Large-scale machine learning and convex optimization

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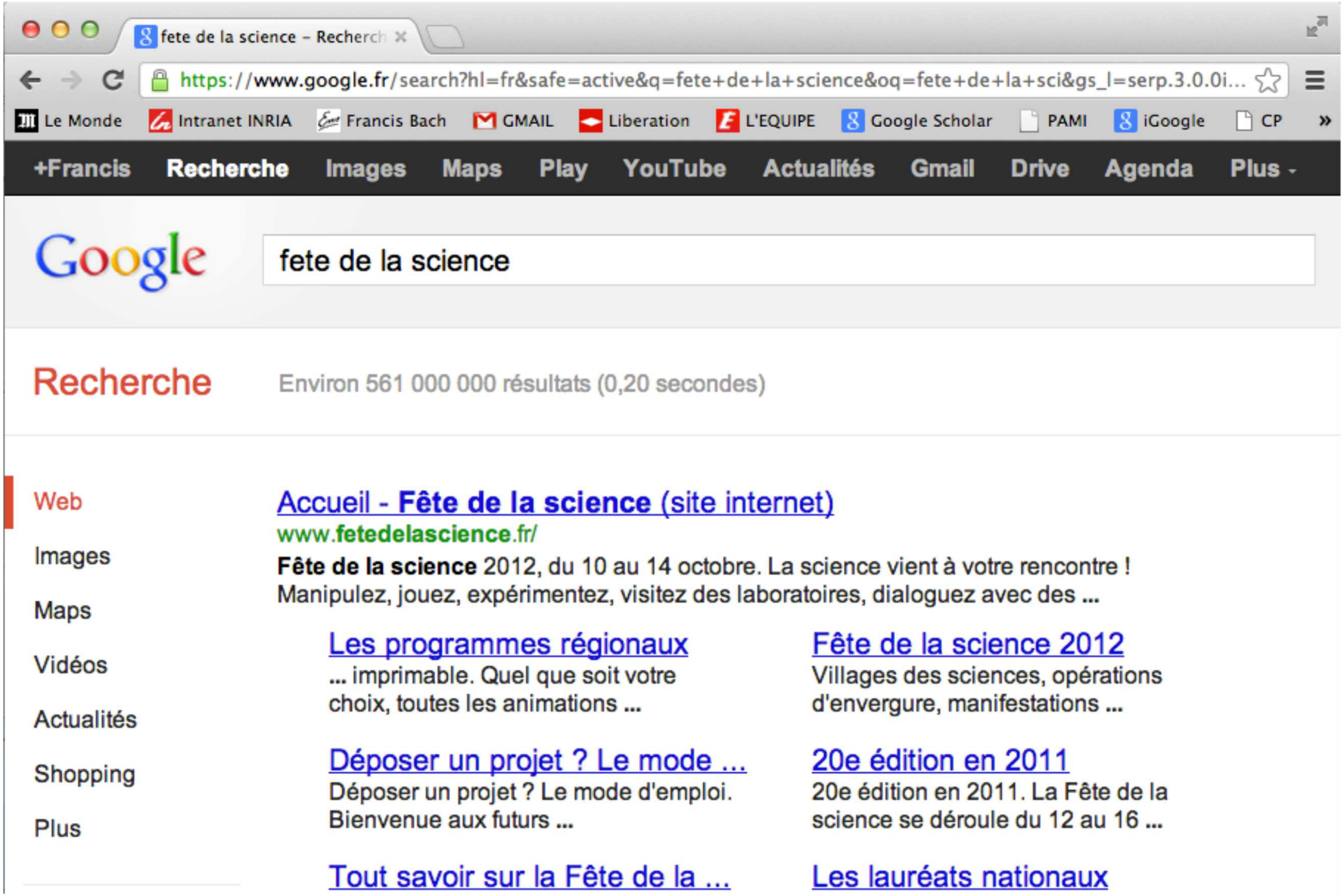
“Big data” revolution?
A new scientific context

- **Data everywhere**: size does not (always) matter
- **Science and industry**
- **Size and variety**
- **Learning from examples**
  - n observations in dimension d

Search engines - advertising



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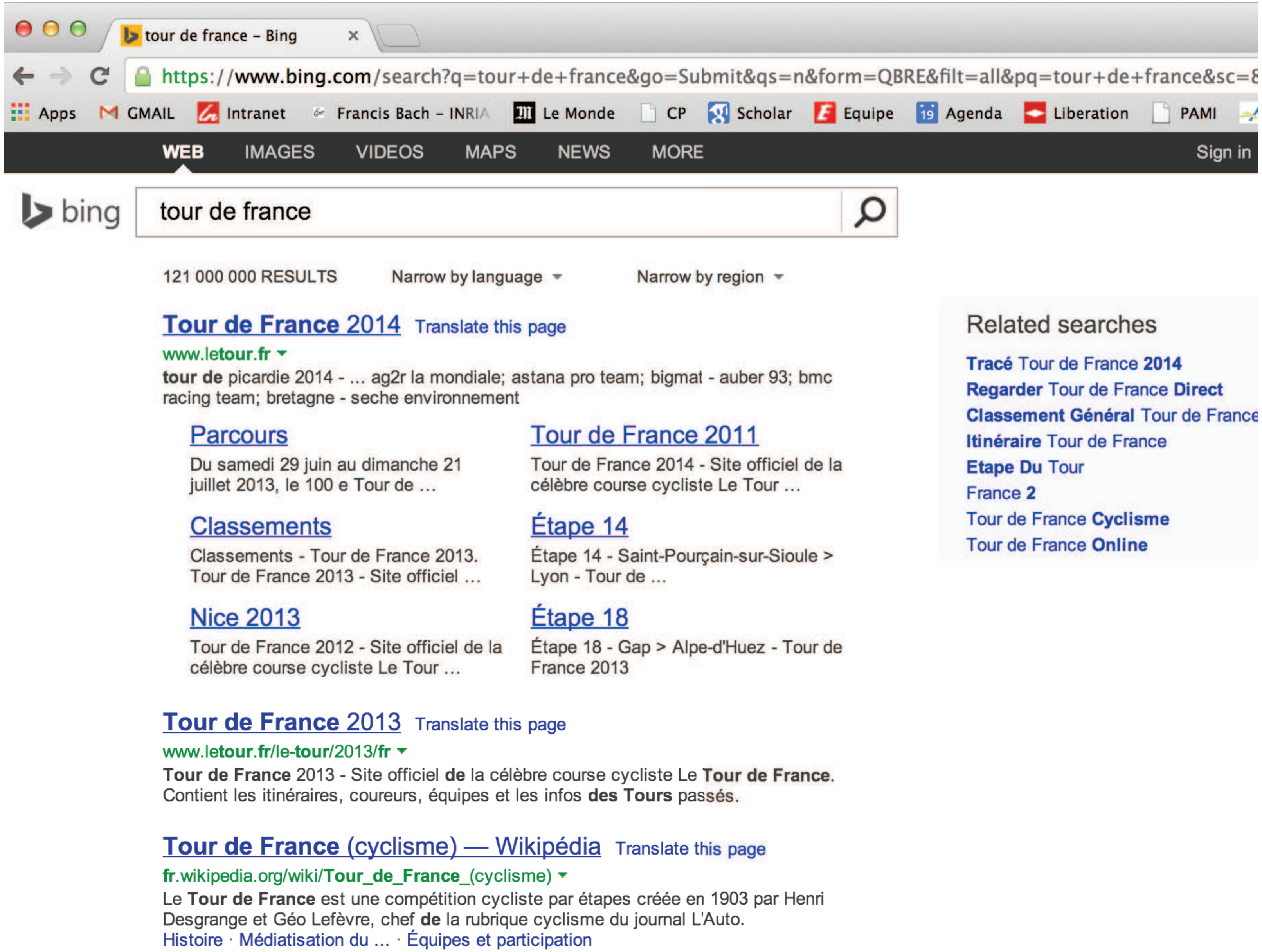
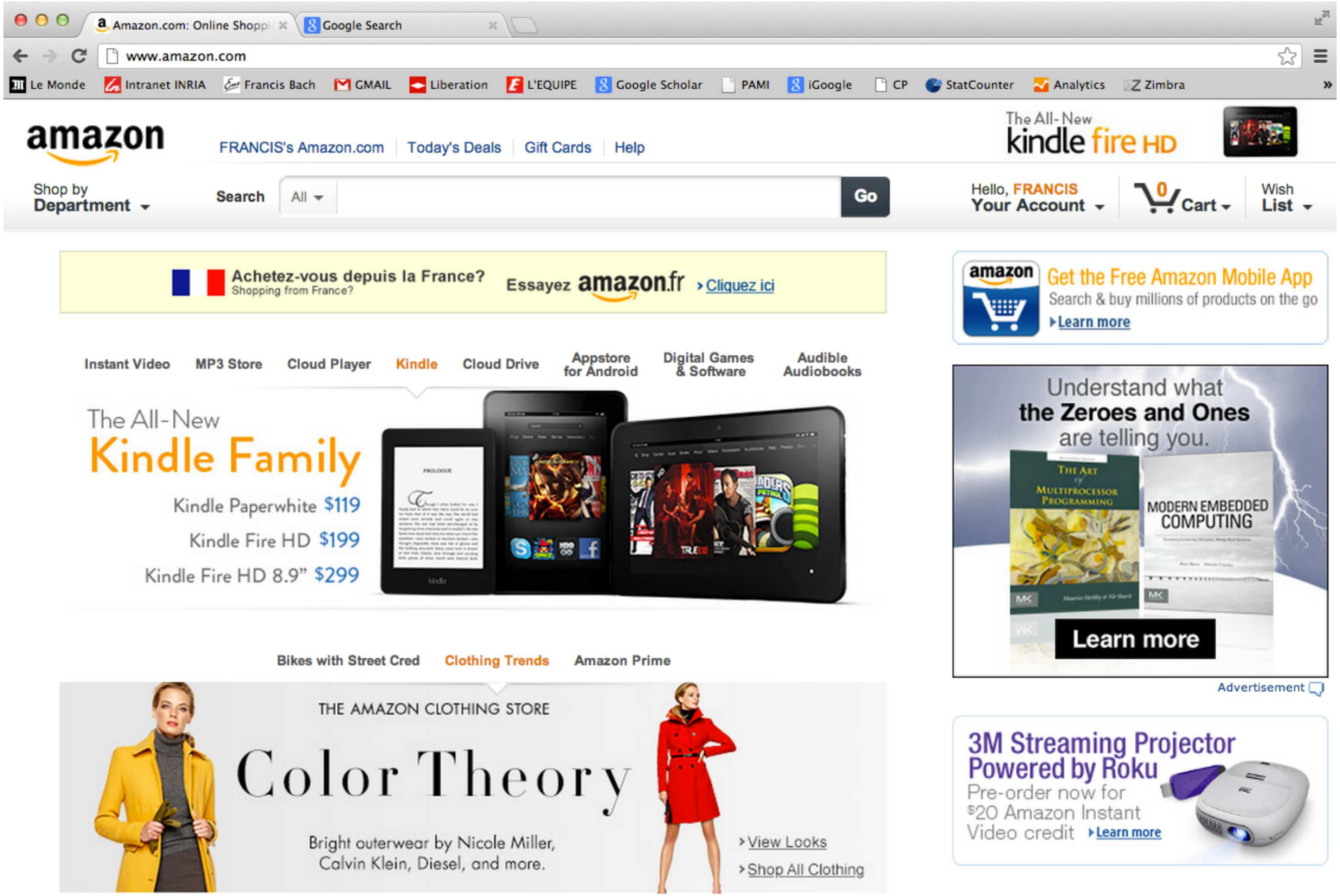
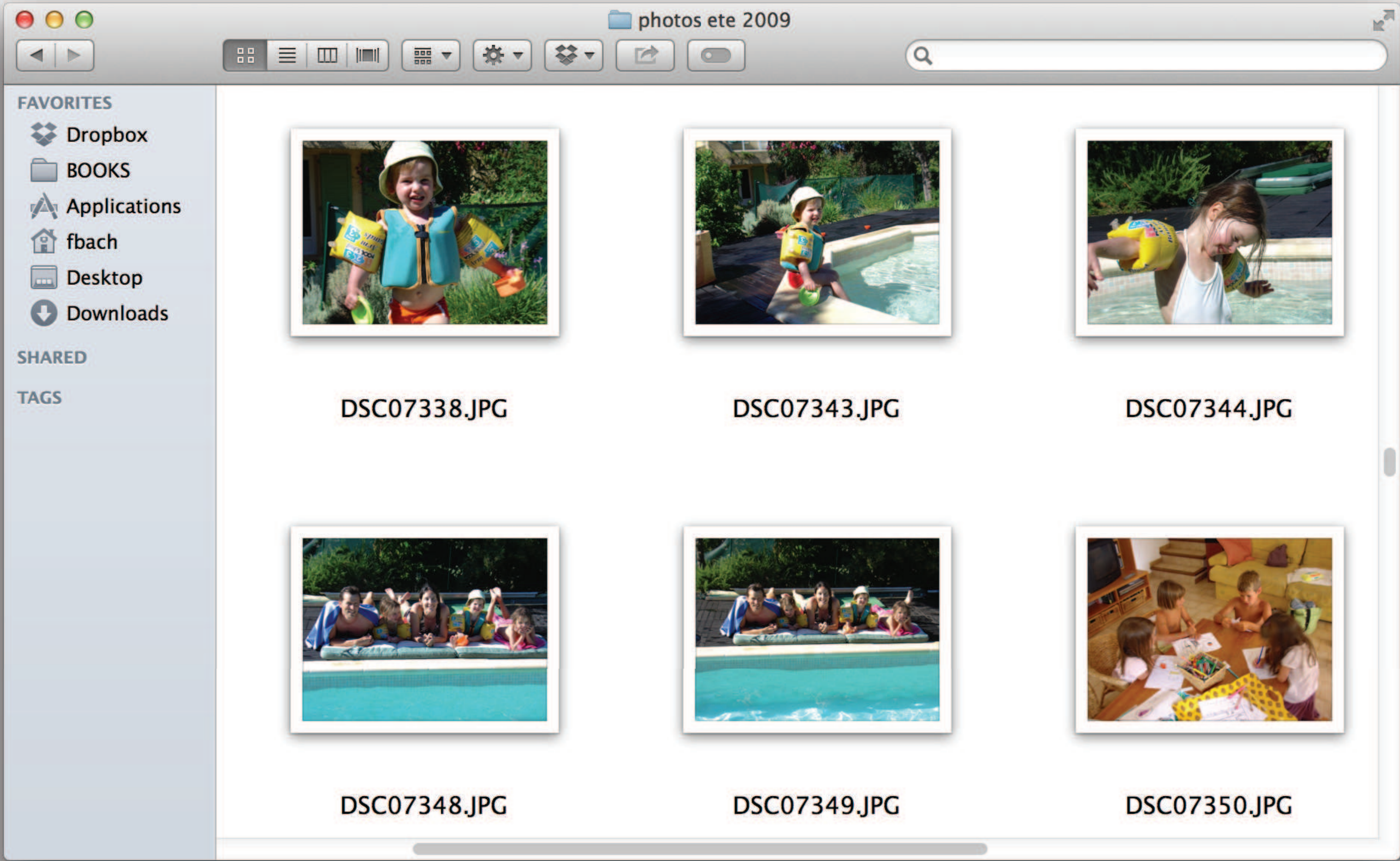


Marketing - Personalized recommendation

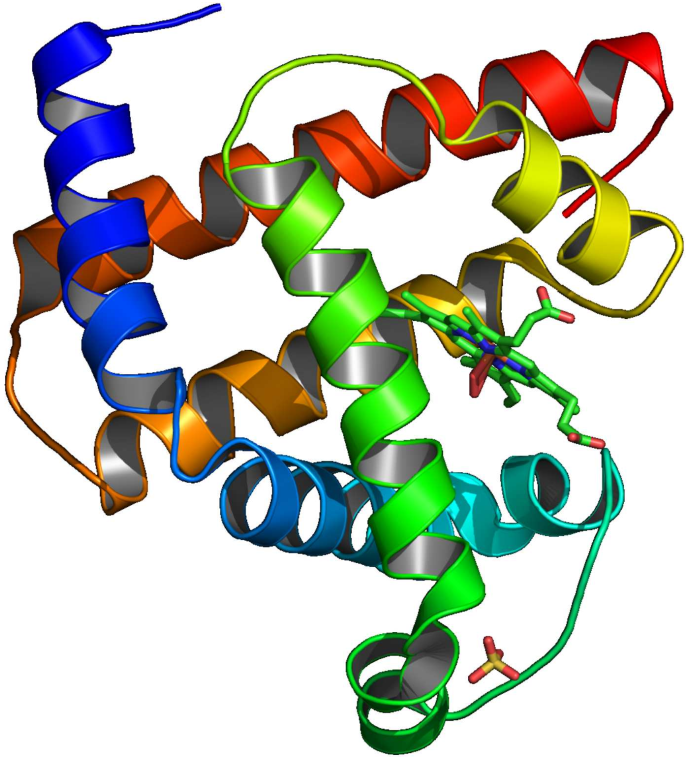


Visual object recognition

Personal photos



Bioinformatics



- **Protein**: Crucial elements of cell life
- **Massive data**: 2 millions for humans
- **Complex data**

Context

Machine learning for "big data"

- **Large-scale machine learning**: **large d, large n**
  - d : dimension of each observation (input)
  - n : number of observations
- **Examples**: computer vision, bioinformatics, advertising

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- **Examples**: computer vision, bioinformatics, advertising
- **Ideal running-time complexity**: $O(dn)$
- **Going back to simple methods**
  - Stochastic gradient methods (Robbins and Monro, 1951)
  - Mixing statistics and optimization

Outline

1. **Large-scale machine learning and optimization**
   - Traditional statistical analysis
   - Classical methods for convex optimization

2. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds
   - Strongly convex vs. non-strongly convex

3. **Smooth stochastic approximation algorithms**
   - Asymptotic and non-asymptotic results

