III – Signatures

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Outline

Basic Security Notions

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Public-Key Encryption

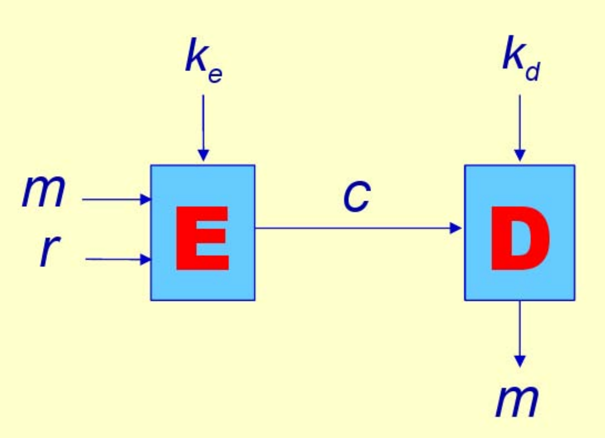
Signatures

Advanced Security for Signature

Forking Lemma

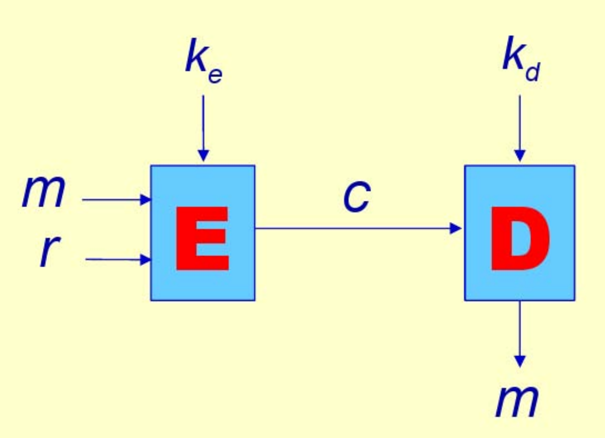
Conclusion

Public-Key Encryption



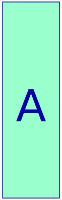
Goal: Privacy/Secrecy of the plaintext

Public-Key Encryption



Goal: Privacy/Secrecy of the plaintext

$\mathrm{OW} - \mathrm{CPA}$ Security Game



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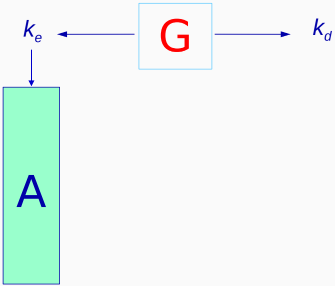
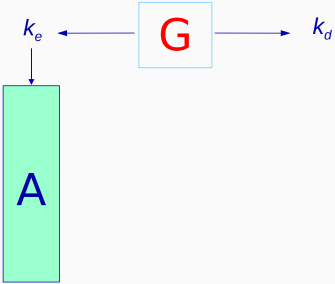


OW − CPA Security Game

m^* random
r^* random



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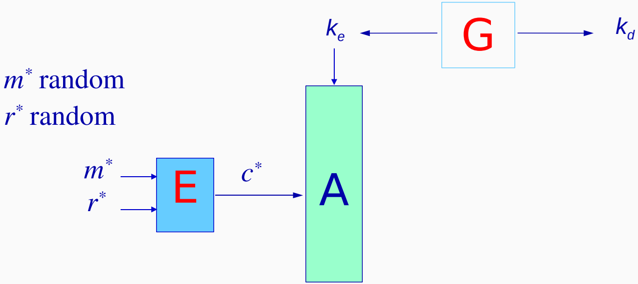

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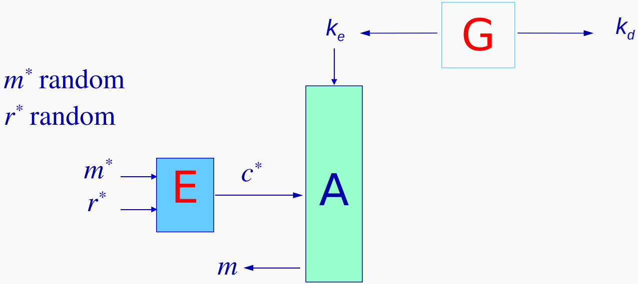

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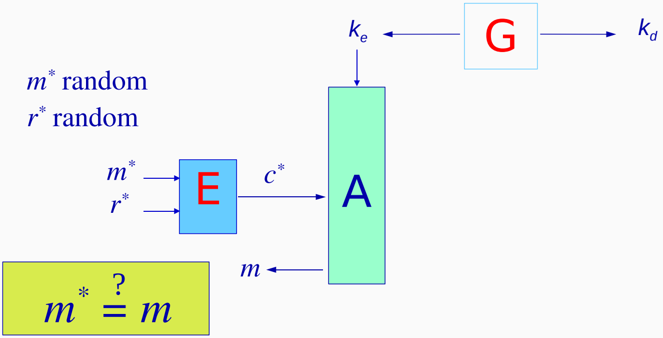

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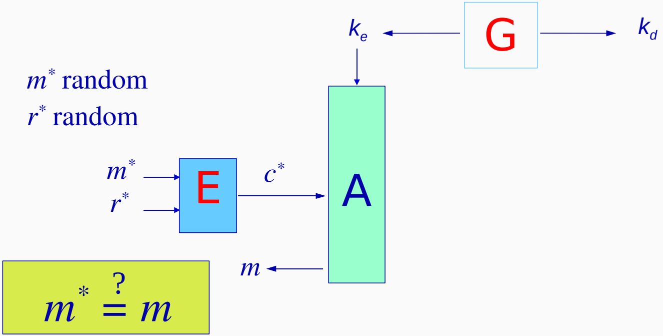
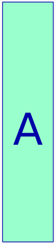
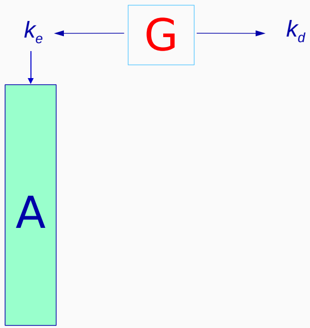


$$\mathbf{Succ}_{\mathcal{S}}^{\mathsf{ow}}(\mathcal{A}) = \Pr[(sk, pk) \leftarrow \mathcal{K}(); m \stackrel{R}{\leftarrow} \mathcal{M}; c = \mathcal{E}_{pk}(m) : \mathcal{A}(pk, c) \rightarrow m]$$

$\mathrm{IND-CPA}$ Security Game



IND − CPA Security Game



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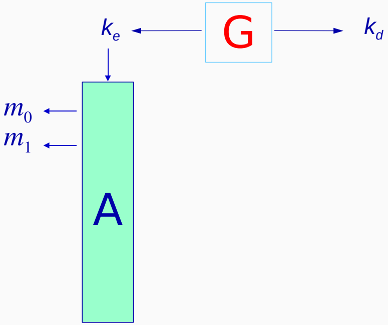
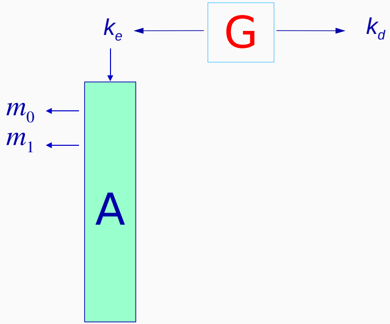


