Sparse Coding and Dictionary Learning for Image Analysis

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What this lecture is about?

- **Why sparsity, what for and how?**
- **Signal and image processing**: Restoration, reconstruction.
- **Machine learning**: Selecting relevant features.
- **Computer vision**: Modelling the local appearance of image patches.
- **Computer vision**: Recent (and intriguing) results in bags of words models.
- **Optimization**: Solving challenging problems.

The Image Denoising Problem

$$\underbrace{\mathbf{y}}_{\text{measurements}} = \underbrace{\mathbf{x}_{orig}}_{\text{original image}} + \underbrace{\mathbf{w}}_{\text{noise}}$$

Sparse representations for image restoration

$$\underbrace{\mathbf{y}}_{\text{measurements}} = \underbrace{\mathbf{x}_{orig}}_{\text{original image}} + \underbrace{\mathbf{w}}_{\text{noise}}$$

Energy minimization problem - MAP estimation

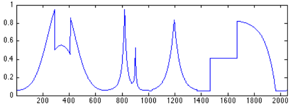
$$E(\mathbf{x}) = \underbrace{\frac{1}{2}\|\mathbf{y}-\mathbf{x}\|_2^2}_{\text{relation to measurements}} + \underbrace{Pr(\mathbf{x})}_{\text{image model (-log prior)}}$$

Some classical priors

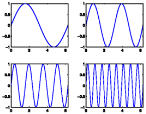
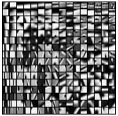
- Smoothness $\lambda\|\mathcal{L}\mathbf{x}\|_2^2$
- Total variation $\lambda\|\nabla\mathbf{x}\|_1^2$
- MRF priors
- ...

What is a Sparse Linear Model?

Let $\mathbf{x}$ in $\mathbb{R}^m$ be a signal.




Let $\mathbf{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_p] \in \mathbb{R}^{m \times p}$ be a set of normalized "basis vectors".
We call it **dictionary**.

$\mathbf{D}$ is "adapted" to $\mathbf{x}$ if it can represent it with a few basis vectors—that is, there exists a **sparse vector** $\boldsymbol{\alpha}$ in $\mathbb{R}^p$ such that $\mathbf{x} \approx \mathbf{D}\boldsymbol{\alpha}$. We call $\boldsymbol{\alpha}$ the **sparse code**.

$$\underbrace{\begin{pmatrix} \mathbf{x} \end{pmatrix}}_{\mathbf{x} \in \mathbb{R}^m} \approx \underbrace{\begin{pmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \cdots & \mathbf{d}_p \end{pmatrix}}_{\mathbf{D} \in \mathbb{R}^{m \times p}} \underbrace{\begin{pmatrix} \boldsymbol{\alpha}[1] \\ \boldsymbol{\alpha}[2] \\ \vdots \\ \boldsymbol{\alpha}[p] \end{pmatrix}}_{\boldsymbol{\alpha} \in \mathbb{R}^p, \textbf{sparse}}$$

First Important Idea

Why Sparsity?

A dictionary can be good for representing a class of signals, but not for representing white Gaussian noise.

The Sparse Decomposition Problem

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \underbrace{\frac{1}{2}\|\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}\|_2^2}_{\text{data fitting term}} + \underbrace{\lambda\psi(\boldsymbol{\alpha})}_{\substack{\text{sparsity-inducing}\\ \text{regularization}}}$$

ψ induces sparsity in $\boldsymbol{\alpha}$. It can be

- the ℓ_0 "pseudo-norm". $\|\boldsymbol{\alpha}\|_0 \triangleq \#\{i \text{ s.t. } \boldsymbol{\alpha}[i] \neq 0\}$ (NP-hard)
- the ℓ_1 norm. $\|\boldsymbol{\alpha}\|_1 \triangleq \sum_{i=1}^{p} |\boldsymbol{\alpha}[i]|$ (convex),
- ...

This is a selection problem. When ψ is the ℓ_1-norm, the problem is called Lasso [Tibshirani, 1996] or basis pursuit [Chen et al., 1999]

Sparse representations for image restoration

Designed dictionaries

[Haar, 1910], [Zweig, Morlet, Grossman ∼70s], [Meyer, Mallat, Daubechies, Coifman, Donoho, Candes ∼80s-today]. . . (see [Mallat, 1999])
Wavelets, Curvelets, Wedgelets, Bandlets, . . . lets

Learned dictionaries of patches

[Olshausen and Field, 1997], [Engan et al., 1999], [Lewicki and Sejnowski, 2000], [Aharon et al., 2006] , [Roth and Black, 2005], [Lee et al., 2007]

$$\min_{\boldsymbol{\alpha}_i, \mathbf{D} \in \mathcal{C}} \sum_i \underbrace{\frac{1}{2}\|\mathbf{x}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2}_{\text{reconstruction}} + \underbrace{\lambda\psi(\boldsymbol{\alpha}_i)}_{\text{sparsity}}$$

- $\psi(\boldsymbol{\alpha}) = \|\boldsymbol{\alpha}\|_0$ (“ℓ_0 pseudo-norm”)
- $\psi(\boldsymbol{\alpha}) = \|\boldsymbol{\alpha}\|_1$ (ℓ_1 norm)

Sparse representations for image restoration

Solving the denoising problem

[Elad and Aharon, 2006]

- Extract all overlapping 8×8 patches $\mathbf{x}_i$.
- Solve a matrix factorization problem:

$$\min_{\boldsymbol{\alpha}_i, \mathbf{D} \in \mathcal{C}} \sum_{i=1}^{n} \underbrace{\frac{1}{2}\|\mathbf{x}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2}_{\text{reconstruction}} + \underbrace{\lambda \psi(\boldsymbol{\alpha}_i)}_{\text{sparsity}},$$

with $n > 100,000$

- Average the reconstruction of each patch.

Sparse representations for image restoration

K-SVD: [Elad and Aharon, 2006]

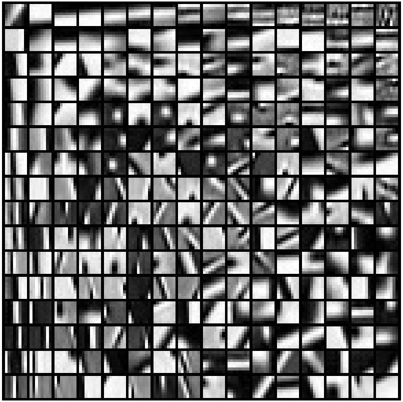


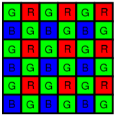
Figure: Dictionary trained on a noisy version of the image boat.

Sparse representations for image restoration

Inpainting, Demosaicking

$$\min_{\mathbf{D}\in\mathcal{C},\boldsymbol{\alpha}} \sum_i \frac{1}{2}\|\boldsymbol{\beta}_i \otimes (\mathbf{x}_i - \mathbf{D}\boldsymbol{\alpha}_i)\|_2^2 + \lambda_i \psi(\boldsymbol{\alpha}_i)$$

RAW Image Processing



White balance. Black substraction.

Denoising

Demosaicking

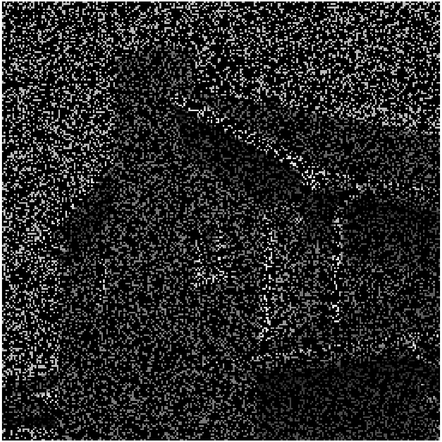
Conversion to sRGB. Gamma correction.

Sparse representations for image restoration

[Mairal, Bach, Ponce, Sapiro, and Zisserman, 2009b]

Sparse representations for image restoration

[Mairal, Sapiro, and Elad, 2008d]



Sparse representations for image restoration

Inpainting, [Mairal, Elad, and Sapiro, 2008b]

Sparse representations for image restoration

Inpainting, [Mairal, Elad, and Sapiro, 2008b]

Sparse representations for video restoration

Key ideas for video processing

[Protter and Elad, 2009]

- Using a 3D dictionary.
- Processing of many frames at the same time.
- Dictionary propagation.

Sparse representations for image restoration

Inpainting, [Mairal, Sapiro, and Elad, 2008d]

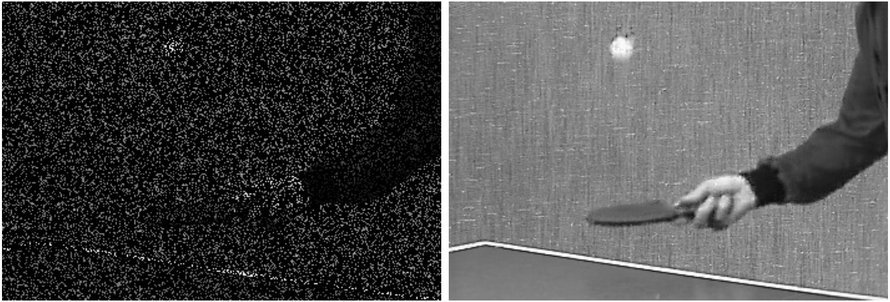
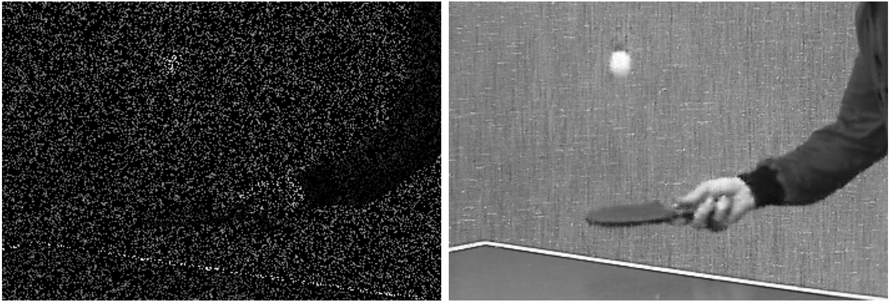


Figure: Inpainting results.

Sparse representations for image restoration
Inpainting, [Mairal, Sapiro, and Elad, 2008d]



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Sparse representations for image restoration
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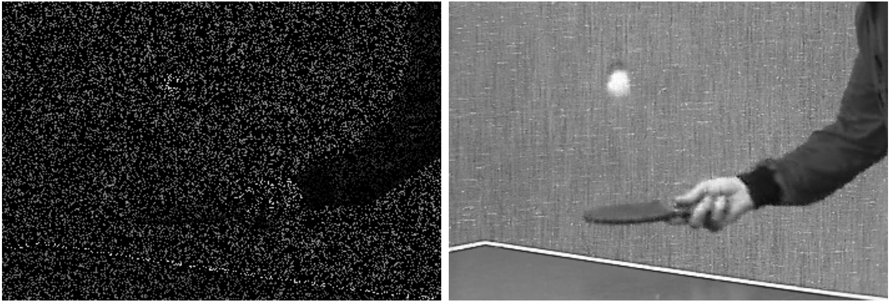

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Sparse representations for image restoration
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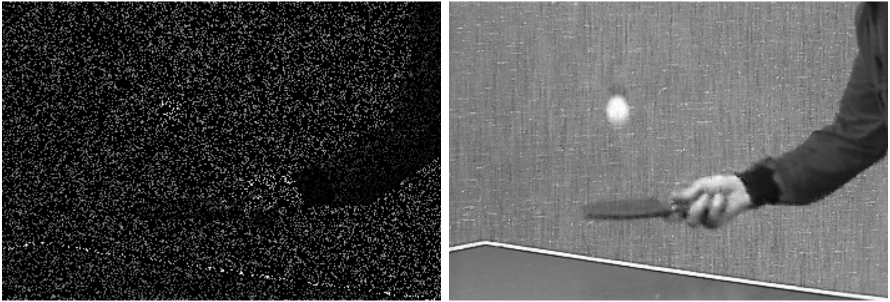

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Sparse representations for image restoration

Inpainting, [Mairal, Sapiro, and Elad, 2008d]

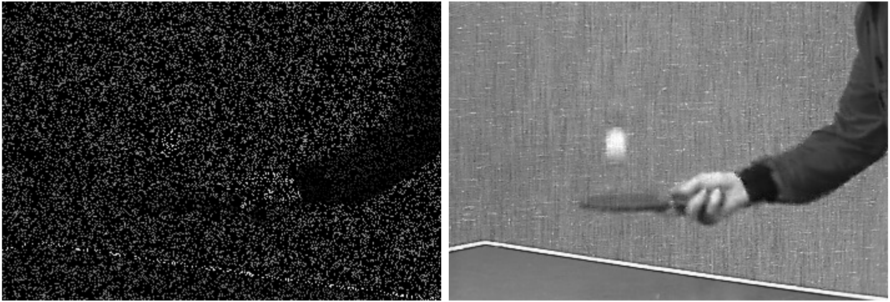

Figure: Inpainting results.

