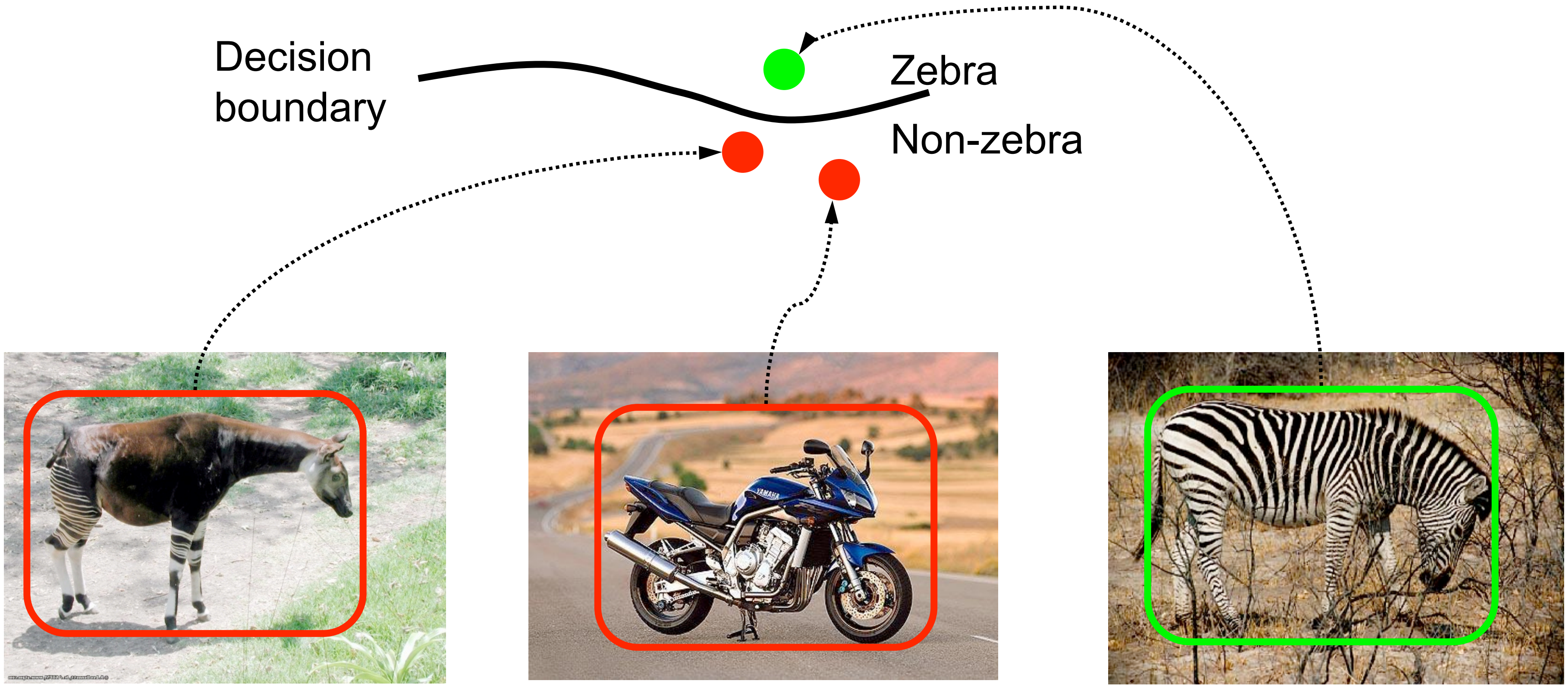
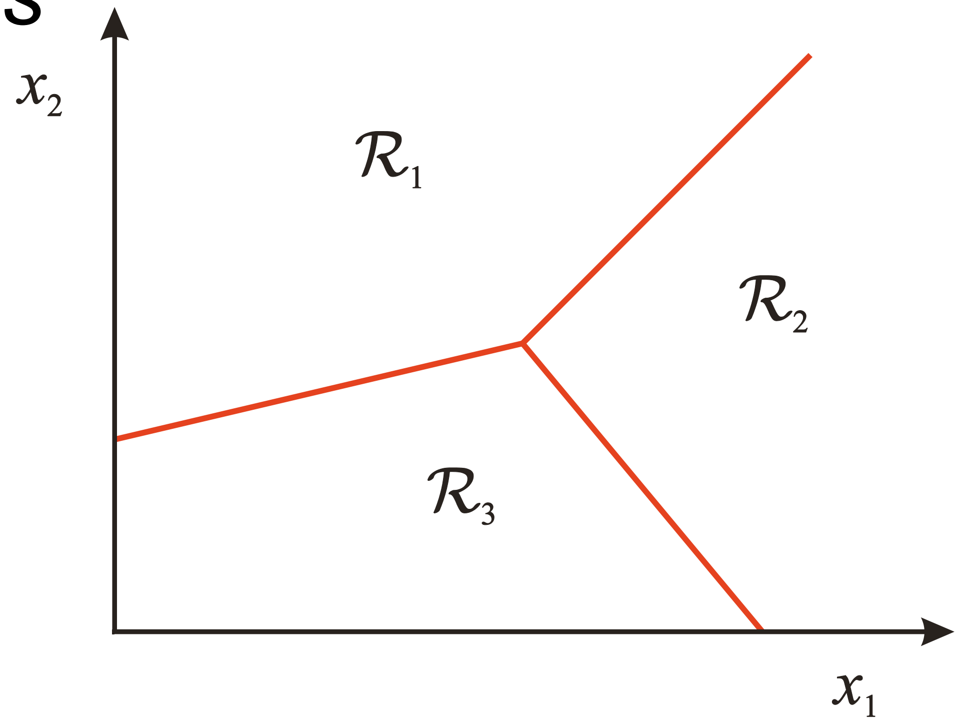
Step 3: Classification

- Learn a decision rule (classifier) assigning bag-of-features representations of images to different classes



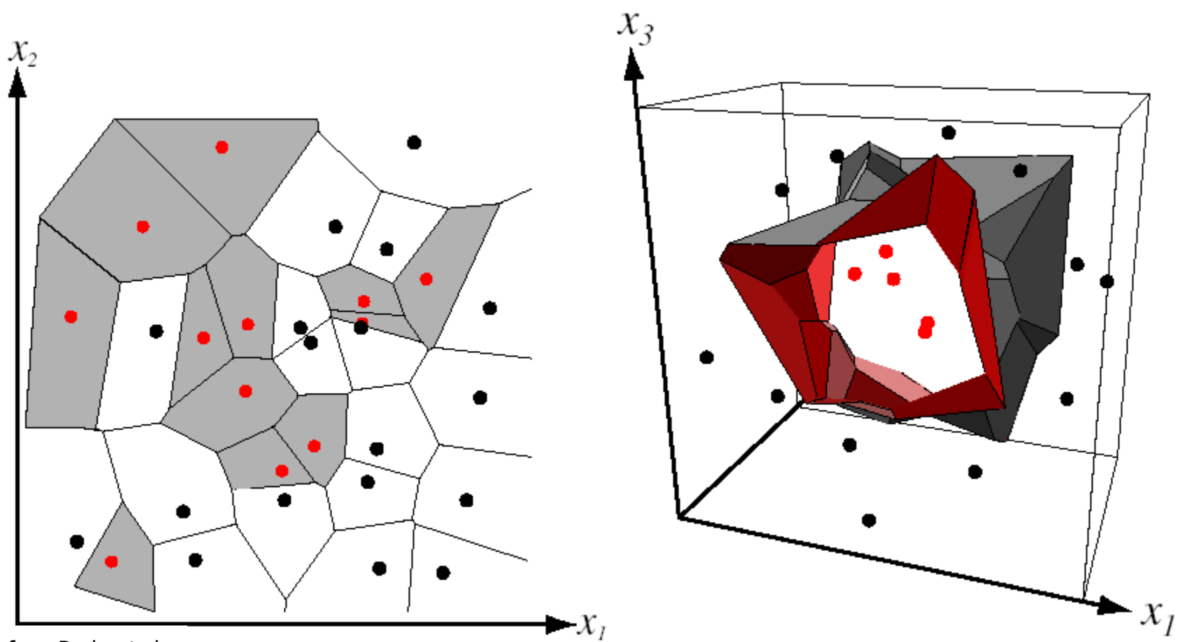
Classification

- Assign input vector to one of two or more classes
- Any decision rule divides input space into *decision regions* separated by *decision boundaries*



Nearest Neighbor Classifier

- Assign label of nearest training data point to each test data point



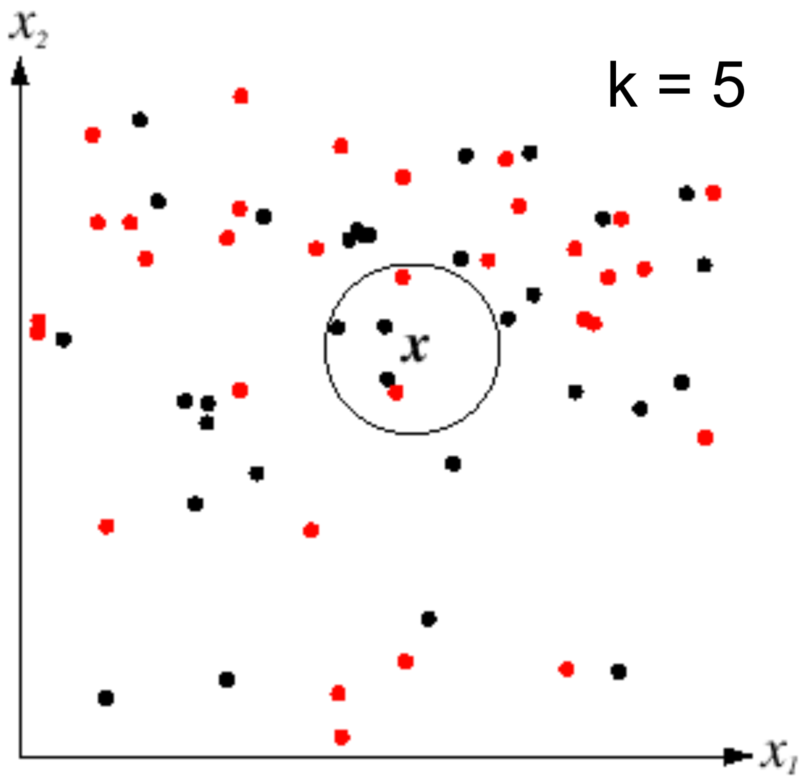
from Duda *et al.*

Voronoi partitioning of feature space
for 2-category 2-D and 3-D data

Source: D. Lowe

K-Nearest Neighbors

- For a new point, find the k closest points from training data
- Labels of the k points "vote" to classify
- Works well provided there is lots of data and the distance function is good