4. **Beyond decaying step-sizes**

5. **Finite data sets**

Supervised machine learning

- **Data**: n observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$, $i = 1, \dots, n$, **i.i.d.**
- Prediction as a linear function $\theta^\top \Phi(x)$ of features $\Phi(x) \in \mathbb{R}^d$
- **(regularized) empirical risk minimization**: find $\hat{\theta}$ solution of

$$\min_{\theta \in \mathbb{R}^d} \quad \frac{1}{n} \sum_{i=1}^{n} \ell\big(y_i, \theta^\top \Phi(x_i)\big) \quad + \quad \mu \Omega(\theta)$$

convex data fitting term $+$ regularizer

Usual losses

- **Regression**: $y \in \mathbb{R}$, prediction $\hat{y} = \theta^\top \Phi(x)$
  - quadratic loss $\frac{1}{2}(y - \hat{y})^2 = \frac{1}{2}(y - \theta^\top \Phi(x))^2$

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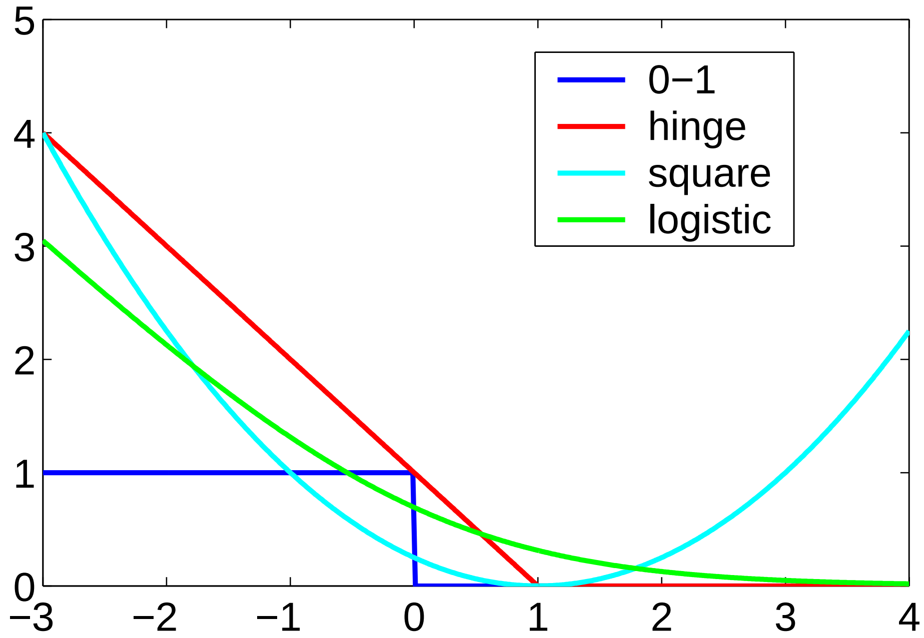
- **Classification** : $y \in \{-1, 1\}$, prediction $\hat{y} = \text{sign}(\theta^\top \Phi(x))$
  - loss of the form $\ell(y\,\theta^\top \Phi(x))$
  - "True" 0-1 loss: $\ell(y\,\theta^\top \Phi(x)) = 1_{y\,\theta^\top \Phi(x) < 0}$
  - Usual convex losses:



Main motivating examples

- **Support vector machine** (hinge loss)

$$\ell(Y, \theta^\top \Phi(X)) = \max\{1 - Y\theta^\top \Phi(X), 0\}$$

- **Logistic regression**

$$\ell(Y, \theta^\top \Phi(X)) = \log(1 + \exp(-Y\theta^\top \Phi(X)))$$

- **Least-squares regression**

$$\ell(Y, \theta^\top \Phi(X)) = \frac{1}{2}(Y - \theta^\top \Phi(X))^2$$

Usual regularizers

- **Main goal**: avoid overfitting

- **(squared) Euclidean norm**: $\|\theta\|_2^2 = \sum_{j=1}^d |\theta_j|^2$
  - Numerically well-behaved
  - Representer theorem and kernel methods : $\theta = \sum_{i=1}^n \alpha_i \Phi(x_i)$
  - See, e.g., Schölkopf and Smola (2001); Shawe-Taylor and Cristianini (2004)

- **Sparsity-inducing norms**
  - Main example: ℓ_1-norm $\|\theta\|_1 = \sum_{j=1}^d |\theta_j|$
  - Perform model selection as well as regularization
  - Non-smooth optimization and structured sparsity
  - See, e.g., Bach, Jenatton, Mairal, and Obozinski (2011, 2012)

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convex data fitting term $+$ regularizer

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convex data fitting term + regularizer

- Empirical risk: $\hat{f}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, \theta^\top \Phi(x_i))$ training cost
- Expected risk: $f(\theta) = \mathbb{E}_{(x,y)} \ell(y, \theta^\top \Phi(x))$ testing cost
- **Two fundamental questions**: (1) computing $\hat{\theta}$ and (2) analyzing $\hat{\theta}$

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$$\min_{\theta \in \mathbb{R}^d} \quad \frac{1}{n} \sum_{i=1}^{n} \ell\big(y_i, \theta^\top \Phi(x_i)\big) \text{ such that } \Omega(\theta) \leqslant D$$

convex data fitting term + constraint

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General assumptions

- **Data**: n observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$, $i = 1, \dots, n$, **i.i.d.**
- Bounded features $\Phi(x) \in \mathbb{R}^d$: $\|\Phi(x)\|_2 \leqslant R$
- Empirical risk: $\hat{f}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, \theta^\top \Phi(x_i))$ training cost
- Expected risk: $f(\theta) = \mathbb{E}_{(x,y)} \ell(y, \theta^\top \Phi(x))$ testing cost
- Loss for a single observation: $f_i(\theta) = \ell(y_i, \theta^\top \Phi(x_i))$
  $\Rightarrow \forall i, \;\; f(\theta) = \mathbb{E} f_i(\theta)$
- **Properties of** $f_i, f, \hat{f}$
  - Convex on $\mathbb{R}^d$
  - Additional regularity assumptions: Lipschitz-continuity, smoothness and strong convexity

Lipschitz continuity

- **Bounded gradients of f (Lipschitz-continuity)**: the function f if convex, differentiable and has (sub)gradients uniformly bounded by B on the ball of center 0 and radius D:

$$\forall \theta \in \mathbb{R}^d, \|\theta\|_2 \leqslant D \Rightarrow \|f'(\theta)\|_2 \leqslant B$$

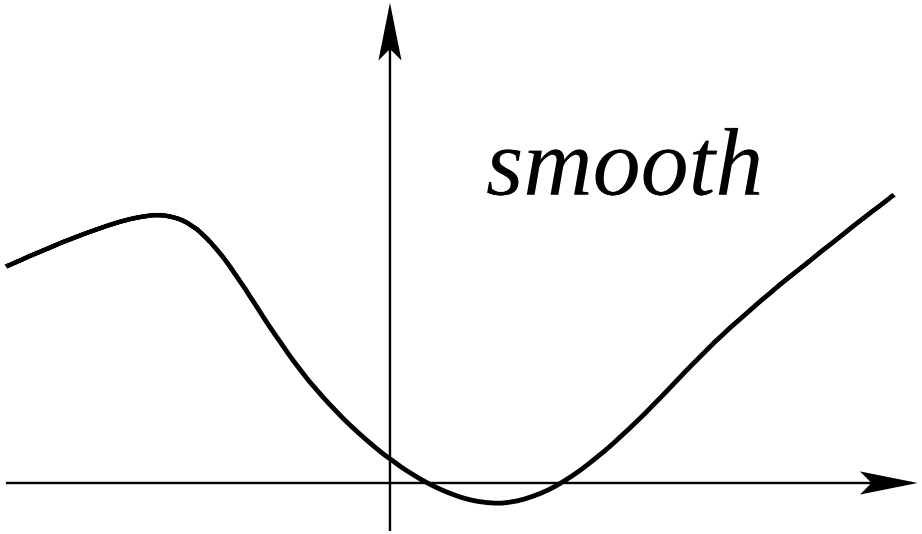
- **Machine learning**
  - with $f(\theta) = \frac{1}{n}\sum_{i=1}^n \ell(y_i, \theta^\top \Phi(x_i))$
  - G-Lipschitz loss and R-bounded data: $B = GR$

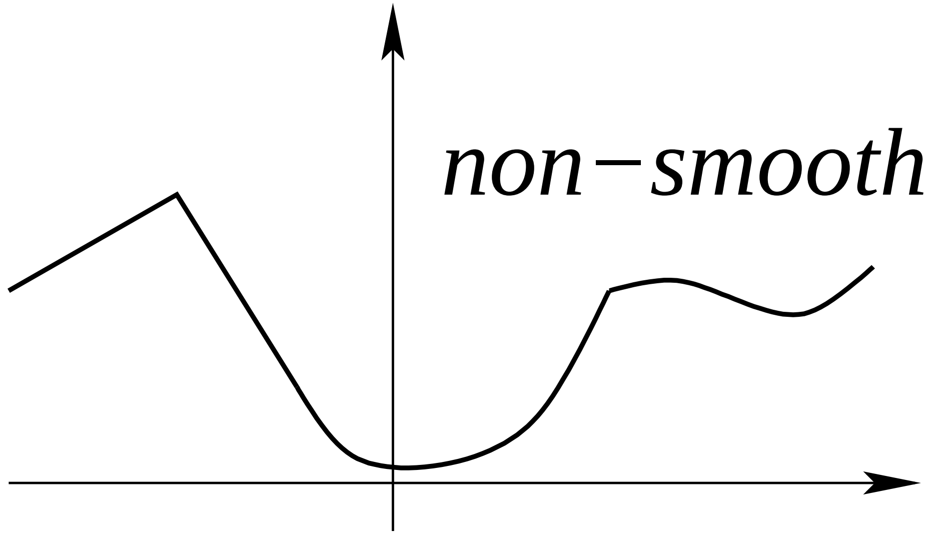
Smoothness and strong convexity

- A function $f : \mathbb{R}^d \to \mathbb{R}$ is L-smooth if and only if it is differentiable and its gradient is L-Lipschitz-continuous

$$\forall \theta_1, \theta_2 \in \mathbb{R}^d, \;\; \|f'(\theta_1) - f'(\theta_2)\|_2 \leqslant L \|\theta_1 - \theta_2\|_2$$

- If f is twice differentiable: $\forall \theta \in \mathbb{R}^d, \;\; f''(\theta) \preccurlyeq L \cdot Id$





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  - with $f(\theta) = \frac{1}{n}\sum_{i=1}^n \ell(y_i, \theta^\top \Phi(x_i))$
  - Hessian $\approx$ covariance matrix $\frac{1}{n}\sum_{i=1}^n \Phi(x_i)\Phi(x_i)^\top$
  - ℓ-smooth loss and R-bounded data: $L = \ell R^2$

Smoothness and strong convexity

- A function $f : \mathbb{R}^d \to \mathbb{R}$ is μ-strongly convex if and only if

$$\forall \theta_1, \theta_2 \in \mathbb{R}^d, \;\; f(\theta_1) \geqslant f(\theta_2) + f'(\theta_2)^\top (\theta_1 - \theta_2) + \frac{\mu}{2} \|\theta_1 - \theta_2\|_2^2$$

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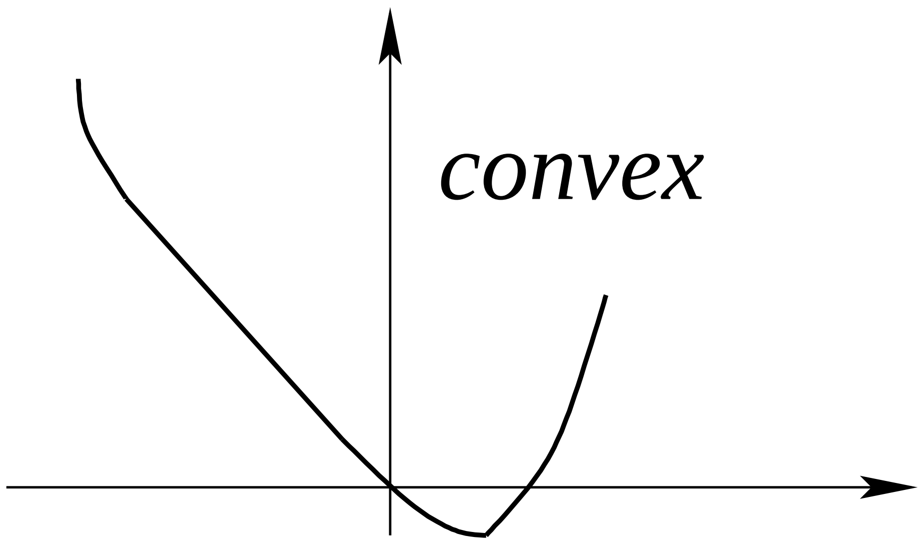

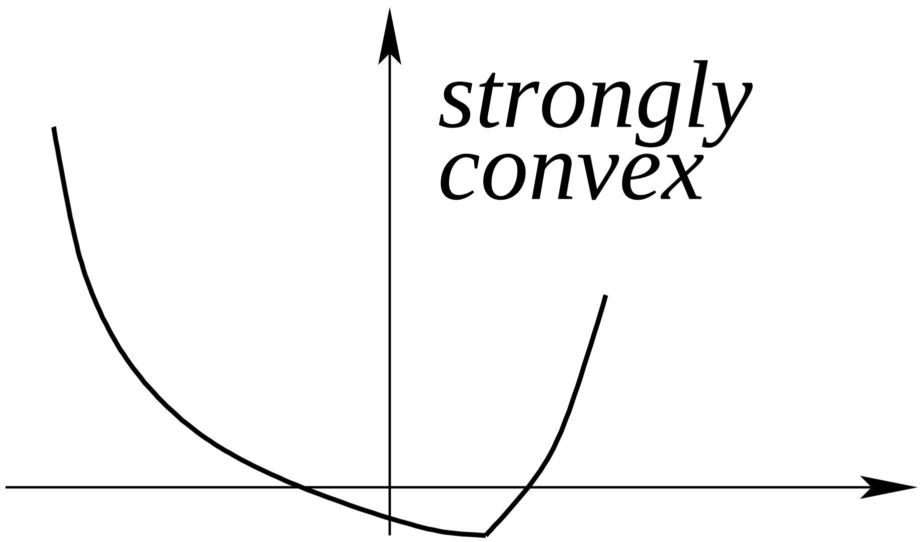

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- **Adding regularization by** $\frac{\mu}{2}\|\theta\|^2$
  - creates additional bias unless μ is small

Summary of smoothness/convexity assumptions

- **Bounded gradients of f (Lipschitz-continuity)**: the function f if convex, differentiable and has (sub)gradients uniformly bounded by B on the ball of center 0 and radius D:

$$\forall \theta \in \mathbb{R}^d, \|\theta\|_2 \leqslant D \Rightarrow \|f'(\theta)\|_2 \leqslant B$$

- **Smoothness of f**: the function f is convex, differentiable with L-Lipschitz-continuous gradient f':

$$\forall \theta_1, \theta_2 \in \mathbb{R}^d, \quad \|f'(\theta_1) - f'(\theta_2)\|_2 \leqslant L\|\theta_1 - \theta_2\|_2$$

- **Strong convexity of f**: The function f is strongly convex with respect to the norm $\|\cdot\|$, with convexity constant $\mu > 0$:

$$\forall \theta_1, \theta_2 \in \mathbb{R}^d, \; f(\theta_1) \geqslant f(\theta_2) + f'(\theta_2)^\top(\theta_1 - \theta_2) + \frac{\mu}{2}\|\theta_1 - \theta_2\|_2^2$$

Analysis of empirical risk minimization

- **Approximation and estimation errors**: $\mathcal{C} = \{\theta \in \mathbb{R}^d, \Omega(\theta) \leqslant D\}$

$$f(\hat\theta) - \min_{\theta\in\mathbb{R}^d} f(\theta) = \Big[f(\hat\theta) - \min_{\theta\in\mathcal{C}} f(\theta) \Big] + \Big[\min_{\theta\in\mathcal{C}} f(\theta) - \min_{\theta\in\mathbb{R}^d} f(\theta) \Big]$$

  – NB: may replace $\min_{\theta\in\mathbb{R}^d} f(\theta)$ by best (non-linear) predictions

1. **Uniform deviation bounds**, with $\boxed{\hat\theta \in \arg\min_{\theta\in\mathcal{C}} \hat f(\theta)}$

$$f(\hat\theta) - \min_{\theta\in\mathcal{C}} f(\theta) \leqslant 2 \sup_{\theta\in\mathcal{C}} |\hat f(\theta) - f(\theta)| \quad (proof)$$

   – Typically slow rate $O\big(\frac{1}{\sqrt{n}}\big)$

2. **More refined concentration results** with faster rates

Motivation from least-squares

- For least-squares, we have $\ell(y, \theta^\top \Phi(x)) = \frac{1}{2}(y - \theta^\top \Phi(x))^2$, and

$$
\begin{aligned}
f(\theta) - \hat{f}(\theta) &= \frac{1}{2}\theta^\top \bigg(\frac{1}{n}\sum_{i=1}^n \Phi(x_i)\Phi(x_i)^\top - \mathbb{E}\Phi(X)\Phi(X)^\top \bigg)\theta \\
&\quad -\theta^\top \bigg(\frac{1}{n}\sum_{i=1}^n y_i \Phi(x_i) - \mathbb{E}Y\Phi(X) \bigg) + \frac{1}{2}\bigg(\frac{1}{n}\sum_{i=1}^n y_i^2 - \mathbb{E}Y^2 \bigg), \\
\sup_{\|\theta\|_2 \leqslant D} |f(\theta) - \hat{f}(\theta)| &\leqslant \frac{D^2}{2}\bigg\| \frac{1}{n}\sum_{i=1}^n \Phi(x_i)\Phi(x_i)^\top - \mathbb{E}\Phi(X)\Phi(X)^\top \bigg\|_{\mathrm{op}} \\
&\quad +D\bigg\| \frac{1}{n}\sum_{i=1}^n y_i \Phi(x_i) - \mathbb{E}Y\Phi(X) \bigg\|_2 + \frac{1}{2}\bigg| \frac{1}{n}\sum_{i=1}^n y_i^2 - \mathbb{E}Y^2 \bigg|, \\
\sup_{\|\theta\|_2 \leqslant D} |f(\theta) - \hat{f}(\theta)| &\leqslant O(1/\sqrt{n}) \text{ with high probability}
\end{aligned}
$$