Sparse representations for image restoration
Color video denoising, [Mairal, Sapiro, and Elad, 2008d]

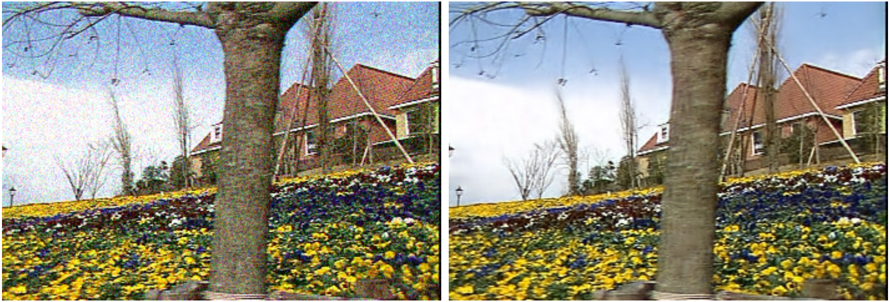

Figure: Denoising results. $\sigma = 25$

Sparse representations for image restoration

Color video denoising, [Mairal, Sapiro, and Elad, 2008d]

Figure: Denoising results. $\sigma = 25$

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Sparse representations for image restoration

Color video denoising, [Mairal, Sapiro, and Elad, 2008d]

Figure: Denoising results. $\sigma = 25$

Digital Zooming

Couzinie-Devy, 2010, Original

Digital Zooming

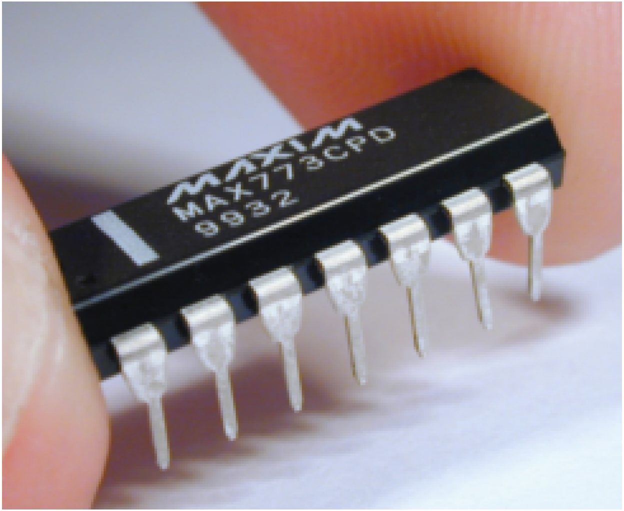
Couzinie-Devy, 2010, Bicubic

Digital Zooming

Couzinie-Devy, 2010, Proposed method

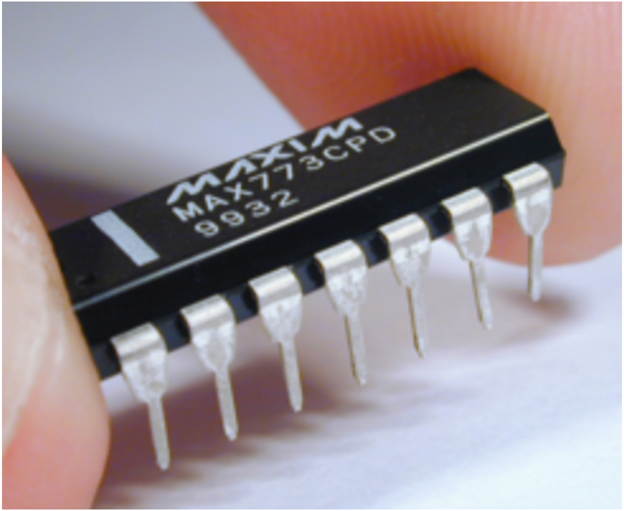
Digital Zooming

Couzinie-Devy, 2010, Original



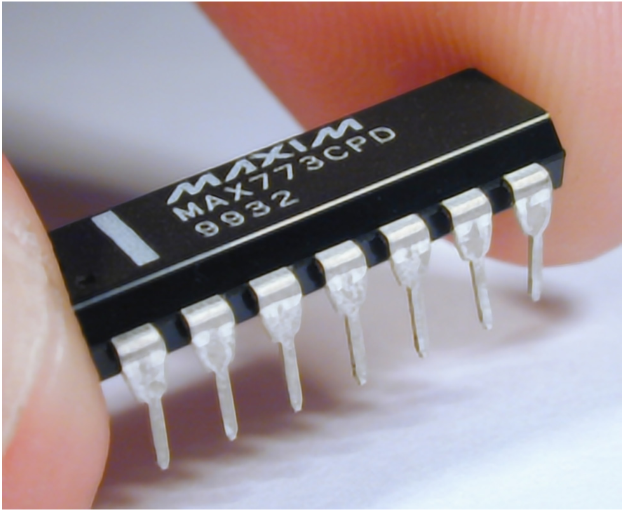
Digital Zooming

Couzinie-Devy, 2010, Bicubic



Digital Zooming

Couzinie-Devy, 2010, Proposed approach



Inverse half-toning

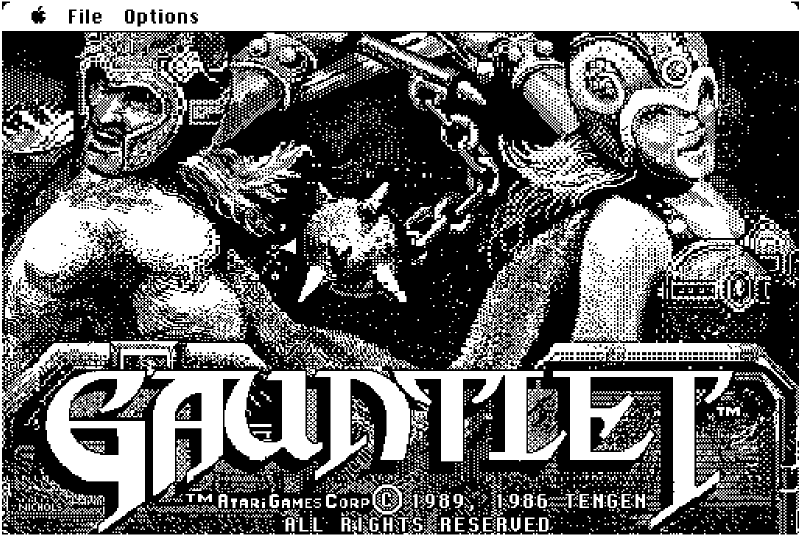
Original

Inverse half-toning

Reconstructed image

Inverse half-toning

Original



Inverse half-toning

Reconstructed image

Inverse half-toning

Original

Inverse half-toning

Reconstructed image

Inverse half-toning

Original

Inverse half-toning

Reconstructed image

Inverse half-toning

Original

Inverse half-toning

Reconstructed image

One short slide on compressed sensing

Important message

Sparse coding is not "compressed sensing".

Compressed sensing is a theory [see Candes, 2006] saying that a sparse signal can be recovered with high probability from a few linear measurements under some conditions.

- Signal Acquisition: $\mathbf{W}^\top \mathbf{x}$, where $\mathbf{W} \in \mathbb{R}^{m \times s}$ is a "sensing" matrix with $s \ll m$.
- Signal Decoding: $\min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \|\boldsymbol{\alpha}\|_1$ s.t. $\mathbf{W}^\top \mathbf{x} = \mathbf{W}^\top \mathbf{D}\boldsymbol{\alpha}$.

with extensions to approximately sparse signals, noisy measurements.

Remark

The dictionaries we are using in this lecture do not satisfy the recovery assumptions of compressed sensing.

Important messages

- Patch-based approaches are achieving state-of-the-art results for many image processing task.
- Dictionary Learning adapts to the data you want to restore.
- Dictionary Learning is well adapted to data that admit sparse representation. **Sparsity is for sparse data only**.

Next topics

- Why does the ℓ_1-norm induce sparsity?
- Some properties of the Lasso.
- Beyond sparsity: Group-sparsity.
- The simplest algorithm for learning dictionaries.
- Links between dictionary learning and matrix factorization techniques.

Why does the ℓ_1-norm induce sparsity?

Exemple: quadratic problem in 1D

$$\min_{\alpha \in \mathbb{R}} \frac{1}{2}(x-\alpha)^2 + \lambda|\alpha|$$

Piecewise quadratic function with a kink at zero.

Derivative at 0_+: $g_+ = -x + \lambda$ and 0_-: $g_- = -x - \lambda$.

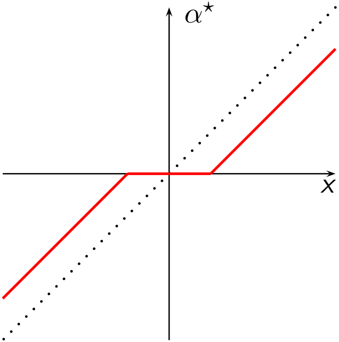
Optimality conditions. α is optimal iff:

- $|\alpha| > 0$ and $(x-\alpha) + \lambda\,\text{sign}(\alpha) = 0$
- $\alpha = 0$ and $g_+ \geq 0$ and $g_- \leq 0$

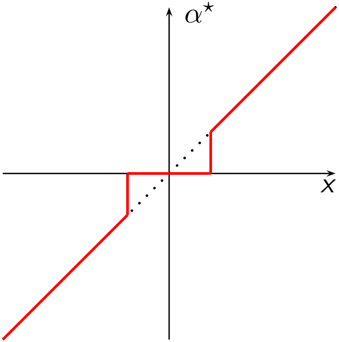
The solution is a **soft-thresholding**:

$$\alpha^\star = \text{sign}(x)(|x| - \lambda)^+.$$

Why does the ℓ_1-norm induce sparsity?



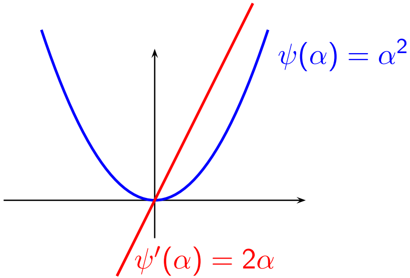
(a) soft-thresholding operator

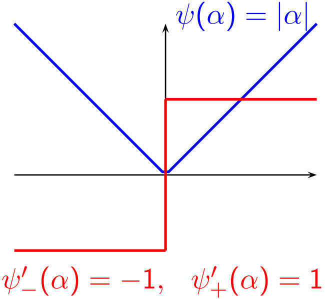


(b) hard-thresholding operator

Why does the ℓ_1-norm induce sparsity?