Source: D. Lowe

Functions for comparing histograms

- L1 distance $$D(h_1, h_2) = \sum_{i=1}^{N} | h_1(i) - h_2(i) |$$

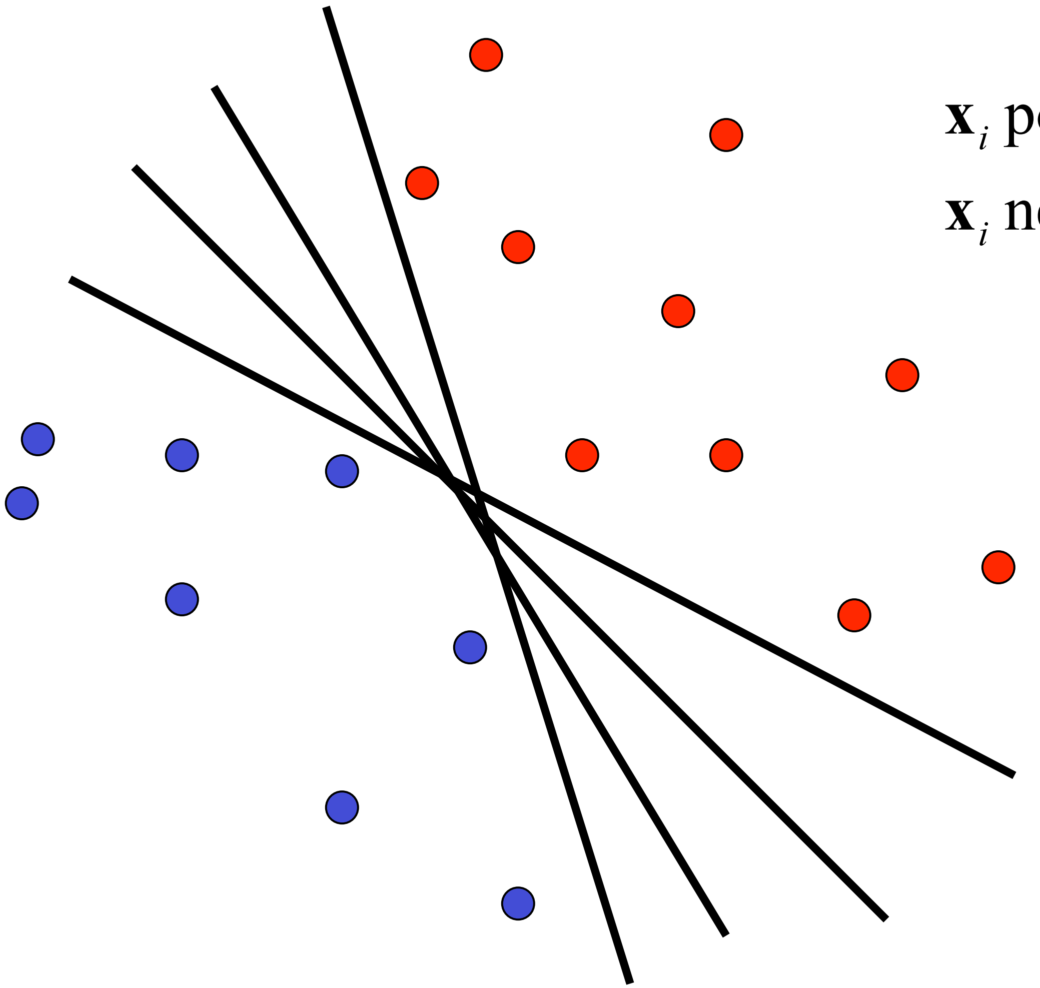
- χ^2 distance $$D(h_1, h_2) = \sum_{i=1}^{N} \frac{\left(h_1(i) - h_2(i)\right)^2}{h_1(i) + h_2(i)}$$

- Quadratic distance (*cross-bin*)

$$D(h_1, h_2) = \sum_{i,j} A_{ij} (h_1(i) - h_2(j))^2$$

Linear classifiers

- Find linear function (*hyperplane*) to separate positive and negative examples



$\mathbf{x}_i$ positive: $\quad \mathbf{x}_i \cdot \mathbf{w} + b \geq 0$

$\mathbf{x}_i$ negative: $\quad \mathbf{x}_i \cdot \mathbf{w} + b < 0$

Which hyperplane is best?

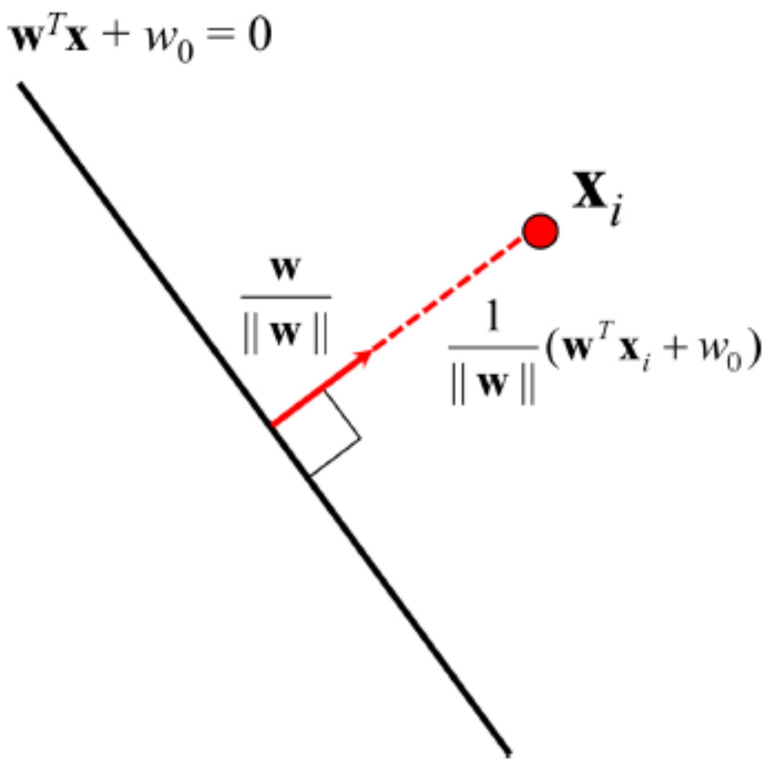
Recall: Geometry of hyperplanes

A **hyperplane** is defined by an equation

$$\mathbf{w}^T\mathbf{x} + w_0 = 0$$

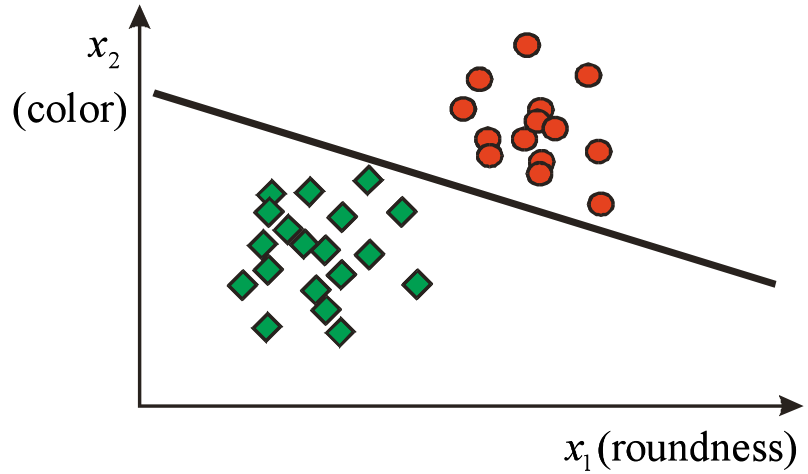
- The unit vector $\mathbf{w}/\|\mathbf{w}\|$ is normal to the hyperplane.
- The signed distance of any point $\mathbf{x}_i$ to the hyperplane is given by

$$\frac{1}{\|\mathbf{w}\|}(\mathbf{w}^T\mathbf{x}_i + w_0)$$

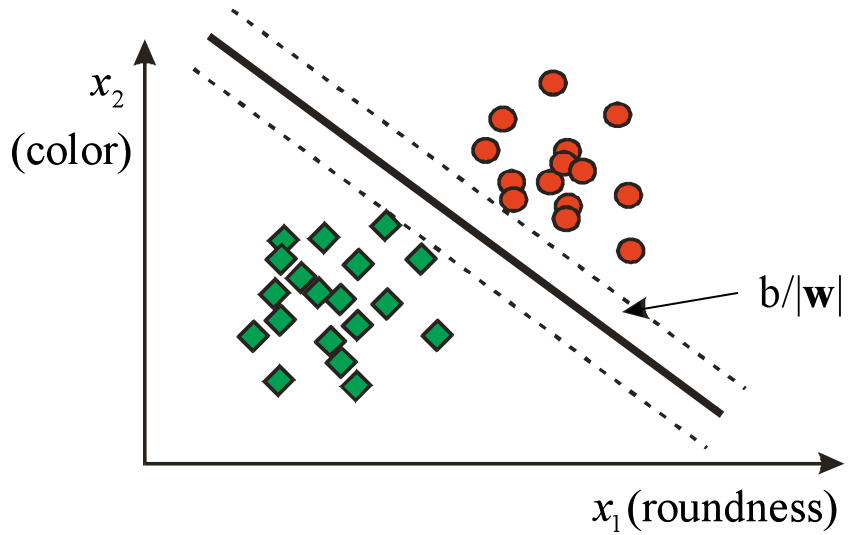


Linear classifiers - margin

Generalization is not good in this case:



Better if a margin is introduced:



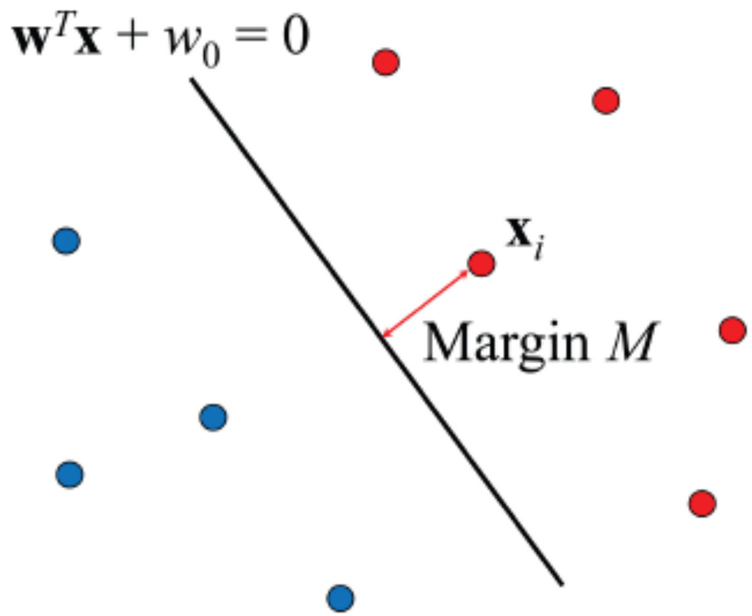
Maximum-margin separating hyperplane

Margin maximization (for linearly separable data) is formulated as follows:

$$\max_{(\mathbf{w}, w_0)} M$$

subject to $y_i(\mathbf{w}^T \mathbf{x}_i + w_0) \geq M \|\mathbf{w}\|$,

$$i = 1, \ldots, n$$



Explanation: $\frac{1}{\|\mathbf{w}\|}(\mathbf{w}^T \mathbf{x}_i + w_0)$ is the signed distance between $\mathbf{x}_i$ and the hyperplane $\mathbf{w}^T \mathbf{x} + w_0 = 0$. The constraints require that each training point is on the correct side of the decision boundary and is at least an *unsigned* distance M from it. The goal is to find the hyperplane with parameters $\mathbf{w}$ and w_0 that would have the largest such M.