Slow rate for supervised learning

- **Assumptions** (f is the expected risk, $\hat{f}$ the empirical risk)
  - $\Omega(\theta) = \|\theta\|_2$ (Euclidean norm)
  - "Linear" predictors: $\theta(x) = \theta^\top \Phi(x)$, with $\|\Phi(x)\|_2 \leqslant R$ a.s.
  - G-Lipschitz loss: f and $\hat{f}$ are GR-Lipschitz on $\mathcal{C} = \{\|\theta\|_2 \leqslant D\}$
  - No assumptions regarding convexity

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- With probability greater than $1 - \delta$

$$\sup_{\theta \in \mathcal{C}} |\hat{f}(\theta) - f(\theta)| \leqslant \frac{GRD}{\sqrt{n}} \left[2 + \sqrt{2 \log \frac{2}{\delta}} \right]$$

- Expectated estimation error: $\mathbb{E}\big[\sup_{\theta \in \mathcal{C}} |\hat{f}(\theta) - f(\theta)|\big] \leqslant \dfrac{4GRD}{\sqrt{n}}$

- Using Rademacher averages (see, e.g., Boucheron et al., 2005)

- **Lipschitz functions $\Rightarrow$ slow rate**

Symmetrization with Rademacher variables

- Let $\mathcal{D}' = \{x'_1, y'_1, \dots, x'_n, y'_n\}$ an independent copy of the data $\mathcal{D} = \{x_1, y_1, \dots, x_n, y_n\}$, with corresponding loss functions $f'_i(\theta)$

$$\begin{aligned}
\mathbb{E}\big[\sup_{\theta\in\Theta}\big|f(\theta)-\hat f(\theta)\big|\big] &= \mathbb{E}\big[\sup_{\theta\in\Theta}\bigg(f(\theta)-\frac{1}{n}\sum_{i=1}^n f_i(\theta)\bigg)\big] \\
&= \mathbb{E}\bigg[\sup_{\theta\in\Theta}\bigg|\frac{1}{n}\sum_{i=1}^n \mathbb{E}\big(f'_i(\theta)-f_i(\theta)|\mathcal{D}\big)\bigg|\bigg] \\
&\leqslant \mathbb{E}\bigg[\mathbb{E}\bigg[\sup_{\theta\in\Theta}\bigg|\frac{1}{n}\sum_{i=1}^n \big(f'_i(\theta)-f_i(\theta)\bigg|\,\bigg|\mathcal{D}\bigg]\bigg] \\
&= \mathbb{E}\bigg[\sup_{\theta\in\Theta}\bigg|\frac{1}{n}\sum_{i=1}^n \big(f'_i(\theta)-f_i(\theta)\big)\bigg|\bigg] \\
&= \mathbb{E}\bigg[\sup_{\theta\in\Theta}\bigg|\frac{1}{n}\sum_{i=1}^n \varepsilon_i\big(f'_i(\theta)-f_i(\theta)\big)\bigg|\bigg] \text{ with } \varepsilon_i \text{ uniform in } \{-1,1\} \\
&\leqslant 2\mathbb{E}\bigg[\sup_{\theta\in\Theta}\bigg|\frac{1}{n}\sum_{i=1}^n \varepsilon_i f_i(\theta)\bigg|\bigg] = \text{Rademacher complexity}
\end{aligned}$$

Rademacher complexity

- Define the Rademacher complexity of the class of functions $(X,Y) \mapsto \ell(Y,\theta^\top\Phi(X))$ as

$$R_n = \mathbb{E}\Big[\sup_{\theta\in\Theta}\Big|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(\theta)\Big|\Big].$$

- Note two expectations, with respect to $\mathcal{D}$ *and* with respect to ε

- **Main property**:

$$\mathbb{E}\big[\sup_{\theta\in\Theta}\big|f(\theta)-\hat{f}(\theta)\big|\big] \leqslant 2R_n$$

From Rademacher complexity to uniform bound

- Let $Z = \sup_{\theta \in \Theta} \left| f(\theta) - \hat{f}(\theta) \right|$

- By changing the pair (x_i, y_i), Z may only change by

$$\frac{2}{n} \sup |\ell(Y, \theta^\top \Phi(X))| \leqslant \frac{2}{n} \big(\sup |\ell(Y, 0)| + GRD \big) \leqslant \frac{2}{n} \big(\ell_0 + GRD \big) = c$$

  with $\sup |\ell(Y, 0)| = \ell_0$

- **MacDiarmid inequality**: with probability greater than $1 - \delta$,

$$Z \leqslant \mathbb{E} Z + \sqrt{\frac{n}{2}} c \cdot \sqrt{\log \frac{1}{\delta}} \leqslant 2 R_n + \frac{\sqrt{2}}{\sqrt{n}} \big(\ell_0 + GRD \big) \sqrt{\log \frac{1}{\delta}}$$

Bounding the Rademacher average - I

- We have, with $\varphi_i(u) = \ell(y_i, u) - \ell(y_i, 0)$ is almost surely B-Lipschitz:

$$\begin{aligned}
R_n &= \mathbb{E}\Big[\sup_{\theta\in\Theta}\Big|\frac{1}{n}\sum_{i=1}^n \varepsilon_i f_i(\theta)\Big|\Big] \\
&\leqslant \mathbb{E}\Big[\sup_{\theta\in\Theta}\Big|\frac{1}{n}\sum_{i=1}^n \varepsilon_i f_i(0)\Big|\Big] + \mathbb{E}\Big[\sup_{\theta\in\Theta}\Big|\frac{1}{n}\sum_{i=1}^n \varepsilon_i \big[f_i(\theta) - f_i(0)\big]\Big|\Big] \\
&\leqslant \frac{\ell_0}{\sqrt{n}} + \mathbb{E}\Big[\sup_{\theta\in\Theta}\frac{1}{n}\sum_{i=1}^n \varepsilon_i \big[f_i(\theta) - f_i(0)\big]\Big] \\
&= \frac{\ell_0}{\sqrt{n}} + \mathbb{E}\Big[\sup_{\theta\in\Theta}\frac{1}{n}\sum_{i=1}^n \varepsilon_i \varphi_i(\theta^\top\Phi(x_i))\Big]
\end{aligned}$$

- Using Ledoux-Talagrand concentration results for Rademacher averages (since φ_i is G-Lipschitz, we get:

$$R_n \leqslant \frac{\ell_0}{\sqrt{n}} + 2G\cdot\mathbb{E}\Big[\sup_{\|\theta\|_2\leqslant D}\Big|\frac{1}{n}\sum_{i=1}^n \varepsilon_i\theta^\top\Phi(x_i)\Big]\Big|$$

Bounding the Rademacher average - II

- We have:

$$\begin{aligned} R_n &\leqslant \frac{\ell_0}{\sqrt{n}} + 2G\mathbb{E}\Big[\sup_{\|\theta\|_2 \leqslant D} \Big|\frac{1}{n}\sum_{i=1}^n \varepsilon_i \theta^\top \Phi(x_i)\Big]\Big| \\ &= \frac{\ell_0}{\sqrt{n}} + 2G\mathbb{E}\Big\|D\frac{1}{n}\sum_{i=1}^n \varepsilon_i \Phi(x_i)\Big\|_2 \\ &\leqslant \frac{\ell_0}{\sqrt{n}} + 2GD\sqrt{\mathbb{E}\Big\|\frac{1}{n}\sum_{i=1}^n \varepsilon_i \Phi(x_i)\Big\|_2^2} \\ &\leqslant \frac{2(\ell_0 + GRD)}{\sqrt{n}} \end{aligned}$$

- Overall, we get, with probability $1 - \delta$:

$$\sup_{\theta \in \Theta} \big|f(\theta) - \hat{f}(\theta)\big| \leqslant \frac{1}{\sqrt{n}}(\ell_0 + GRD)(4 + \sqrt{2\log\frac{1}{\delta}})$$

Putting it all together

- We have, with probability $1-\delta$, for all $\theta \in \Theta$:

$$
\begin{aligned}
f(\theta) - f(\theta_*) &\leqslant \big[f(\theta) - \hat{f}(\theta)\big] + \big[\hat{f}(\theta) - \min_{\theta' \in \Theta} \hat{f}(\theta')\big] + \big[\min_{\theta' \in \Theta} \hat{f}(\theta') - \hat{f}(\theta_*)\big] \\
&\leqslant \frac{2}{\sqrt{n}}(\ell_0 + GRD)\Big(4 + \sqrt{2\log\frac{1}{\delta}}\Big) + \big[\hat{f}(\theta) - \min_{\theta' \in \Theta} \hat{f}(\theta')\big]
\end{aligned}
$$

- Only need to optimize with precision $\frac{2}{\sqrt{n}}(\ell_0 + GRD)$

Slow rate for supervised learning (summary)

- **Assumptions** (f is the expected risk, $\hat{f}$ the empirical risk)
  - $\Omega(\theta) = \|\theta\|_2$ (Euclidean norm)
  - "Linear" predictors: $\theta(x) = \theta^\top \Phi(x)$, with $\|\Phi(x)\|_2 \leqslant R$ a.s.
  - G-Lipschitz loss: f and $\hat{f}$ are GR-Lipschitz on $\mathcal{C} = \{\|\theta\|_2 \leqslant D\}$
  - No assumptions regarding convexity

- With probability greater than $1 - \delta$

$$\sup_{\theta \in \mathcal{C}} |\hat{f}(\theta) - f(\theta)| \leqslant \frac{(\ell_0 + GRD)}{\sqrt{n}} \left[2 + \sqrt{2 \log \frac{2}{\delta}} \right]$$

- Expectated estimation error: $\mathbb{E}\big[\sup_{\theta \in \mathcal{C}} |\hat{f}(\theta) - f(\theta)|\big] \leqslant \dfrac{4(\ell_0 + GRD)}{\sqrt{n}}$

- Using Rademacher averages (see, e.g., Boucheron et al., 2005)

- **Lipschitz functions $\Rightarrow$ slow rate**

Motivation from mean estimation

- Estimator $\hat{\theta} = \frac{1}{n}\sum_{i=1}^n z_i = \arg\min_{\theta \in \mathbb{R}} \frac{1}{2n}\sum_{i=1}^n (\theta - z_i)^2 = \hat{f}(\theta)$

- From before:
  - $f(\theta) = \frac{1}{2}\mathbb{E}(\theta - z)^2 = \frac{1}{2}(\theta - \mathbb{E}z)^2 + \frac{1}{2}\operatorname{var}(z) = \hat{f}(\theta) + O(1/\sqrt{n})$
  - $f(\hat{\theta}) = \frac{1}{2}(\hat{\theta} - \mathbb{E}z)^2 + \frac{1}{2}\operatorname{var}(z) = f(\mathbb{E}z) + O(1/\sqrt{n})$

Motivation from mean estimation

- Estimator $\hat{\theta} = \frac{1}{n}\sum_{i=1}^n z_i = \arg\min_{\theta\in\mathbb{R}} \frac{1}{2n}\sum_{i=1}^n (\theta - z_i)^2 = \hat{f}(\theta)$

- From before:
  - $f(\theta) = \frac{1}{2}\mathbb{E}(\theta - z)^2 = \frac{1}{2}(\theta - \mathbb{E}z)^2 + \frac{1}{2}\,\text{var}(z) = \hat{f}(\theta) + O(1/\sqrt{n})$
  - $f(\hat{\theta}) = \frac{1}{2}(\hat{\theta} - \mathbb{E}z)^2 + \frac{1}{2}\,\text{var}(z) = f(\mathbb{E}z) + O(1/\sqrt{n})$

- More refined/direct bound:

$$\begin{aligned} f(\hat{\theta}) - f(\mathbb{E}z) &= \frac{1}{2}(\hat{\theta} - \mathbb{E}z)^2 \\ \mathbb{E}\big[f(\hat{\theta}) - f(\mathbb{E}z)\big] &= \frac{1}{2}\mathbb{E}\bigg(\frac{1}{n}\sum_{i=1}^n z_i - \mathbb{E}z\bigg)^2 = \frac{1}{2n}\,\text{var}(z) \end{aligned}$$

- Bound only at $\hat{\theta}$ + strong convexity

Fast rate for supervised learning

- **Assumptions** (f is the expected risk, $\hat{f}$ the empirical risk)
  – Same as before (bounded features, Lipschitz loss)
  – Regularized risks: $f^\mu(\theta) = f(\theta) + \frac{\mu}{2}\|\theta\|_2^2$ and $\hat{f}^\mu(\theta) = \hat{f}(\theta) + \frac{\mu}{2}\|\theta\|_2^2$
  – Convexity

- For any $a > 0$, with probability greater than $1 - \delta$, for all $\theta \in \mathbb{R}^d$,

$$f^\mu(\theta) - \min_{\eta \in \mathbb{R}^d} f^\mu(\eta) \leqslant (1+a)\big(\hat{f}^\mu(\theta) - \min_{\eta \in \mathbb{R}^d} \hat{f}^\mu(\eta)\big) + \frac{8(1+\frac{1}{a})G^2R^2(32 + \log\frac{1}{\delta})}{\mu n}$$

- Results from Sridharan, Srebro, and Shalev-Shwartz (2008)
  – see also Boucheron and Massart (2011) and references therein

- **Strongly convex functions $\Rightarrow$ fast rate**
  – Warning: μ should decrease with n to reduce approximation error

Outline

1. **Large-scale machine learning and optimization**
   - Traditional statistical analysis
   - Classical methods for convex optimization

2. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds
   - Strongly convex vs. non-strongly convex

3. **Smooth stochastic approximation algorithms**
   - Asymptotic and non-asymptotic results

4. **Beyond decaying step-sizes**

5. **Finite data sets**

Complexity results in convex optimization

- **Assumption**: f convex on $\mathbb{R}^d$
- **Classical generic algorithms**
  - (sub)gradient method/descent
  - Accelerated gradient descent
  - Newton method
- **Key additional properties of f**
  - Lipschitz continuity, smoothness or strong convexity
- **Key insight from Bottou and Bousquet (2008)**
  - In machine learning, no need to optimize below estimation error
- **Key reference**: Nesterov (2004)

Subgradient method/descent

- **Assumptions**
  - f convex and B-Lipschitz-continuous on $\{\|\theta\|_2 \leqslant D\}$
- **Algorithm**: $\theta_t = \Pi_D\left(\theta_{t-1} - \frac{2D}{B\sqrt{t}} f'(\theta_{t-1})\right)$
  - Π_D : orthogonal projection onto $\{\|\theta\|_2 \leqslant D\}$
- **Bound**:

$$f\left(\frac{1}{t}\sum_{k=0}^{t-1}\theta_k\right) - f(\theta_*) \leqslant \frac{2DB}{\sqrt{t}}$$

- Three-line proof
- Best possible convergence rate after $O(d)$ iterations

Subgradient method/descent - proof - I

- Iteration: $\theta_t = \Pi_D(\theta_{t-1} - \gamma_t f'(\theta_{t-1}))$ with $\gamma_t = \frac{2D}{B\sqrt{t}}$
- Assumption: $\|f'(\theta)\|_2 \leqslant B$ and $\|\theta\|_2 \leqslant D$

$$\begin{aligned}
\|\theta_t - \theta_*\|_2^2 &\leqslant \|\theta_{t-1} - \theta_* - \gamma_t f'(\theta_{t-1})\|_2^2 \text{ by contractivity of projections} \\
&\leqslant \|\theta_{t-1} - \theta_*\|_2^2 + B^2\gamma_t^2 - 2\gamma_t(\theta_{t-1} - \theta_*)^\top f'(\theta_{t-1}) \text{ because } \|f'(\theta_{t-1})\|_2 \leqslant B \\
&\leqslant \|\theta_{t-1} - \theta_*\|_2^2 + B^2\gamma_t^2 - 2\gamma_t\big[f(\theta_{t-1}) - f(\theta_*)\big] \text{ (property of subgradients)}
\end{aligned}$$

- leading to

$$f(\theta_{t-1}) - f(\theta_*) \leqslant \frac{B^2\gamma_t}{2} + \frac{1}{2\gamma_t}\big[\|\theta_{t-1} - \theta_*\|_2^2 - \|\theta_t - \theta_*\|_2^2\big]$$

Subgradient method/descent - proof - II

- Starting from $\quad f(\theta_{t-1}) - f(\theta_*) \leqslant \dfrac{B^2\gamma_t}{2} + \dfrac{1}{2\gamma_t}\big[\|\theta_{t-1}-\theta_*\|_2^2 - \|\theta_t-\theta_*\|_2^2\big]$

$$\begin{aligned}
\sum_{u=1}^{t}\big[f(\theta_{u-1}) - f(\theta_*)\big] &\leqslant \sum_{u=1}^{t}\frac{B^2\gamma_u}{2} + \sum_{u=1}^{t}\frac{1}{2\gamma_u}\big[\|\theta_{u-1}-\theta_*\|_2^2 - \|\theta_u-\theta_*\|_2^2\big] \\
&= \sum_{u=1}^{t}\frac{B^2\gamma_u}{2} + \sum_{u=1}^{t-1}\|\theta_u-\theta_*\|_2^2\Big(\frac{1}{2\gamma_{u+1}} - \frac{1}{2\gamma_u}\Big) + \frac{\|\theta_0-\theta_*\|_2^2}{2\gamma_1} - \frac{\|\theta_t-\theta_*\|_2^2}{2\gamma_t} \\
&\leqslant \sum_{u=1}^{t}\frac{B^2\gamma_u}{2} + \sum_{u=1}^{t-1}4D^2\Big(\frac{1}{2\gamma_{u+1}} - \frac{1}{2\gamma_u}\Big) + \frac{4D^2}{2\gamma_1} \\
&= \sum_{u=1}^{t}\frac{B^2\gamma_u}{2} + \frac{4D^2}{2\gamma_t} \leqslant 2DB\sqrt{t} \text{ with } \gamma_t = \frac{2D}{B\sqrt{t}}
\end{aligned}$$

- Using convexity: $\quad f\Big(\dfrac{1}{t}\displaystyle\sum_{k=0}^{t-1}\theta_k\Big) - f(\theta_*) \leqslant \dfrac{2DB}{\sqrt{t}}$

Subgradient descent for machine learning

- **Assumptions** (f is the expected risk, $\hat{f}$ the empirical risk)
  - "Linear" predictors: $\theta(x) = \theta^\top \Phi(x)$, with $\|\Phi(x)\|_2 \leqslant R$ a.s.
  - $\hat{f}(\theta) = \frac{1}{n}\sum_{i=1}^n \ell(y_i, \Phi(x_i)^\top \theta)$
  - G-Lipschitz loss: f and $\hat{f}$ are GR-Lipschitz on $\mathcal{C} = \{\|\theta\|_2 \leqslant D\}$