Analysis of the norms in 1D

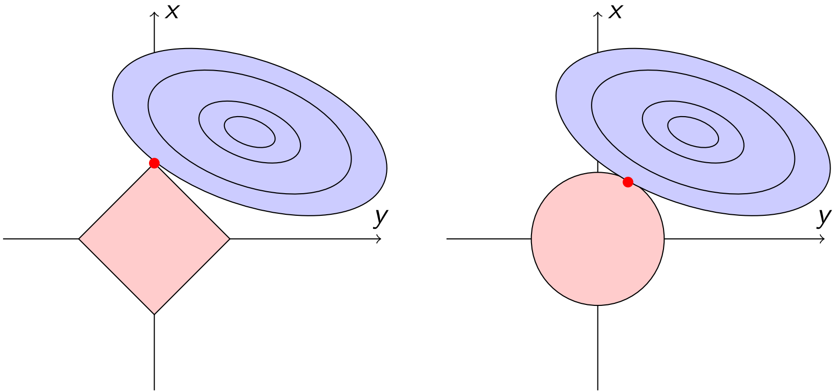




The gradient of the ℓ_2-norm vanishes when α get close to 0. On its differentiable part, the norm of the gradient of the ℓ_1-norm is constant.

Why does the ℓ_1-norm induce sparsity?

Geometric explanation



$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \frac{1}{2}\|\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}\|_2^2 + \lambda\|\boldsymbol{\alpha}\|_1$$

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \|\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}\|_2^2 \;\; \text{s.t.} \;\; \|\boldsymbol{\alpha}\|_1 \leq T.$$

Important property of the Lasso

Piecewise linearity of the regularization path

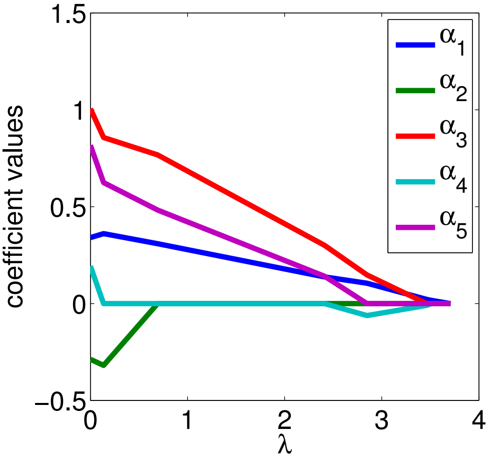


Figure: Regularization path of the Lasso

Sparsity-Inducing Norms (1/2)

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \overbrace{f(\boldsymbol{\alpha})}^{\text{data fitting term}} + \lambda \underbrace{\psi(\boldsymbol{\alpha})}_{\textbf{sparsity-inducing norm}}$$

Standard approach to enforce sparsity in learning procedures:

- Regularizing by a **sparsity-inducing norm** ψ.
- The effect of ψ is to set some $\boldsymbol{\alpha}_j$'s to zero, depending on the regularization parameter $\lambda \geq 0$.

The most popular choice for ψ:

- The ℓ_1 norm, $\|\boldsymbol{\alpha}\|_1 = \sum_{j=1}^{p} |\boldsymbol{\alpha}_j|$.
- For the square loss, Lasso [Tibshirani, 1996].
- However, the ℓ_1 norm encodes poor information, just **cardinality**!

Sparsity-Inducing Norms (2/2)

Another popular choice for ψ:

- The ℓ_1-ℓ_2 norm,

$$\sum_{G \in \mathcal{G}} \|\boldsymbol{\alpha}_G\|_2 = \sum_{G \in \mathcal{G}} \Big(\sum_{j \in \mathcal{G}} \boldsymbol{\alpha}_j^2 \Big)^{1/2}, \text{ with } \mathcal{G} \text{ a } \textbf{partition} \text{ of } \{1, \ldots, p\}.$$

- The ℓ_1-ℓ_2 norm sets to zero **groups of non-overlapping variables** (as opposed to single variables for the ℓ_1 norm).
- For the square loss, group Lasso [Yuan and Lin, 2006].
- However, the ℓ_1-ℓ_2 norm encodes fixed/static prior information, requires to know in advance how to group the variables !

Applications:

- Selecting groups of features instead of individual variables.
- Multi-task learning.

Optimization for Dictionary Learning

$$\min_{\substack{\boldsymbol{\alpha} \in \mathbb{R}^{p \times n} \\ \mathbf{D} \in \mathcal{C}}} \sum_{i=1}^{n} \frac{1}{2} \|\mathbf{x}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda \|\boldsymbol{\alpha}_i\|_1$$

$$\mathcal{C} \triangleq \{\mathbf{D} \in \mathbb{R}^{m \times p} \ \text{ s.t. } \ \forall j = 1, \ldots, p, \ \ \|\mathbf{d}_j\|_2 \leq 1\}.$$

- Classical optimization alternates between $\mathbf{D}$ and $\boldsymbol{\alpha}$.
- Good results, but slow!
- Instead use online learning [Mairal et al., 2009a]

Optimization for Dictionary Learning

Inpainting a 12-Mpixel photograph

Optimization for Dictionary Learning
Inpainting a 12-Mpixel photograph

Optimization for Dictionary Learning
Inpainting a 12-Mpixel photograph

Optimization for Dictionary Learning
Inpainting a 12-Mpixel photograph

Matrix Factorization Problems and Dictionary Learning

$$\min_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^{p\times n}\\ \mathbf{D}\in\mathcal{C}}} \sum_{i=1}^{n} \frac{1}{2}\|\mathbf{x}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda\|\boldsymbol{\alpha}_i\|_1$$

can be rewritten

$$\min_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^{p\times n}\\ \mathbf{D}\in\mathcal{C}}} \frac{1}{2}\|\mathbf{X} - \mathbf{D}\boldsymbol{\alpha}\|_F^2 + \lambda\|\boldsymbol{\alpha}\|_1,$$

where $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ and $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n]$.

Matrix Factorization Problems and Dictionary Learning
PCA

$$\min_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^{p\times n}\\ \mathbf{D}\in\mathbb{R}^{m\times p}}} \|\mathbf{X} - \mathbf{D}\boldsymbol{\alpha}\|_F^2,$$

with the additional constraints that $\mathbf{D}$ is orthonormal and $\boldsymbol{\alpha}^\top$ is orthogonal.

$\mathbf{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_p]$ are the principal components.

Matrix Factorization Problems and Dictionary Learning

Hard clustering

$$\min_{\substack{\boldsymbol{\alpha} \in \mathbb{R}^{p \times n} \\ \mathbf{D} \in \mathbb{R}^{m \times p}}} \|\mathbf{X} - \mathbf{D}\boldsymbol{\alpha}\|_F^2,$$

with the additional constraints that $\boldsymbol{\alpha}$ is binary and its columns sum to one.

Matrix Factorization Problems and Dictionary Learning

Soft clustering

$$\min_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^{p\times n}\\ \mathbf{D}\in\mathbb{R}^{m\times p}}} \|\mathbf{X} - \mathbf{D}\boldsymbol{\alpha}\|_F^2,$$

with the additional constraints that the columns of $\boldsymbol{\alpha}$ sum to one.

Matrix Factorization Problems and Dictionary Learning

Non-negative matrix factorization [Lee and Seung, 2001]

$$\min_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^{p\times n}\\ \mathbf{D}\in\mathbb{R}^{m\times p}}} \|\mathbf{X}-\mathbf{D}\boldsymbol{\alpha}\|_F^2,$$

with the additional constraints that the entries of $\mathbf{D}$ and $\boldsymbol{\alpha}$ are non-negative.

Matrix Factorization Problems and Dictionary Learning

NMF+sparsity?

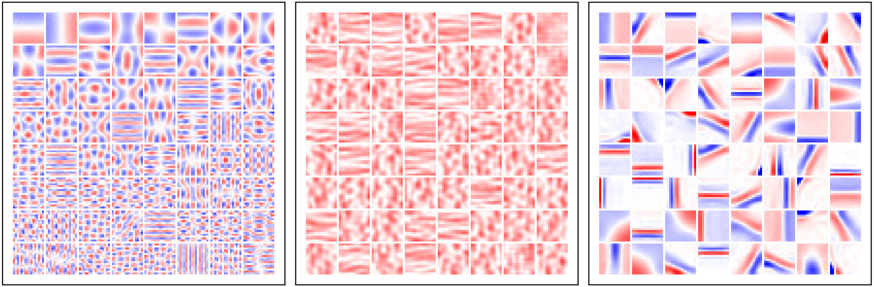
$$\min_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^{p\times n}\\ \mathbf{D}\in\mathbb{R}^{m\times p}}} \|\mathbf{X}-\mathbf{D}\boldsymbol{\alpha}\|_F^2 + \lambda\|\boldsymbol{\alpha}\|_1$$

with the additional constraints that the entries of $\mathbf{D}$ and $\boldsymbol{\alpha}$ are non-negative.

Most of these formulations can be addressed the same types of algorithms.

Matrix Factorization Problems and Dictionary Learning

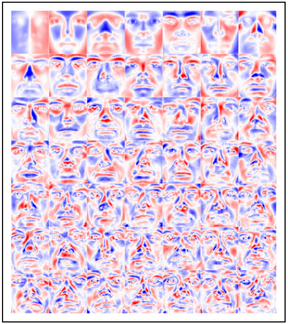
Natural Patches



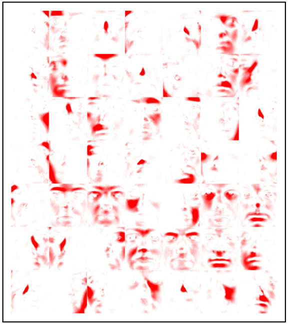
(a) PCA (b) NNMF (c) DL

Matrix Factorization Problems and Dictionary Learning

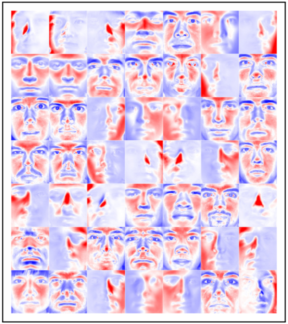
Faces



(d) PCA



(e) NNMF



(f) DL

Important messages

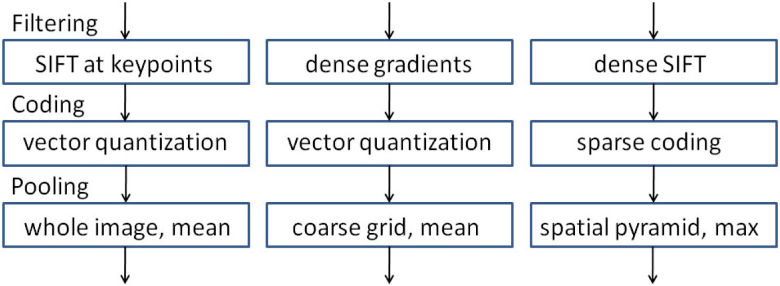
- The ℓ_1-norm induces sparsity and shrinks the coefficients (soft-thresholding)
- The regularization path of the Lasso is piecewise linear.
- Sparsity can be induced at the group level.
- Learning the dictionary is simple, fast and scalable.
- Dictionary learning is related to several matrix factorization problems.

Software SPAMS is available for all of this:
`www.di.ens.fr/willow/SPAMS/`.

Next topics: Computer Vision

- Intriguing results on the use of dictionary learning for bags of words.
- Modelling the local appearance of image patches.

Learning Codebooks for Image Classification



Idea

Replacing Vector Quantization by Learned Dictionaries!