Maximum-margin separating hyperplane

Constrained optimization problem:

$$\max_{(\mathbf{w}, w_0)} M$$

$$\text{subject to } y_i(\mathbf{w}^T\mathbf{x}_i + w_0) \geq M\|\mathbf{w}\|, \qquad i = 1, \ldots, n$$

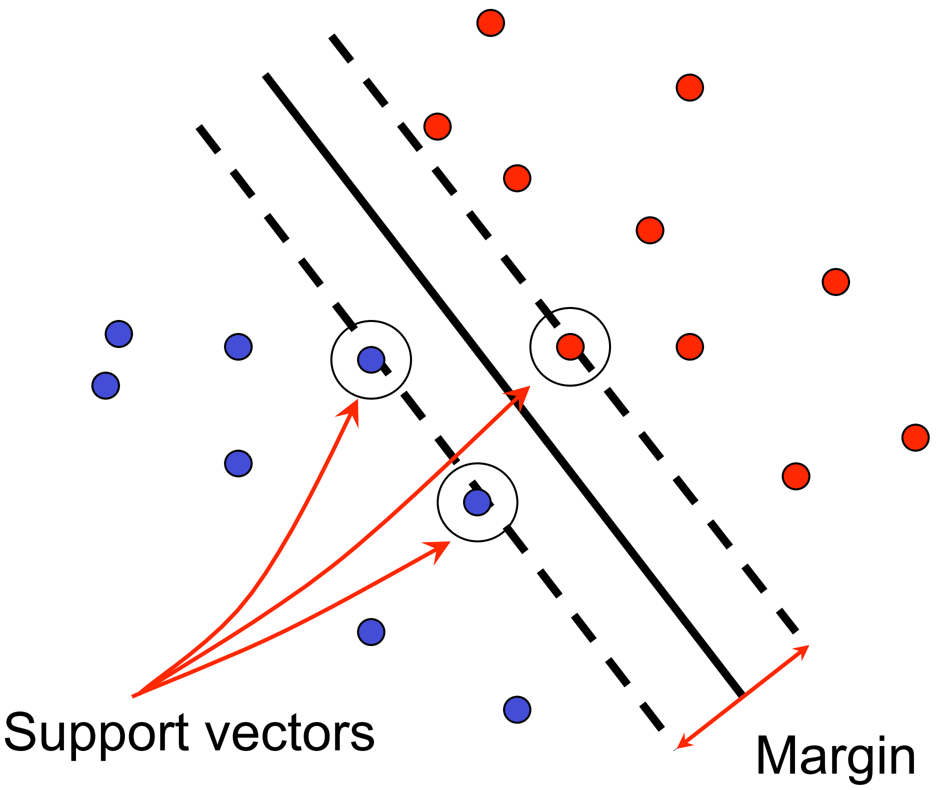
We can choose $M = 1/\|\mathbf{w}\|$ and instead solve

$$\min_{(\mathbf{w}, w_0)} \frac{1}{2}\|\mathbf{w}\|^2$$

$$\text{subject to } y_i(\mathbf{w}^T\mathbf{x}_i + w_0) \geq 1, \qquad i = 1, \ldots, n$$

Support vector machines

- Find hyperplane that maximizes the *margin* between the positive and negative examples



$\mathbf{x}_i$ positive $(y_i = 1)$: $\quad \mathbf{x}_i \cdot \mathbf{w} + b \geq 1$

$\mathbf{x}_i$ negative $(y_i = -1)$: $\quad \mathbf{x}_i \cdot \mathbf{w} + b \leq -1$

For support, vectors, $\quad \mathbf{x}_i \cdot \mathbf{w} + b = \pm 1$

The margin is $2 \,/\, \|\mathbf{w}\|$

Finding the maximum margin hyperplane

1. Maximize margin $2/||\mathbf{w}||$
2. Correctly classify all training data:

$$\mathbf{x}_i \text{ positive } (y_i = 1): \quad \mathbf{x}_i \cdot \mathbf{w} + b \geq 1$$

$$\mathbf{x}_i \text{ negative } (y_i = -1): \quad \mathbf{x}_i \cdot \mathbf{w} + b \leq -1$$

Quadratic optimization problem:

Minimize $\frac{1}{2}\mathbf{w}^T\mathbf{w}$

Subject to $y_i(\mathbf{w}\cdot\boldsymbol{x}_i+b) \geq 1$

Solution based on Lagrange multipliers

Finding the maximum margin hyperplane

- Solution: $\mathbf{w} = \sum_i \alpha_i y_i \mathbf{x}_i$

learned weight

Support vector

Finding the maximum margin hyperplane

- Solution: $\mathbf{w} = \sum_i \alpha_i y_i \mathbf{x}_i$

  $b = y_i - \mathbf{w} \cdot \mathbf{x}_i$ for any support vector

- Classification function (decision boundary):

$$\mathbf{w} \cdot \mathbf{x} + b = \sum_i \alpha_i y_i \mathbf{x}_i \cdot \mathbf{x} + b$$

- Notice that it relies on an *inner product* between the test point $\boldsymbol{x}$ and the support vectors $\boldsymbol{x}_i$
- Solving the optimization problem also involves computing the inner products $\boldsymbol{x}_i \cdot \boldsymbol{x}_j$ between all pairs of training points

Non-separable case

- What if the training data are not linearly separable? We can no longer require exact margin constraints.
- One idea: minimize

$$\min_{\mathbf{w}} \frac{1}{2}\|\mathbf{w}\|^2 + C(\#\text{mistakes}).$$

- This is the 0-1 loss.
- The parameter C determines the penalty paid for violating margin constraints. (Tradeoff: number of mistakes and margin.)
- Problem: not QP anymore, also does not distinguish between "near misses" and bad mistakes.

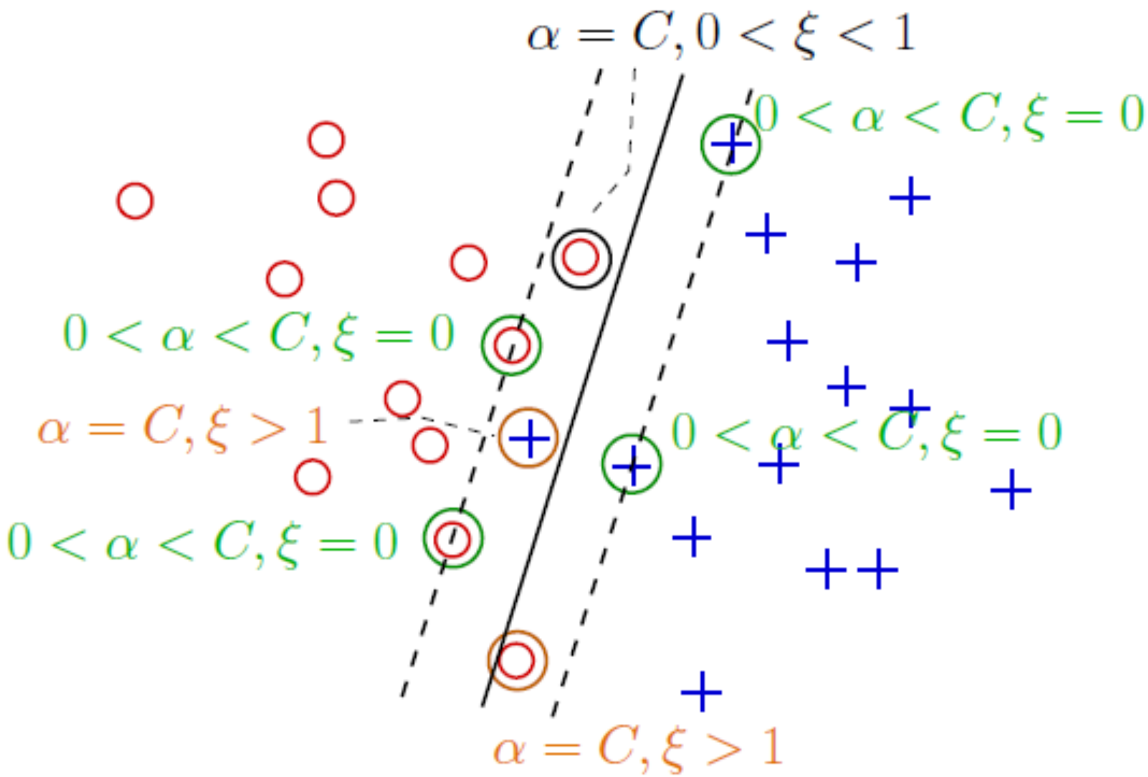
Non-separable case

- Another idea: rewrite the constraints with slack variables $\xi_i \geq 0$:

$$\min_{(\mathbf{w}, w_0)} \frac{1}{2}\|\mathbf{w}\| + C \sum_{i=1}^{n} \xi_i$$

$$\text{subject to } y_i \left(w_0 + \mathbf{w}^T \mathbf{x}_i\right) - 1 + \xi_i \geq 0.$$

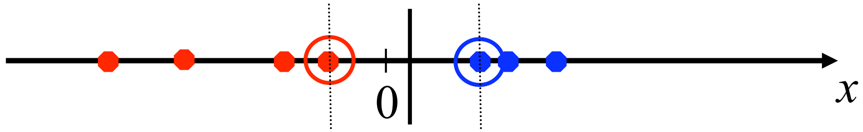
- Whenever margin is ≥ 1 (original constraint is satisfied), $\xi_i = 0$.
- Whenever margin is < 1 (constraint violated), pay linear penalty.



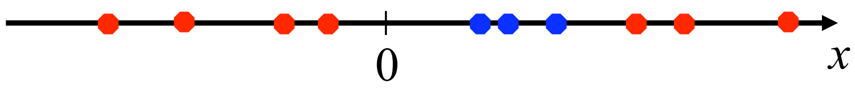
- Support vectors: points with $\alpha > 0$
- If $0 < \alpha < C$: SVs on the margin, $\xi = 0$.
- If $0 < \alpha = C$: over the margin, either misclassified ($\xi > 1$) or not ($0 < \xi \leq 1$).

Nonlinear SVMs

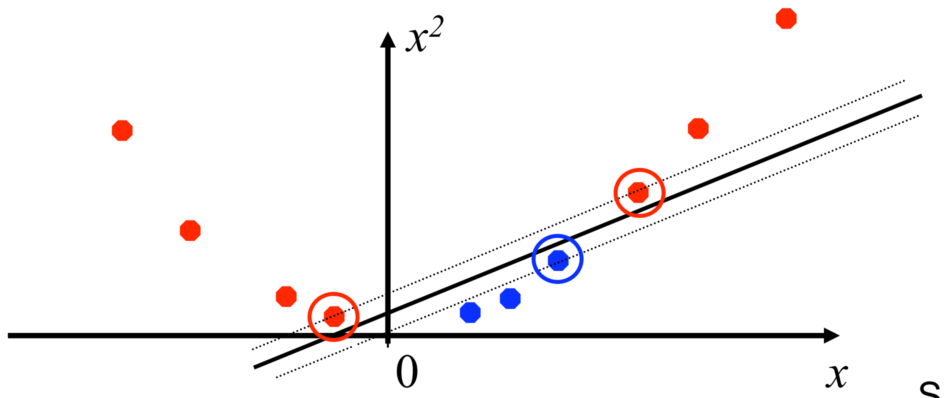
- Datasets that are linearly separable work out great:



- But what if the dataset is just too hard?