$\mathrm{IND} - \mathrm{CPA}$ Security Game

$b \in \{0,1\}$

r random



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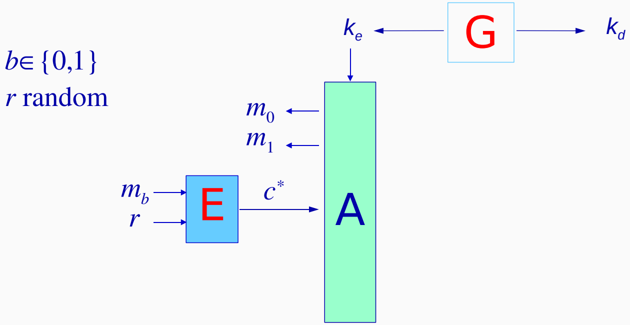

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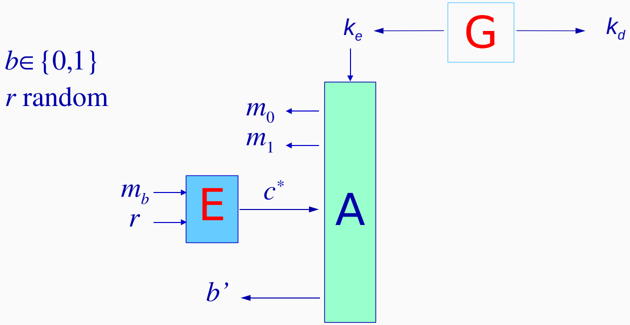

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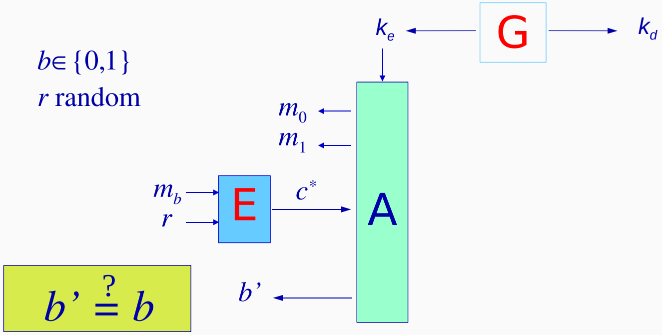

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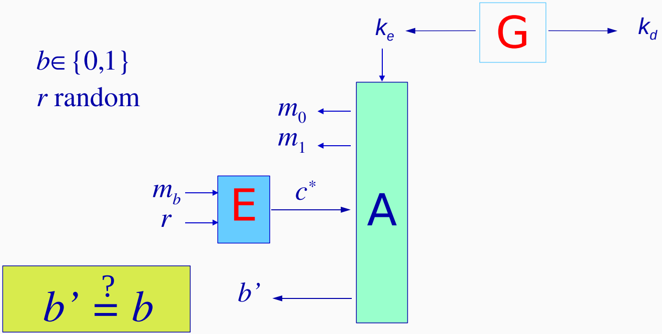


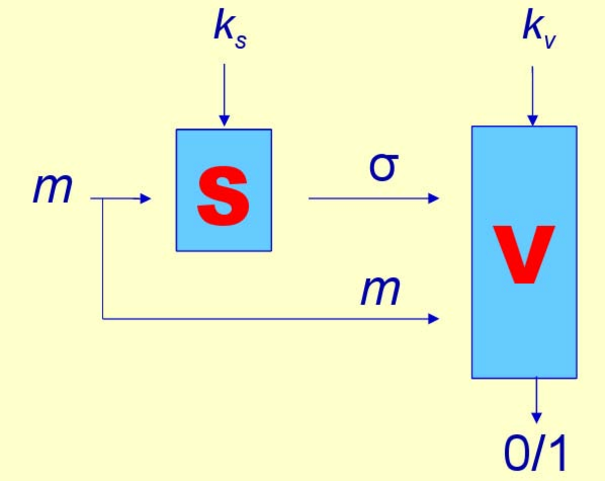
$$(sk, pk) \leftarrow \mathcal{K}(); (m_0, m_1, \text{state}) \leftarrow \mathcal{A}(pk);$$

$$b \stackrel{R}{\leftarrow} \{0,1\}; c = \mathcal{E}_{pk}(m_b); b' \leftarrow \mathcal{A}(\text{state}, c)$$

$$\mathbf{Adv}_{\mathcal{S}}^{\text{ind-cpa}}(\mathcal{A}) = \left|\Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0]\right| = \left|2 \times \Pr[b' = b] - 1\right|$$

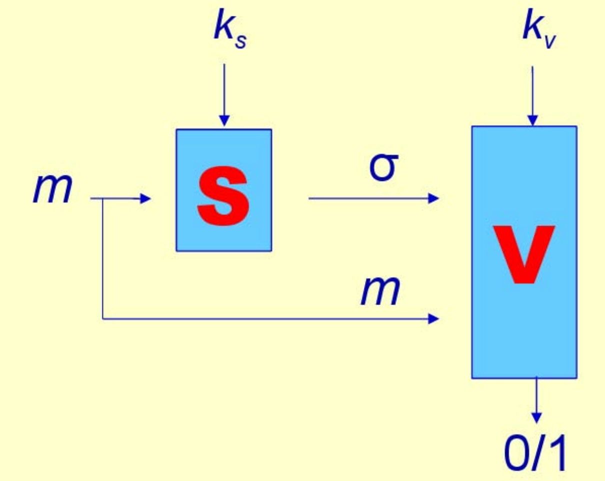
Outline

Signature



Goal: Authentication of the sender

Signature



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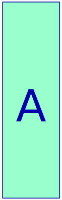
EUF − NMA



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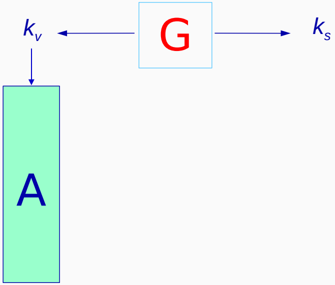

