) = u^0(z,j) = u^1(z,i) = u^1(z,j),$$

while their preferences with respect to x and y are described in Table IV. We want to show that, if xIy is false, we must have a contradiction. Suppose xPy. By IR, this implies xP^0y. Now, by A, xP^0y implies xP^1y. However, by Lemma 2, xP^1y implies yP^0x, which contradicts xP^0y.

TABLE IV

	u		u^0		u^1	
	i	j	i	j	i	j
x	α	β	α	β	β	α
y	β	α	β	α	α	β

Next, we move to general permutations. Consider the ordering R^* on E^N which is associated with f. We have just shown that if any $b, d \in E^N$ are the same except for an exchange of elements between two columns, one must have bI^*d. As N is finite, any permutation of elements of b may be obtained by a sequence of at most n successive permutations, each involving only two columns. As R^* is an ordering by Lemma 3, social indifference is preserved along the sequence. By Lemma 2, x and y must be socially indifferent, if they satisfy the condition mentioned in the theorem. ∎

Our next lemma deals with a list of properties which the ordering or the utility space R^* inherits from f.

SP*: *For every* $a, b \in E^N$*,* aR^*b *if,* $\forall\, i \in N$*,* $a_i \geq b_i$*; and if, moreover* $a_j > b_j$ *for some* $j \in N$*, then* aP^*b*.*

IOU*: *For every* $a^0, a^1, b^0, b^1 \in E^N$*,* $a^0R^*b^0$ *if, and only if,* $a^1R^*b^1$*, whenever there exist* $n+1$ *numbers* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *and* $\beta > 0$*, such that,* $\forall\, i \in N$*,*

$$a_i^0 = \alpha_i + \beta\, a_i^1 \text{ and } b_i^0 = \alpha_i + \beta b_i^1.$$

CO*: *For every* $a^*, a^1, b^*, b^1 \in E^N$*,* $a^0R^*b^0$ *if and only if,* $a^1R^*b^1$*, whenever there exists a strictly increasing numerical function* ϕ *such that,* $\forall\, i \in N$*,*

$$a_i^0 = \phi(a_i^1) \text{ and } b_i^0 = \phi(b_i^1).$$

A*: *For every* $a, b \in E^N$*,* aI^*b *if there exists a permutation* σ *of* N *such that,* $\forall\, i \in N$*,* $a_{\sigma(i)} = b_i$*.*

In view of the above discussion, we leave it to the reader to prove:

Lemma 5 *The ordering* R* *satisfies* SP* *if* f *satisfies* IR *and* SP*; if* f *satisfies also* IOU *(respectively* CO, A*), then* R^* *satisfies* IOU* *(respectively* CO**,* A^**).*

It is now a straightforward task to use Milnor's [7] argument together with the above lemmas, in order to characterize utilitarianism.

Proof of Theorem 3. The proof of necessity is left to the reader. To prove sufficiency, consider any $a, b \in E^N$ such that $\sum_{i\in N} a_i = \sum_{i\in N} b_i$.

Let a^0 (respectively b^0) $\in E^N$ be obtained by permuting the elements of a (respectively b) so that they are in order of increasing size. By Lemma 5, we get aI^*a^* and bI^*b^0. Select next a^1 (respectively b^1) $\in E^N$ by subtracting, $\forall\, i \in N$, $\min\{a_i^0, b_i^0\}$ from a_i^0 (respectively b_i^0). By Lemma 5, $a^0I^*b^0$ if $a^1I^*b^1$.

By repeating this process a finite number m of steps, one gets $a^m = b^m = (0, \cdots, 0)$. By reflexivity (Lemma 3), $a^m I^* b^m$. Moving back along the chain of implications, we get, by transitivity (Lemma 3), aI^*b. We conclude that $\forall\, u \in U$, $\forall\, x, y \in X$, xIy if

$$\sum_{i \in N} u(x,i) = \sum_{i \in N} u(y,i).$$

If $\sum_{i \in N} u(x,i) > \sum_{i \in N} u(y,i)$, one has to consider R^* again. By Lemmas 5 and 3, we may use SP* and transitivity of R^* to get xPy. ■

Further properties with R^* inherits from f are SE*, XE*, and IN*.