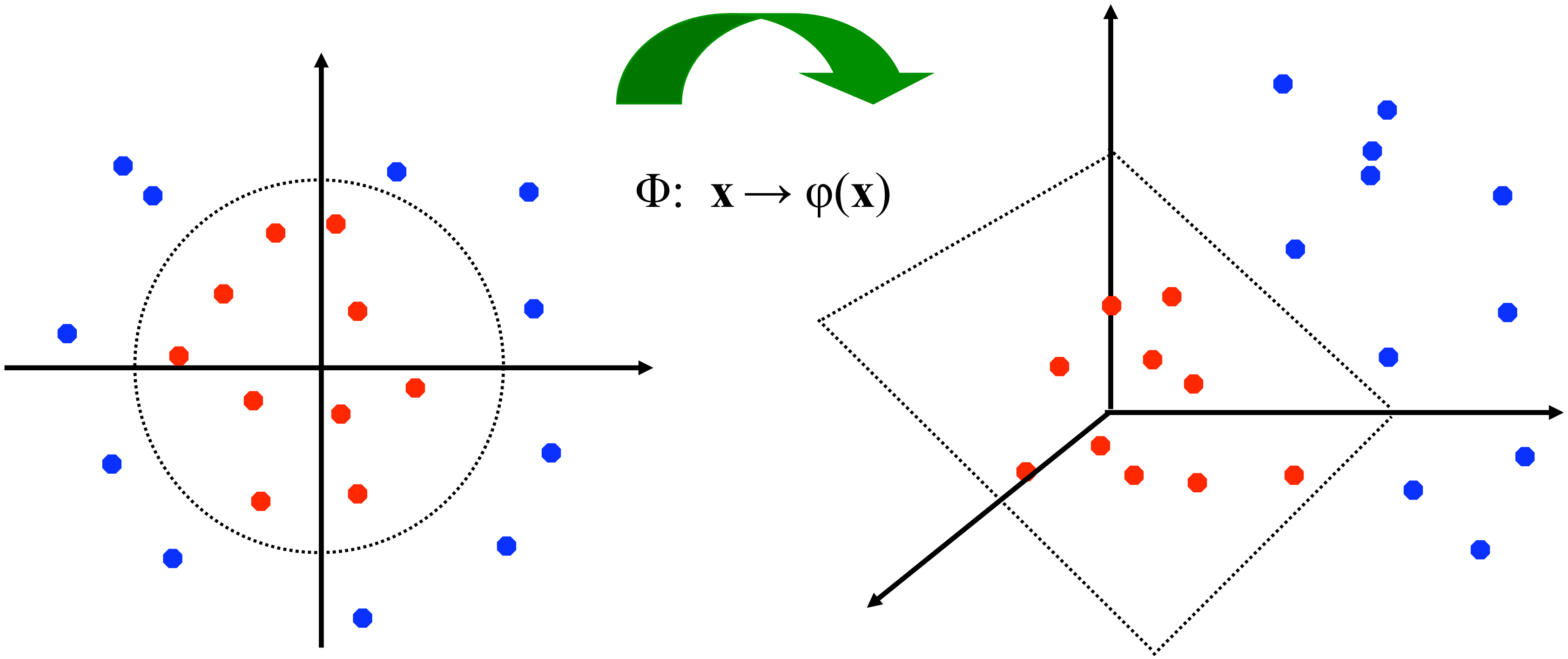
- We can map it to a higher-dimensional space:



Slide credit: Andrew Moore

Nonlinear SVMs

- General idea: the original input space can always be mapped to some higher-dimensional feature space where the training set is separable:



Slide credit: Andrew Moore

Nonlinear SVMs

- *The kernel trick*: instead of explicitly computing the lifting transformation $\boldsymbol{\varphi}(\mathbf{x})$, define a kernel function K such that

$$K(\mathbf{x}_i, \mathbf{x}_j) = \boldsymbol{\varphi}(\mathbf{x}_i) \cdot \boldsymbol{\varphi}(\mathbf{x}_j)$$

  (to be valid, the kernel function must satisfy *Mercer's condition*)
- This gives a nonlinear decision boundary in the original feature space:

$$\sum_i \alpha_i y_i K(\mathbf{x}_i, \mathbf{x}) + b$$

Mercer's kernels

Source: G. Shakhnarovich

What kind of function K is a valid kernel, i.e. such that there exists a feature space $\Phi(\mathbf{x})$ in which $K(\mathbf{x}, \mathbf{z}) = \phi(\mathbf{x})^T \phi(\mathbf{z})$?

Theorem due to Mercer (1930s)

K must be

- continuous;
- symmetric: $K(\mathbf{x}, \mathbf{z}) = K(\mathbf{z}, \mathbf{x})$;
- positive definite: for any $\mathbf{x}_1, \ldots, \mathbf{x}_N$, the *kernel matrix*

$$K = \begin{bmatrix} K(\mathbf{x}_1, \mathbf{x}_1) & K(\mathbf{x}_1, \mathbf{x}_2) & K(\mathbf{x}_1, \mathbf{x}_N) \\ \cdots & \cdots & \cdots \\ K(\mathbf{x}_N, \mathbf{x}_1) & K(\mathbf{x}_N, \mathbf{x}_2) & K(\mathbf{x}_N, \mathbf{x}_N) \end{bmatrix}$$

must be positive definite.

Radial basis function kernel

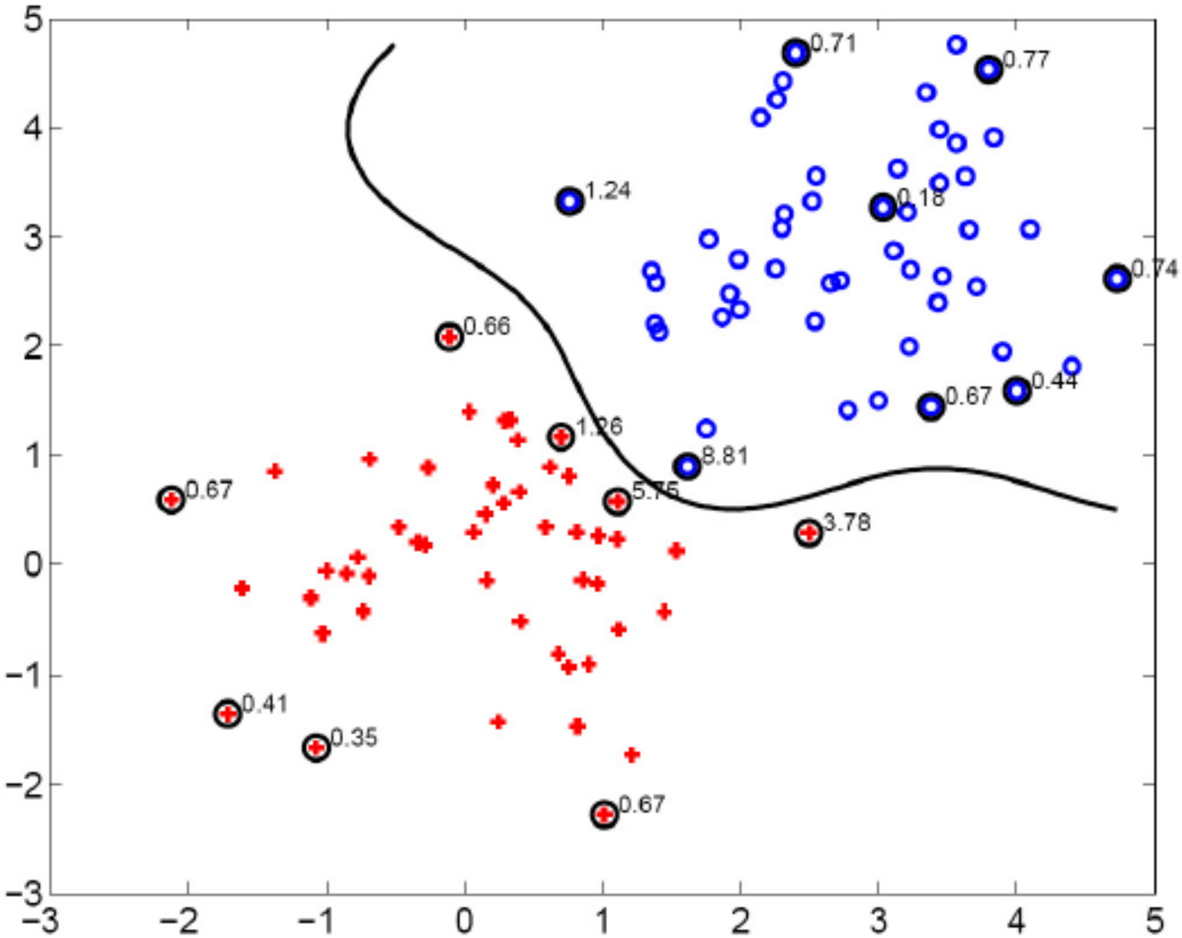
Source: G. Shakhnarovich

$$K(\mathbf{x}, \mathbf{z}; \sigma) = \exp\left(-\frac{1}{\sigma^2}\|\mathbf{x} - \mathbf{z}\|^2\right).$$

- The RBF kernel is a measure of similarity between two examples.
  - The mapping $\phi(\mathbf{x})$ is infinite-dimensional!
- What is the role of parameter σ?
  - Consider $\sigma \to 0$. Then $K(\mathbf{x}_i, \mathbf{x}; \sigma) \to 1$ if $\mathbf{x} = \mathbf{z}$ or 0 if $\mathbf{x} \neq \mathbf{z}$. The SVM simply "memorizes" the training data (overfitting, lack of generalization).
  - What about $\sigma \to \infty$? Then $K(\mathbf{x}, \mathbf{z}) \to 1$ for all $\mathbf{x}$, $\mathbf{z}$. The SVM underfits.

SVM with RBF (Gaussian) kernels

Source: G. Shakhnarovich



- Note: some SV here not close to the boundary

Multi-class SVMs

- Various "direct" formulations exist, but they are not widely used in practice. It is more common to obtain multi-class classifiers by combining two-class SVMs in various ways.
- One vs. others:
    - Traning: learn an SVM for each class vs. the others
    - Testing: apply each SVM to test example and assign to it the class of the SVM that returns the highest decision value
- One vs. one:
    - Training: learn an SVM for each pair of classes
    - Testing: each learned SVM "votes" for a class to assign to the test example

Kernels for bags of features

- Histogram intersection kernel:

$$I(h_1, h_2) = \sum_{i=1}^{N} \min(h_1(i), h_2(i))$$

- Generalized Gaussian kernel:

$$K(h_1, h_2) = \exp\left(-\frac{1}{A} D(h_1, h_2)^2 \right)$$

- *D* can be Euclidean distance, χ^2 distance, Earth Mover's Distance, etc.