- unsupervised: [Yang et al., 2009]
- supervised: [Boureau et al., 2010, Yang et al., 2010]

Learning Codebooks for Image Classification

Let an image be represented by a set of low-level descriptors $\mathbf{x}_i$ at N locations identified with their indices $i = 1, \ldots, N$.

- hard-quantization:

$$\mathbf{x}_i \approx \mathbf{D}\boldsymbol{\alpha}_i, \quad \boldsymbol{\alpha}_i \in \{0,1\}^p \text{ and } \sum_{j=1}^{p} \boldsymbol{\alpha}_i[j] = 1$$

- soft-quantization:

$$\boldsymbol{\alpha}_i[j] = \frac{e^{-\beta\|\mathbf{x}_i - \mathbf{d}_j\|_2^2}}{\sum_{k=1}^{p} e^{-\beta\|\mathbf{x}_i - \mathbf{d}_k\|_2^2}}$$

- sparse coding:

$$\mathbf{x}_i \approx \mathbf{D}\boldsymbol{\alpha}_i, \quad \boldsymbol{\alpha}_i = \arg\min_{\boldsymbol{\alpha}} \frac{1}{2}\|\mathbf{x}_i - \mathbf{D}\boldsymbol{\alpha}\|_2^2 + \lambda\|\boldsymbol{\alpha}\|_1$$

Learning Codebooks for Image Classification

Table from Boureau et al. [2010]

Method	Caltech-101, 30 training examples		15 Scenes, 100 training examples	
	Average Pool	Max Pool	Average Pool	Max Pool
	Results with basic features, SIFT extracted each 8 pixels			
Hard quantization, linear kernel	51.4 ± 0.9 [256]	64.3 ± 0.9 [256]	73.9 ± 0.9 [1024]	80.1 ± 0.6 [1024]
Hard quantization, intersection kernel	64.2 ± 1.0 [256] **(1)**	64.3 ± 0.9 [256]	80.8 ± 0.4 [256] **(1)**	80.1 ± 0.6 [1024]
Soft quantization, linear kernel	57.9 ± 1.5 [1024]	69.0 ± 0.8 [256]	75.6 ± 0.5 [1024]	81.4 ± 0.6 [1024]
Soft quantization, intersection kernel	66.1 ± 1.2 [512] **(2)**	70.6 ± 1.0 [1024]	81.2 ± 0.4 [1024] **(2)**	83.0 ± 0.7 [1024]
Sparse codes, linear kernel	61.3 ± 1.3 [1024]	$\mathbf{71.5 \pm 1.1}$ [1024] **(3)**	76.9 ± 0.6 [1024]	83.1 ± 0.6 [1024] **(3)**
Sparse codes, intersection kernel	70.3 ± 1.3 [1024]	$\mathbf{71.8 \pm 1.0}$ [1024] **(4)**	83.2 ± 0.4 [1024]	$\mathbf{84.1 \pm 0.5}$ [1024] **(4)**
	Results with macrofeatures and denser SIFT sampling			
Hard quantization, linear kernel	55.6 ± 1.6 [256]	70.9 ± 1.0 [1024]	74.0 ± 0.5 [1024]	80.1 ± 0.5 [1024]
Hard quantization, intersection kernel	68.8 ± 1.4 [512]	70.9 ± 1.0 [1024]	81.0 ± 0.5 [1024]	80.1 ± 0.5 [1024]
Soft quantization, linear kernel	61.6 ± 1.6 [1024]	71.5 ± 1.0 [1024]	76.4 ± 0.7 [1024]	81.5 ± 0.4 [1024]
Soft quantization, intersection kernel	70.1 ± 1.3 [1024]	73.2 ± 1.0 [1024]	81.8 ± 0.4 [1024]	83.0 ± 0.4 [1024]
Sparse codes, linear kernel	65.7 ± 1.4 [1024]	$\mathbf{75.1 \pm 0.9}$ [1024]	78.2 ± 0.7 [1024]	83.6 ± 0.4 [1024]
Sparse codes, intersection kernel	73.7 ± 1.3 [1024]	$\mathbf{75.7 \pm 1.1}$ [1024]	83.5 ± 0.4 [1024]	$\mathbf{84.3 \pm 0.5}$ [1024]

	Unsup	Discr
Linear	83.6 ± 0.4	$\mathbf{84.9 \pm 0.3}$
Intersect	84.3 ± 0.5	$\mathbf{84.7 \pm 0.4}$

Yang et al. [2009] have won the PASCAL VOC'09 challenge using this kind of techniques.

Learning dictionaries with a discriminative cost function

Idea:

Let us consider 2 sets S_-, S_+ of signals representing 2 different classes. Each set should admit a dictionary best adapted to its reconstruction.

Classification procedure for a signal $\mathbf{x} \in \mathbb{R}^n$:

$$\min(\mathbf{R}^\star(\mathbf{x}, \mathbf{D}_-), \mathbf{R}^\star(\mathbf{x}, \mathbf{D}_+))$$

where

$$\mathbf{R}^\star(\mathbf{x}, \mathbf{D}) = \min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \|\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}\|_2^2 \text{ s.t. } \|\boldsymbol{\alpha}\|_0 \leq L.$$

"Reconstructive" training

$$\begin{cases} \min_{\mathbf{D}_-} \sum_{i \in S_-} \mathbf{R}^\star(\mathbf{x}_i, \mathbf{D}_-) \\ \min_{\mathbf{D}_+} \sum_{i \in S_+} \mathbf{R}^\star(\mathbf{x}_i, \mathbf{D}_+) \end{cases}$$

[Grosse et al., 2007], [Huang and Aviyente, 2006],
[Sprechmann et al., 2010] for unsupervised clustering (CVPR '10)

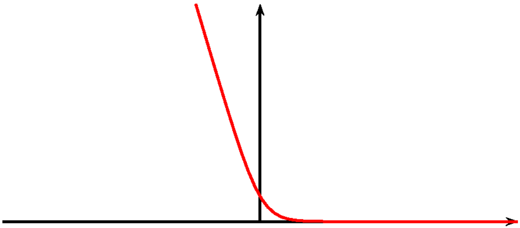
Learning dictionaries with a discriminative cost function

"Discriminative" training

[Mairal, Bach, Ponce, Sapiro, and Zisserman, 2008a]

$$\min_{\mathbf{D}_-,\mathbf{D}_+} \sum_i \mathcal{C}\Big(\lambda z_i \big(\mathbf{R}^\star(\mathbf{x}_i,\mathbf{D}_-) - \mathbf{R}^\star(\mathbf{x}_i,\mathbf{D}_+)\big)\Big),$$

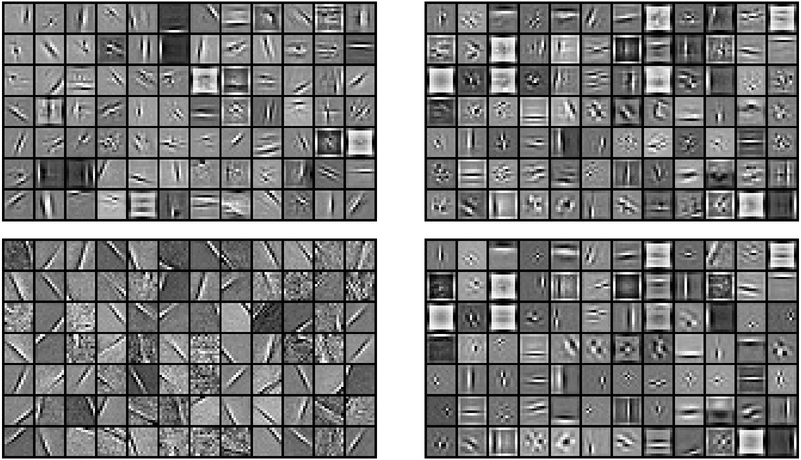
where $z_i \in \{-1,+1\}$ is the label of $\mathbf{x}_i$.



Logistic regression function

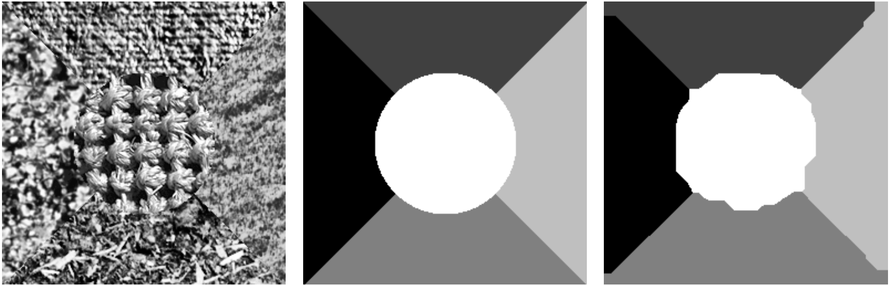
Learning dictionaries with a discriminative cost function

Examples of dictionaries



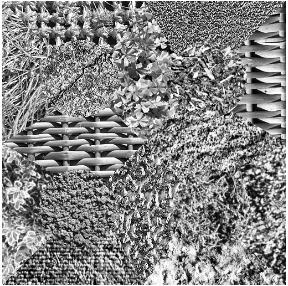
Top: reconstructive, Bottom: discriminative, Left: Bicycle, Right: Background.

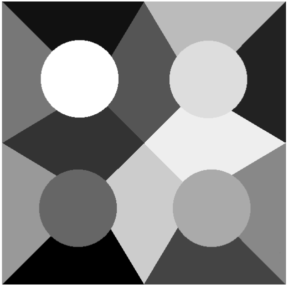
Learning dictionaries with a discriminative cost function
Texture segmentation



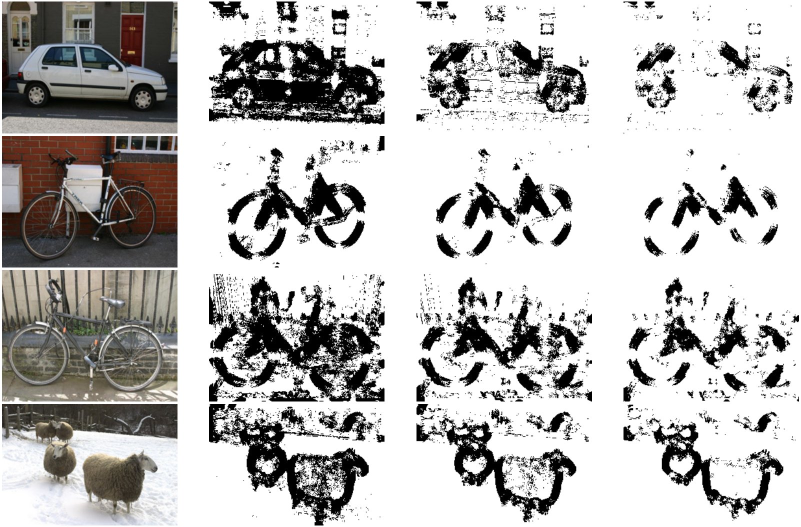
Learning dictionaries with a discriminative cost function

Texture segmentation





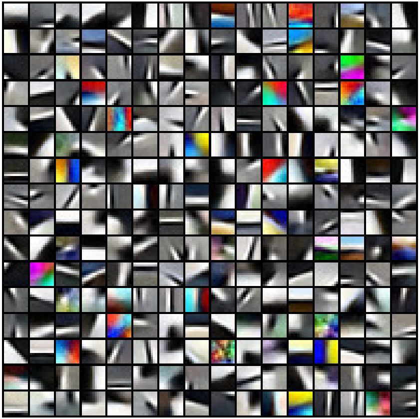
Learning dictionaries with a discriminative cost function
Pixelwise classification