SE*: *For every $a^0, a^1, b^0, b^1 \in E^N$, $a^0 R^* b^0$ if $a^1 R^* b^1$ whenever there exists $M \subset N$, for which, $\forall\, i \in M, a_i^0 = a_i^1$ and $b_i^0 = b_i^1$, while $\forall\, h \in (N \setminus M), a_h^0 = b_h^0$ and $a_h^1 = b_h^1$.*

XE*: *For every $a, b \in E^N$, $\forall\, i, j \in N$, aP^*b whenever, $\forall\, g \in (N \setminus \{i,j\}), a_g = b_g$ and*

$$b_i < a_i < a_j < b_j.$$

IN*: *For every $a, b \in E^N$, $\forall\, i, j \in N$, bP^*a whenever, $\forall\, g \in (N \setminus \{i,j\}), a_g = g_g$ and*

$$b_i < a_i < a_j < b_j.$$

We leave it to the reader to prove:

Lemma 6 *Whenever f satisfies* IR *and* SP*, R^* satisfies* SE* *(respectively* XE*, IN**) if, and only if, f satisfies* SE *(respectively* XE, IN*).*

Given any utility vector, we shall need to find out which individual occupies which welfare rank. For this purpose, for every $a \in E^N$, we define a function i from $\underline{N}$ to N satisfying

$$a_{i(h)} < a_{i(k)} \to h < k.$$

Lemma 7 *If f satisfies* IR, SP, CO, A *and* SE*, either, $\forall\, h, k \in \underline{N}$, $h < k$, $\forall\, a, b \in E^N$ such that $b_{i(h)} < a_{i(h)} < a_{i(k)} < b_{i(k)}$ and $\forall\, g \in (\underline{N} \setminus \{h,k\}), b_{i(g)} = a_{i(g)}$, aP^*b, or, under the same conditions, bP^*a.*

Proof. Select any $a^0, b^0, a^1, b^1 \in E^N$ satisfying the conditions of the lemma.

By Lemma 3, we know that either $a^0 P^* b^0$, or $b^0 P^* a^0$, or $a^0 I^* b^0$. Suppose $a^0 P^* b^0$. By the conditions of the lemma, there exists a strictly increasing function ϕ such that,

$$\forall\, g \in \{h, k\},\ a^0_{i(g)} = \phi(a^1_{i(g)}) \text{ and } b^0_{i(g)} = \phi(b^1_{i(g)}).$$

Select now $a^2, b^2 \in E^N$ such that,

$$\forall\, g \in \underline{N},\ \phi(a^2_{i(g)}) = a^0_{i(g)} \text{ and } \phi(b^2_{i(g)}) = b^0_{i(g)}.$$

By Lemma 5, R^* satisfies A^* and CO^*, so that, $a^0 P^* b^0$ implies $a^2 P^* b^2$. By Lemma 6, R^* satisfies SE*. Thus by A* and SE*, $a^2 P^* b^2$ implies $a^1 P^* b^1$, so that $a^0 P^* b^0$ implies $a^1 P^* b^1$.

One could show equally easily that $a^0 I^* b^0$ implies $a^1 I^* b^1$, and that $b^0 P^* a^0$ implies $b^1 P^* a^1$. By A* and SE*, if these results hold for some $h, k \in \underline{N}, h < k$, it is also true for all $r,\ s \in \underline{N}$ such that $r < s$.

To complete the proof, there remains to eliminate the possibility of indifference. Thus, assume that $a^0 I^* b^0$ and select $c \in E^N$ as follows:

$$b^0_{i(h)} < c_{i(h)} < a^0_{i(h)}$$

and $\forall\, g \in (\underline{N} \setminus \{h\}), c_{i(g)} = a^0_{i(g)}$.