SVM classifier

SMV with multi-channel chi-square kernel

$$K(H_i, H_j) = \exp\left(-\sum_{c \in \mathcal{C}} \frac{1}{A_c} D_c(H_i, H_j)\right)$$

- Channel *c* is a combination of detector, descriptor
- $D_c(H_i, H_j)$ is the chi-square distance between histograms

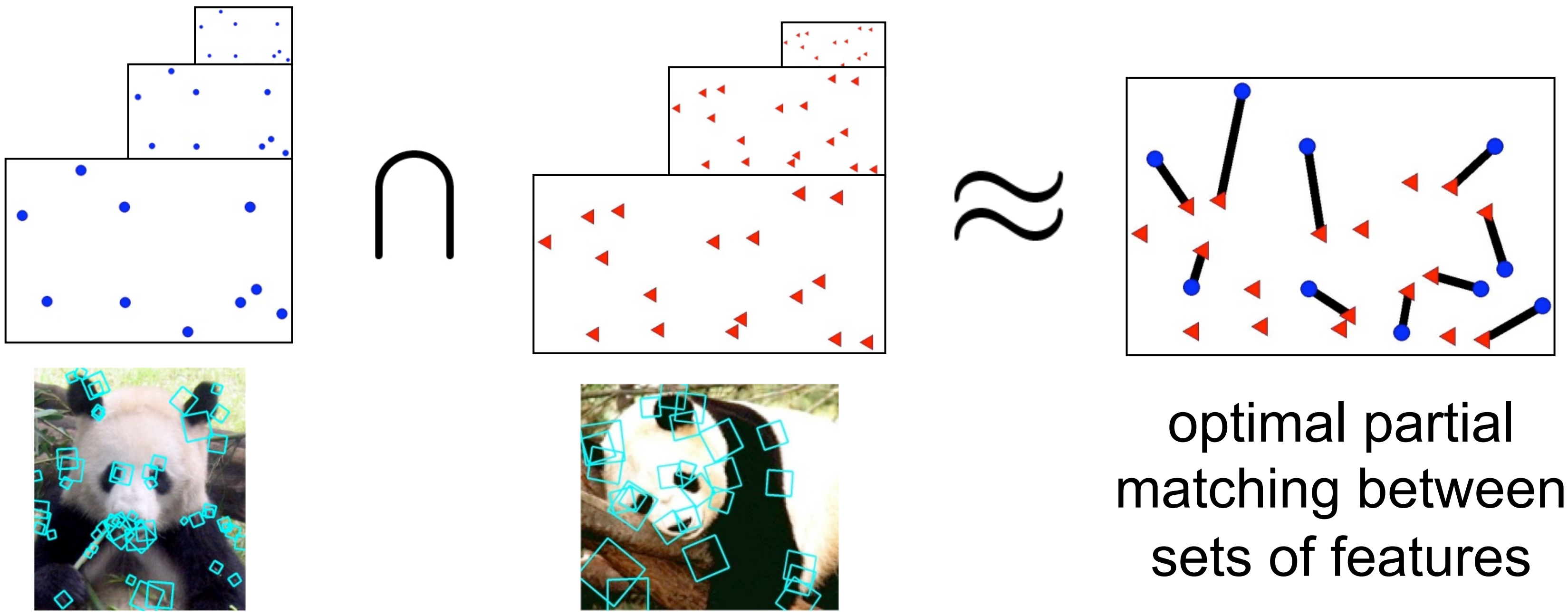
$$D_c(H_1, H_2) = \frac{1}{2} \sum_{i=1}^{m} [(h_{1i} - h_{2i})^2 / (h_{1i} + h_{2i})]$$

- A_c is the mean value of the distances between all training sample
- Extension: learning of the weights, for example with MKL

J. Zhang, M. Marszalek, S. Lazebnik, and C. Schmid, Local Features and Kernels for Classifcation of Texture and Object Categories: A Comprehensive Study, IJCV 2007

Pyramid match kernel

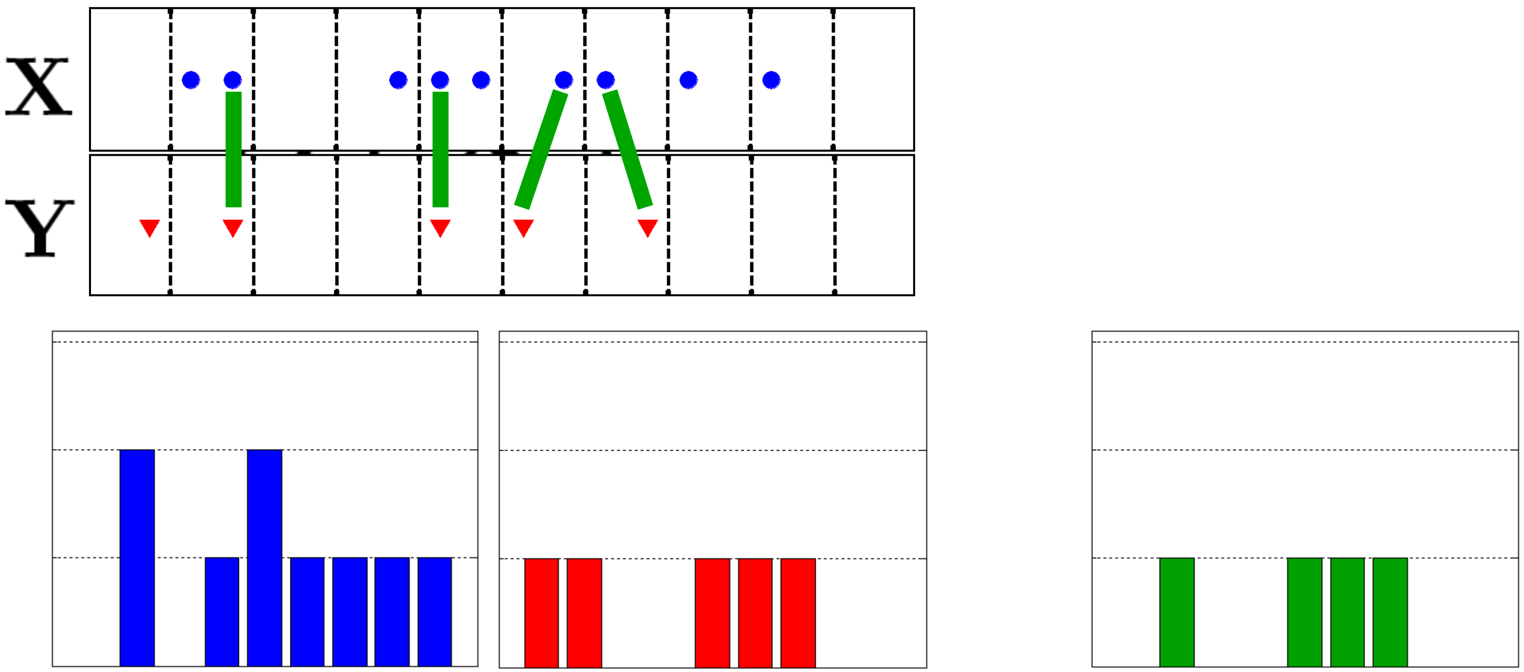
- Weighted sum of histogram intersections at mutliple resolutions (linear in the number of features instead of cubic)



K. Grauman and T. Darrell.
The Pyramid Match Kernel: Discriminative Classification with Sets of Image Features, ICCV 2005.

Pyramid Match

Histogram intersection $\mathcal{I}\left(H(\mathbf{X}), H(\mathbf{Y})\right) = \sum_{j=1}^{r} \min\left(H(\mathbf{X})_j, H(\mathbf{Y})_j\right)$



$H(\mathbf{X})$ $H(\mathbf{Y})$ $\mathcal{I}\left(H(\mathbf{X}), H(\mathbf{Y})\right) = 4$

Pyramid Match

Histogram intersection

$$\mathcal{I}\left(H(\mathbf{X}), H(\mathbf{Y})\right) = \sum_{j=1}^{r} \min\left(H(\mathbf{X})_j, H(\mathbf{Y})_j\right)$$

$$N_i = \overbrace{\mathcal{I}\left(H_i(\mathbf{X}), H_i(\mathbf{Y})\right)}^{\text{matches at this level}} - \overbrace{\mathcal{I}\left(H_{i-1}(\mathbf{X}), H_{i-1}(\mathbf{Y})\right)}^{\text{matches at previous level}}$$

Difference in histogram intersections across levels counts *number of new pairs* matched

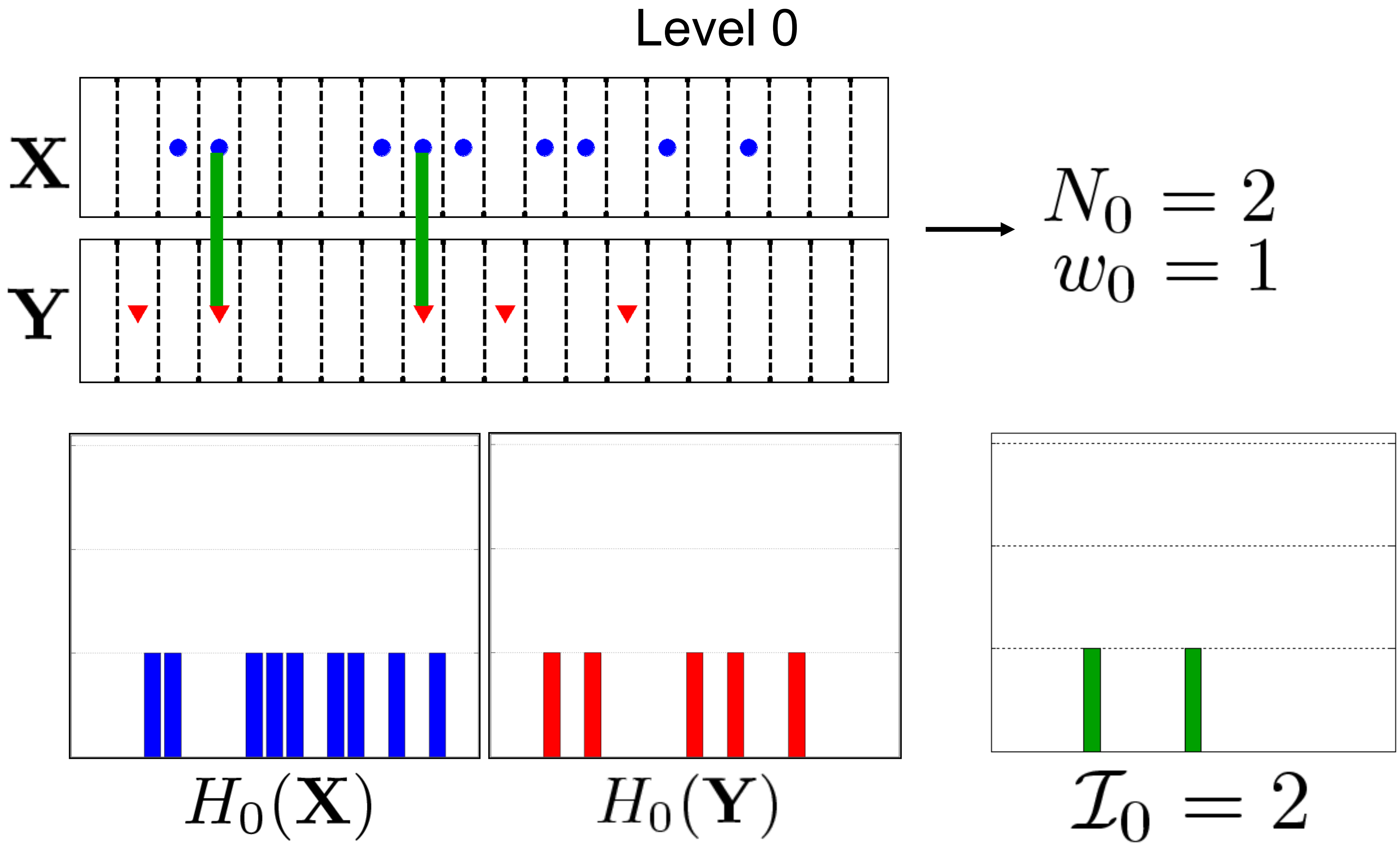
Pyramid match kernel

$$K_\Delta\left(\overbrace{\Psi(\mathbf{X}), \Psi(\mathbf{Y})}^{\text{histogram pyramids}}\right) =$$