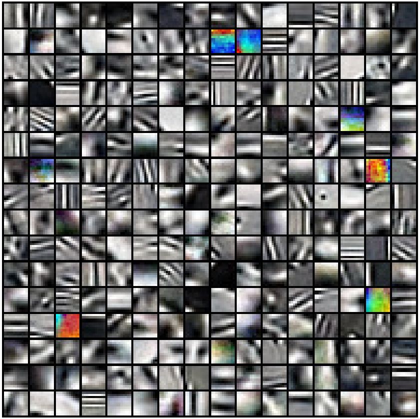
Learning dictionaries with a discriminative cost function
weakly-supervised pixel classification

Application to edge detection and classification

[Mairal, Leordeanu, Bach, Hebert, and Ponce, 2008c]

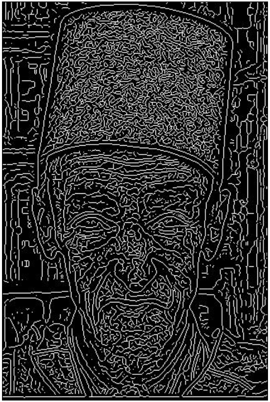


Good edges



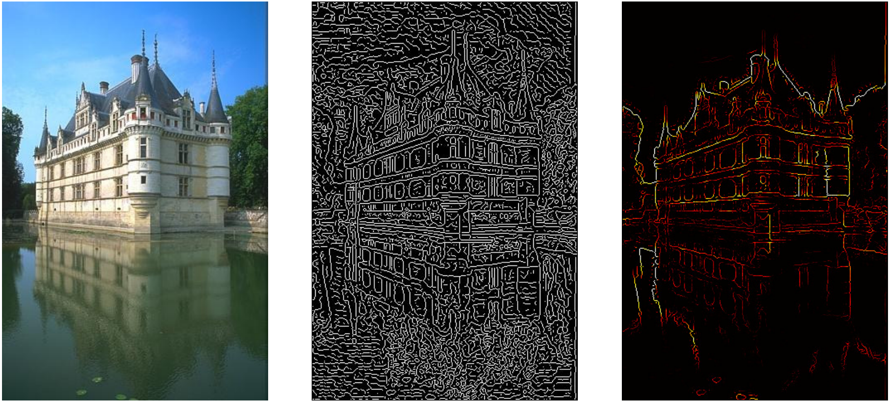
Bad edges

Application to edge detection and classification
Berkeley segmentation benchmark

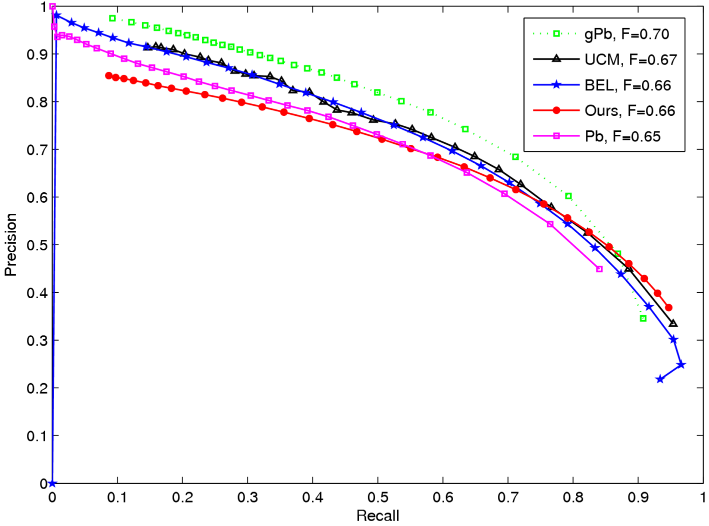

Raw edge detection on the right

Application to edge detection and classification
Berkeley segmentation benchmark



Raw edge detection on the right

Application to edge detection and classification
Berkeley segmentation benchmark

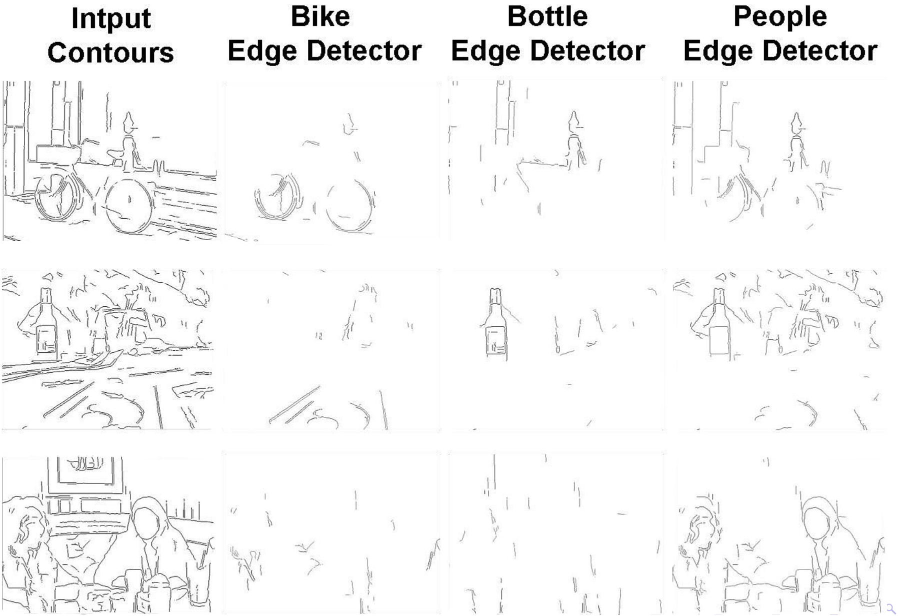


Application to edge detection and classification

Contour-based classifier: [Leordeanu, Hebert, and Sukthankar, 2007]

Is there a bike, a motorbike, a car or a person on this image?

Application to edge detection and classification



Application to edge detection and classification

Performance gain due to the prefiltering

Ours + [Leordeanu '07]	[Leordeanu '07]	[Winn '05]
96.8%	89.4%	76.9%

Recognition rates for the same experiment as [Winn et al., 2005] on VOC 2005.

Category	Ours+[Leordeanu '07]	[Leordeanu '07]
Aeroplane	71.9%	61.9%
Boat	67.1%	56.4%
Cat	82.6%	53.4%
Cow	68.7%	59.2%
Horse	76.0%	67%
Motorbike	80.6%	73.6%
Sheep	72.9%	58.4%
Tvmonitor	87.7%	83.8%
Average	75.9%	64.2 %

Recognition performance at equal error rate for 8 classes on a subset of images from Pascal 07.

Digital Art Authentification

Data Courtesy of Hugues, Graham, and Rockmore [2009]

Authentic

Fake

Digital Art Authentification

Data Courtesy of Hugues, Graham, and Rockmore [2009]

Authentic

Fake

Fake

Digital Art Authentification

Data Courtesy of Hugues, Graham, and Rockmore [2009]

Authentic

Fake

Authentic

Important messages

- Learned dictionaries are well adapted to model the local appearance of images and edges.
- They can be used to learn dictionaries of SIFT features.

Next topics

- Optimization for solving sparse decomposition problems
- Optimization for dictionary learning

Recall: The Sparse Decomposition Problem

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \underbrace{\frac{1}{2}\|\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}\|_2^2}_{\text{data fitting term}} + \underbrace{\lambda\psi(\boldsymbol{\alpha})}_{\substack{\text{sparsity-inducing}\\\text{regularization}}}$$

ψ induces sparsity in $\boldsymbol{\alpha}$. It can be

- the ℓ_0 "pseudo-norm". $\|\boldsymbol{\alpha}\|_0 \triangleq \#\{i \text{ s.t. } \boldsymbol{\alpha}[i] \neq 0\}$ (NP-hard)
- the ℓ_1 norm. $\|\boldsymbol{\alpha}\|_1 \triangleq \sum_{i=1}^{p} |\boldsymbol{\alpha}[i]|$ (convex)
- ...

This is a selection problem.

Finding your way in the sparse coding literature...

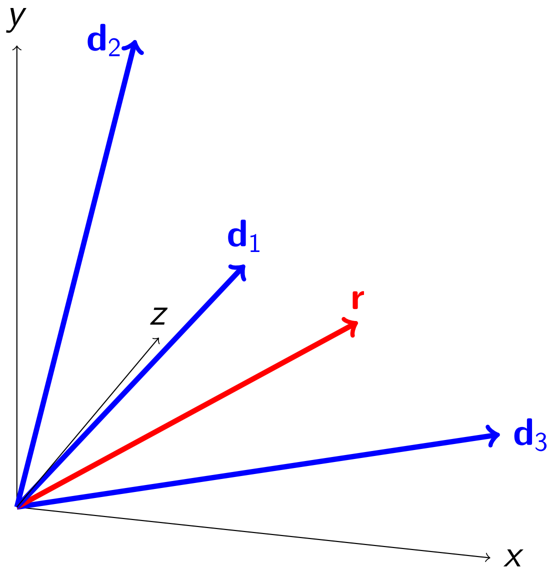
...is not easy. The literature is vast, redundant, sometimes confusing and many papers are claiming victory...

The main class of methods are

- greedy procedures [Mallat and Zhang, 1993], [Weisberg, 1980]
- homotopy [Osborne et al., 2000], [Efron et al., 2004], [Markowitz, 1956]
- soft-thresholding based methods [Fu, 1998], [Daubechies et al., 2004], [Friedman et al., 2007], [Nesterov, 2007], [Beck and Teboulle, 2009], ...
- reweighted-ℓ_2 methods [Daubechies et al., 2009],...
- active-set methods [Roth and Fischer, 2008].
- ...

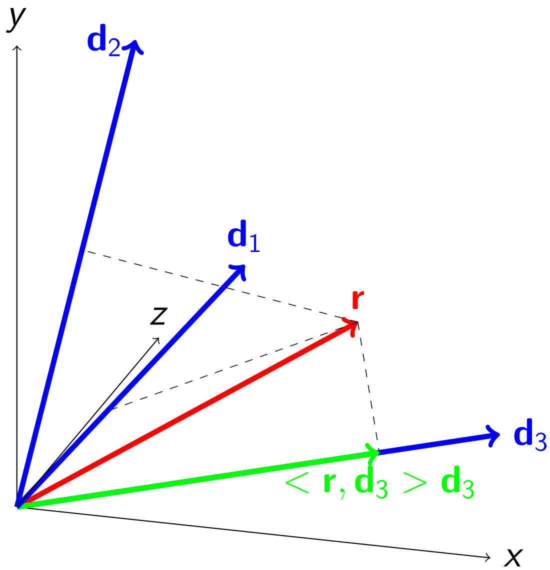
Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0, 0)$$



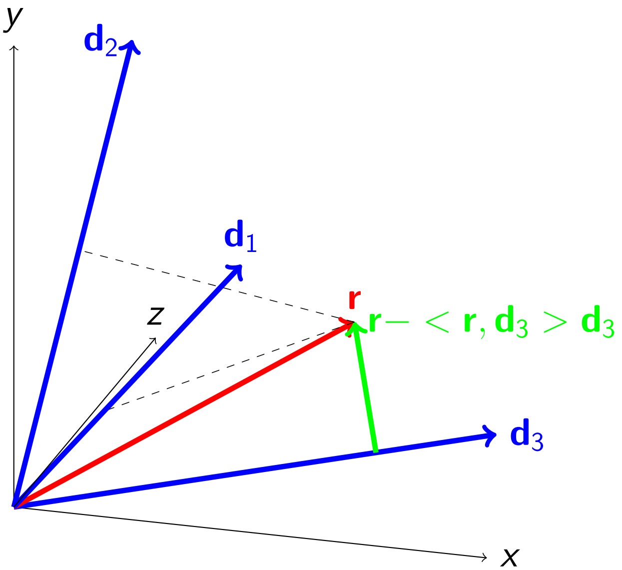
Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0, 0)$$