By our previous argument, we get $cI^* b^0$. By transitivity, $cI^* a^0$, which contradicts SP*. Thus, the lemma is established. ■

Proof of Theorem 4. Consider any $i, j \in N$, any $a, b \in E^N$ such that,

$$\forall\, g \in (N \setminus \{i, j\}),\ a_g = b_g \text{ and } b_i < a_i < a_j < b_j.$$

Select $a^0, b^0 \in E^N$ such that

$$b^0_i = b_i, b^0_j = b_j, a^0_i = a_i, a^0_j = a_j \text{ and } \forall\, g \in (N \setminus \{i, j\}), a^0_g = b^0_g = \beta < b_i.$$

By Lemma 6, SE* holds; thus, aR^*b if, and only if, $a^0 R^* b^0$ and bR^*a if, and only if, $b^0 R^* a^0$. As a^0, b^0 satisfy the conditions of Lemma 7, we conclude that either XE* or IN* holds. By Lemma 6 again, the theorem is established. ■

Rather than to prove Theorem 5, we refer the reader to Lemma A.6 and to Theorem 7.2 in Hammond [3]. Of course, the proofs must be put in terms of utility levels. It is worth point out that XE is actually stronger than Hammond's equity axiom and that our Lemma 4 corresponds to his condition S. Moreover, no invariance axiom is required to carry out the argument as it is translated.

To conclude, we observe that Theorem 6 is symmetrical to Theorem 5. As pointed out, Theorem 7 is a corollary of Theorems 4, 5 and 6.

First version received August 1975; *final version accepted October* 1976 (*Eds.*).

This paper is an extensively modified version of one presented at the third world congress of the Econometric Society, Toronto, August 1975. The authors are much indebted to P.J. Hammond, A.K. Sen, and an anonymous referee for useful suggestions. They are indebted to Ch. Duchateau, L.A. Gerard-Varet and J.C. Harsanyi for stimulating conversations. They are responsible for any remaining errors.

References

[1] Arrow, K.J. *Social Choice and Individual Values*, 2nd ed. (New York: John WIley and Sons, 1963).

[2] Blau, J.H. "Arrow's Theorem with Weak Independence", *Economica*, **NS 38** (1971), 413-420.

[3] Hammond, P.J. "Equity, Arrow's Conditions and Rawls' Difference Principle", *Econometrica*, **44** (1976), 793-804.

[4] Harsanyi, J.C. "Can the Maximin Principle Serve as a Basis for Morality: A Critique of Joh, Rawls's Theory", Working Paper CP-351, Center for Research in Management Science, University of California (1973).

[5] Kolm, S.Ch. *Justice et Equité* (Paris: C.N.R.S., 1973).

[6] May, K.O. “A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision”, *Econometrica*, **20** (1952), 680-689.

[7] Milnor, J. “Games against Nature”, in Thrall, R.M., Coombs, C.H. and Davis, R.L. (eds.), *Decision Processes* (New York: John Wiley and Sons, 1971).

[8] Rawls, J. *A Theory of Justice* (Cambridge: Harward University Press, 1971).

[9] Sen, A.K. *Collective CHoice and Social Welfare* (San Francisco: Holden-Day, 1970).

[10] Sen, A.K. “Informational Basis of Alternative Welfare Approaches, Aggregation and Income Distribution”, Journal of Public Economics, **3** (1974), 387-403.

[11] Sen, A.K. “Social Choice Theory: A Re-examination”, *Econometrica*, **45** (1977), 53-88.

[12] Sen, A.K. “Interpersonal Comparisons of Welfare”, in Boskin, M. J. (Ed.), *Economics and Human Welfare: Essays in honor of Tibor Scitovsky* (Academic Press, 1979, pp. 183-201).

[13] Strasnick, S. “Arrow’s Paradox and Beyond”, Mimeo (1975).