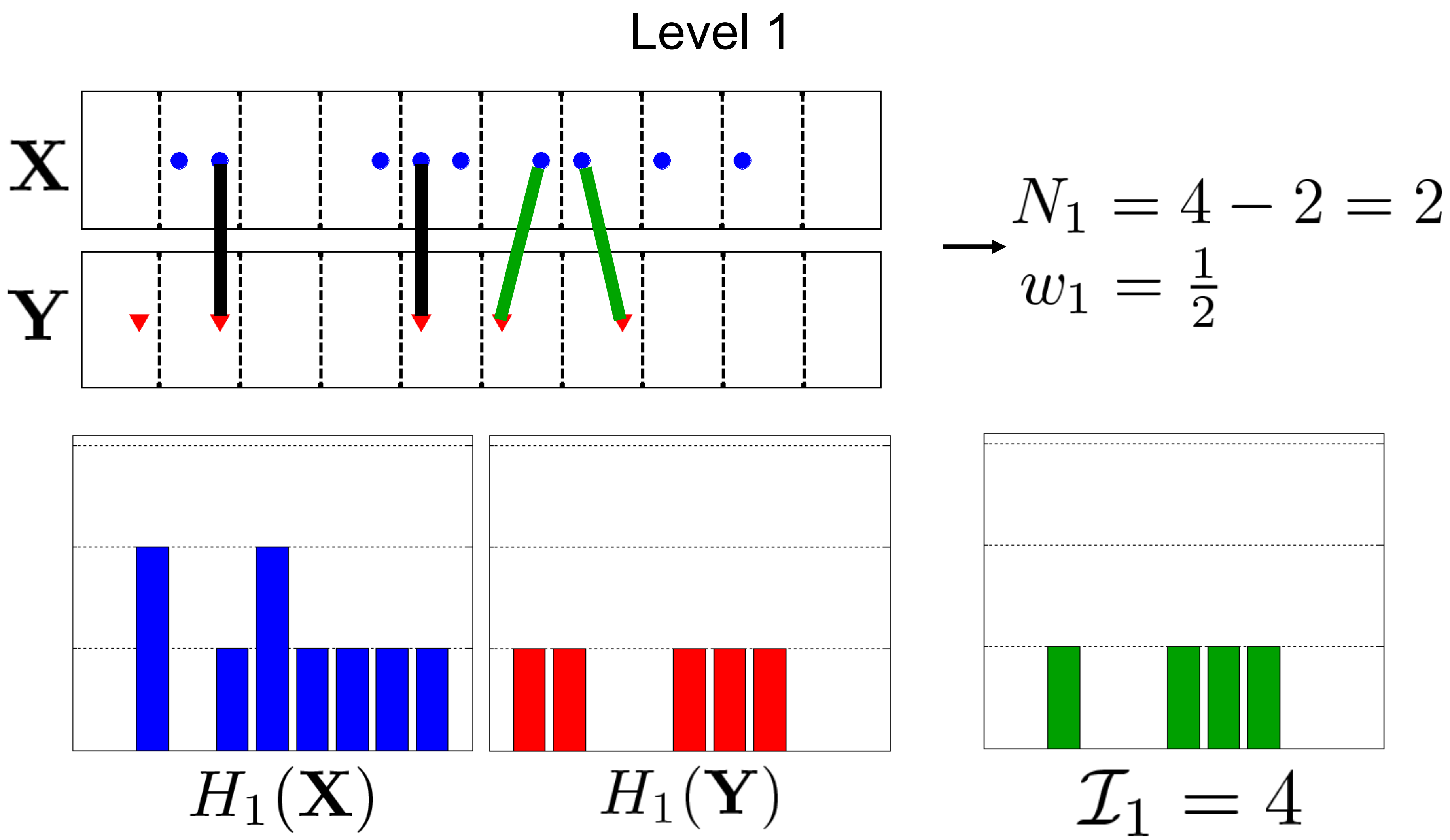
$$\sum_{i=0}^{L} \underset{\substack{\uparrow \\ \text{measure of difficulty of} \\ \text{a match at level } i}}{\frac{1}{2^i}} \Big(\underbrace{\mathcal{I}\left(H_i(\mathbf{X}), H_i(\mathbf{Y})\right) - \mathcal{I}\left(H_{i-1}(\mathbf{X}), H_{i-1}(\mathbf{Y})\right)}_{\text{number of newly matched pairs at level } i} \Big)$$

- Weights inversely proportional to bin size
- Normalize kernel values to avoid favoring large sets

Example pyramid match

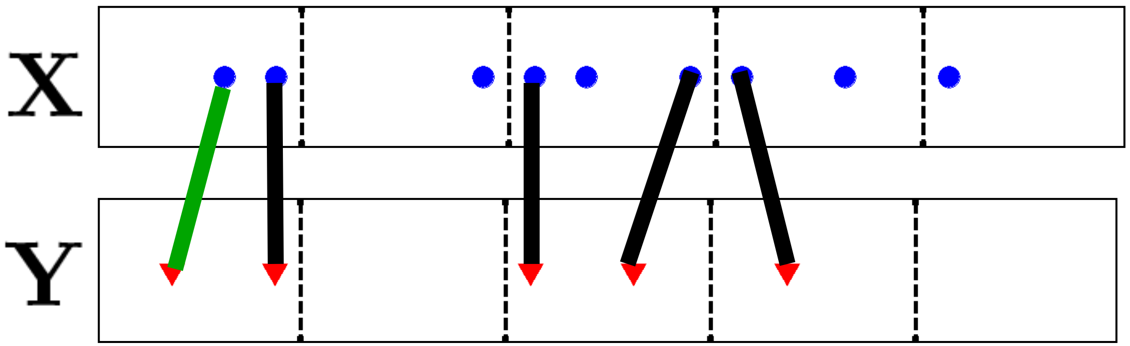


Example pyramid match

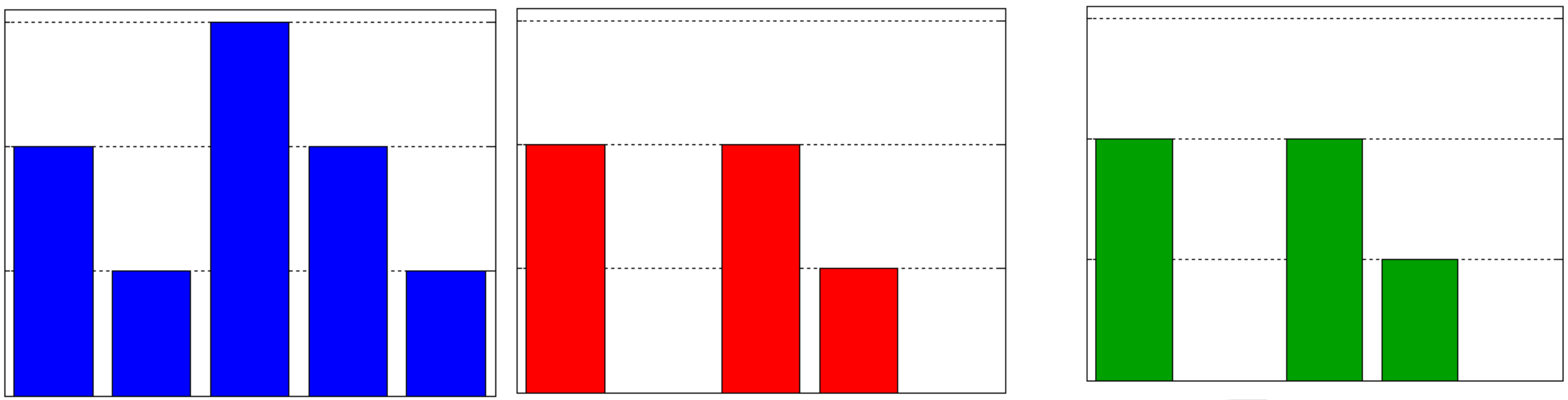


Example pyramid match

Level 2



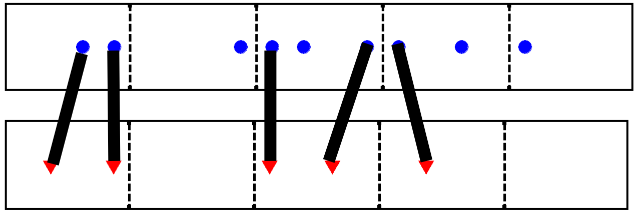
$$N_2 = 5 - 4 = 1$$
$$w_2 = \frac{1}{4}$$