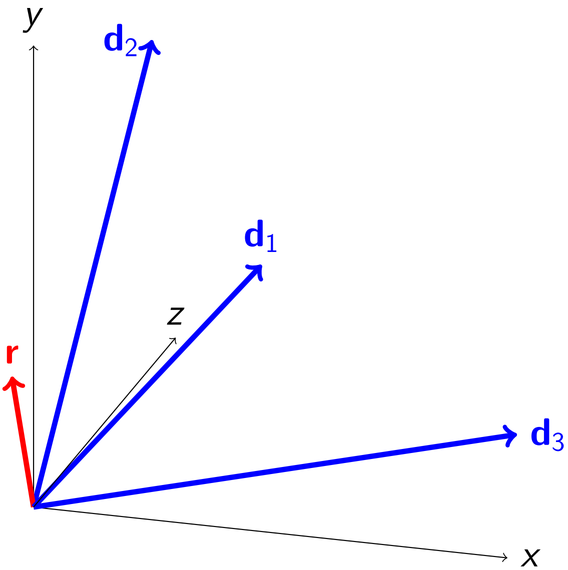
Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0, 0)$$



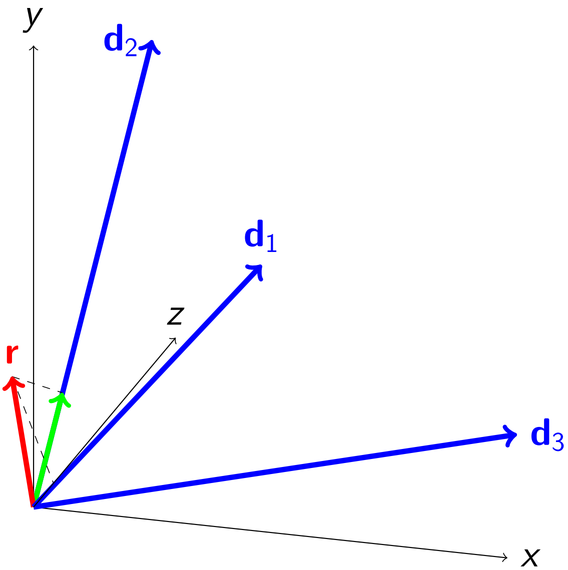
Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0, 0.75)$$



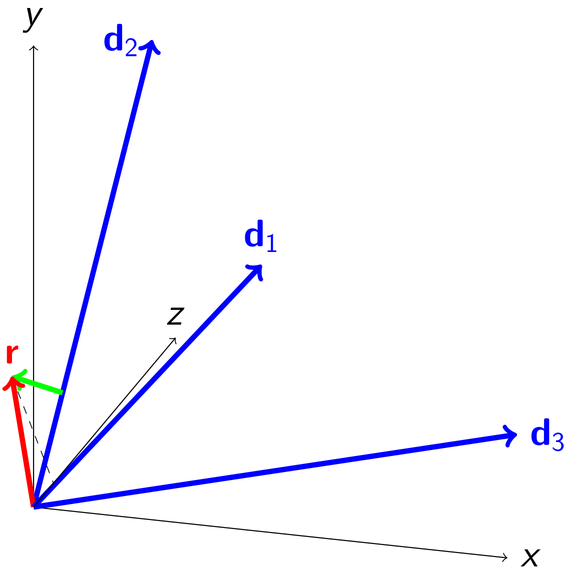
Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0, 0.75)$$



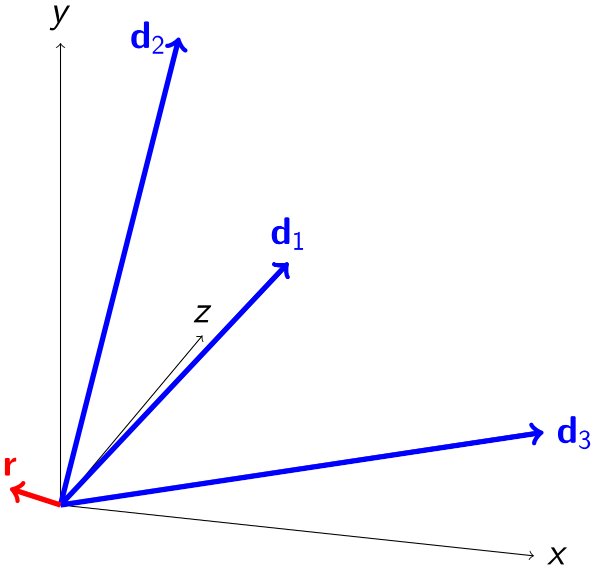
Matching Pursuit

$\boldsymbol{\alpha} = (0, 0, 0.75)$



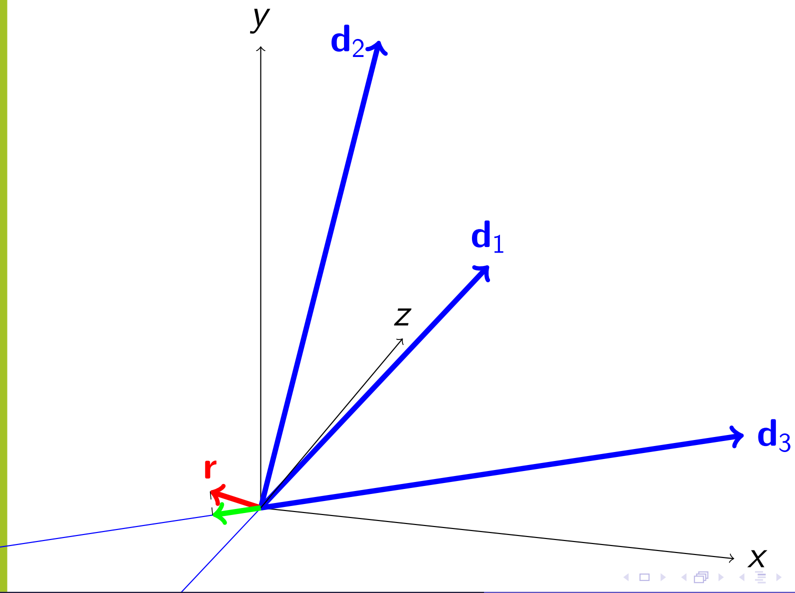
Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0.24, 0.75)$$



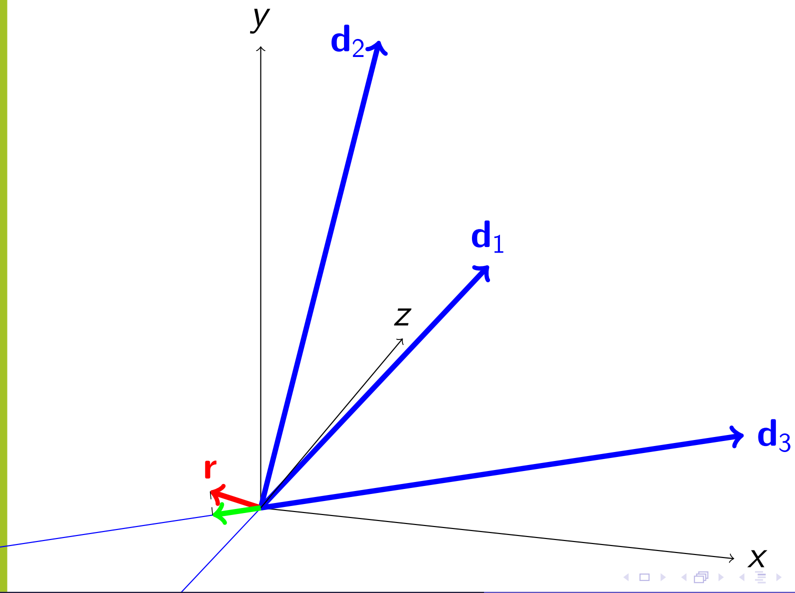
Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0.24, 0.75)$$



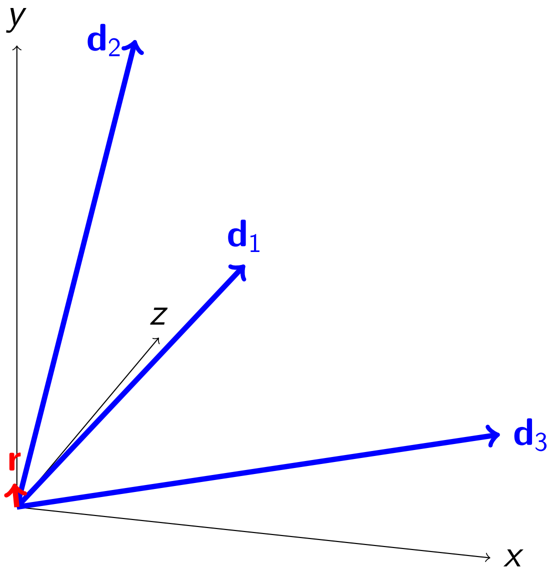
Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0.24, 0.75)$$



Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0.24, 0.65)$$



Matching Pursuit

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \|\underbrace{\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}}_{\mathbf{r}}\|_2^2 \;\text{ s.t. }\; \|\boldsymbol{\alpha}\|_0 \leq L$$

1: $\boldsymbol{\alpha} \leftarrow 0$
2: $\mathbf{r} \leftarrow \mathbf{x}$ (residual).
3: **while** $\|\boldsymbol{\alpha}\|_0 < L$ **do**
4: Select the atom with maximum correlation with the residual

$$\hat{\imath} \leftarrow \arg\max_{i=1,\ldots,p} |\mathbf{d}_i^T \mathbf{r}|$$

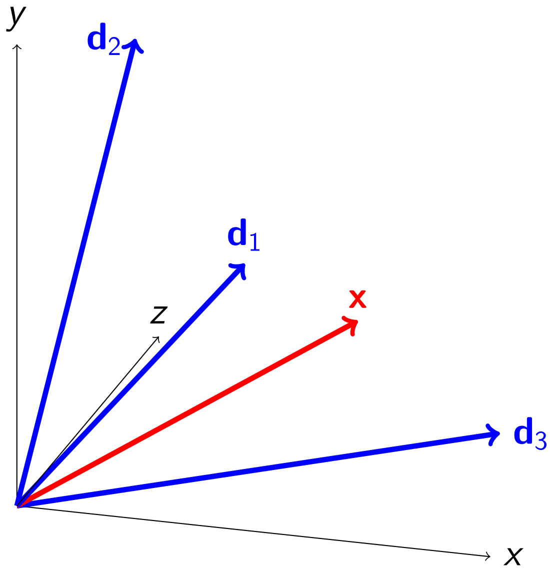
5: Update the residual and the coefficients

$$\begin{aligned} \boldsymbol{\alpha}[\hat{\imath}] &\leftarrow \boldsymbol{\alpha}[\hat{\imath}] + \mathbf{d}_{\hat{\imath}}^T \mathbf{r} \\ \mathbf{r} &\leftarrow \mathbf{r} - (\mathbf{d}_{\hat{\imath}}^T \mathbf{r})\mathbf{d}_{\hat{\imath}} \end{aligned}$$