- **Statistics**: with probability greater than $1-\delta$

$$\sup_{\theta \in \mathcal{C}} |\hat{f}(\theta) - f(\theta)| \leqslant \frac{GRD}{\sqrt{n}}\bigg[2 + \sqrt{2\log\frac{2}{\delta}}\,\bigg]$$

- **Optimization**: after t iterations of subgradient method

$$\hat{f}(\hat{\theta}) - \min_{\eta \in \mathcal{C}} \hat{f}(\eta) \leqslant \frac{GRD}{\sqrt{t}}$$

- $t = n$ iterations, with total running-time complexity of $O(n^2 d)$

Subgradient descent - strong convexity

- **Assumptions**
  - f convex and B-Lipschitz-continuous on $\{\|\theta\|_2 \leqslant D\}$
  - f μ-strongly convex

- **Algorithm**: $\theta_t = \Pi_D\left(\theta_{t-1} - \dfrac{2}{\mu(t+1)} f'(\theta_{t-1})\right)$

- **Bound**:

$$f\left(\frac{2}{t(t+1)}\sum_{k=1}^{t} k\theta_{k-1}\right) - f(\theta_*) \leqslant \frac{2B^2}{\mu(t+1)}$$

- Three-line proof

- Best possible convergence rate after $O(d)$ iterations

Subgradient method - strong convexity - proof - I

- Iteration: $\theta_t = \Pi_D(\theta_{t-1} - \gamma_t f'(\theta_{t-1}))$ with $\gamma_t = \frac{2}{\mu(t+1)}$
- Assumption: $\|f'(\theta)\|_2 \leqslant B$ and $\|\theta\|_2 \leqslant D$ and μ-strong convexity of f

$$
\begin{aligned}
\|\theta_t - \theta_*\|_2^2 &\leqslant \|\theta_{t-1} - \theta_* - \gamma_t f'(\theta_{t-1})\|_2^2 \text{ by contractivity of projections} \\
&\leqslant \|\theta_{t-1} - \theta_*\|_2^2 + B^2\gamma_t^2 - 2\gamma_t(\theta_{t-1} - \theta_*)^\top f'(\theta_{t-1}) \text{ because } \|f'(\theta_{t-1})\|_2 \leqslant B \\
&\leqslant \|\theta_{t-1} - \theta_*\|_2^2 + B^2\gamma_t^2 - 2\gamma_t\big[f(\theta_{t-1}) - f(\theta_*) + \frac{\mu}{2}\|\theta_{t-1} - \theta_*\|_2^2\big] \\
&\qquad \text{(property of subgradients and strong convexity)}
\end{aligned}
$$

- leading to

$$
\begin{aligned}
f(\theta_{t-1}) - f(\theta_*) &\leqslant \frac{B^2\gamma_t}{2} + \frac{1}{2}\Big[\frac{1}{\gamma_t} - \mu\Big]\|\theta_{t-1} - \theta_*\|_2^2 - \frac{1}{2\gamma_t}\|\theta_t - \theta_*\|_2^2 \\
&\leqslant \frac{B^2}{\mu(t+1)} + \frac{\mu}{2}\Big[\frac{t-1}{2}\Big]\|\theta_{t-1} - \theta_*\|_2^2 - \frac{\mu(t+1)}{4}\|\theta_t - \theta_*\|_2^2
\end{aligned}
$$

Subgradient method - strong convexity - proof - II

- From $f(\theta_{t-1}) - f(\theta_*) \leqslant \dfrac{B^2}{\mu(t+1)} + \dfrac{\mu}{2}\Big[\dfrac{t-1}{2}\Big]\|\theta_{t-1} - \theta_*\|_2^2 - \dfrac{\mu(t+1)}{4}\|\theta_t - \theta_*\|_2^2$

$$\sum_{u=1}^{t} u\big[f(\theta_{u-1}) - f(\theta_*)\big] \leqslant \sum_{t=1}^{u} \frac{B^2 u}{\mu(u+1)} + \frac{1}{4}\sum_{u=1}^{t}\big[u(u-1)\|\theta_{u-1} - \theta_*\|_2^2 - u(u+1)\|\theta_u - \theta_*\|_2^2\big]$$

$$\leqslant \quad \frac{B^2 t}{\mu} + \frac{1}{4}\big[0 - t(t+1)\|\theta_t - \theta_*\|_2^2\big] \leqslant \frac{B^2 t}{\mu}$$

- Using convexity: $f\bigg(\dfrac{2}{t(t+1)}\displaystyle\sum_{u=1}^{t} u\theta_{u-1}\bigg) - f(\theta_*) \leqslant \dfrac{2B^2}{t+1}$

(smooth) gradient descent

- **Assumptions**
  - f convex with L-Lipschitz-continuous gradient
  - Minimum attained at θ_*

- **Algorithm**:

$$\theta_t = \theta_{t-1} - \frac{1}{L} f'(\theta_{t-1})$$

- **Bound**:

$$f(\theta_t) - f(\theta_*) \leqslant \frac{2L\|\theta_0 - \theta_*\|^2}{t+4}$$

- Three-line proof

- Not best possible convergence rate after $O(d)$ iterations

(smooth) gradient descent - strong convexity

- **Assumptions**
  - f convex with L-Lipschitz-continuous gradient
  - f μ-strongly convex

- **Algorithm**:

$$\theta_t = \theta_{t-1} - \frac{1}{L} f'(\theta_{t-1})$$

- **Bound**:

$$f(\theta_t) - f(\theta_*) \leqslant (1 - \mu/L)^t \big[f(\theta_0) - f(\theta_*) \big]$$

- Three-line proof

- **Adaptivity of gradient descent to problem difficulty**

- Line search

Accelerated gradient methods (Nesterov, 1983)

- **Assumptions**
  - f convex with L-Lipschitz-cont. gradient , min. attained at θ_*

- **Algorithm**:

$$\begin{aligned} \theta_t &= \eta_{t-1} - \frac{1}{L} f'(\eta_{t-1}) \\ \eta_t &= \theta_t + \frac{t-1}{t+2}(\theta_t - \theta_{t-1}) \end{aligned}$$

- **Bound**:

$$f(\theta_t) - f(\theta_*) \leqslant \frac{2L\|\theta_0 - \theta_*\|^2}{(t+1)^2}$$

- Ten-line proof (see, e.g., Schmidt, Le Roux, and Bach, 2011)

- Not improvable

- Extension to strongly convex functions

Optimization for sparsity-inducing norms
(see Bach, Jenatton, Mairal, and Obozinski, 2011)

- Gradient descent as a **proximal method** (differentiable functions)
  - $\theta_{t+1} = \arg\min_{\theta \in \mathbb{R}^d} f(\theta_t) + (\theta - \theta_t)^\top \nabla f(\theta_t) + \frac{L}{2}\|\theta - \theta_t\|_2^2$
  - $\theta_{t+1} = \theta_t - \frac{1}{L}\nabla f(\theta_t)$

Optimization for sparsity-inducing norms (see Bach, Jenatton, Mairal, and Obozinski, 2011)

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  - $\theta_{t+1} = \arg\min_{\theta \in \mathbb{R}^d} f(\theta_t) + (\theta - \theta_t)^\top \nabla f(\theta_t) + \frac{L}{2}\|\theta - \theta_t\|_2^2$
  - $\theta_{t+1} = \theta_t - \frac{1}{L}\nabla f(\theta_t)$

- Problems of the form: $\boxed{\min_{\theta \in \mathbb{R}^d} f(\theta) + \mu\Omega(\theta)}$
  - $\theta_{t+1} = \arg\min_{\theta \in \mathbb{R}^d} f(\theta_t) + (\theta - \theta_t)^\top \nabla f(\theta_t) + \mu\Omega(\theta) + \frac{L}{2}\|\theta - \theta_t\|_2^2$
  - $\Omega(\theta) = \|\theta\|_1 \Rightarrow$ **Thresholded gradient descent**

- Similar convergence rates than smooth optimization
  - Acceleration methods (Nesterov, 2007; Beck and Teboulle, 2009)

Summary: minimizing convex functions

- **Assumption**: f convex
- **Gradient descent**: $\theta_t = \theta_{t-1} - \gamma_t f'(\theta_{t-1})$
  - $O(1/\sqrt{t})$ convergence rate for non-smooth convex functions
  - $O(1/t)$ convergence rate for smooth convex functions
  - $O(e^{-\rho t})$ convergence rate for strongly smooth convex functions
- **Newton method**: $\theta_t = \theta_{t-1} - f''(\theta_{t-1})^{-1} f'(\theta_{t-1})$
  - $O\big(e^{-\rho 2^t}\big)$ convergence rate

Summary: minimizing convex functions

- **Assumption**: f convex

- **Gradient descent**: $\theta_t = \theta_{t-1} - \gamma_t \, f'(\theta_{t-1})$
  - $O(1/\sqrt{t})$ convergence rate for non-smooth convex functions
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- **Newton method**: $\theta_t = \theta_{t-1} - f''(\theta_{t-1})^{-1} f'(\theta_{t-1})$
  - $O\big(e^{-\rho 2^t}\big)$ convergence rate

- **Key insights from Bottou and Bousquet (2008)**

1. In machine learning, no need to optimize below statistical error
2. In machine learning, cost functions are averages

$\Rightarrow$ **Stochastic approximation**

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Stochastic approximation

- **Goal**: Minimizing a function f defined on $\mathbb{R}^d$
  - given only unbiased estimates $f'_n(\theta_n)$ of its gradients $f'(\theta_n)$ at certain points $\theta_n \in \mathbb{R}^d$

Stochastic approximation

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- **Machine learning - statistics**
  - **loss for a single pair of observations**: $\boxed{f_n(\theta) = \ell(y_n, \theta^\top \Phi(x_n))}$
  - $f(\theta) = \mathbb{E} f_n(\theta) = \mathbb{E}\, \ell(y_n, \theta^\top \Phi(x_n))$ = **generalization error**
  - Expected gradient: $f'(\theta) = \mathbb{E} f'_n(\theta) = \mathbb{E}\left\{\ell'(y_n, \theta^\top \Phi(x_n))\, \Phi(x_n)\right\}$
  - Non-asymptotic results

- **Number of iterations = number of observations**

Stochastic approximation

- **Goal**: Minimizing a function f defined on $\mathbb{R}^d$
  - given only unbiased estimates $f'_n(\theta_n)$ of its gradients $f'(\theta_n)$ at certain points $\theta_n \in \mathbb{R}^d$

- **Stochastic approximation**
  - (much) broader applicability beyond convex optimization

$$\theta_n = \theta_{n-1} - \gamma_n h_n(\theta_{n-1}) \text{ with } \mathbb{E}\big[h_n(\theta_{n-1})|\theta_{n-1}\big] = h(\theta_{n-1})$$

  - Beyond convex problems, i.i.d assumption, finite dimension, etc.
  - Typically asymptotic results
  - See, e.g., Kushner and Yin (2003); Borkar (2008); Benveniste et al. (2012)

Relationship to online learning

- **Stochastic approximation**
  - Minimize $f(\theta) = \mathbb{E}_z \ell(\theta, z)$ = **generalization error** of θ
  - Using the gradients of single i.i.d. observations

Relationship to online learning

- **Stochastic approximation**
  - Minimize $f(\theta) = \mathbb{E}_z \ell(\theta, z)$ = **generalization error** of θ
  - Using the gradients of single i.i.d. observations

- **Batch learning**
  - Finite set of observations: $z_1, \dots, z_n$
  - Empirical risk: $\hat{f}(\theta) = \frac{1}{n} \sum_{k=1}^{n} \ell(\theta, z_i)$
  - Estimator $\hat{\theta}$ = Minimizer of $\hat{f}(\theta)$ over a certain class Θ
  - Generalization bound using uniform concentration results

Relationship to online learning

- **Stochastic approximation**
  - Minimize $f(\theta) = \mathbb{E}_z \ell(\theta, z)$ = **generalization error** of θ
  - Using the gradients of single i.i.d. observations

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  - Estimator $\hat{\theta}$ = Minimizer of $\hat{f}(\theta)$ over a certain class Θ
  - Generalization bound using uniform concentration results

- **Online learning**
  - Update $\hat{\theta}_n$ after each new (potentially adversarial) observation z_n
  - Cumulative loss: $\frac{1}{n} \sum_{k=1}^{n} \ell(\hat{\theta}_{k-1}, z_k)$
  - Online to batch through averaging (Cesa-Bianchi et al., 2004)

Convex stochastic approximation

- **Key properties of f and/or f_n**
  - Smoothness: f B-Lipschitz continuous, f' L-Lipschitz continuous
  - Strong convexity: f μ-strongly convex

Convex stochastic approximation

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  - Smoothness: f B-Lipschitz continuous, f' L-Lipschitz continuous
  - Strong convexity: f μ-strongly convex

- **Key algorithm:** Stochastic gradient descent (a.k.a. Robbins-Monro)

$$\boxed{\theta_n = \theta_{n-1} - \gamma_n f_n'(\theta_{n-1})}$$

  - Polyak-Ruppert averaging: $\bar{\theta}_n = \frac{1}{n}\sum_{k=0}^{n-1}\theta_k$
  - Which learning rate sequence γ_n? Classical setting: $\boxed{\gamma_n = Cn^{-\alpha}}$

Convex stochastic approximation

- **Key properties of f and/or f_n**
  - Smoothness: f B-Lipschitz continuous, f' L-Lipschitz continuous
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  - Which learning rate sequence γ_n? Classical setting: $\boxed{\gamma_n = Cn^{-\alpha}}$

- **Desirable practical behavior**
  - Applicable (at least) to classical supervised learning problems
  - Robustness to (potentially unknown) constants (L,B,μ)
  - Adaptivity to difficulty of the problem (e.g., strong convexity)

Stochastic subgradient descent/method

- **Assumptions**
  - f_n convex and B-Lipschitz-continuous on $\{\|\theta\|_2 \leqslant D\}$
  - (f_n) i.i.d. functions such that $\mathbb{E}f_n = f$
  - θ_* global optimum of f on $\{\|\theta\|_2 \leqslant D\}$

- **Algorithm**: $\theta_n = \Pi_D\bigg(\theta_{n-1} - \dfrac{2D}{B\sqrt{n}} f'_n(\theta_{n-1})\bigg)$

- **Bound**:
$$\mathbb{E}f\bigg(\frac{1}{n}\sum_{k=0}^{n-1}\theta_k\bigg) - f(\theta_*) \leqslant \frac{2DB}{\sqrt{n}}$$

- "Same" three-line proof as in the deterministic case

- Minimax convergence rate

- Running-time complexity: $O(dn)$ after n iterations

Stochastic subgradient method - proof - I

- Iteration: $\theta_n = \Pi_D(\theta_{n-1} - \gamma_n f_n'(\theta_{n-1}))$ with $\gamma_n = \frac{2D}{B\sqrt{n}}$

- $\mathcal{F}_n$: information up to time n

- $\|f_n'(\theta)\|_2 \leqslant B$ and $\|\theta\|_2 \leqslant D$, unbiased gradients/functions $\mathbb{E}(f_n|\mathcal{F}_{n-1}) = f$

$$\begin{aligned}
\|\theta_n - \theta_*\|_2^2 &\leqslant \|\theta_{n-1} - \theta_* - \gamma_n f_n'(\theta_{n-1})\|_2^2 \text{ by contractivity of projections} \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f_n'(\theta_{n-1}) \text{ because } \|f_n'(\theta_{n-1})\|_2 \leqslant B
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\big[\|\theta_n - \theta_*\|_2^2|\mathcal{F}_{n-1}\big] &\leqslant \|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f'(\theta_{n-1}) \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n\big[f(\theta_{n-1}) - f(\theta_*)\big] \text{ (subgradient property)} \\
\mathbb{E}\|\theta_n - \theta_*\|_2^2 &\leqslant \mathbb{E}\|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n\big[\mathbb{E}f(\theta_{n-1}) - f(\theta_*)\big]
\end{aligned}$$

- leading to $\mathbb{E}f(\theta_{n-1}) - f(\theta_*) \leqslant \dfrac{B^2\gamma_n}{2} + \dfrac{1}{2\gamma_n}\big[\mathbb{E}\|\theta_{n-1} - \theta_*\|_2^2 - \mathbb{E}\|\theta_n - \theta_*\|_2^2\big]$

Stochastic subgradient method - proof - II

- Starting from $\mathbb{E}f(\theta_{n-1}) - f(\theta_*) \leqslant \frac{B^2\gamma_n}{2} + \frac{1}{2\gamma_n}\big[\mathbb{E}\|\theta_{n-1} - \theta_*\|_2^2 - \mathbb{E}\|\theta_n - \theta_*\|_2^2\big]$

$$\begin{aligned}
\sum_{u=1}^{n}\big[\mathbb{E}f(\theta_{u-1}) - f(\theta_*)\big] &\leqslant \sum_{u=1}^{n}\frac{B^2\gamma_u}{2} + \sum_{u=1}^{n}\frac{1}{2\gamma_u}\big[\mathbb{E}\|\theta_{u-1} - \theta_*\|_2^2 - \mathbb{E}\|\theta_u - \theta_*\|_2^2\big] \\
&\leqslant \sum_{u=1}^{n}\frac{B^2\gamma_u}{2} + \frac{4D^2}{2\gamma_n} \leqslant \frac{2DB}{\sqrt{n}} \text{ with } \gamma_n = \frac{2D}{B\sqrt{n}}
\end{aligned}$$

- Using convexity: $\mathbb{E}f\left(\frac{1}{n}\sum_{k=0}^{n-1}\theta_k\right) - f(\theta_*) \leqslant \frac{2DB}{\sqrt{n}}$

Stochastic subgradient descent - strong convexity - I

- **Assumptions**
  - f_n convex and B-Lipschitz-continuous
  - (f_n) i.i.d. functions such that $\mathbb{E} f_n = f$
  - f μ-strongly convex on $\{\|\theta\|_2 \leqslant D\}$
  - θ_* global optimum of f over $\{\|\theta\|_2 \leqslant D\}$

- **Algorithm**: $\theta_n = \Pi_D\bigg(\theta_{n-1} - \dfrac{2}{\mu(n+1)} f_n'(\theta_{n-1})\bigg)$