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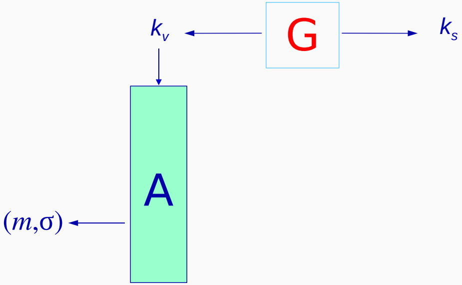
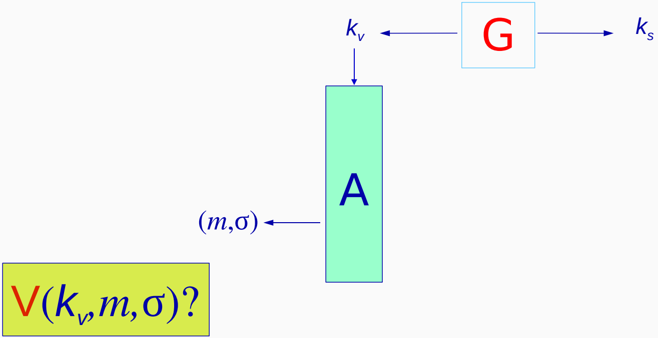


EUF – NMA



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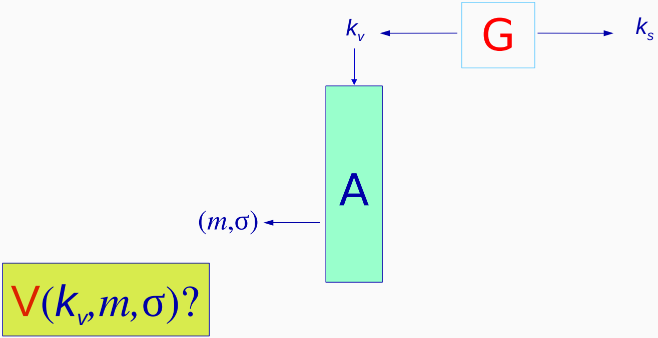
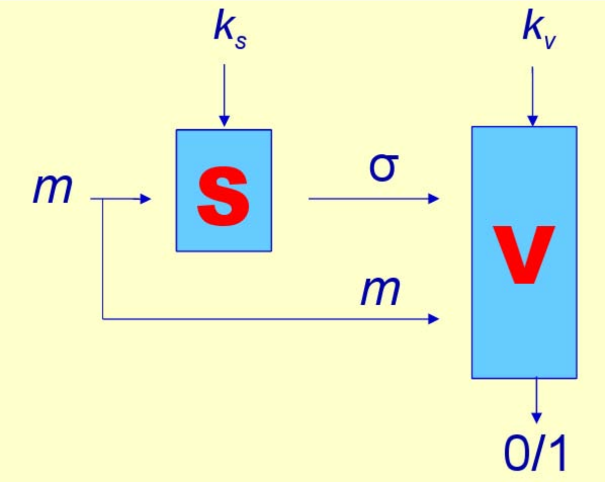


$$\mathbf{Succ}^{\mathsf{euf}}_{\mathcal{SG}}(\mathcal{A}) = \Pr[(sk, pk) \leftarrow \mathcal{K}(); (m, \sigma) \leftarrow \mathcal{A}(pk) : \mathcal{V}_{pk}(m, \sigma) = 1]$$

Advanced Security for Signature

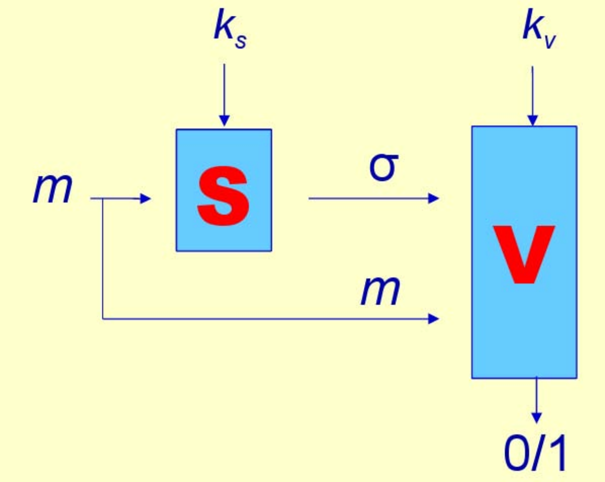
Outline

Signature



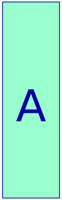
Goal: Authentication of the sender

Signature

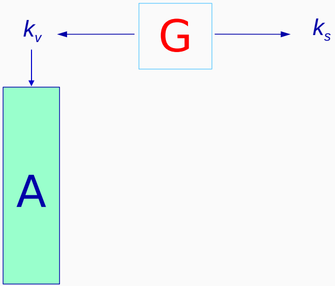


Goal: Authentication of the sender

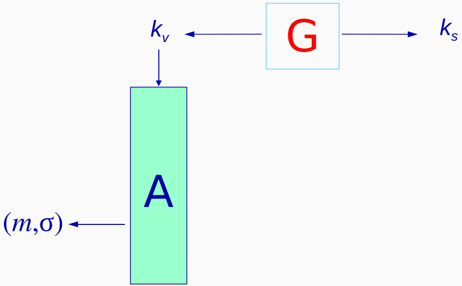
EUF − NMA



EUF – NMA



EUF − NMA



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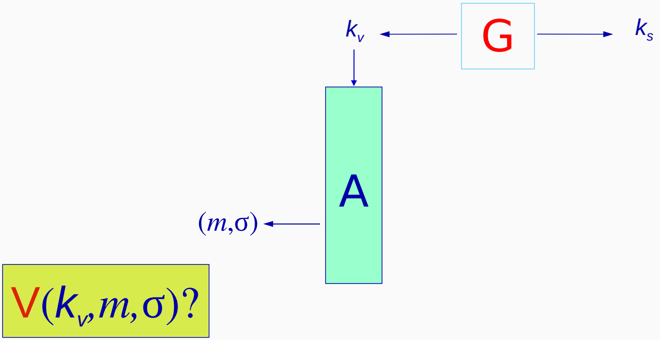

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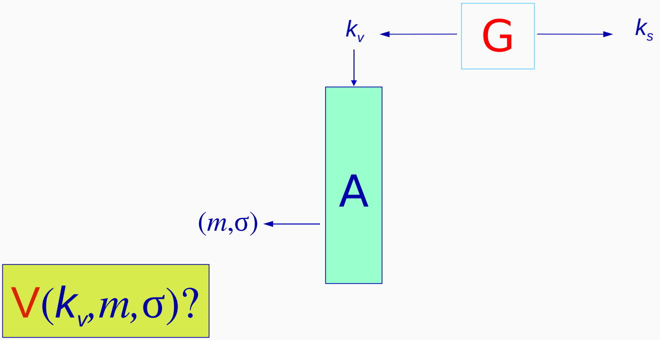
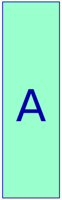
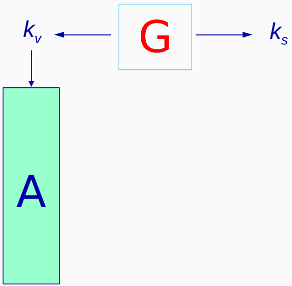


The adversary knows the public key only,
whereas signatures are not private!