6: **end while**

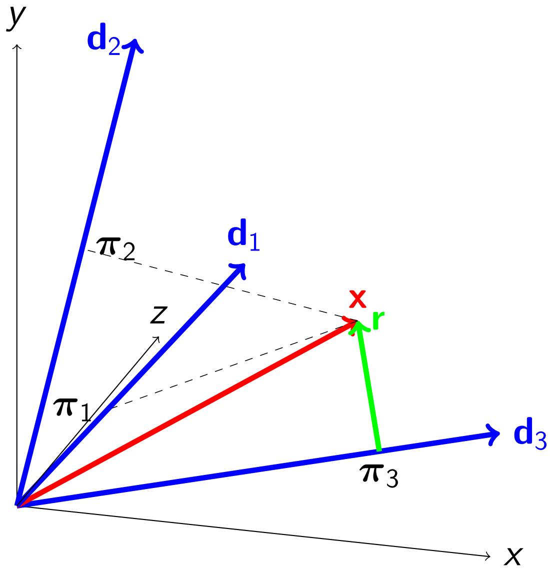
Orthogonal Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0, 0)$$
$$\Gamma = \emptyset$$



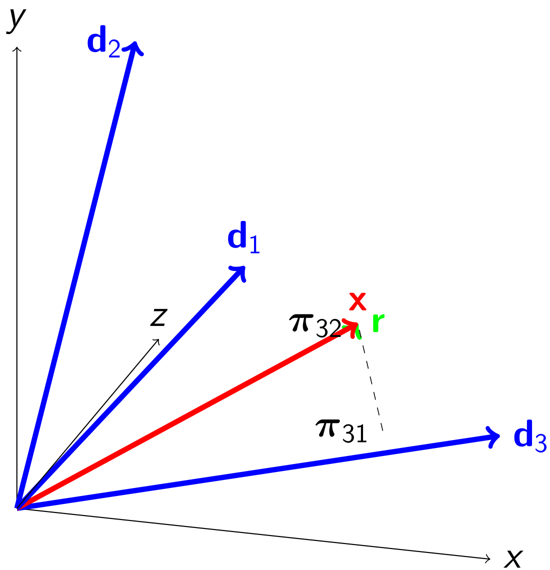
Orthogonal Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0, 0.75)$$
$$\Gamma = \{3\}$$



Orthogonal Matching Pursuit

$$\boldsymbol{\alpha} = (0, 0.29, 0.63)$$
$$\Gamma = \{3, 2\}$$



Orthogonal Matching Pursuit

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \|\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}\|_2^2 \;\text{ s.t. }\; \|\boldsymbol{\alpha}\|_0 \leq L$$

1: $\Gamma = \emptyset$.
2: **for** $iter = 1, \ldots, L$ **do**
3: Select the atom which most reduces the objective

$$\hat{\imath} \leftarrow \arg\min_{i\in\Gamma^C} \left\{ \min_{\boldsymbol{\alpha}'} \|\mathbf{x}-\mathbf{D}_{\Gamma\cup\{i\}}\boldsymbol{\alpha}'\|_2^2 \right\}$$

4: Update the active set: $\Gamma \leftarrow \Gamma \cup \{\hat{\imath}\}$.
5: Update the residual (orthogonal projection)

$$\mathbf{r} \leftarrow (\mathbf{I} - \mathbf{D}_\Gamma(\mathbf{D}_\Gamma^T\mathbf{D}_\Gamma)^{-1}\mathbf{D}_\Gamma^T)\mathbf{x}.$$

6: Update the coefficients

$$\boldsymbol{\alpha}_\Gamma \leftarrow (\mathbf{D}_\Gamma^T\mathbf{D}_\Gamma)^{-1}\mathbf{D}_\Gamma^T\mathbf{x}.$$

7: **end for**

Orthogonal Matching Pursuit

Contrary to MP, an atom can only be selected one time with OMP. It is, however, more difficult to implement efficiently. The keys for a good implementation in the case of a large number of signals are

- Precompute the Gram matrix $\mathbf{G} = \mathbf{D}^T\mathbf{D}$ once in for all,
- Maintain the computation of $\mathbf{D}^T\mathbf{r}$ for each signal,
- Maintain a Cholesky decomposition of $(\mathbf{D}_\Gamma^T\mathbf{D}_\Gamma)^{-1}$ for each signal.

The total complexity for decomposing n L-sparse signals of size m with a dictionary of size p is

$$\underbrace{O(p^2m)}_{\text{Gram matrix}} + \underbrace{O(nL^3)}_{\text{Cholesky}} + \underbrace{O(n(pm + pL^2))}_{\mathbf{D}^T\mathbf{r}} = O(np(m + L^2))$$

It is also possible to use the matrix inversion lemma instead of a Cholesky decomposition (same complexity, but less numerical stability)

Example with the software SPAMS

Software available at http://www.di.ens.fr/willow/SPAMS/

```
>> I=double(imread('data/lena.eps'))/255;
>> %extract all patches of I
>> X=im2col(I,[8 8],'sliding');
>> %load a dictionary of size 64 x 256
>> D=load('dict.mat');
>>
>> %set the sparsity parameter L to 10
>> param.L=10;
>> alpha=mexOMP(X,D,param);
```

On a 8-cores 2.83Ghz machine: **230000 signals processed per second!**

Optimality conditions of the Lasso

Nonsmooth optimization

Directional derivatives and subgradients are useful tools for studying ℓ_1-decomposition problems:

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \frac{1}{2}\|\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}\|_2^2 + \lambda\|\boldsymbol{\alpha}\|_1$$

In this tutorial, we use the **directional derivatives** to derive simple optimality conditions of the Lasso.

For more information on convex analysis and nonsmooth optimization, see the following books: [Boyd and Vandenberghe, 2004], [Nocedal and Wright, 2006], [Borwein and Lewis, 2006], [Bonnans et al., 2006], [Bertsekas, 1999].

Optimality conditions of the Lasso

Directional derivatives

- **Directional derivative** in the direction $\mathbf{u}$ at $\boldsymbol{\alpha}$:

$$\nabla f(\boldsymbol{\alpha}, \mathbf{u}) = \lim_{t \to 0^+} \frac{f(\boldsymbol{\alpha} + t\mathbf{u}) - f(\boldsymbol{\alpha})}{t}$$

- Main idea: in non smooth situations, one may need to look at all directions $\mathbf{u}$ and not simply p independent ones!
- **Proposition 1:** if f is differentiable in $\boldsymbol{\alpha}$, $\nabla f(\boldsymbol{\alpha}, \mathbf{u}) = \nabla f(\boldsymbol{\alpha})^T \mathbf{u}$.
- **Proposition 2:** $\boldsymbol{\alpha}$ is optimal iff for all $\mathbf{u}$ in $\mathbb{R}^p$, $\nabla f(\boldsymbol{\alpha}, \mathbf{u}) \geq 0$.

Optimality conditions of the Lasso

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \frac{1}{2}\|\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}\|_2^2 + \lambda\|\boldsymbol{\alpha}\|_1$$

$\boldsymbol{\alpha}^\star$ is optimal iff for all $\mathbf{u}$ in $\mathbb{R}^p$, $\nabla f(\boldsymbol{\alpha},\mathbf{u}) \geq 0$—that is,

$$-\mathbf{u}^T\mathbf{D}^T(\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}^\star) + \lambda \sum_{i,\boldsymbol{\alpha}^\star[i]\neq 0} \text{sign}(\boldsymbol{\alpha}^\star[i])\mathbf{u}[i] + \lambda \sum_{i,\boldsymbol{\alpha}^\star[i]=0} |\mathbf{u}_i| \geq 0,$$

which is equivalent to the following conditions:

$$\forall i = 1,\ldots,p, \quad \begin{cases} |\mathbf{d}_i^T(\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}^\star)| & \leq & \lambda & \text{if } \boldsymbol{\alpha}^\star[i]=0 \\ \mathbf{d}_i^T(\mathbf{x}-\mathbf{D}\boldsymbol{\alpha}^\star) & = & \lambda\,\text{sign}(\boldsymbol{\alpha}^\star[i]) & \text{if } \boldsymbol{\alpha}^\star[i]\neq 0 \end{cases}$$

Homotopy

- A homotopy method provides a set of solutions indexed by a parameter.
- The regularization path $(\lambda, \boldsymbol{\alpha}^\star(\lambda))$ for instance!!
- It can be useful when the path has some "nice" properties (piecewise linear, piecewise quadratic).
- LARS [Efron et al., 2004] starts from a trivial solution, and follows the regularization path of the Lasso, which is is **piecewise linear**.

Homotopy, LARS

[Osborne et al., 2000], [Efron et al., 2004]

$$\forall i = 1, \ldots, p, \quad \begin{cases} |\mathbf{d}_i^T(\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}^\star)| & \leq & \lambda & \text{if } \boldsymbol{\alpha}^\star[i] = 0 \\ \mathbf{d}_i^T(\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}^\star) & = & \lambda \operatorname{sign}(\boldsymbol{\alpha}^\star[i]) & \text{if } \boldsymbol{\alpha}^\star[i] \neq 0 \end{cases} \tag{1}$$

The regularization path is piecewise linear:

$$\mathbf{D}_\Gamma^T(\mathbf{x} - \mathbf{D}_\Gamma \boldsymbol{\alpha}_\Gamma^\star) = \lambda \operatorname{sign}(\boldsymbol{\alpha}_\Gamma^\star)$$

$$\boldsymbol{\alpha}_\Gamma^\star(\lambda) = (\mathbf{D}_\Gamma^T \mathbf{D}_\Gamma)^{-1}(\mathbf{D}_\Gamma^T \mathbf{x} - \lambda \operatorname{sign}(\boldsymbol{\alpha}_\Gamma^\star)) = \mathbf{A} + \lambda \mathbf{B}$$

A simple interpretation of LARS

- Start from the trivial solution $(\lambda = \|\mathbf{D}^T\mathbf{x}\|_\infty, \boldsymbol{\alpha}^\star(\lambda) = 0)$.
- Maintain the computations of $|\mathbf{d}_i^T(\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}^\star(\lambda))|$ for all i.
- Maintain the computation of the current direction $\mathbf{B}$.
- Follow the path by reducing λ until the next kink.

Example with the software SPAMS

http://www.di.ens.fr/willow/SPAMS/

```
>> I=double(imread('data/lena.eps'))/255;
>> %extract all patches of I
>> X=normalize(im2col(I,[8 8],'sliding'));
>> %load a dictionary of size 64 x 256
>> D=load('dict.mat');
>>
>> %set the sparsity parameter lambda to 0.15
>> param.lambda=0.15;
>> alpha=mexLasso(X,D,param);
```

On a 8-cores 2.83Ghz machine: **77000 signals processed per second!**
Note that it can also solve **constrained** version of the problem. The complexity is more or less the same as OMP and uses the same tricks (Cholesky decomposition).

Coordinate Descent

- Coordinate descent + nonsmooth objective: **WARNING: not convergent in general**
- Here, the problem is equivalent to a convex smooth optimization problem with **separable** constraints

$$\min_{\boldsymbol{\alpha}_+,\boldsymbol{\alpha}_-} \frac{1}{2}\|\mathbf{x} - \mathbf{D}_+\boldsymbol{\alpha}_+ + \mathbf{D}_-\boldsymbol{\alpha}_-\|_2^2 + \lambda\boldsymbol{\alpha}_+^T\mathbf{1} + \lambda\boldsymbol{\alpha}_-^T\mathbf{1} \;\text{ s.t. }\; \boldsymbol{\alpha}_-, \boldsymbol{\alpha}_+ \geq 0.$$

- For this **specific** problem, coordinate descent is **convergent**.
- Supposing $\|\mathbf{d}_i\|_2 = 1$, updating the coordinate i:

$$\boldsymbol{\alpha}[i] \leftarrow \arg\min_{\beta} \frac{1}{2}\|\underbrace{\mathbf{x} - \sum_{j\neq i}\boldsymbol{\alpha}[j]\mathbf{d}_j}_{\mathbf{r}} - \beta\mathbf{d}_i\|_2^2 + \lambda|\beta|$$

$$\leftarrow \operatorname{sign}(\mathbf{d}_i^T\mathbf{r})(|\mathbf{d}_i^T\mathbf{r}| - \lambda)^+$$

- $\Rightarrow$ **soft-thresholding**!