- **Bound**:

$$\mathbb{E} f\bigg(\frac{2}{n(n+1)}\sum_{k=1}^{n} k\theta_{k-1}\bigg) - f(\theta_*) \leqslant \frac{2B^2}{\mu(n+1)}$$

- "Same" three-line proof than in the deterministic case

- Minimax convergence rate

Stochastic subgradient descent - strong convexity - II

- **Assumptions**
  - f_n convex and B-Lipschitz-continuous
  - (f_n) i.i.d. functions such that $\mathbb{E} f_n = f$
  - θ_* global optimum of $g = f + \frac{\mu}{2}\|\cdot\|_2^2$
  - No compactness assumption - no projections

- **Algorithm**:

$$\theta_n = \theta_{n-1} - \frac{2}{\mu(n+1)} g_n'(\theta_{n-1}) = \theta_{n-1} - \frac{2}{\mu(n+1)} \big[f_n'(\theta_{n-1}) + \mu \theta_{n-1} \big]$$

- **Bound**: $\mathbb{E} g\bigg(\frac{2}{n(n+1)} \sum_{k=1}^{n} k\theta_{k-1} \bigg) - g(\theta_*) \leqslant \frac{2B^2}{\mu(n+1)}$

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Convex stochastic approximation
Existing work

- **Known global minimax rates of convergence for non-smooth problems** (Nemirovsky and Yudin, 1983; Agarwal et al., 2012)
  - Strongly convex: $O((\mu n)^{-1})$
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- **Many contributions in optimization and online learning:** Bottou and Le Cun (2005); Bottou and Bousquet (2008); Hazan et al. (2007); Shalev-Shwartz and Srebro (2008); Shalev-Shwartz et al. (2007, 2009); Xiao (2010); Duchi and Singer (2009); Nesterov and Vial (2008); Nemirovski et al. (2009)

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- **Non-asymptotic analysis for smooth problems?**

Smoothness/convexity assumptions

- Iteration: $\boxed{\theta_n = \theta_{n-1} - \gamma_n f_n'(\theta_{n-1})}$
  - Polyak-Ruppert averaging: $\bar{\theta}_n = \frac{1}{n}\sum_{k=0}^{n-1}\theta_k$
- **Smoothness of f_n**: For each $n \geqslant 1$, the function f_n is a.s. convex, differentiable with L-Lipschitz-continuous gradient f_n':
  - Smooth loss and bounded data
- **Strong convexity of f**: The function f is strongly convex with respect to the norm $\|\cdot\|$, with convexity constant $\mu > 0$:
  - Invertible population covariance matrix
  - or regularization by $\frac{\mu}{2}\|\theta\|^2$

Summary of new results (Bach and Moulines, 2011)

- Stochastic gradient descent with learning rate $\gamma_n = Cn^{-\alpha}$
- **Strongly convex smooth objective functions**
  - Old: $O(n^{-1})$ rate achieved without averaging for $\alpha = 1$
  - New: $O(n^{-1})$ rate achieved with averaging for $\alpha \in [1/2, 1]$
  - Non-asymptotic analysis with explicit constants
  - Forgetting of initial conditions
  - Robustness to the choice of C

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  - Robustness to the choice of C
- **Convergence rates** for $\mathbb{E}\|\theta_n - \theta^*\|^2$ and $\mathbb{E}\|\bar{\theta}_n - \theta^*\|^2$
  - no averaging: $O\Big(\dfrac{\sigma^2\gamma_n}{\mu}\Big) + O(e^{-\mu n\gamma_n})\|\theta_0 - \theta^*\|^2$
  - averaging: $\dfrac{\operatorname{tr} H(\theta^*)^{-1}}{n} + \mu^{-1}O(n^{-2\alpha}+n^{-2+\alpha}) + O\Big(\dfrac{\|\theta_0 - \theta^*\|^2}{\mu^2 n^2}\Big)$

Classical proof sketch (no averaging)

$$\begin{aligned}
\|\theta_n - \theta_*\|_2^2 &= \|\theta_{n-1} - \gamma_n f_n'(\theta_{n-1}) - \theta_*\|_2^2 \\
&= \|\theta_{n-1} - \theta_*\|_2^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f_n'(\theta_{n-1}) + \gamma_n^2 \|f_n'(\theta_{n-1})\|_2^2 \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f_n'(\theta_{n-1}) \\
&\qquad + 2\gamma_n^2 \|f_n'(\theta_*)\|_2^2 + 2\gamma_n^2 \|f_n'(\theta_{n-1}) - f_n'(\theta_*)\|_2^2 \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f_n'(\theta_{n-1}) \\
&\qquad + 2\gamma_n^2 \|f_n'(\theta_*)\|_2^2 + 2\gamma_n^2 L [f_n'(\theta_{n-1}) - f_n'(\theta_*)]^\top (\theta_{n-1} - \theta_*) \\
\mathbb{E}\big[\|\theta_n - \theta_*\|_2^2 | \mathcal{F}_{n-1}\big] &\leqslant \|\theta_{n-1} - \theta_*\|_2^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f'(\theta_{n-1}) \\
&\qquad + 2\gamma_n^2 \mathbb{E}\|f_n'(\theta_*)\|_2^2 + 2\gamma_n^2 L [f'(\theta_{n-1}) - 0]^\top (\theta_{n-1} - \theta_*) \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 - 2\gamma_n(1 - \gamma_n L)(\theta_{n-1} - \theta_*)^\top f'(\theta_{n-1}) + 2\gamma_n^2 \sigma^2 \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 - 2\gamma_n(1 - \gamma_n L)\frac{1}{2}\mu \|\theta_{n-1} - \theta_*\|_2^2 + 2\gamma_n^2 \sigma^2 \\
&= \big[1 - \mu\gamma_n(1 - \gamma_n L)\big] \|\theta_{n-1} - \theta_*\|_2^2 + 2\gamma_n^2 \sigma^2 \\
\mathbb{E}\big[\|\theta_{n-1} - \theta_*\|_2^2\big] &\leqslant \big[1 - \mu\gamma_n(1 - \gamma_n L)\big] \mathbb{E}\big[\|\theta_{n-1} - \theta_*\|_2^2\big] + 2\gamma_n^2 \sigma^2
\end{aligned}$$

Proof sketch (averaging)

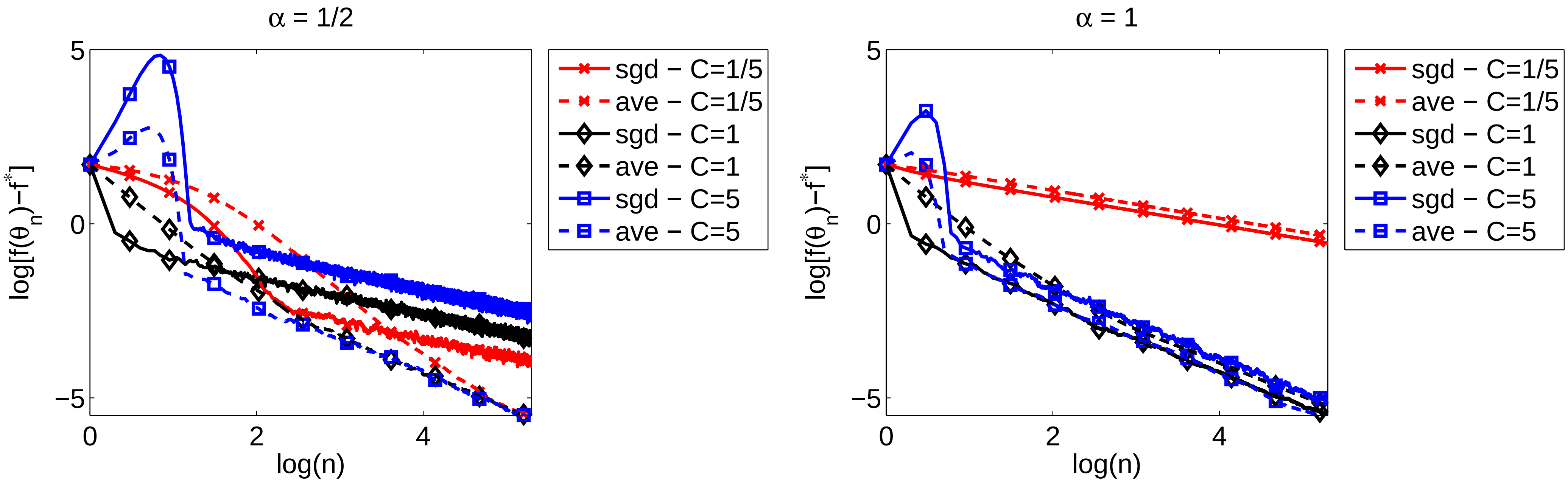
- From Polyak and Juditsky (1992):

$$
\begin{aligned}
& \theta_n = \theta_{n-1} - \gamma_n f_n'(\theta_{n-1}) \\
\Leftrightarrow \quad & f_n'(\theta_{n-1}) = \frac{1}{\gamma_n}(\theta_{n-1} - \theta_n) \\
\Leftrightarrow \quad & f_n'(\theta_*) + f_n''(\theta_*)(\theta_{n-1} - \theta_*) = \frac{1}{\gamma_n}(\theta_{n-1} - \theta_n) + O(\|\theta_{n-1} - \theta_*\|^2) \\
\Leftrightarrow \quad & f_n'(\theta_*) + f''(\theta_*)(\theta_{n-1} - \theta_*) = \frac{1}{\gamma_n}(\theta_{n-1} - \theta_n) + O(\|\theta_{n-1} - \theta_*\|^2) \\
& \qquad + O(\|\theta_{n-1} - \theta_*\|)\varepsilon_n \\
\Leftrightarrow \quad & \theta_{n-1} - \theta_* = -f''(\theta_*)^{-1} f_n'(\theta_*) + \frac{1}{\gamma_n} f''(\theta_*)^{-1}(\theta_{n-1} - \theta_n) \\
& \qquad + O(\|\theta_{n-1} - \theta_*\|^2) + O(\|\theta_{n-1} - \theta_*\|)\varepsilon_n
\end{aligned}
$$

- Averaging to cancel the term $\frac{1}{\gamma_n} f''(\theta_*)^{-1}(\theta_{n-1} - \theta_n)$

Robustness to wrong constants for $\gamma_n = Cn^{-\alpha}$

- $f(\theta) = \frac{1}{2}|\theta|^2$ with i.i.d. Gaussian noise ($d = 1$)
- Left: $\alpha = 1/2$
- Right: $\alpha = 1$



- See also `http://leon.bottou.org/projects/sgd`

Summary of new results (Bach and Moulines, 2011)

- Stochastic gradient descent with learning rate $\gamma_n = Cn^{-\alpha}$
- **Strongly convex smooth objective functions**
  - Old: $O(n^{-1})$ rate achieved without averaging for $\alpha = 1$
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- **Non-strongly convex smooth objective functions**
  - Old: $O(n^{-1/2})$ rate achieved with averaging for $\alpha = 1/2$
  - New: $O(\max\{n^{1/2-3\alpha/2}, n^{-\alpha/2}, n^{\alpha-1}\})$ rate achieved without averaging for $\alpha \in [1/3, 1]$
- **Take-home message**
  - Use $\alpha = 1/2$ with averaging to be adaptive to strong convexity

Beyond stochastic gradient method

- **Adding a proximal step**
  - Goal: $\min_{\theta \in \mathbb{R}^d} f(\theta) + \Omega(\theta) = \mathbb{E} f_n(\theta) + \Omega(\theta)$
  - Replace recursion $\theta_n = \theta_{n-1} - \gamma_n f_n'(\theta_n)$ by

$$\theta_n = \min_{\theta \in \mathbb{R}^d} \big\| \theta - \theta_{n-1} + \gamma_n f_n'(\theta_n) \big\|_2^2 + C\Omega(\theta)$$

  - Xiao (2010); Hu et al. (2009)
  - May be accelerated (Ghadimi and Lan, 2013)

- **Related frameworks**
  - Regularized dual averaging (Nesterov, 2009; Xiao, 2010)
  - Mirror descent (Nemirovski et al., 2009; Lan et al., 2012)

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Adaptive algorithm for logistic regression

- **Logistic regression**: $(\Phi(x_n), y_n) \in \mathbb{R}^d \times \{-1, 1\}$
  - Single data point: $f_n(\theta) = \log(1 + \exp(-y_n \theta^\top \Phi(x_n)))$
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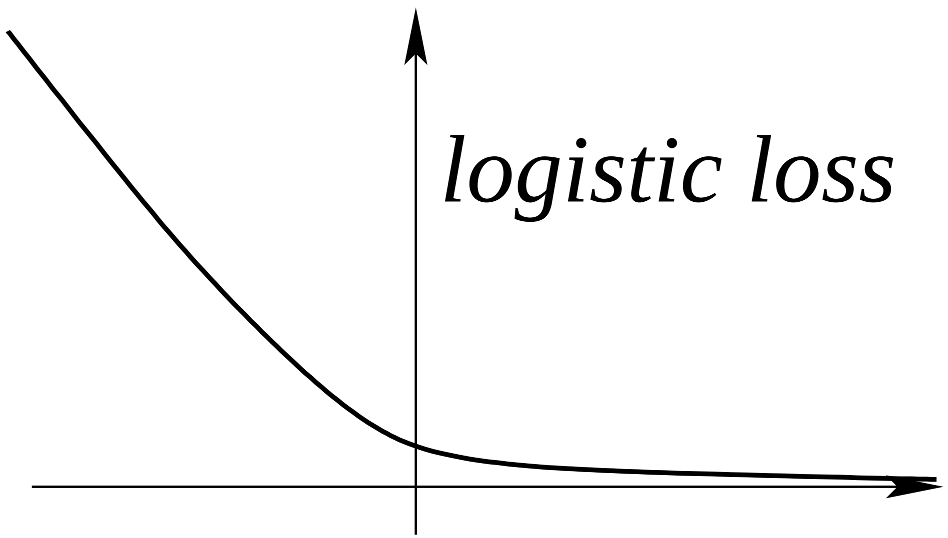

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  - with $R =$ radius of data (Bach, 2013):

$$\mathbb{E} f(\bar{\theta}_n) - f(\theta_*) \leqslant \min \left\{ \frac{1}{\sqrt{n}}, \frac{R^2}{n\mu} \right\} \big(15 + 5R\|\theta_0 - \theta_*\|\big)^4$$

  - Proof based on self-concordance (Nesterov and Nemirovski, 1994)

Self-concordance

- Usual definition for convex $\varphi : \mathbb{R} \to \mathbb{R}$: $|\varphi'''(t)| \leqslant 2\varphi''(t)^{3/2}$
  - Affine invariant
  - Extendable to all convex functions on $\mathbb{R}^d$ by looking at rays
  - Used for the sharp proof of quadratic convergence of Newton method (Nesterov and Nemirovski, 1994)
- Generalized notion: $|\varphi'''(t)| \leqslant \varphi''(t)$
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- **Important properties**
  - Allows global Taylor expansions
  - Relates expansions of derivatives of different orders

Adaptive algorithm for logistic regression
Proof sketch

- Step 1: use existing result $f(\bar{\theta}_n) - f(\theta_*) + \frac{R^2}{\sqrt{n}}\|\theta_0 - \theta_*\|_2^2 = O(1/\sqrt{n})$
- Step 2: $f_n'(\theta_{n-1}) = \frac{1}{\gamma}(\theta_{n-1} - \theta_n) \Rightarrow \frac{1}{n}\sum_{k=1}^n f_k'(\theta_{k-1}) = \frac{1}{n\gamma}(\theta_0 - \theta_n)$
- Step 3: $\Big\| f'\big(\frac{1}{n}\sum_{k=1}^n \theta_{k-1}\big) - \frac{1}{n}\sum_{k=1}^n f'(\theta_{k-1})\Big\|_2$
  $= O\big(f(\bar{\theta}_n) - f(\theta_*)\big) = O(1/\sqrt{n})$ using self-concordance
- Step 4a: if f μ-strongly convex, $f(\bar{\theta}_n) - f(\theta_*) \leqslant \frac{1}{2\mu}\big\| f'(\bar{\theta}_n)\big\|_2^2$
- Step 4b: if f self-concordant, "locally true" with $\mu = \lambda_{\min}(f''(\theta_*))$

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Least-mean-square algorithm

- **Least-squares**: $f(\theta) = \frac{1}{2}\mathbb{E}\big[(y_n - \langle \Phi(x_n), \theta\rangle)^2\big]$ with $\theta \in \mathbb{R}^d$
  - SGD = least-mean-square algorithm (see, e.g., Macchi, 1995)
  - usually studied without averaging and decreasing step-sizes
  - with strong convexity assumption $\mathbb{E}\big[\Phi(x_n) \otimes \Phi(x_n)\big] = H \succcurlyeq \mu \cdot \mathrm{Id}$

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- **New analysis for averaging and constant step-size** $\gamma = 1/(4R^2)$
  - Assume $\|\Phi(x_n)\| \leqslant R$ and $|y_n - \langle \Phi(x_n), \theta_*\rangle| \leqslant \sigma$ almost surely
  - No assumption regarding lowest eigenvalues of H
  - Main result: $\boxed{\mathbb{E} f(\bar{\theta}_{n-1}) - f(\theta_*) \leqslant \dfrac{4\sigma^2 d}{n} + \dfrac{4R^2\|\theta_0 - \theta_*\|^2}{n}}$

- **Matches statistical lower bound** (Tsybakov, 2003)
  - Non-asymptotic robust version of Györfi and Walk (1996)

Least-squares - Proof technique

- LMS recursion:

$$\theta_n - \theta_* = \big[I - \gamma \Phi(x_n) \otimes \Phi(x_n)\big](\theta_{n-1} - \theta_*) + \gamma\, \varepsilon_n \Phi(x_n)$$

- Simplified LMS recursion: with $H = \mathbb{E}\big[\Phi(x_n) \otimes \Phi(x_n)\big]$

$$\theta_n - \theta_* = \big[I - \gamma H\big](\theta_{n-1} - \theta_*) + \gamma\, \varepsilon_n \Phi(x_n)$$

  – Direct proof technique of Polyak and Juditsky (1992), e.g.,

$$\theta_n - \theta_* = \big[I - \gamma H\big]^n(\theta_0 - \theta_*) + \gamma \sum_{k=1}^{n} \big[I - \gamma H\big]^{n-k} \varepsilon_k \Phi(x_k)$$