$H_2(\mathbf{X})$ $H_2(\mathbf{Y})$ $\mathcal{I}_2 = 5$

Example pyramid match

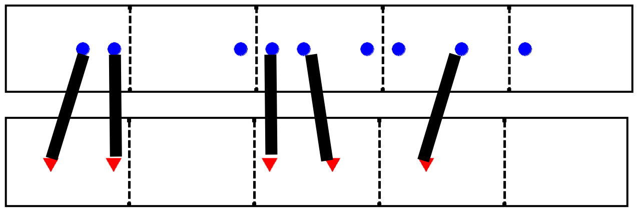
pyramid match



$$K_\Delta = \sum_{i=0}^{L} w_i N_i$$

$$= 1(2) + \tfrac{1}{2}(2) + \tfrac{1}{4}(1) = 3.25$$

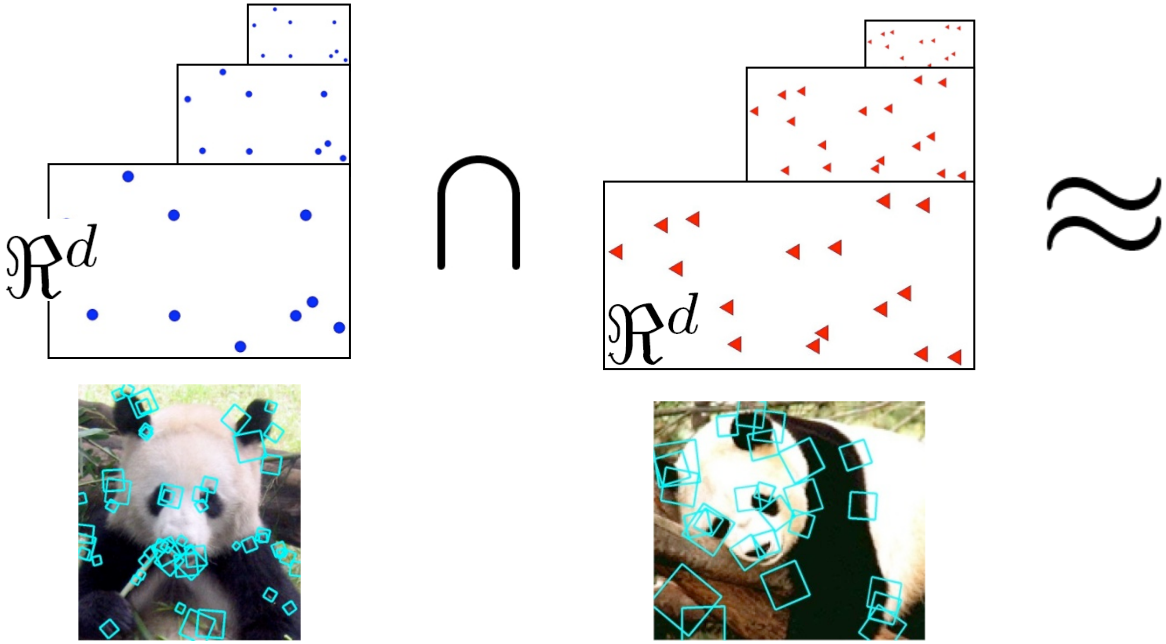
optimal match

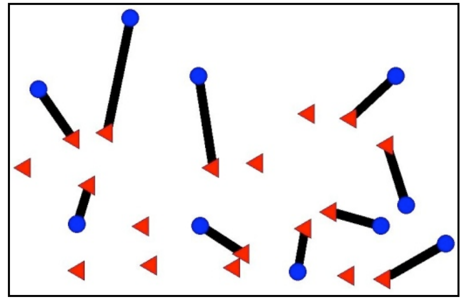


$$K = \max_{\pi : \mathbf{X} \to \mathbf{Y}} \sum_{\mathbf{x}_i \in \mathbf{X}} \mathcal{S}(\mathbf{x}_i, \pi(\mathbf{x}_i))$$

$$= 1(2) + \tfrac{1}{2}(3) = 3.5$$

Summary: Pyramid match kernel





optimal partial matching between sets of features

$$K_{\Delta}\left(\Psi(\mathbf{X}), \Psi(\mathbf{Y})\right) = \sum_{i=0}^{L} \underbrace{\frac{1}{2^i}}_{\text{difficulty of a match at level i}} \Big(\underbrace{\mathcal{I}\left(H_i(\mathbf{X}), H_i(\mathbf{Y})\right) - \mathcal{I}\left(H_{i-1}(\mathbf{X}), H_{i-1}(\mathbf{Y})\right)}_{\text{number of new matches at level i}} \Big)$$

Review: Discriminative methods

- Nearest-neighbor and k-nearest-neighbor classifiers
  - L1 distance, χ^2 distance, quadratic distance,
- Support vector machines
  - Linear classifiers
  - Margin maximization
  - The kernel trick
  - Kernel functions: histogram intersection, generalized Gaussian, pyramid match
- Of course, there are many other classifiers out there
  - Neural networks, boosting, decision trees, …

Summary: SVMs for image classification

1. Pick an image representation (in our case, bag of features)
2. Pick a kernel function for that representation
3. Compute the matrix of kernel values between every pair of training examples
4. Feed the kernel matrix into your favorite SVM solver to obtain support vectors and weights
5. At test time: compute kernel values for your test example and each support vector, and combine them with the learned weights to get the value of the decision function

SVMs: Pros and cons

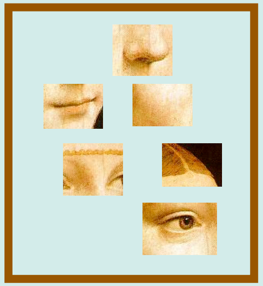
- Pros
  - Many publicly available SVM packages: http://www.kernel-machines.org/software
  - Kernel-based framework is very powerful, flexible
  - SVMs work very well in practice, even with very small training sample sizes

- Cons
  - No “direct” multi-class SVM, must combine two-class SVMs
  - Computation, memory
    - During training time, must compute matrix of kernel values for every pair of examples
    - Learning can take a very long time for large-scale problems

Generative methods

- Model the probability distribution that produced a given bag of features
- We will cover two models, both inspired by text document analysis:
  - Naïve Bayes
  - Probabilistic Latent Semantic Analysis

The Naïve Bayes model

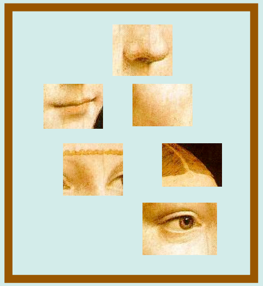


- Assume that each feature is conditionally independent *given the class*

$$p(w_1,\ldots,w_N \mid c) = \prod_{i=1}^{N} p(w_i \mid c)$$

Csurka et al. 2004

The Naïve Bayes model



- Assume that each feature is conditionally independent *given the class*

$$c^* = \arg\max_c \; p(c) \prod_{i=1}^{N} p(w_i \mid c)$$

MAP decision

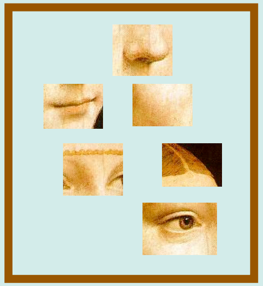
Prior prob. of the object classes

Likelihood of ith visual word given the class

Estimated by empirical frequencies of visual words in images from a given class

Csurka et al. 2004

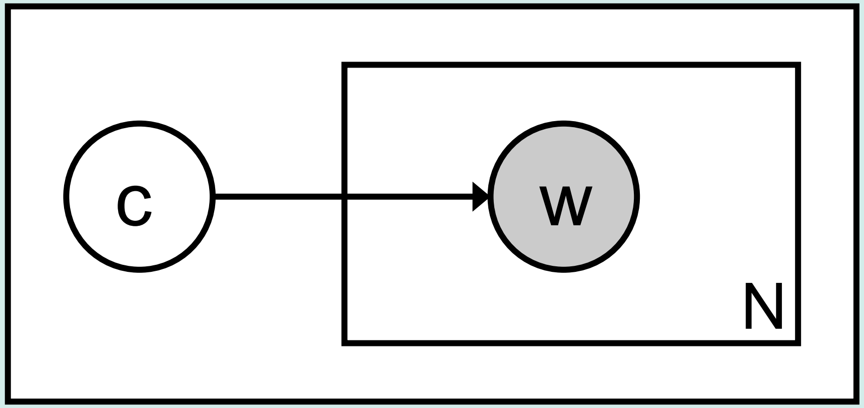
The Naïve Bayes model



- Assume that each feature is conditionally independent *given the class*

$$c^* = \arg\max_c \; p(c) \prod_{i=1}^{N} p(w_i \mid c)$$

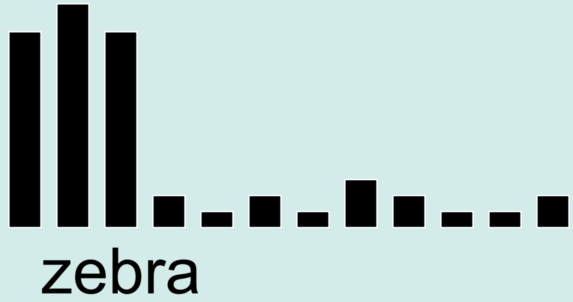
- "Graphical model":

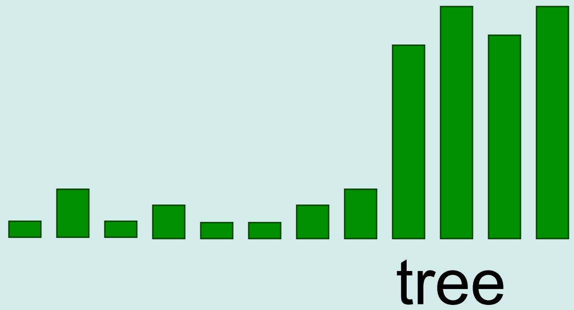


Csurka et al. 2004

Probabilistic Latent Semantic Analysis

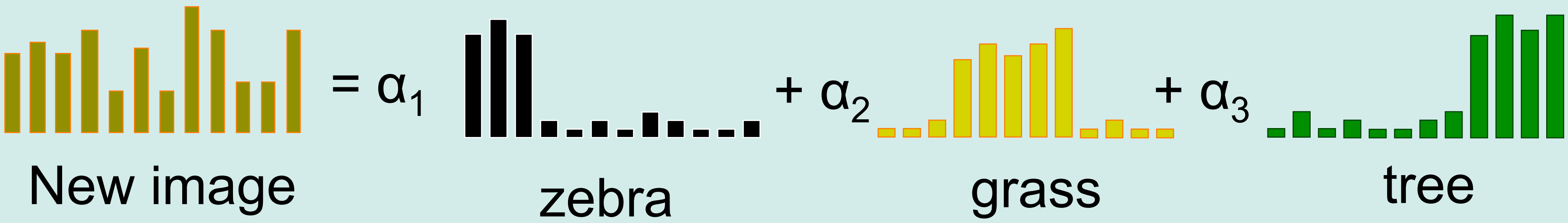
“visual topics”





T. Hofmann, Probabilistic Latent Semantic Analysis, UAI 1999

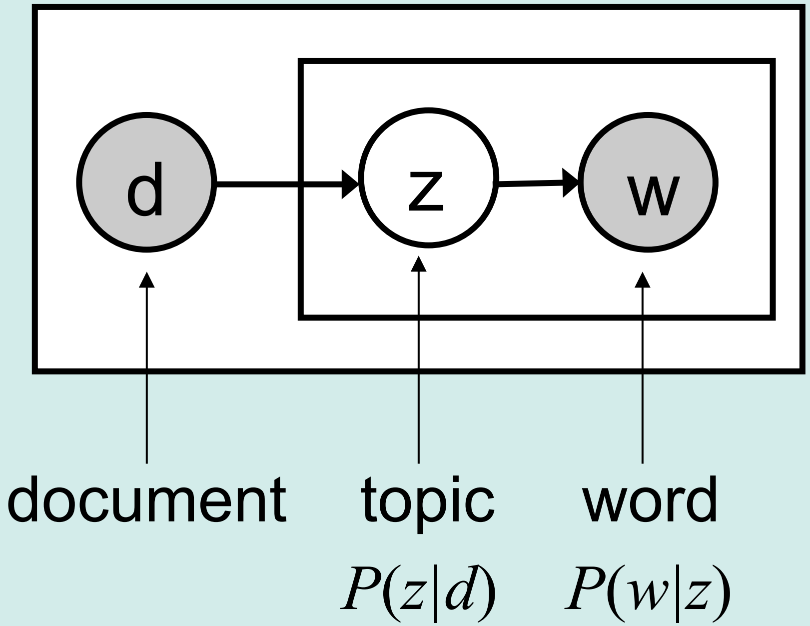
Probabilistic Latent Semantic Analysis



T. Hofmann, [Probabilistic Latent Semantic Analysis](), UAI 1999

Probabilistic Latent Semantic Analysis

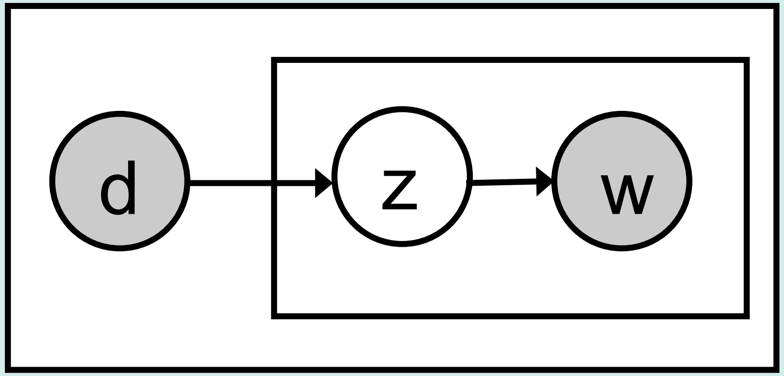
- Unsupervised technique
- Two-level generative model: a document is a mixture of topics, and each topic has its own characteristic word distribution