Example with the software SPAMS

http://www.di.ens.fr/willow/SPAMS/

```
>> I=double(imread('data/lena.eps'))/255;
>> %extract all patches of I
>> X=normalize(im2col(I,[8 8],'sliding'));
>> %load a dictionary of size 64 x 256
>> D=load('dict.mat');
>>
>> %set the sparsity parameter lambda to 0.15
>> param.lambda=0.15;
>> param.tol=1e-2;
>> param.itermax=200;
>> alpha=mexCD(X,D,param);
```

On a 8-cores 2.83Ghz machine: **93000 signals processed per second!**

first-order/proximal methods

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \; f(\boldsymbol{\alpha}) + \lambda \psi(\boldsymbol{\alpha})$$

- f is strictly convex and continuously differentiable with a Lipshitz gradient.
- Generalize the idea of gradient descent

$$\begin{aligned}\boldsymbol{\alpha}_{k+1} &\leftarrow \arg\min_{\boldsymbol{\alpha} \in \mathbb{R}} f(\boldsymbol{\alpha}_k) + \nabla f(\boldsymbol{\alpha}_k)^T(\boldsymbol{\alpha} - \boldsymbol{\alpha}_k) + \frac{L}{2}\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_k\|_2^2 + \lambda\psi(\boldsymbol{\alpha}) \\ &\leftarrow \arg\min_{\boldsymbol{\alpha} \in \mathbb{R}} \frac{1}{2}\|\boldsymbol{\alpha} - (\boldsymbol{\alpha}_k - \frac{1}{L}\nabla f(\boldsymbol{\alpha}_k))\|_2^2 + \frac{\lambda}{L}\psi(\boldsymbol{\alpha})\end{aligned}$$

When $\lambda = 0$, this is equivalent to a classical gradient descent step.

first-order/proximal methods

- They require solving efficiently the proximal operator

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \frac{1}{2}\|\mathbf{u} - \boldsymbol{\alpha}\|_2^2 + \lambda\psi(\boldsymbol{\alpha})$$

- For the ℓ_1-norm, this amounts to a soft-thresholding:

$$\boldsymbol{\alpha}^\star[i] = \text{sign}(\mathbf{u}[i])(\mathbf{u}[i] - \lambda)^+.$$

- There exists accelerated versions based on Nesterov optimal first-order method (gradient method with "extrapolation") [Beck and Teboulle, 2009, Nesterov, 2007, 1983]
- suited for large-scale experiments.

Optimization for Grouped Sparsity

The formulation:

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \underbrace{\frac{1}{2}\|\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}\|_2^2}_{\text{data fitting term}} + \underbrace{\lambda \sum_{g \in \mathcal{G}} \|\boldsymbol{\alpha}_g\|_q}_{\text{group-sparsity-inducing regularization}}$$

The main class of algorithms for solving grouped-sparsity problems are

- Greedy approaches
- Block-coordinate descent
- Proximal methods

Optimization for Grouped Sparsity

The proximal operator:

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \frac{1}{2}\|\mathbf{u}-\boldsymbol{\alpha}\|_2^2 + \lambda \sum_{g\in\mathcal{G}} \|\boldsymbol{\alpha}_g\|_q$$

For $q=2$,

$$\boldsymbol{\alpha}_g^\star = \frac{\mathbf{u}_g}{\|\mathbf{u}_g\|_2}(\|\mathbf{u}_g\|_2 - \lambda)^+, \quad \forall g \in \mathcal{G}$$

For $q=\infty$,

$$\boldsymbol{\alpha}_g^\star = \mathbf{u}_g - \Pi_{\|.\|_1\leq\lambda}[\mathbf{u}_g], \quad \forall g \in \mathcal{G}$$

These formula generalize soft-thrsholding to groups of variables. They are used in block-coordinate descent and proximal algorithms.

Reweighted ℓ_2

Let us start from something simple

$$a^2 - 2ab + b^2 \geq 0.$$

Then

$$a \leq \frac{1}{2}\left(\frac{a^2}{b} + b\right) \;\text{ with equality iff }\; a = b$$

and

$$\|\boldsymbol{\alpha}\|_1 = \min_{\eta_j \geq 0} \frac{1}{2} \sum_{j=1}^{p} \frac{\boldsymbol{\alpha}[j]^2}{\eta_j} + \eta_j.$$

The formulation becomes

$$\min_{\boldsymbol{\alpha}, \eta_j \geq \varepsilon} \frac{1}{2}\|\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}\|_2^2 + \frac{\lambda}{2} \sum_{j=1}^{p} \frac{\boldsymbol{\alpha}[j]^2}{\eta_j} + \eta_j.$$

Important messages

- Greedy methods directly address the NP-hard ℓ_0-decomposition problem.
- Homotopy methods can be extremely efficient for small or medium-sized problems, or when the solution is very sparse.
- Coordinate descent provides in general quickly a solution with a small/medium precision, but gets slower when there is a lot of correlation in the dictionary.
- First order methods are very attractive in the large scale setting.
- Other good alternatives exists, active-set, reweighted ℓ_2 methods, stochastic variants, variants of OMP,. . .

Optimization for Dictionary Learning

$$\min_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^{p\times n}\\ \mathbf{D}\in\mathcal{C}}} \sum_{i=1}^{n} \frac{1}{2}\|\mathbf{x}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda\|\boldsymbol{\alpha}_i\|_1$$

$$\mathcal{C} \triangleq \{\mathbf{D} \in \mathbb{R}^{m\times p} \;\text{ s.t. }\; \forall j = 1,\ldots,p, \;\; \|\mathbf{d}_j\|_2 \leq 1\}.$$

- Classical optimization alternates between $\mathbf{D}$ and $\boldsymbol{\alpha}$.
- Good results, but very slow!

Optimization for Dictionary Learning

[Mairal, Bach, Ponce, and Sapiro, 2009a]

Classical formulation of dictionary learning

$$\min_{\mathbf{D}\in\mathcal{C}} f_n(\mathbf{D}) = \min_{\mathbf{D}\in\mathcal{C}} \frac{1}{n}\sum_{i=1}^{n} l(\mathbf{x}_i, \mathbf{D}),$$

where

$$l(\mathbf{x}, \mathbf{D}) \triangleq \min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \frac{1}{2}\|\mathbf{x} - \mathbf{D}\boldsymbol{\alpha}\|_2^2 + \lambda\|\boldsymbol{\alpha}\|_1.$$

Which formulation are we interested in?

$$\min_{\mathbf{D}\in\mathcal{C}} \left\{ f(\mathbf{D}) = \mathbb{E}_x[l(\mathbf{x}, \mathbf{D})] \approx \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} l(\mathbf{x}_i, \mathbf{D}) \right\}$$

[Bottou and Bousquet, 2008]: Online learning can

- handle potentially infinite or dynamic datasets,
- be dramatically faster than batch algorithms.

Optimization for Dictionary Learning

Require: $\mathbf{D}_0 \in \mathbb{R}^{m \times p}$ (initial dictionary); $\lambda \in \mathbb{R}$

1: $\mathbf{A}_0 = 0$, $\mathbf{B}_0 = 0$.

2: **for** t=1,...,T **do**

3: Draw $\mathbf{x}_t$

4: Sparse Coding

$$\boldsymbol{\alpha}_t \leftarrow \underset{\boldsymbol{\alpha} \in \mathbb{R}^p}{\arg\min} \frac{1}{2}\|\mathbf{x}_t - \mathbf{D}_{t-1}\boldsymbol{\alpha}\|_2^2 + \lambda\|\boldsymbol{\alpha}\|_1,$$

5: Aggregate sufficient statistics
$\mathbf{A}_t \leftarrow \mathbf{A}_{t-1} + \boldsymbol{\alpha}_t\boldsymbol{\alpha}_t^T$, $\mathbf{B}_t \leftarrow \mathbf{B}_{t-1} + \mathbf{x}_t\boldsymbol{\alpha}_t^T$

6: Dictionary Update (block-coordinate descent)

$$\mathbf{D}_t \leftarrow \underset{\mathbf{D} \in \mathcal{C}}{\arg\min} \frac{1}{t}\sum_{i=1}^{t}\left(\frac{1}{2}\|\mathbf{x}_i - \mathbf{D}\boldsymbol{\alpha}_i\|_2^2 + \lambda\|\boldsymbol{\alpha}_i\|_1\right).$$

7: **end for**

Optimization for Dictionary Learning

Which guarantees do we have?

Under a few reasonable assumptions,

- we build a surrogate function $\hat{f}_t$ of the expected cost f verifying

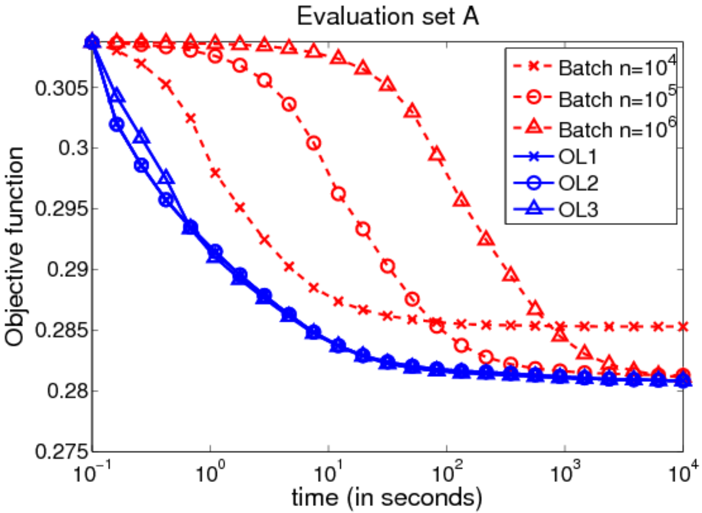
$$\lim_{t \to +\infty} \hat{f}_t(\mathbf{D}_t) - f(\mathbf{D}_t) = 0,$$

- $\mathbf{D}_t$ is asymptotically close to a stationary point.

Extensions (all implemented in SPAMS)

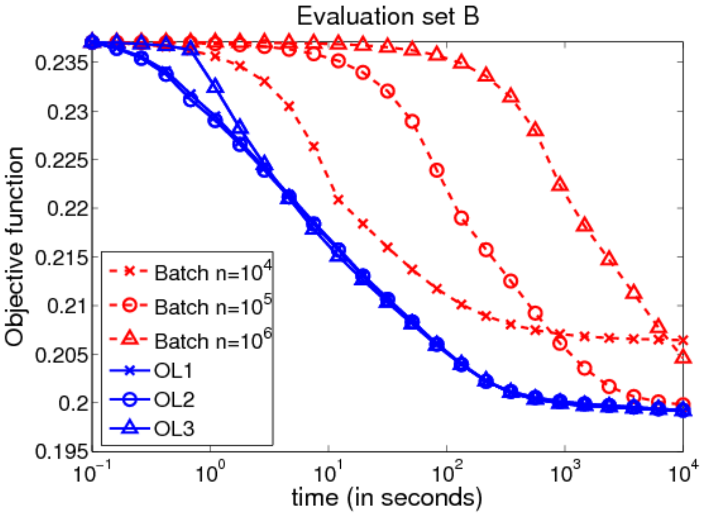
- non-negative matrix decompositions.
- sparse PCA (sparse dictionaries).
- fused-lasso regularizations (piecewise constant dictionaries)

Optimization for Dictionary Learning
Experimental results, batch vs online



$$m = 8 \times 8, \ p = 256$$

Optimization for Dictionary Learning
Experimental results, batch vs online



$$m = 12 \times 12 \times 3,\ p = 512$$

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