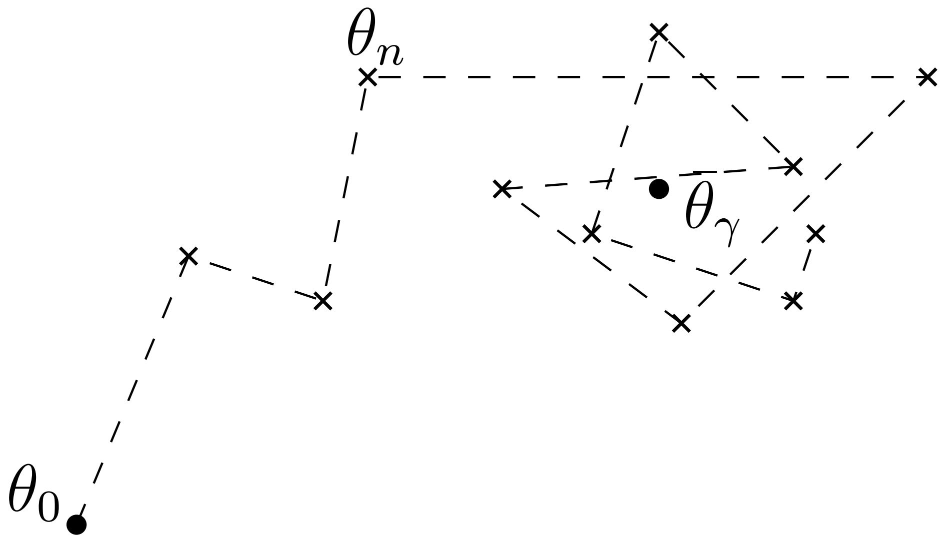
- Infinite expansion of Aguech, Moulines, and Priouret (2000) in powers of γ

Markov chain interpretation of constant step sizes

- LMS recursion for $f_n(\theta) = \frac{1}{2}\big(y_n - \langle \Phi(x_n), \theta\rangle\big)^2$

$$\theta_n = \theta_{n-1} - \gamma\big(\langle \Phi(x_n), \theta_{n-1}\rangle - y_n\big)\Phi(x_n)$$

- The sequence $(\theta_n)_n$ is a homogeneous Markov chain
  - convergence to a stationary distribution π_γ
  - with expectation $\bar{\theta}_\gamma \stackrel{\text{def}}{=} \int \theta \pi_\gamma(\mathrm{d}\theta)$



Markov chain interpretation of constant step sizes

- LMS recursion for $f_n(\theta) = \frac{1}{2}\big(y_n - \langle \Phi(x_n), \theta \rangle\big)^2$

$$\theta_n = \theta_{n-1} - \gamma\big(\langle \Phi(x_n), \theta_{n-1} \rangle - y_n\big)\Phi(x_n)$$

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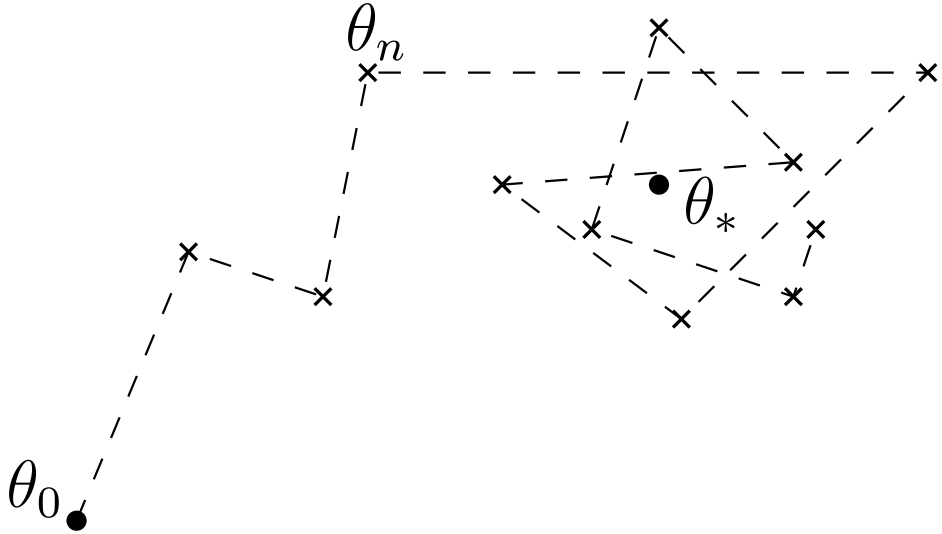
- **For least-squares, $\bar{\theta}_\gamma = \theta_*$**



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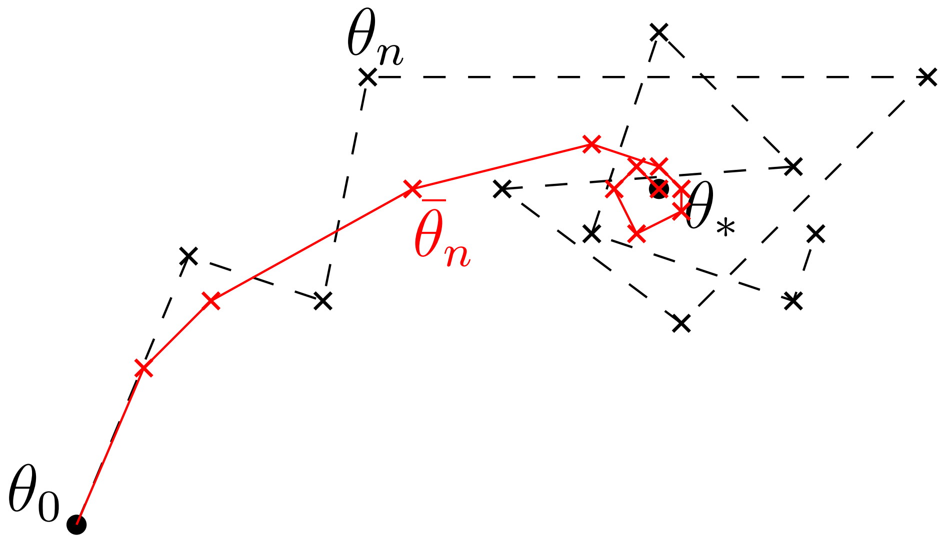

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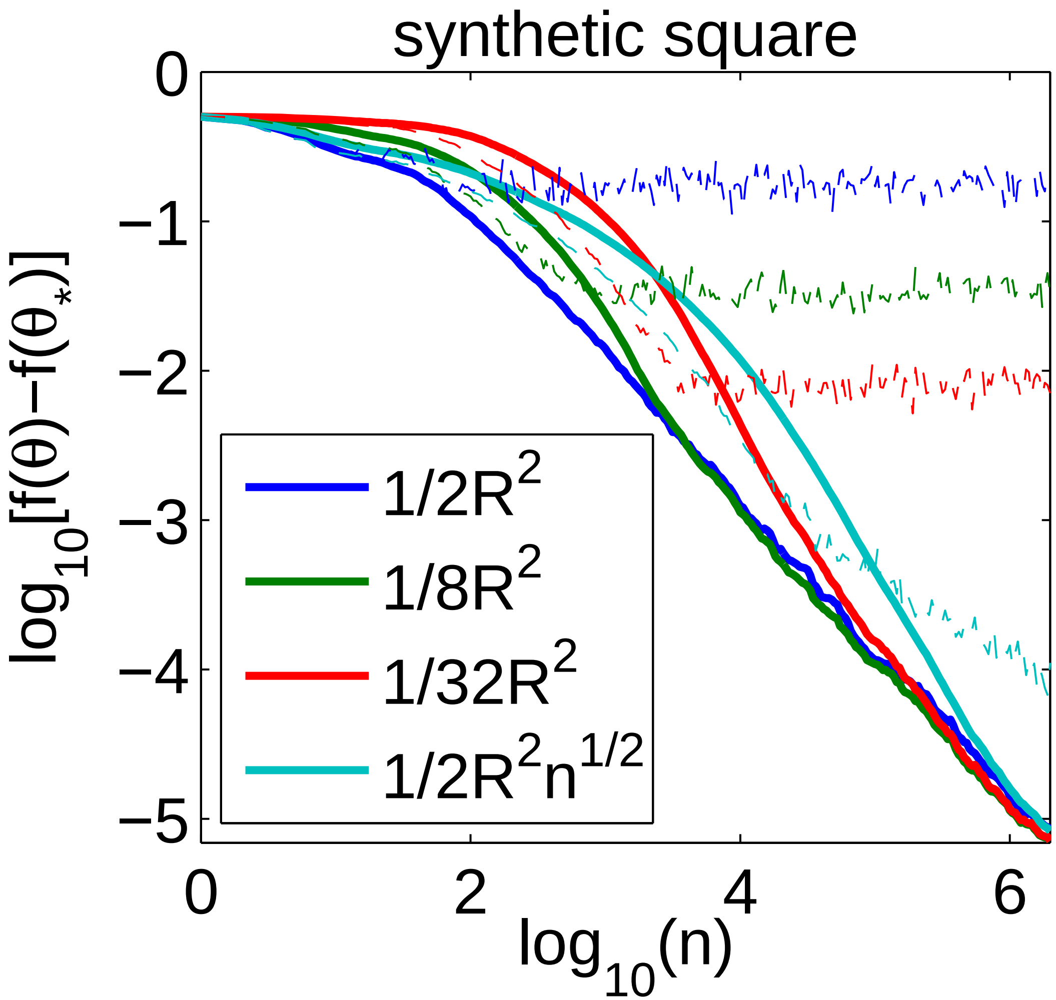
- **For least-squares, $\bar{\theta}_\gamma = \theta_*$**
  - θ_n does not converge to θ_* but oscillates around it
  - oscillations of order $\sqrt{\gamma}$

- **Ergodic theorem:**
  - Averaged iterates converge to $\bar{\theta}_\gamma = \theta_*$ at rate $O(1/n)$

Simulations - synthetic examples

- Gaussian distributions - $p = 20$



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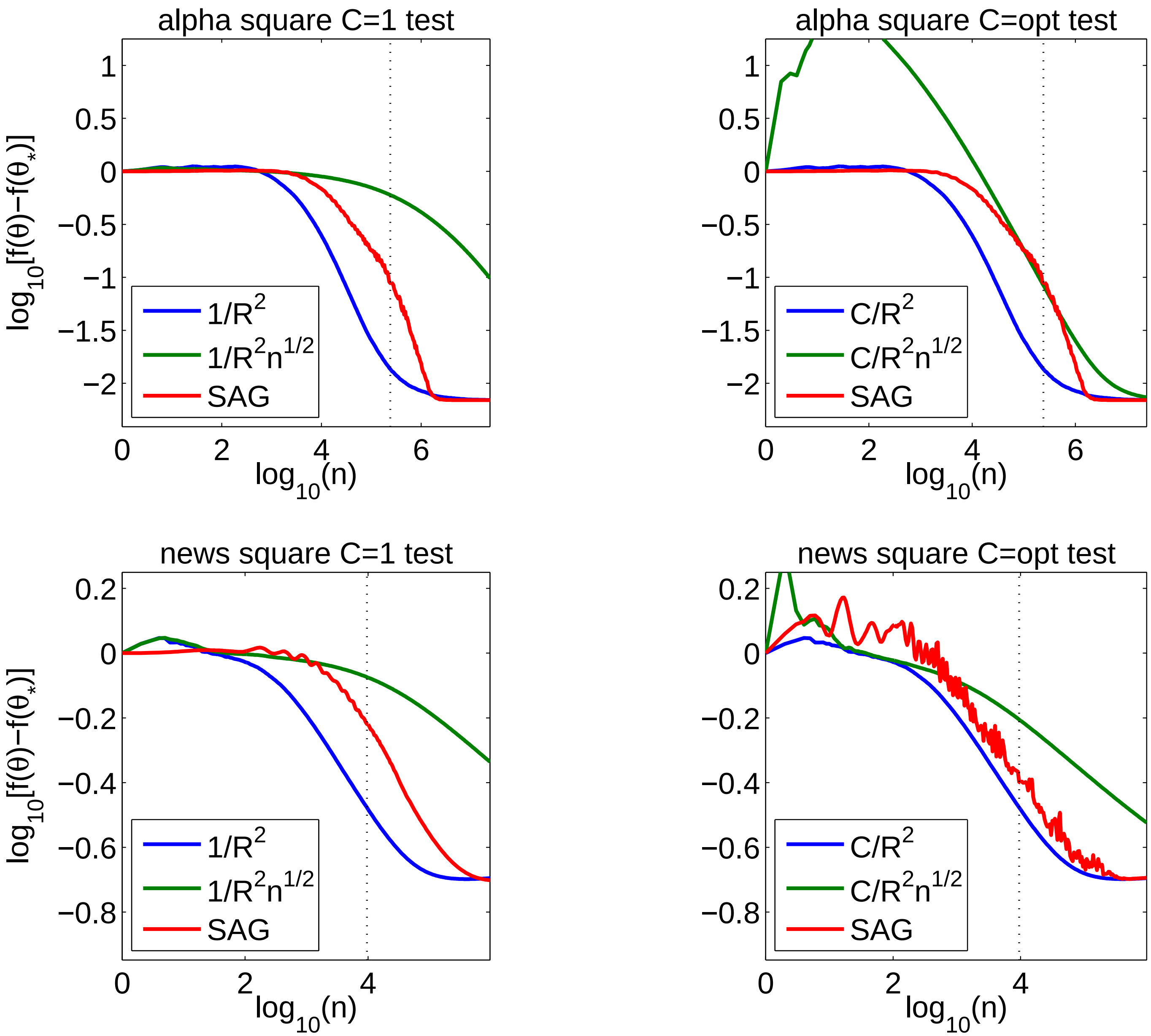

Beyond least-squares - Markov chain interpretation

- Recursion $\theta_n = \theta_{n-1} - \gamma f_n'(\theta_{n-1})$ also defines a Markov chain
  - Stationary distribution π_γ such that $\int f'(\theta)\pi_\gamma(\mathrm{d}\theta) = 0$
  - When f' is not linear, $f'(\int \theta \pi_\gamma(\mathrm{d}\theta)) \neq \int f'(\theta)\pi_\gamma(\mathrm{d}\theta) = 0$

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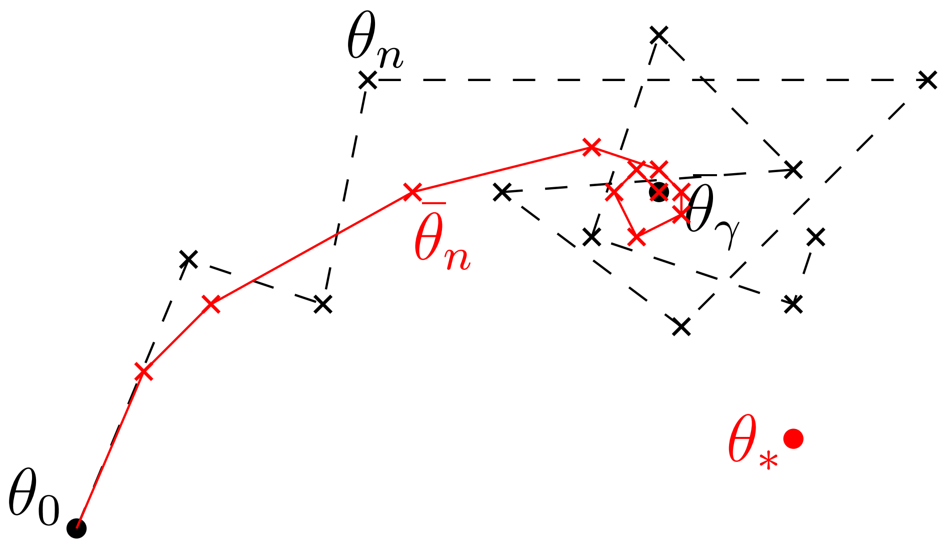

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  - moreover, $\|\theta_* - \theta_n\| = O_p(\sqrt{\gamma})$

- **Ergodic theorem**
  - averaged iterates converge to $\bar{\theta}_\gamma \neq \theta_*$ at rate $O(1/n)$
  - moreover, $\|\theta_* - \bar{\theta}_\gamma\| = O(\gamma)$ (Bach, 2013)

Simulations - synthetic examples

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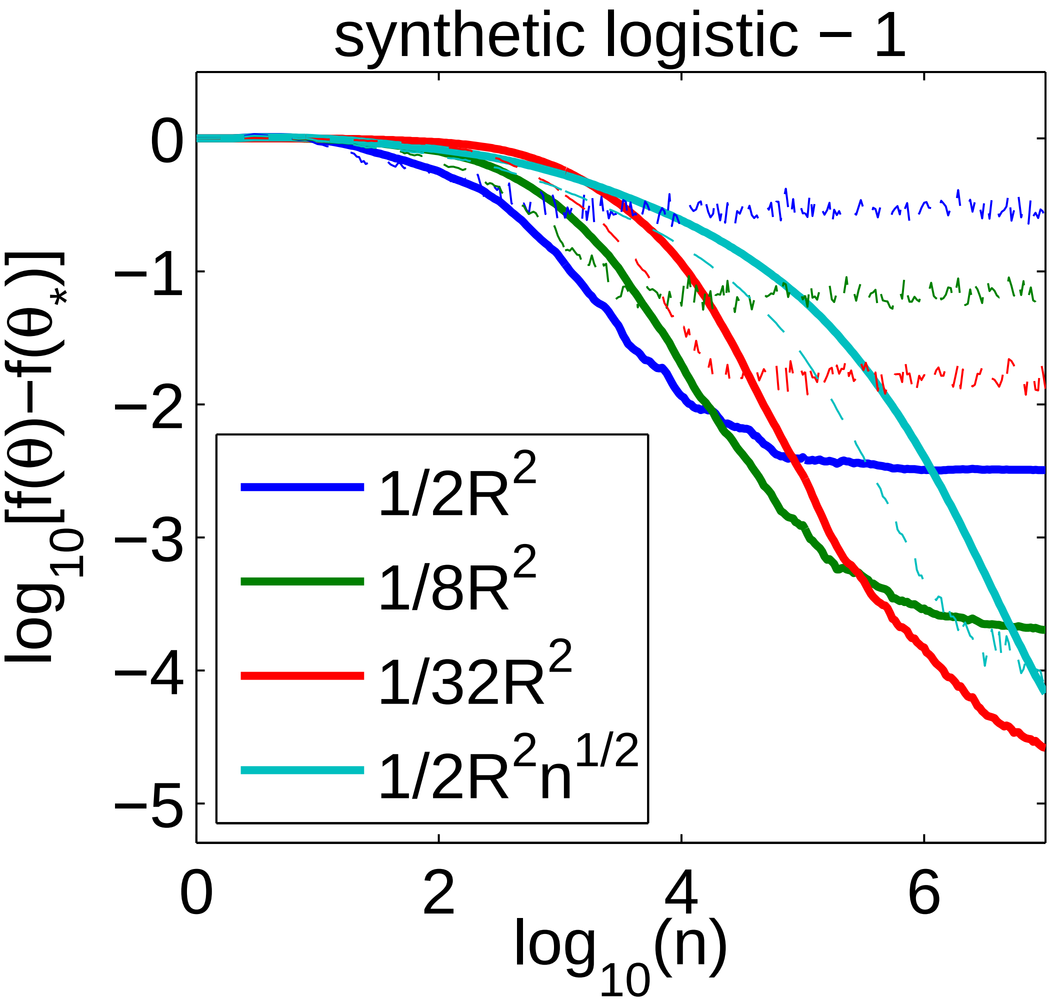