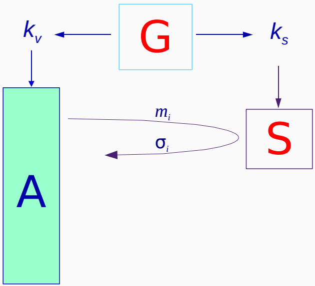
EUF − CMA



$\mathrm{EUF}-\mathrm{CMA}$



$\mathsf{EUF} - \mathsf{CMA}$



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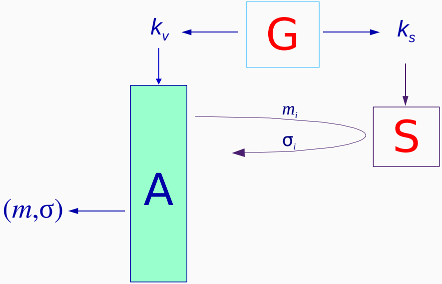

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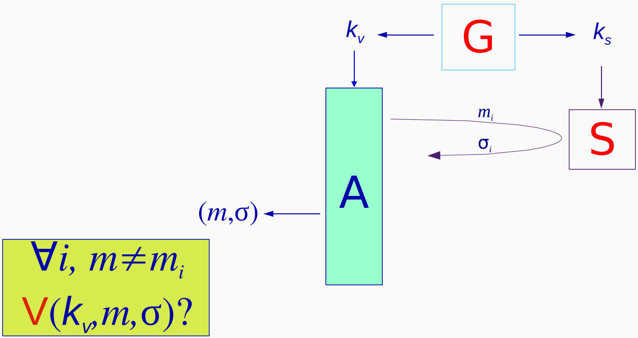
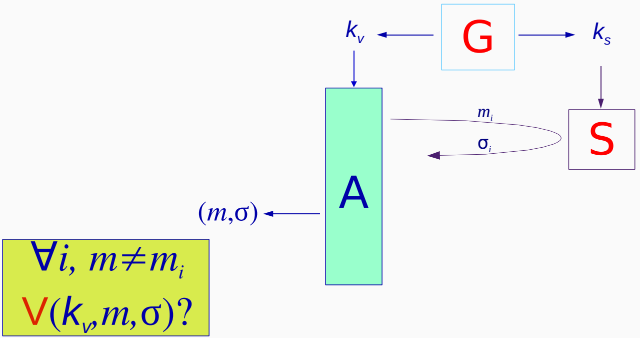
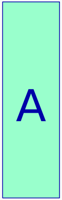
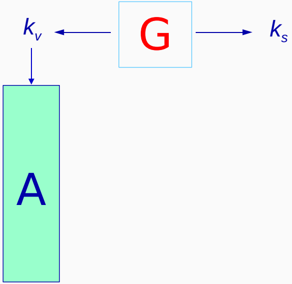




The adversary has access to any signature of its choice:
Chosen-Message Attacks (oracle access):

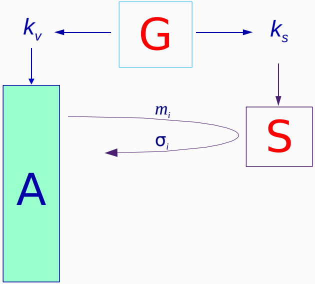
$$\mathbf{Succ}^{\text{euf}-\text{cma}}_{\mathcal{SG}}(\mathcal{A}) = \Pr\left[\begin{array}{l}(sk, pk) \leftarrow \mathcal{K}(); (m, \sigma) \leftarrow \mathcal{A}^{\mathcal{S}}(pk) : \\ \forall i, m \neq m_i \wedge \mathcal{V}_{pk}(m, \sigma) = 1\end{array}\right]$$

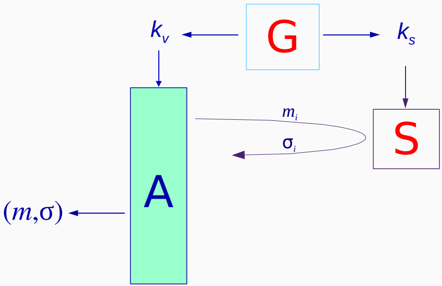

A

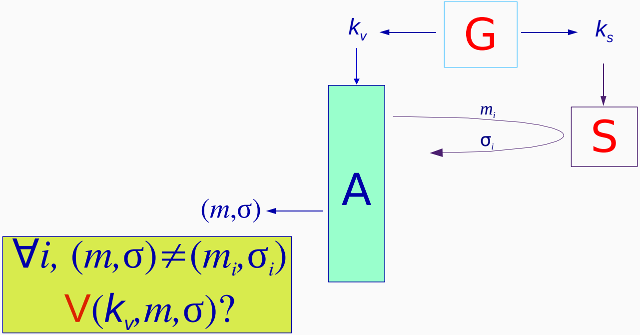

k_v
G
k_s
A

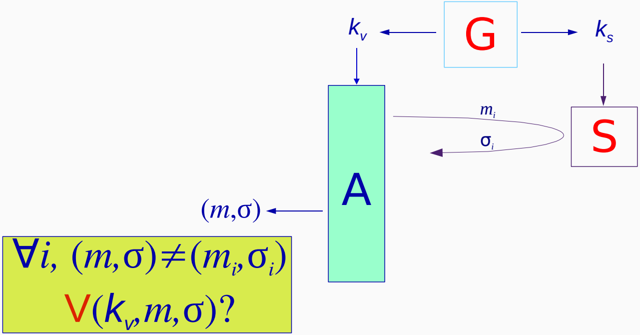
SUF − CMA

[Stern-Pointcheval-Malone-Lee-Smart – Crypto '01]




k_v
G
k_s
m_i
σ_i
S
A
(m,σ)


k_v
G
k_s
A
m_i
σ_i
S
(m,σ)
$\forall i,\ (m,\sigma) \neq (m_i,\sigma_i)$
$V(k_v,m,\sigma)?$



The notion is even stronger (in case of probabilistic signature): also known as non-malleability:

$$\mathbf{Succ}^{\mathsf{suf-cma}}_{\mathcal{SG}}(\mathcal{A}) = \Pr\left[\begin{array}{l}(sk,pk) \leftarrow \mathcal{K}();\ (m,\sigma) \leftarrow \mathcal{A}^{\mathcal{S}}(pk):\\ \forall i,\ (m,\sigma) \neq (m_i,\sigma_i) \wedge \mathcal{V}_{pk}(m,\sigma) = 1\end{array}\right]$$

Outline

Signature Scheme

- Key generation: the public key $f \stackrel{R}{\leftarrow} \mathcal{P}$ is a trapdoor one-way bijection from X onto Y; the private key is the inverse $g : Y \to X$;
- Signature of $M \in Y$: $\sigma = g(M)$;
- Verification of (M, σ): check $f(\sigma) = M$

Full-Domain Hash (Hash-and-Invert)

$$\mathcal{H} : \{0,1\}^\star \to Y$$

- in order to sign m, one computes $M = \mathcal{H}(m) \in Y$, and $\sigma = g(M)$
- and the verification consists in checking whether $f(\sigma) = H(m)$