T. Hofmann, Probabilistic Latent Semantic Analysis, UAI 1999

Probabilistic Latent Semantic Analysis

- Unsupervised technique
- Two-level generative model: a document is a mixture of topics, and each topic has its own characteristic word distribution



$$p(w_i \mid d_j) = \sum_{k=1}^{K} p(w_i \mid z_k) p(z_k \mid d_j)$$

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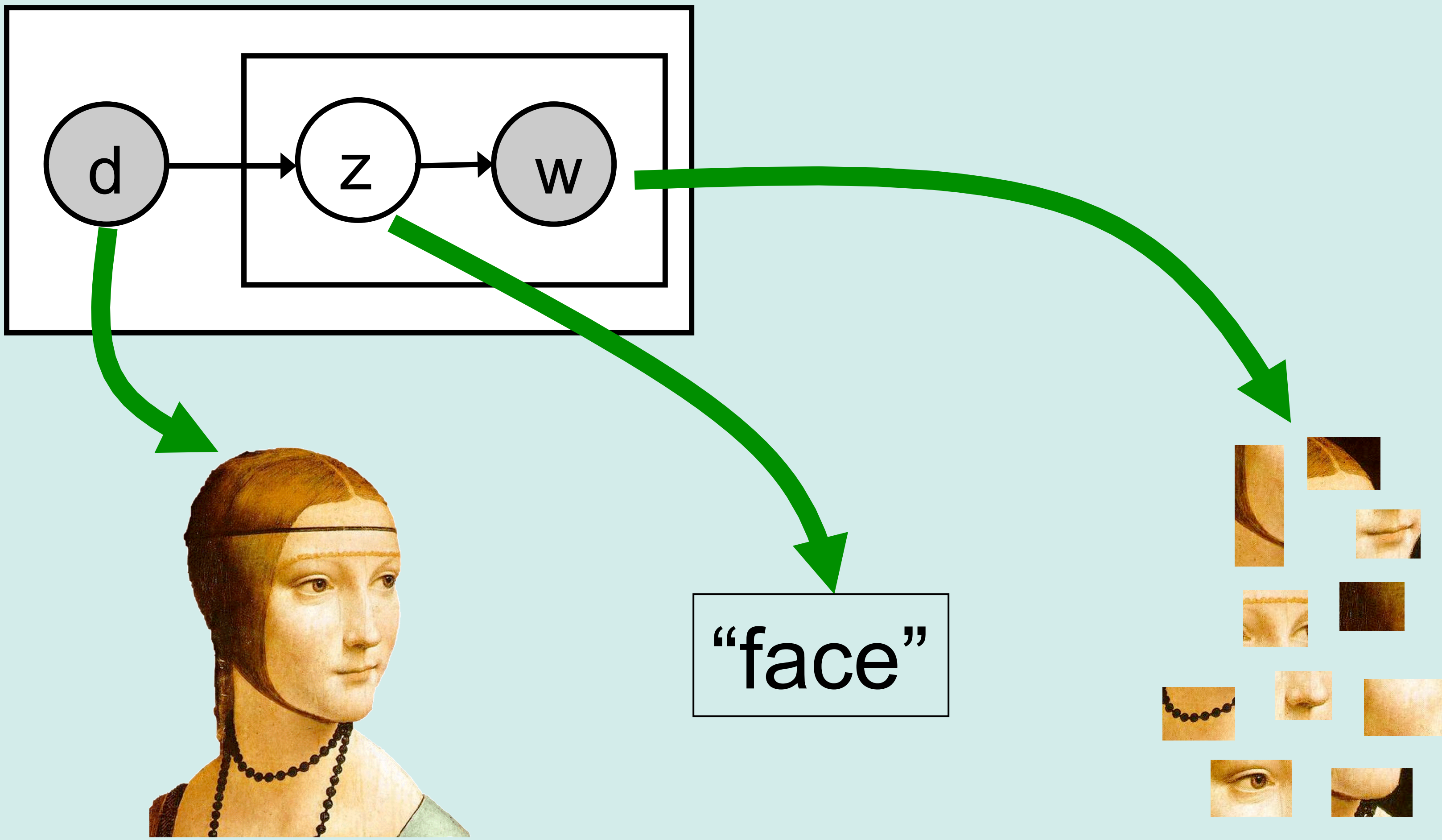
pLSA for images

Document = image, topic = class, word = quantized feature



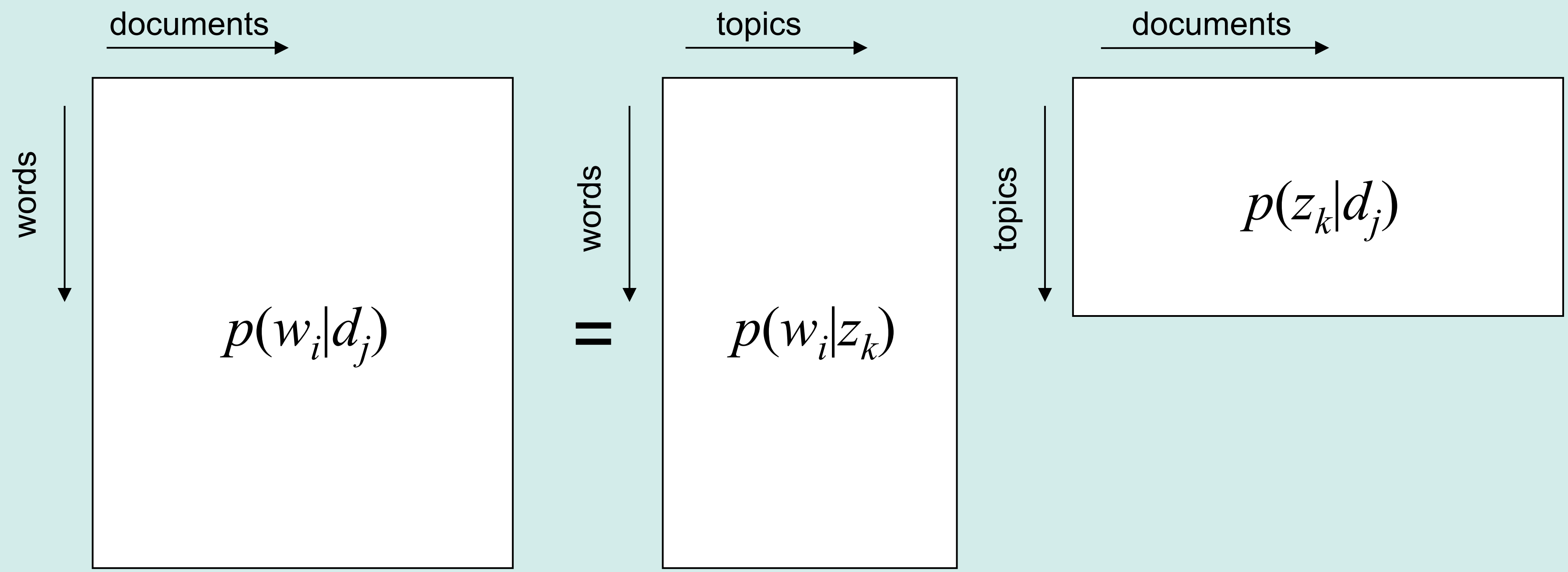
J. Sivic, B. Russell, A. Efros, A. Zisserman, B. Freeman,
Discovering Objects and their Location in Images, *ICCV 2005*

The pLSA model

$$\underbrace{p(w_i \mid d_j)}_{\substack{\text{Probability of word i}\\ \text{in document j}\\ \text{(known)}}} = \sum_{k=1}^{K} \underbrace{p(w_i \mid z_k)}_{\substack{\text{Probability of}\\ \text{word i given}\\ \text{topic k}\\ \text{(unknown)}}} \underbrace{p(z_k \mid d_j)}_{\substack{\text{Probability of}\\ \text{topic k given}\\ \text{document j}\\ \text{(unknown)}}}$$

The pLSA model

$$p(w_i \mid d_j) = \sum_{k=1}^{K} p(w_i \mid z_k)\, p(z_k \mid d_j)$$



Observed codeword distributions ($M \times N$)

Codeword distributions per topic (class) ($M \times K$)

Class distributions per image ($K \times N$)

Learning pLSA parameters

Maximize likelihood of data using EM:

Observed counts of word *i* in document *j*

$$L = \prod_{i=1}^{M} \prod_{j=1}^{N} P(w_i|d_j)^{n(w_i,d_j)}$$

$$\sum_{k=1}^{K} P(z_k|d_j) P(w_i|z_k)$$

M … number of codewords

N … number of images

Slide credit: Josef Sivic

Recognition

- Finding the most likely topic (class) for an image:

$$z^{*} = \arg\max_{z} p(z \mid d)$$

Recognition

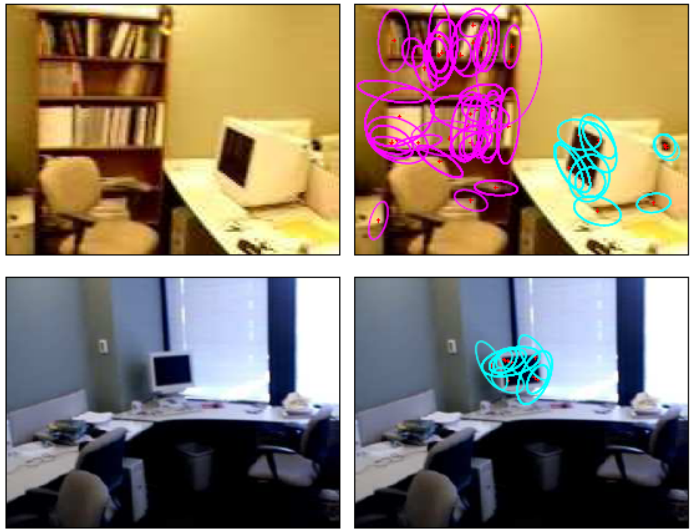
- Finding the most likely topic (class) for an image:

$$z^* = \arg\max_z p(z \mid d)$$

- Finding the most likely topic (class) for a visual word in a given image:

$$z^* = \arg\max_z p(z \mid w, d) = \arg\max_z \frac{p(w \mid z)\, p(z \mid d)}{\sum_{z'} p(w \mid z')\, p(z' \mid d)}$$

Topic discovery in images



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Summary: Generative models

- Naïve Bayes
  - *Unigram models* in document analysis
  - Assumes conditional independence of words given class
  - Parameter estimation: frequency counting
- Probabilistic Latent Semantic Analysis
  - Unsupervised technique
  - Each document is a mixture of topics (image is a mixture of classes)
  - Can be thought of as matrix decomposition
  - Parameter estimation: EM