Restoring convergence through online Newton steps

- **Known facts**

  1. Averaged SGD with $\gamma_n \propto n^{-1/2}$ leads to *robust* rate $O(n^{-1/2})$ for all convex functions
  2. Averaged SGD with γ_n constant leads to *robust* rate $O(n^{-1})$ for all convex *quadratic* functions
  3. Newton's method squares the error at each iteration for smooth functions
  4. A single step of Newton's method is equivalent to minimizing the quadratic Taylor expansion

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  3. Newton's method squares the error at each iteration for smooth functions $\Rightarrow O((n^{-1/2})^2)$
  4. A single step of Newton's method is equivalent to minimizing the quadratic Taylor expansion

- **Online Newton step**
  - Rate: $O((n^{-1/2})^2 + n^{-1}) = O(n^{-1})$
  - Complexity: $O(p)$ per iteration

Restoring convergence through online Newton steps

- The Newton step for $f = \mathbb{E} f_n(\theta) \stackrel{\text{def}}{=} \mathbb{E}\big[\ell(y_n, \langle \theta, \Phi(x_n)\rangle)\big]$ at $\tilde{\theta}$ is equivalent to minimizing the quadratic approximation

$$
\begin{aligned}
g(\theta) &= f(\tilde{\theta}) + \langle f'(\tilde{\theta}), \theta - \tilde{\theta}\rangle + \tfrac{1}{2}\langle \theta - \tilde{\theta}, f''(\tilde{\theta})(\theta - \tilde{\theta})\rangle \\
&= f(\tilde{\theta}) + \langle \mathbb{E} f_n'(\tilde{\theta}), \theta - \tilde{\theta}\rangle + \tfrac{1}{2}\langle \theta - \tilde{\theta}, \mathbb{E} f_n''(\tilde{\theta})(\theta - \tilde{\theta})\rangle \\
&= \mathbb{E}\Big[f(\tilde{\theta}) + \langle f_n'(\tilde{\theta}), \theta - \tilde{\theta}\rangle + \tfrac{1}{2}\langle \theta - \tilde{\theta}, f_n''(\tilde{\theta})(\theta - \tilde{\theta})\rangle\Big]
\end{aligned}
$$

Restoring convergence through online Newton steps

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- **Complexity of least-mean-square recursion for g is $O(p)$**

$$\theta_n = \theta_{n-1} - \gamma\big[f_n'(\tilde{\theta}) + f_n''(\tilde{\theta})(\theta_{n-1} - \tilde{\theta})\big]$$

  - $f_n''(\tilde{\theta}) = \ell''(y_n, \langle \tilde{\theta}, \Phi(x_n)\rangle)\Phi(x_n) \otimes \Phi(x_n)$ has rank one
  - New online Newton step without computing/inverting Hessians

Choice of support point for online Newton step

- **Two-stage procedure**

(1) Run $n/2$ iterations of averaged SGD to obtain $\tilde{\theta}$
(2) Run $n/2$ iterations of averaged constant step-size LMS

  - Reminiscent of one-step estimators (see, e.g., Van der Vaart, 2000)
  - Provable convergence rate of $O(p/n)$ for logistic regression
  - Additional assumptions but no strong convexity

Logistic regression - Proof technique

- Using generalized self-concordance of $\varphi : u \mapsto \log(1+e^{-u})$:

$$|\varphi'''(u)| \leqslant \varphi''(u)$$

  – NB: difference with regular self-concordance: $|\varphi'''(u)| \leqslant 2\varphi''(u)^{3/2}$

- Using novel high-probability convergence results for regular averaged stochastic gradient descent

- Requires assumption on the kurtosis in every direction, i.e.,

$$\mathbb{E}\langle \Phi(x_n), \eta\rangle^4 \leqslant \kappa \big[\mathbb{E}\langle \Phi(x_n), \eta\rangle^2\big]^2$$

Choice of support point for online Newton step

- **Two-stage procedure**

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- **Update at each iteration using the current averaged iterate**

  - Recursion: $$\boxed{\theta_n = \theta_{n-1} - \gamma\big[f_n'(\bar{\theta}_{n-1}) + f_n''(\bar{\theta}_{n-1})(\theta_{n-1} - \bar{\theta}_{n-1})\big]}$$
  - No provable convergence rate (yet) but best practical behavior
  - Note (dis)similarity with regular SGD: $\theta_n = \theta_{n-1} - \gamma f_n'(\theta_{n-1})$

Online Newton algorithm
Current proof (Flammarion et al., 2014)

- Recursion

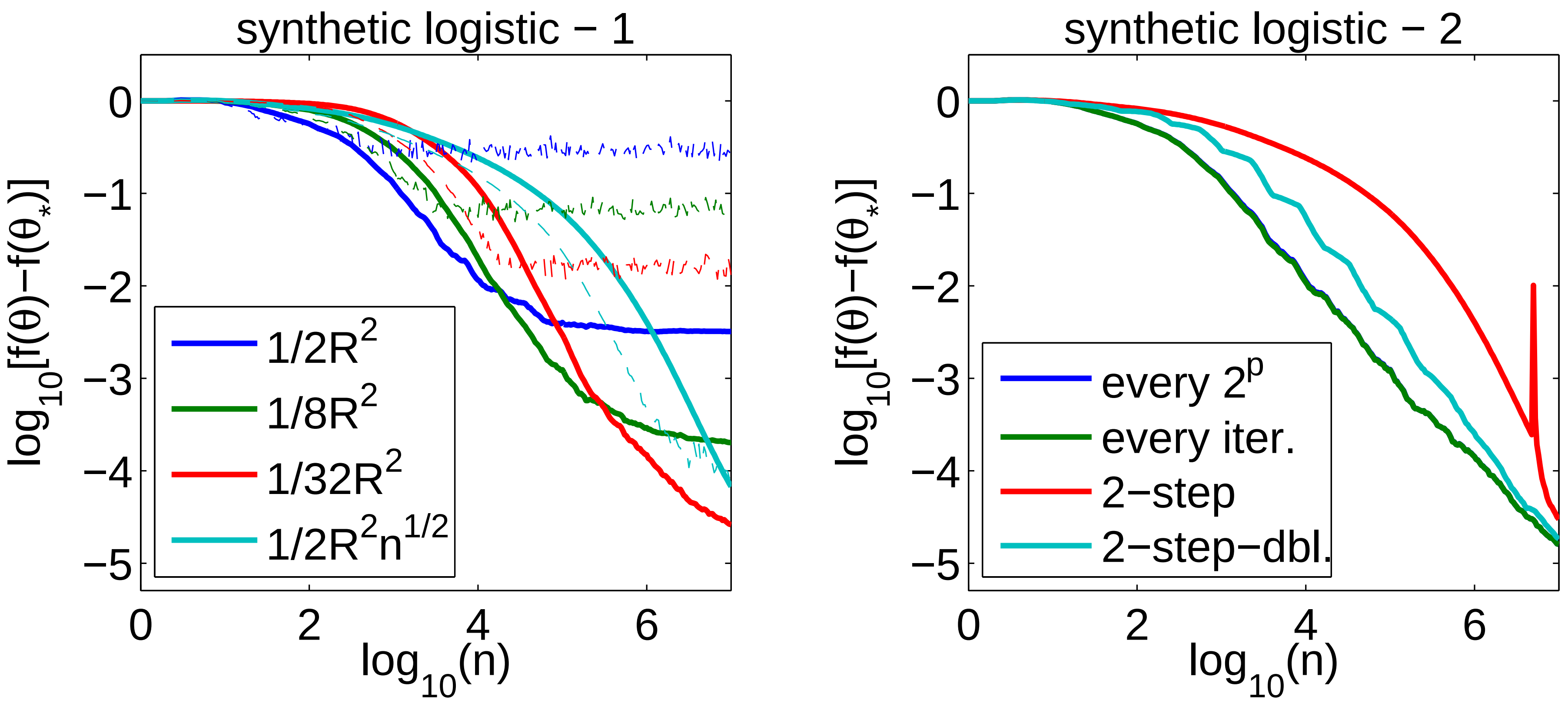
$$\begin{cases} \theta_n &= \theta_{n-1} - \gamma\big[f_n'(\bar\theta_{n-1}) + f_n''(\bar\theta_{n-1})(\theta_{n-1} - \bar\theta_{n-1})\big] \\ \bar\theta_n &= \bar\theta_{n-1} + \frac{1}{n}(\theta_n - \bar\theta_{n-1}) \end{cases}$$

- Instance of two-time-scale stochastic approximation (Borkar, 1997)
  - Given $\bar\theta$, $\theta_n = \theta_{n-1} - \gamma\big[f_n'(\bar\theta) + f_n''(\bar\theta)(\theta_{n-1} - \bar\theta)\big]$ defines a homogeneous Markov chain (fast dynamics)
  - $\bar\theta_n$ is updated at rate $1/n$ (slow dynamics)

- **Difficulty**: preserving robustness to ill-conditioning

Simulations - synthetic examples

- Gaussian distributions - $p = 20$



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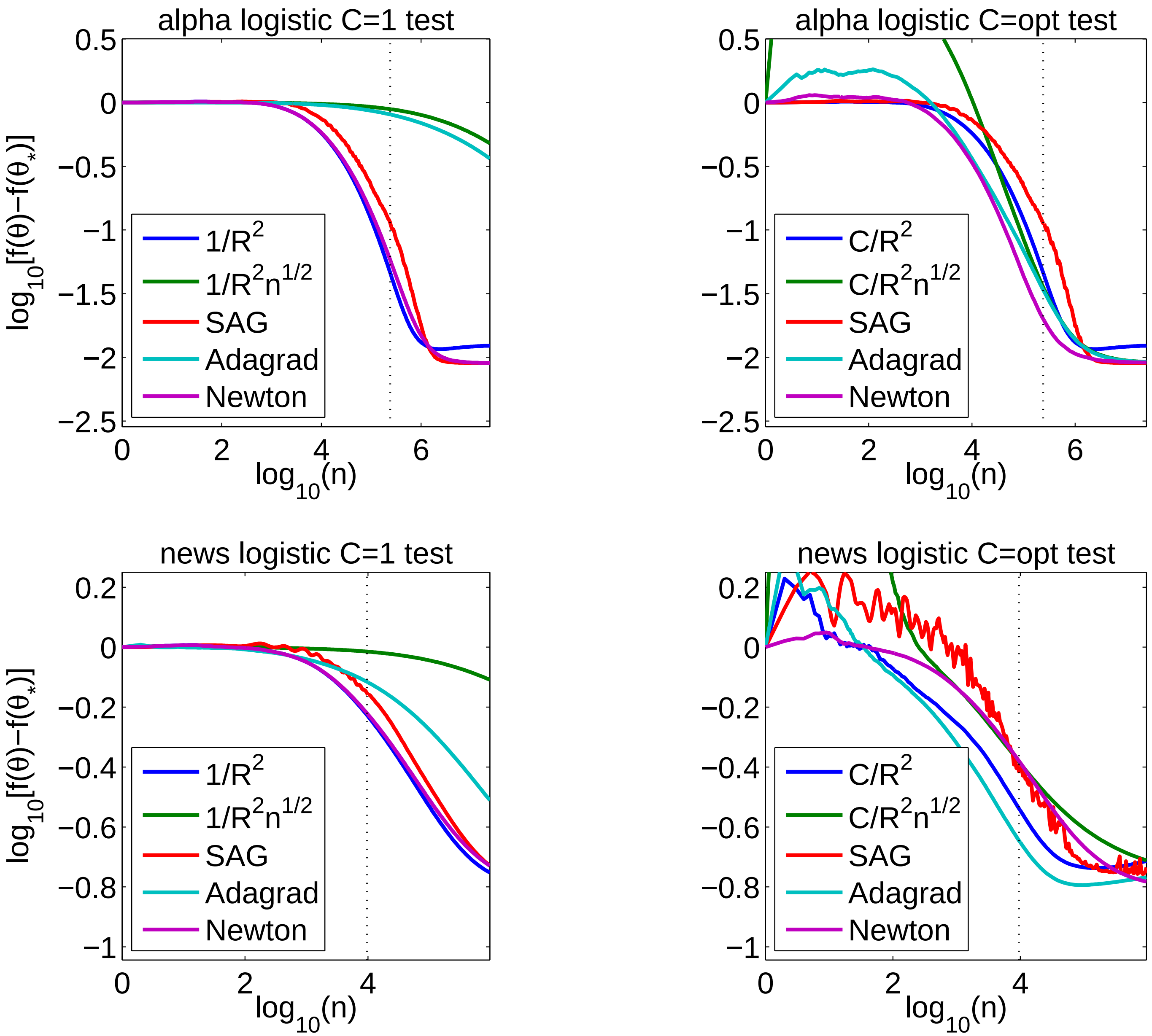

Outline

1. **Large-scale machine learning and optimization**
   - Traditional statistical analysis
   - Classical methods for convex optimization
2. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds
   - Strongly convex vs. non-strongly convex
3. **Smooth stochastic approximation algorithms**
   - Asymptotic and non-asymptotic results
4. **Beyond decaying step-sizes**
5. **Finite data sets**

Going beyond a single pass over the data

- **Stochastic approximation**
  - Assumes infinite data stream
  - Observations are used only once
  - Directly minimizes testing cost $\mathbb{E}_{(x,y)}\, \ell(y, \theta^\top \Phi(x))$

Going beyond a single pass over the data

- **Stochastic approximation**
  - Assumes infinite data stream
  - Observations are used only once
  - Directly minimizes testing cost $\mathbb{E}_{(x,y)}\, \ell(y, \theta^\top \Phi(x))$
- **Machine learning practice**
  - Finite data set $(x_1, y_1, \dots, x_n, y_n)$
  - Multiple passes
  - Minimizes training cost $\frac{1}{n}\sum_{i=1}^n \ell(y_i, \theta^\top \Phi(x_i))$
  - Need to regularize (e.g., by the ℓ_2-norm) to avoid overfitting
- **Goal**: minimize $g(\theta) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n f_i(\theta)$

Stochastic vs. deterministic methods

- Minimizing $g(\theta) = \frac{1}{n}\sum_{i=1}^{n} f_i(\theta)$ with $f_i(\theta) = \ell\big(y_i, \theta^\top \Phi(x_i)\big) + \mu\Omega(\theta)$

- Batch gradient descent: $\theta_t = \theta_{t-1} - \gamma_t g'(\theta_{t-1}) = \theta_{t-1} - \frac{\gamma_t}{n}\sum_{i=1}^{n} f_i'(\theta_{t-1})$

  - Linear (e.g., exponential) convergence rate in $O(e^{-\alpha t})$
  - Iteration complexity is linear in n *(with line search)*

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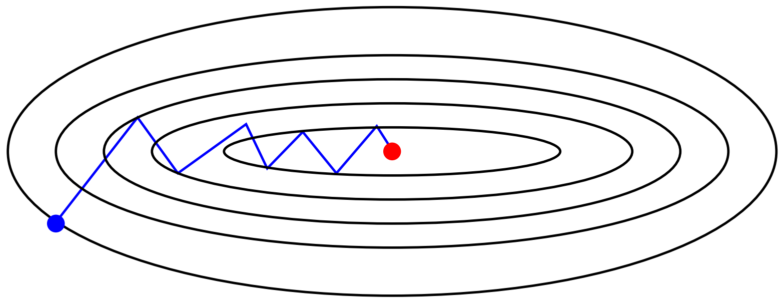

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- Stochastic gradient descent: $\theta_t = \theta_{t-1} - \gamma_t f'_{i(t)}(\theta_{t-1})$
  - Sampling with replacement: $i(t)$ random element of $\{1, \dots, n\}$
  - Convergence rate in $O(1/t)$
  - Iteration complexity is independent of n *(step size selection?)*

Stochastic vs. deterministic methods

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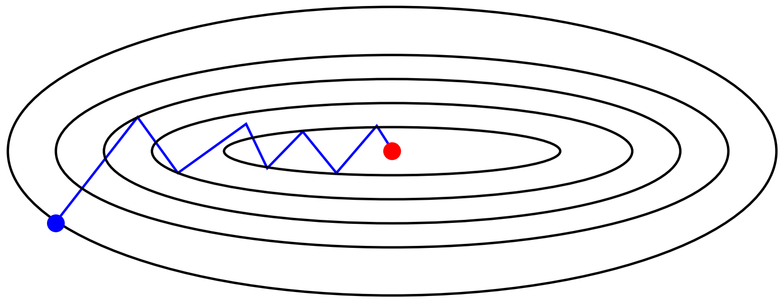