Full-Domain Hash Signature

[Bellare-Rogaway – Eurocrypt '94]

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Random Oracle Model

Random Oracle

- $\mathcal{H}$ is modelled as a truly random function, from $\{0,1\}^*$ into Y.
- Formally, $\mathcal{H}$ is chosen at random at the beginning of the game.
- More concretely, for any new query, a random element in Y is uniformly and independently drawn

Any security game becomes:

$$\mathbf{Succ}_{\mathcal{SG}}^{\mathsf{euf-cma}}(\mathcal{A}) = \Pr\left[\begin{array}{l}\mathcal{H} \stackrel{R}{\leftarrow} Y^{\infty}; (sk, pk) \leftarrow \mathcal{K}(); (m, \sigma) \leftarrow \mathcal{A}^{\mathcal{S},\mathcal{H}}(pk) : \\ \forall i, m \neq m_i \wedge \mathcal{V}_{pk}(m, \sigma) = 1\end{array}\right]$$

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Security of the FDH Signature

Theorem

The FDH signature achieves $\mathbf{EUF-CMA}$ *security, under the One-Wayness of* $\mathcal{P}$*, in the Random Oracle Model:*

$$\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(t) \leq q_H \times \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H \tau_f)$$

Assumptions:

- any signing query has been first asked to $\mathcal{H}$
- the forgery has been asked to $\mathcal{H}$
- τ_f is the maximal time to evaluate $f \in \mathcal{P}$

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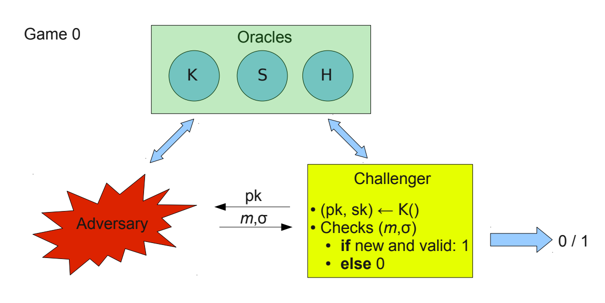
Real Attack Game



Random Oracle

$\mathcal{H}(m)$: $M \stackrel{R}{\leftarrow} Y$, output M

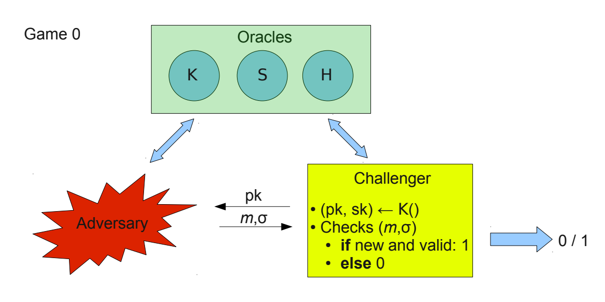
Key Generation Oracle

$\mathcal{K}()$: $(f, g) \stackrel{R}{\leftarrow} \mathcal{P}$, $sk \leftarrow g$, $pk \leftarrow f$

Signing Oracle

$\mathcal{S}(m)$: $M = \mathcal{H}(m)$, output $\sigma = g(M)$

Real Attack Game



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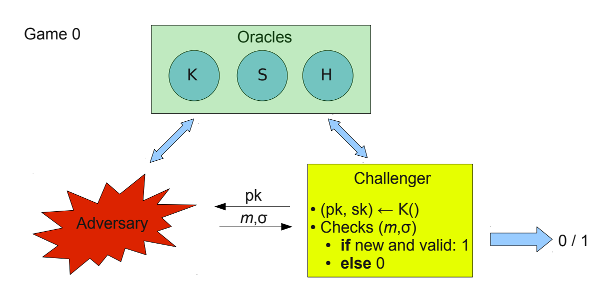

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Signing Oracle

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Simulations

- **Game$_0$**: use of the oracles $\mathcal{K}$, $\mathcal{S}$ and $\mathcal{H}$
- **Game$_1$**: use of the *simulation of the Random Oracle*

Simulation of $\mathcal{H}$

$\mathcal{H}(m)$: $\mu \stackrel{R}{\leftarrow} X$, output $M = f(\mu)$

- **Game$_2$**: use of the *simulation of the Signing Oracle*

Simulation of $\mathcal{S}$

$\mathcal{S}(m)$: find μ such that $M = \mathcal{H}(m) = f(\mu)$, output $\sigma = \mu$

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$\Longrightarrow$ **Hop-D-Perfect**: $\Pr_{\mathbf{Game}_1}[1] = \Pr_{\mathbf{Game}_0}[1]$

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Simulation of $\mathcal{S}$

$\mathcal{S}(m)$: find μ such that $M = \mathcal{H}(m) = f(\mu)$, output $\sigma = \mu$

$\Longrightarrow$ **Hop-S-Perfect**: $\Pr_{\textbf{Game}_2}[1] = \Pr_{\textbf{Game}_1}[1]$

Simulations

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- **Game**$_2$: use of the *simulation of the Signing Oracle*

Simulation of $\mathcal{S}$

$\mathcal{S}(m)$: find μ such that $M = \mathcal{H}(m) = f(\mu)$, output $\sigma = \mu$

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Simulations

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$\mathcal{H}$-Query Selection

- **Game$_3$**: random index $t \overset{R}{\leftarrow} \{1, \ldots, q_H\}$

Event Ev

If the t-th query to $\mathcal{H}$ is not the output forgery

We terminate the game and output 0 if **Ev** happens

$\Longrightarrow$ **Hop-S-Non-Negl**

Then, clearly

$$\Pr_{\textbf{Game}_3}[1] = \Pr_{\textbf{Game}_2}[1] \times \Pr[\neg\textbf{Ev}] \qquad \Pr[\textbf{Ev}] = 1 - 1/q_H$$

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$\mathbf{OW}$ Instance

- **Game**$_4$: $\mathcal{P} - \mathbf{OW}$ instance (f, y) (where $f \overset{R}{\leftarrow} \mathcal{P}, x \overset{R}{\leftarrow} X, y = f(x)$)
  Use of the *simulation of the Key Generation Oracle*

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$\mathcal{K}()$: set $pk \leftarrow f$

Modification of the *simulation of the Random Oracle*

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Summary

In **Game**$_4$, when the output is 1, $\sigma = g(y) = g(f(x)) = x$
and the simulator computes one exponentiation per hashing:

$$\Pr_{\textbf{Game}_4}[1] \quad \leq \quad \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H \tau_f)$$

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Key Size

$$\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(\mathcal{A}) \leq q_H \times \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H \tau_f)$$

- If one wants $\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(t) \leq \varepsilon$ with $t/\varepsilon \approx 2^{80}$
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If one uses FDH-RSA: at least 3072 bit keys are needed.