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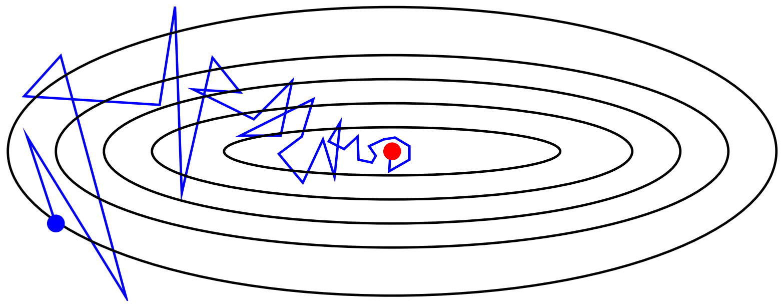
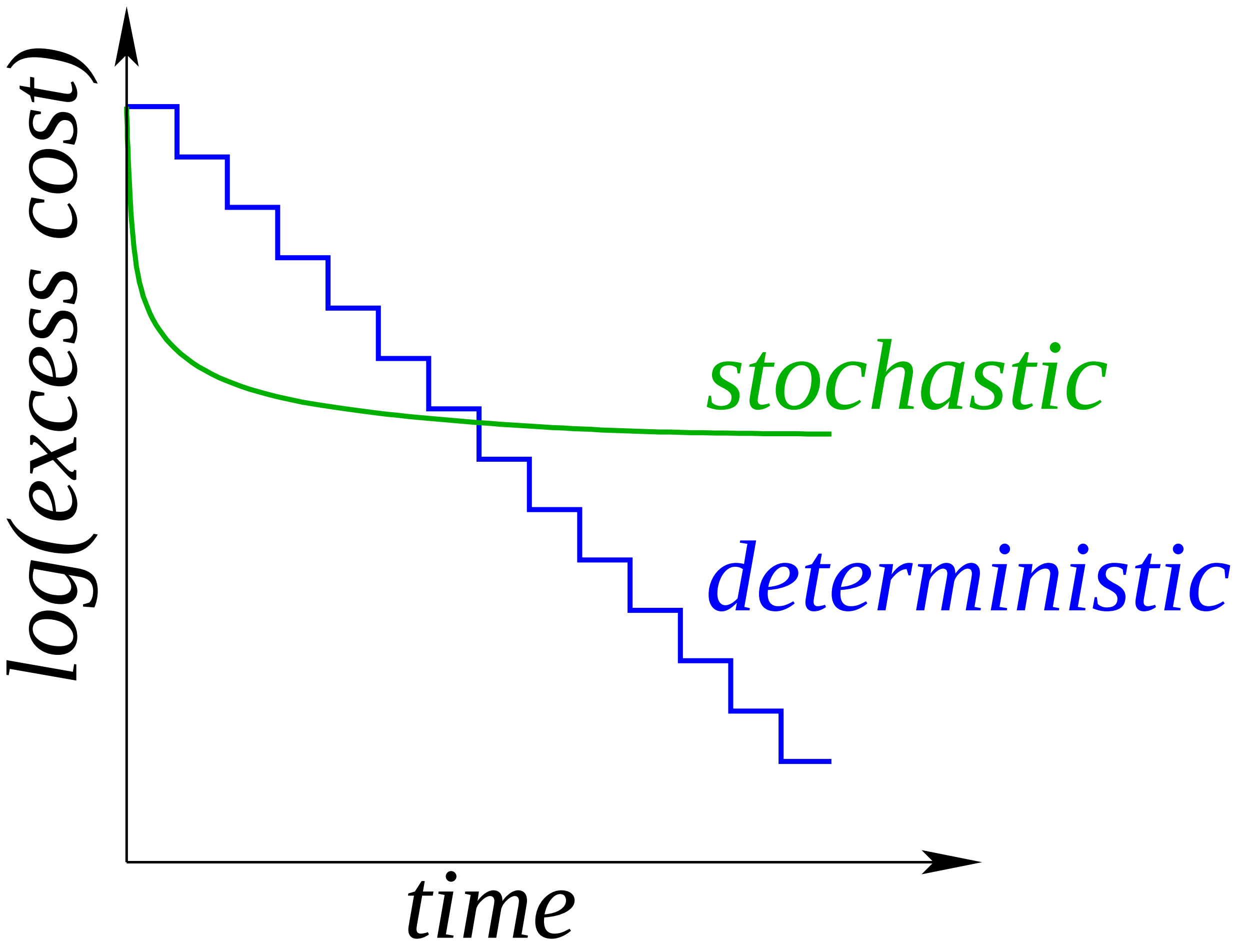


Stochastic vs. deterministic methods

- **Goal** = **best of both worlds**: Linear rate with $O(1)$ iteration cost
  Robustness to step size



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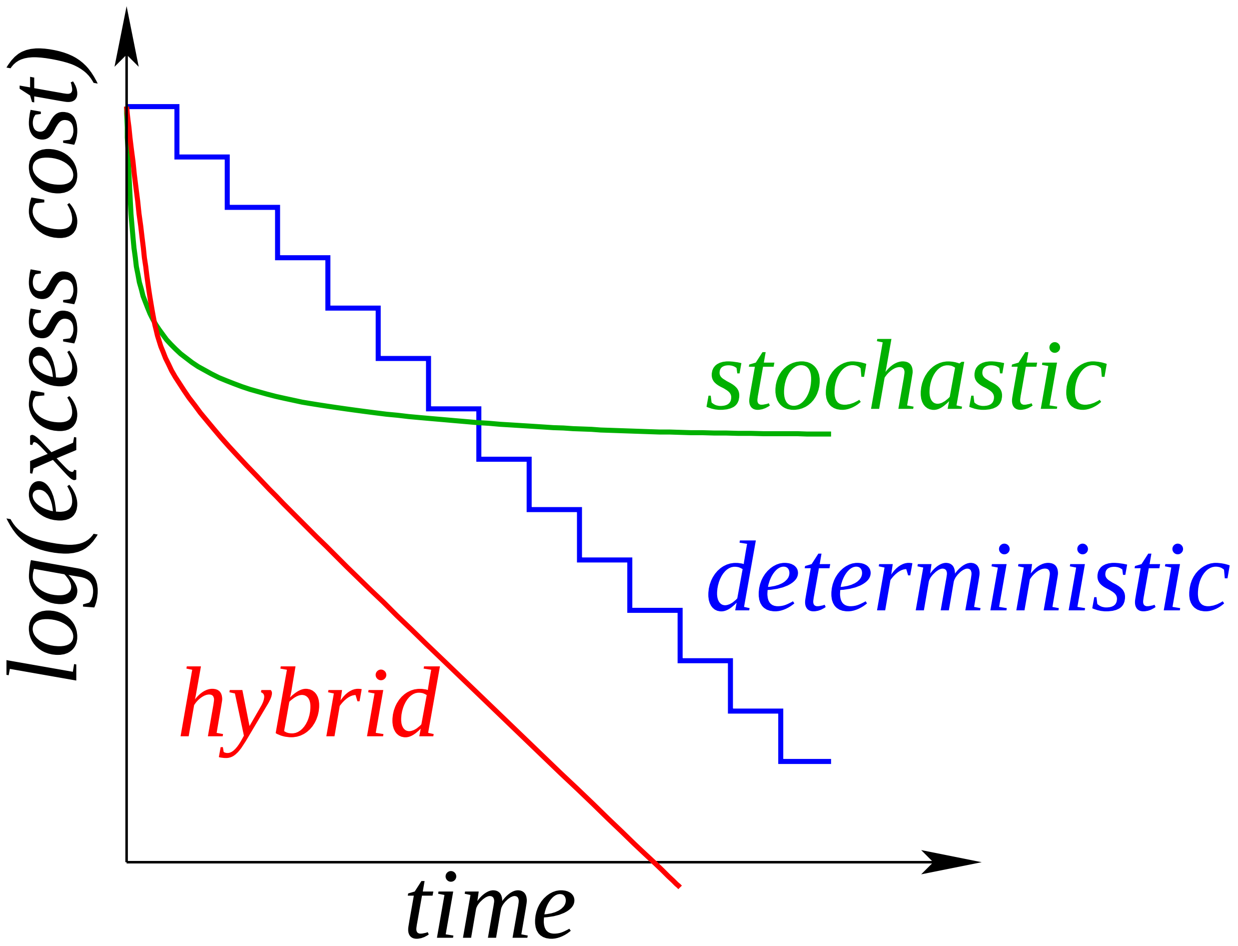

Accelerating gradient methods - Related work

- **Nesterov acceleration**
  - Nesterov (1983, 2004)
  - Better linear rate but still $O(n)$ iteration cost

- **Hybrid methods, incremental average gradient, increasing batch size**
  - Bertsekas (1997); Blatt et al. (2008); Friedlander and Schmidt (2011)
  - Linear rate, but iterations make full passes through the data.

Accelerating gradient methods - Related work

- **Momentum, gradient/iterate averaging, stochastic version of accelerated batch gradient methods**
  - Polyak and Juditsky (1992); Tseng (1998); Sunehag et al. (2009); Ghadimi and Lan (2010); Xiao (2010)
  - Can improve constants, but still have sublinear $O(1/t)$ rate

- **Constant step-size stochastic gradient (SG), accelerated SG**
  - Kesten (1958); Delyon and Juditsky (1993); Solodov (1998); Nedic and Bertsekas (2000)
  - Linear convergence, but only up to a fixed tolerance.

- **Stochastic methods in the dual**
  - Shalev-Shwartz and Zhang (2012)
  - Similar linear rate but limited choice for the f_i's

Stochastic average gradient (Le Roux, Schmidt, and Bach, 2012)

- **Stochastic average gradient** (SAG) iteration
  - Keep in memory the gradients of all functions f_i, $i = 1, \ldots, n$
  - Random selection $i(t) \in \{1, \ldots, n\}$ with replacement
  - Iteration: $\theta_t = \theta_{t-1} - \frac{\gamma_t}{n} \sum_{i=1}^{n} y_i^t$ with $y_i^t = \begin{cases} f_i'(\theta_{t-1}) & \text{if } i = i(t) \\ y_i^{t-1} & \text{otherwise} \end{cases}$

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- Stochastic version of incremental average gradient (Blatt et al., 2008)

- Extra memory requirement
  - Supervised machine learning
    - If $f_i(\theta) = \ell_i(y_i, \Phi(x_i)^\top \theta)$, then $f_i'(\theta) = \ell_i'(y_i, \Phi(x_i)^\top \theta)\, \Phi(x_i)$
    - Only need to store n real numbers

Stochastic average gradient - Convergence analysis

- **Assumptions**
  - Each f_i is L-smooth, $i = 1, \dots, n$
  - $g = \frac{1}{n} \sum_{i=1}^{n} f_i$ is μ-strongly convex (with potentially $\mu = 0$)
  - constant step size $\gamma_t = 1/(16L)$
  - initialization with one pass of averaged SGD

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- **Strongly convex case** (Le Roux et al., 2012, 2013)

$$\mathbb{E}\big[g(\theta_t) - g(\theta_*)\big] \leqslant \Big(\frac{8\sigma^2}{n\mu} + \frac{4L\|\theta_0 - \theta_*\|^2}{n}\Big) \exp\Big(-t \min\Big\{\frac{1}{8n}, \frac{\mu}{16L}\Big\}\Big)$$

  - Linear (exponential) convergence rate with $O(1)$ iteration cost
  - After one pass, reduction of cost by $\exp\Big(-\min\Big\{\frac{1}{8}, \frac{n\mu}{16L}\Big\}\Big)$

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- **Non-strongly convex case** (Le Roux et al., 2013)

$$\mathbb{E}\big[g(\theta_t) - g(\theta_*)\big] \leqslant 48 \frac{\sigma^2 + L\|\theta_0 - \theta_*\|^2}{\sqrt{n}} \frac{n}{t}$$

  - Improvement over regular batch and stochastic gradient
  - Adaptivity to potentially hidden strong convexity

Convergence analysis - Proof sketch

- **Main step**: find "good" Lyapunov function $J(\theta_t, y_1^t, \dots, y_n^t)$
  - such that $\mathbb{E}\big[J(\theta_t, y_1^t, \dots, y_n^t)|\mathcal{F}_{t-1}\big] < J(\theta_{t-1}, y_1^{t-1}, \dots, y_n^{t-1})$
  - no natural candidates

- **Computer-aided proof**
  - Parameterize function $J(\theta_t, y_1^t, \dots, y_n^t) = g(\theta_t) - g(\theta_*) + \text{quadratic}$
  - Solve semidefinite program to obtain candidates (that depend on n, μ, L)
  - Check validity with symbolic computations

Rate of convergence comparison

- Assume that $L = 100$, $\mu = .01$, and $n = 80000$
  - Full gradient method has rate
  $$\left(1 - \frac{\mu}{L}\right) = 0.9999$$
  - Accelerated gradient method has rate
  $$\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.9900$$
  - Running n iterations of SAG for the same cost has rate
  $$\left(1 - \frac{1}{8n}\right)^n = 0.8825$$
  - *Fastest possible* first-order method has rate
  $$\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.9608$$
- **Beating two lower bounds** (with additional assumptions)
  - (1) stochastic gradient and (2) full gradient

Stochastic average gradient
Implementation details and extensions

- The algorithm can use sparsity in the features to reduce the storage and iteration cost

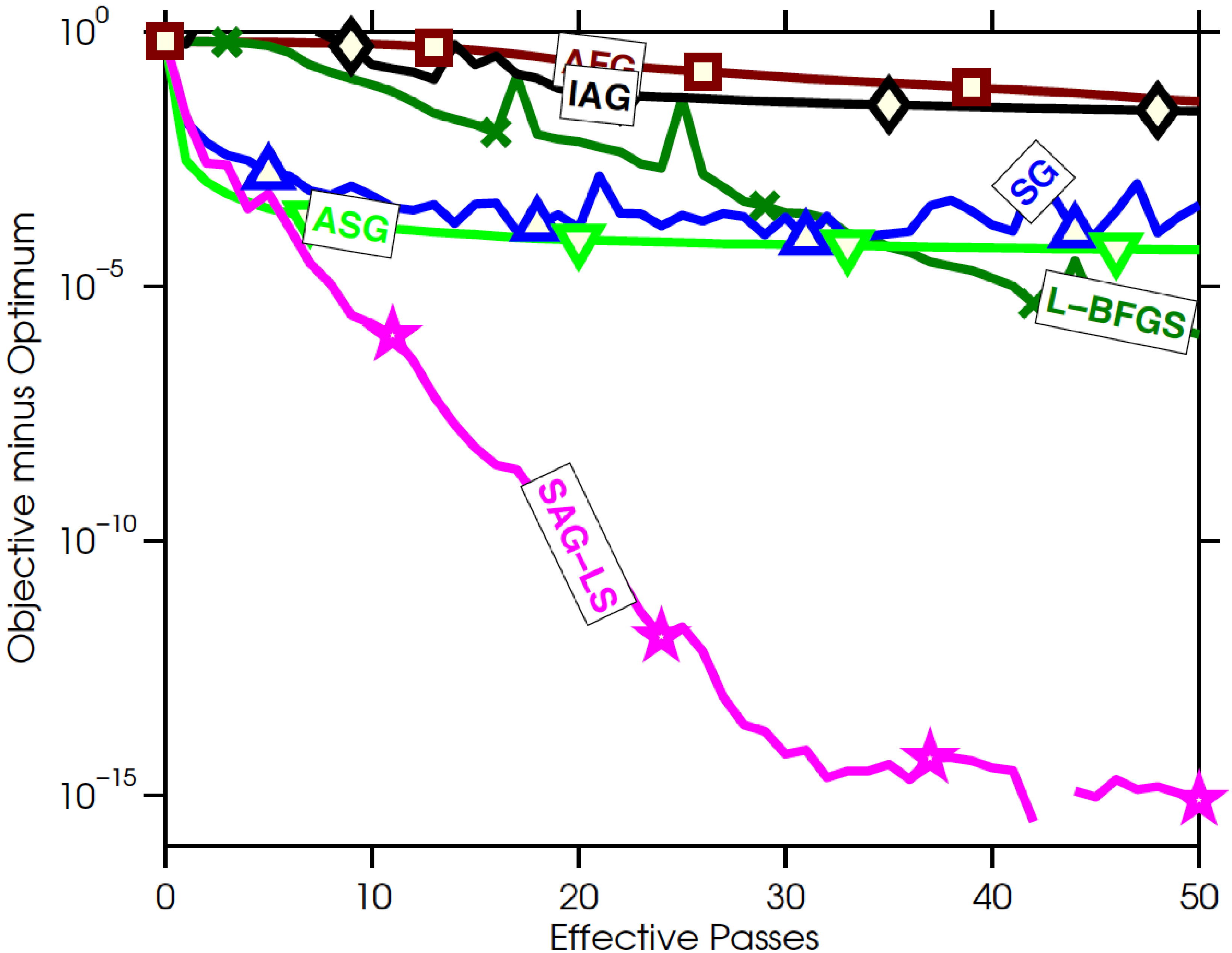
- Grouping functions together can further reduce the memory requirement

- We have obtained good performance when L is not known with a heuristic line-search

- Algorithm allows non-uniform sampling

- Possibility of making proximal, coordinate-wise, and Newton-like variants

spam dataset (n = 92 189, d = 823 470)



Summary and future work

- **Constant-step-size averaged stochastic gradient descent**
  - Reaches convergence rate $O(1/n)$ in all regimes
  - Improves on the $O(1/\sqrt{n})$ lower-bound of non-smooth problems
  - Efficient online Newton step for non-quadratic problems
  - Robustness to step-size selection

- **Going beyond a single pass through the data**

Summary and future work

- **Constant-step-size averaged stochastic gradient descent**
  - Reaches convergence rate $O(1/n)$ in all regimes
  - Improves on the $O(1/\sqrt{n})$ lower-bound of non-smooth problems
  - Efficient online Newton step for non-quadratic problems
  - Robustness to step-size selection

- **Going beyond a single pass through the data**

- **Extensions and future work**
  - Pre-conditioning
  - Proximal extensions fo non-differentiable terms
  - kernels and non-parametric estimation
  - line-search
  - parallelization

Outline

1. **Large-scale machine learning and optimization**
   - Traditional statistical analysis
   - Classical methods for convex optimization

2. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds
   - Strongly convex vs. non-strongly convex

3. **Smooth stochastic approximation algorithms**
   - Asymptotic and non-asymptotic results

4. **Beyond decaying step-sizes**

5. **Finite data sets**

Conclusions
Machine learning and convex optimization

- **Statistics with or without optimization?**
  - Significance of mixing algorithms with analysis
  - Benefits of mixing algorithms with analysis

- **Open problems**
  - Non-parametric stochastic approximation
  - Going beyond a single pass over the data (testing performance)
  - Characterization of implicit regularization of online methods
  - Further links between convex optimization and online learning/bandits

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