In the case that f is homomorphic (as RSA): $f(ab) = f(a)f(b)$

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Improvement [Coron – Crypto '00]

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In **Game**$_3$, when the output is 1, with probability $1 - p$:

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The maximal for $p \mapsto (1-p)p^{q_S}$ is reached for

$$p = 1 - \frac{1}{q_S+1} \quad \rightarrow \quad \frac{1}{q_S+1} \times \left(1 - \frac{1}{q_S+1}\right)^{q_S} \approx \frac{e^{-1}}{q_S}$$

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- If one wants $\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(t) \leq \varepsilon$ with $t/\varepsilon \approx 2^{80}$
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If one uses FDH-RSA: 2048 bit keys are enough.

Forking Lemma

Outline

Basic Security Notions

Advanced Security for Signature

Forking Lemma

- Zero-Knowledge Proofs
- The Forking Lemma

Conclusion

Proof of Knowledge

How do I prove that I know a solution s to a problem P?

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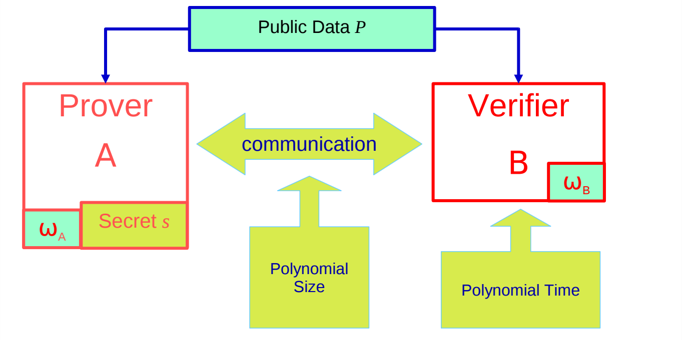

Proof of Knowledge: Soundness

If I can be accepted, I really know a solution: extractor

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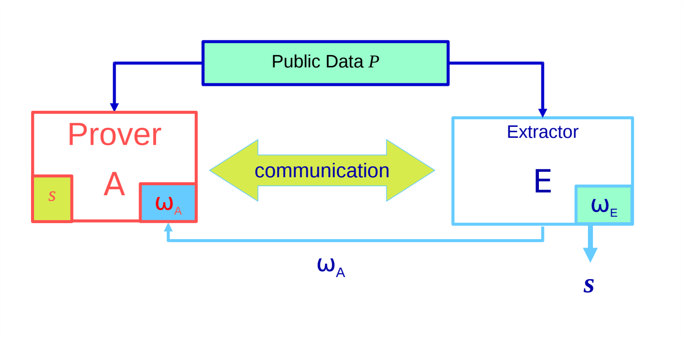

Proof of Knowledge: Zero-Knowledge

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I reveal the solution…

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Zero-knowledge: simulator

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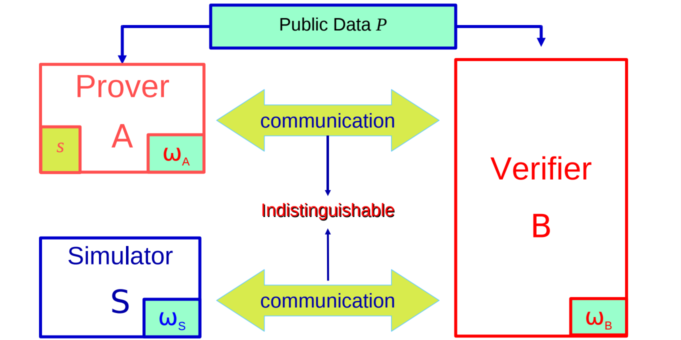

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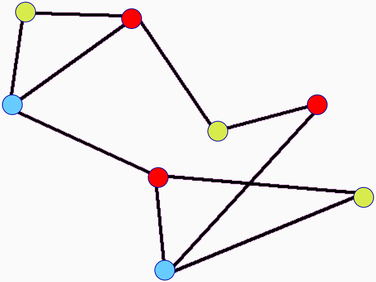
How do I prove that I know a 3-color covering, without revealing any information?



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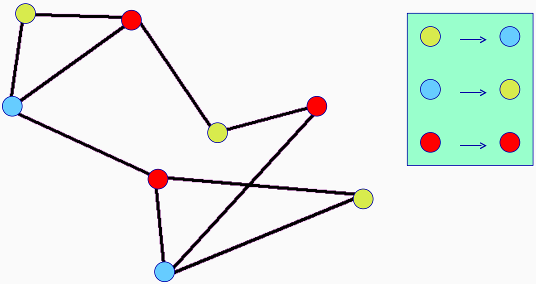

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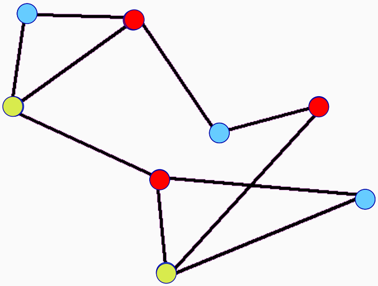

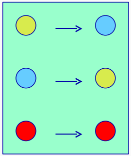

I choose a random permutation on the colors
and I apply it to the vertices

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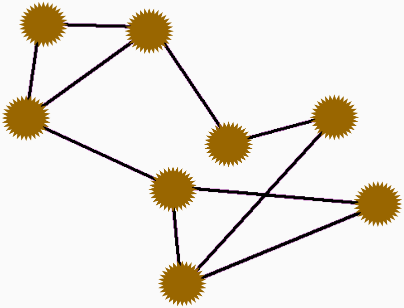

I mask the vertices
and send it to the verifier

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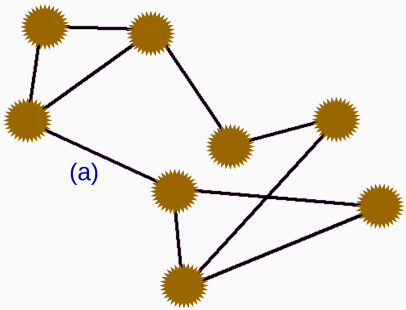

The verifier chooses an edge

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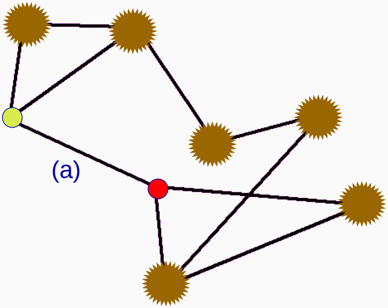

I open it

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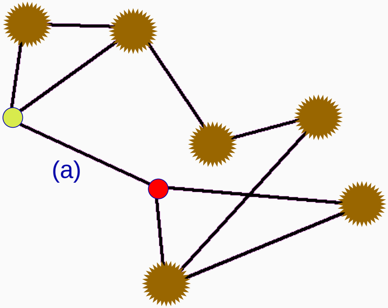

I open it

The verifier checks the validity: 2 different colors

Secure Multiple Proofs of Knowledge: Authentication

If there exists an efficient adversary,
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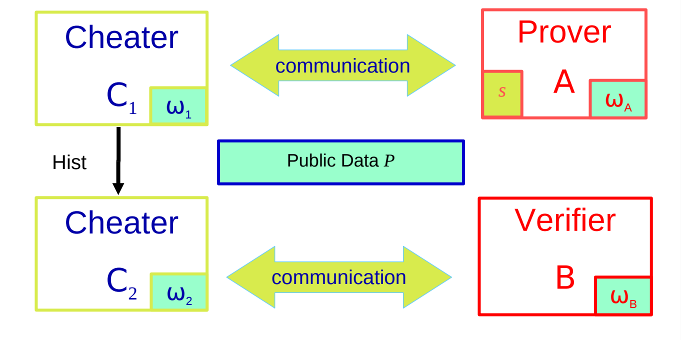

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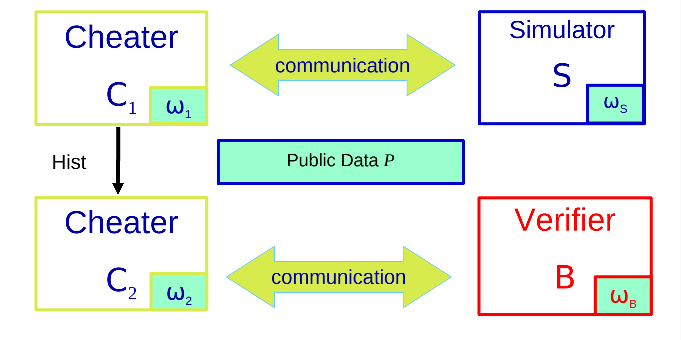

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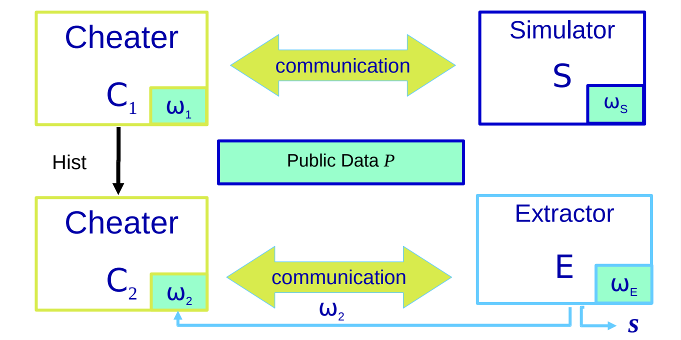

Schnorr Proofs

[Schnorr – Eurocrypt '89 - Crypto '89]

Zero-Knowledge Proof

- Setting: ($\mathbb{G} = \langle g \rangle$) of order q
  $\mathcal{P}$ knows x, such that $y = g^{-x}$
  and wants to prove it to $\mathcal{V}$
- $\mathcal{P}$ chooses $K \stackrel{R}{\leftarrow} \mathbb{Z}_q^\star$
  sets and sends $r = g^K$
- $\mathcal{V}$ chooses $h \stackrel{R}{\leftarrow} \{0,1\}^k$
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- $\mathcal{P}$ computes and sends
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Signature

- ($\mathbb{G} = \langle g \rangle$) of order q
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Signature

$\mathcal{H}$ viewed as a random oracle

- Key Generation $\rightarrow (y, x)$
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Generic Zero-Knowledge Proofs

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Zero-Knowledge Proof

- Proof of knowledge of x, such that $\mathcal{R}(x, y)$
- $\mathcal{P}$ builds a commitment r and sends it to $\mathcal{V}$
- $\mathcal{V}$ chooses a challenge $h \xleftarrow{R} \{0,1\}^k$ for $\mathcal{P}$
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$\mathcal{H}$ viewed as a random oracle

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Outline

Splitting Lemma

Idea

When a subset A is "large" in a product space $X \times Y$,
it has many "large" sections.

The Splitting Lemma

Let $A \subset X \times Y$ such that $\Pr[(x, y) \in A] \geq \varepsilon$. For any $\alpha < \varepsilon$, define

$$B_\alpha = \left\{ (x, y) \in X \times Y \mid \Pr_{y' \in Y}[(x, y') \in A] \geq \varepsilon - \alpha \right\}, \qquad \text{then}$$

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Splitting Lemma – Proof

(*i*) we argue by contradiction, using the notation $\bar{B}$ for the complement of B in $X \times Y$. Assume that $\Pr[B_\alpha] < \alpha$. Then,

$$\varepsilon \leq \Pr[B] \cdot \Pr[A \mid B] + \Pr[\bar{B}] \cdot \Pr[A \mid \bar{B}] < \alpha \cdot 1 + 1 \cdot (\varepsilon - \alpha) = \varepsilon.$$

(*ii*) straightforward.

(*iii*) using Bayes' law:

$$\begin{aligned} \Pr[B \mid A] &= 1 - \Pr[\bar{B} \mid A] \\ &= 1 - \Pr[A \mid \bar{B}] \cdot \Pr[\bar{B}] / \Pr[A] \geq 1 - (\varepsilon - \alpha)/\varepsilon = \alpha/\varepsilon. \end{aligned}$$

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Theorem (The Forking Lemma)

Let $(\mathcal{K}, \mathcal{S}, \mathcal{V})$ be a digital signature scheme with security parameter k, with a signature as above, of the form (m, r, h, s), where $h = \mathcal{H}(m, r)$ and s depends on r and h only.

Let $\mathcal{A}$ be a probabilistic polynomial time Turing machine whose input only consists of public data and which can ask q_H queries to the random oracle, with $q_H > 0$.

We assume that, within the time bound T, $\mathcal{A}$ produces, with probability $\varepsilon \geq 7q_H/2^k$, a valid signature (m, r, h, s).

Then, within time $T' \leq 16q_H T/\varepsilon$, and with probability $\varepsilon' \geq 1/9$, a replay of this machine outputs two valid signatures (m, r, h, s) and (m, r, h', s') such that $h \neq h'$.

Forking Lemma – Proof

- $\mathcal{A}$ is a PPTM with random tape ω.
- During the attack, $\mathcal{A}$ asks a polynomial number of queries to $\mathcal{H}$.
- We may assume that these questions are distinct:
  - $\mathcal{Q}_1, \ldots, \mathcal{Q}_{q_H}$ are the q_H distinct questions
  - and let $H = (h_1, \ldots, h_{q_H})$ be the list of the q_H answers of $\mathcal{H}$.

  Note: a random choice of $\mathcal{H}$ = a random choice of H.
- For a random choice of $(\omega, \mathcal{H})$, with probability ε, $\mathcal{A}$ outputs a valid signature (m, r, h, s).
- Since $\mathcal{H}$ is a random oracle, the probability for h to be equal to $\mathcal{H}(m, r)$ is less than $1/2^k$, unless it has been asked during the attack.

Accordingly, we define $Ind_{\mathcal{H}}(\omega)$ to be the index of this question:
$(m, r) = \mathcal{Q}_{Ind_{\mathcal{H}}(\omega)}$ $\quad$ ($Ind_{\mathcal{H}}(\omega) = \infty$ if the question is never asked).

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We then define the sets

$$\begin{aligned}\mathcal{S} &= \left\{(\omega, \mathcal{H}) \mid \mathcal{A}^{\mathcal{H}}(\omega) \text{ succeeds \& } \mathit{Ind}_{\mathcal{H}}(\omega) \neq \infty\right\}, \\ \mathcal{S}_i &= \left\{(\omega, \mathcal{H}) \mid \mathcal{A}^{\mathcal{H}}(\omega) \text{ succeeds \& } \mathit{Ind}_{\mathcal{H}}(\omega) = i\right\} \quad i \in \{1, \ldots, q_H\}.\end{aligned}$$

Note: the set $\{\mathcal{S}_i\}$ is a partition of $\mathcal{S}$.

$$\nu = \Pr[\mathcal{S}] \geq \varepsilon - 1/2^k.$$

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$$\begin{array}{rcl} \mathcal{S} & = & \left\{(\omega, \mathcal{H}) \,|\, \mathcal{A}^{\mathcal{H}}(\omega) \text{ succeeds \& } \mathit{Ind}_{\mathcal{H}}(\omega) \neq \infty\right\}, \\ \mathcal{S}_i & = & \left\{(\omega, \mathcal{H}) \,|\, \mathcal{A}^{\mathcal{H}}(\omega) \text{ succeeds \& } \mathit{Ind}_{\mathcal{H}}(\omega) = i\right\} \quad i \in \{1, \ldots, q_H\}. \end{array}$$

Note: the set $\{\mathcal{S}_i\}$ is a partition of $\mathcal{S}$.

$$\nu = \Pr[\mathcal{S}] \geq \varepsilon - 1/2^k.$$

Since $\varepsilon \geq 7q_H/2^k \geq 7/2^k$, then

$$\nu \geq 6\varepsilon/7.$$

Forking Lemma – Proof

Let I be the set consisting of the most likely indices i,

$$I = \{i \mid \Pr[\mathcal{S}_i \mid \mathcal{S}] \geq 1/2q_H\}.$$

Lemma

$$\Pr[Ind_{\mathcal{H}}(\omega) \in I \mid \mathcal{S}] \geq \frac{1}{2}.$$

By definition of $\mathcal{S}_i$,

$$\Pr[Ind_{\mathcal{H}}(\omega) \in I \mid \mathcal{S}] = \sum_{i \in I} \Pr[\mathcal{S}_i \mid \mathcal{S}] = 1 - \sum_{i \notin I} \Pr[\mathcal{S}_i \mid \mathcal{S}].$$

Since the complement of I contains fewer than q_H elements,

$$\sum_{i \notin I} \Pr[\mathcal{S}_i \mid \mathcal{S}] \leq q_H \times 1/2q_H \leq 1/2.$$

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Forking Lemma – Proof

- Run $2/\varepsilon$ times $\mathcal{A}$, with independent random ω and random $\mathcal{H}$. Since $\nu = \Pr[\mathcal{S}] \geq 6\varepsilon/7$, with probability greater than $1 - (1-\nu)^{2/\varepsilon} \geq 4/5$, we get at least one pair $(\omega, \mathcal{H})$ in $\mathcal{S}$.
- Apply the Splitting Lemma, with $\varepsilon = \nu/2q_h$ and $\alpha = \varepsilon/2$, for $i \in I$. We denote by $\mathcal{H}_{|i}$ the restriction of $\mathcal{H}$ to queries of index $< i$. Since $\Pr[\mathcal{S}_i] \geq \nu/2q_h$, there exists a subset Ω_i such that

$$\forall(\omega,\mathcal{H}) \in \Omega_i, \quad \Pr_{\mathcal{H}'}[(\omega,\mathcal{H}') \in \mathcal{S} \mid \mathcal{H}'_{|i} = \mathcal{H}_{|i}] \geq \frac{\nu}{4q_h}$$

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Since all the subsets $\mathcal{S}_i$ are disjoint,

$$\Pr_{\omega,\mathcal{H}}[(\exists i \in I)\ (\omega,\mathcal{H}) \in \Omega_i \cap \mathcal{S}_i \,|\, \mathcal{S}]$$

$$\begin{aligned} &= \Pr\left[\bigcup_{i\in I}(\Omega_i \cap \mathcal{S}_i) \,|\, \mathcal{S}\right] = \sum_{i\in I} \Pr[\Omega_i \cap \mathcal{S}_i \,|\, \mathcal{S}] \\ &= \sum_{i\in I} \Pr[\Omega_i \,|\, \mathcal{S}_i] \cdot \Pr[\mathcal{S}_i \,|\, \mathcal{S}] \geq \left(\sum_{i\in I} \Pr[\mathcal{S}_i \,|\, \mathcal{S}]\right)/2 \geq \frac{1}{4}. \end{aligned}$$

Let β denote the index $Ind_{\mathcal{H}}(\omega)$ of to the successful pair.

With prob. at least $1/4$, $\beta \in I$ and $(\omega,\mathcal{H}) \in \mathcal{S}_\beta \cap \Omega_\beta$.

With prob. greater than $4/5 \times 1/4 = 1/5$, the $2/\varepsilon$ attacks provided a successful pair $(\omega,\mathcal{H})$, with $\beta = Ind_{\mathcal{H}}(\omega) \in I$ and $(\omega,\mathcal{H}) \in \mathcal{S}_\beta$.

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Using the assumption that $\varepsilon \geq 7q_H/2^k$, the above prob. is $\geq \varepsilon/14q_H$.

Replay the attack $14q_H/\varepsilon$ times with a new random oracle $\mathcal{H}'$ such that $\mathcal{H}'_{|\beta} = \mathcal{H}_{|\beta}$, and get another success with probability greater than

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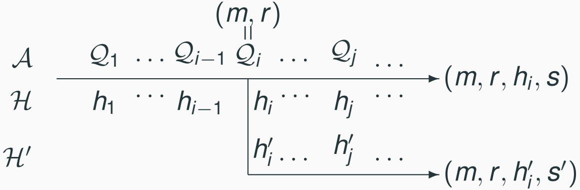

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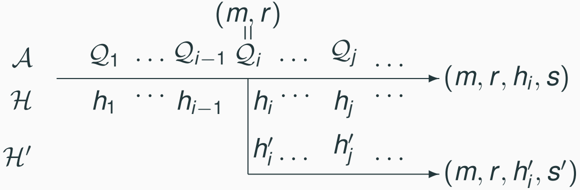

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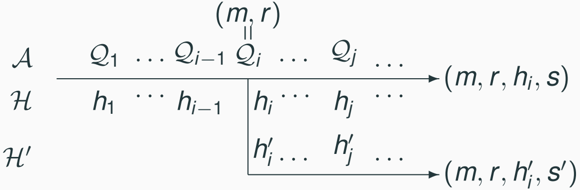

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In order to answer signing queries, one simply uses the simulator of the zero-knowledge proof: (r, h, s), and we set $\mathcal{H}(m, r) \leftarrow h$.

The random oracle programming may fail, but with negligible probability.

Conclusion

Outline

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- Based on Pairings
- etc

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