I – Basic Notions

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Outline

Cryptography

Provable Security

Basic Security Notions

Conclusion

Cryptography

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Introduction

Kerckhoffs' Principles

Formal Notations

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Secrecy of Communications

One ever wanted to communicate secretly

Alice

Bob

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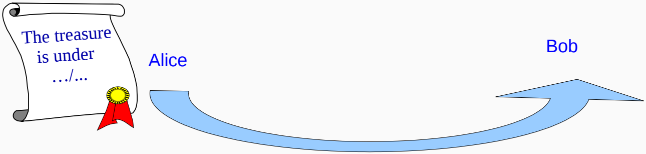

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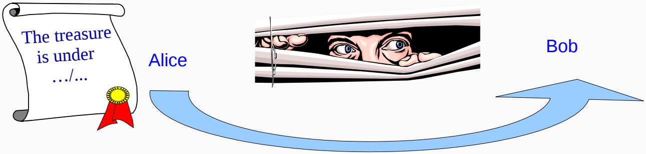

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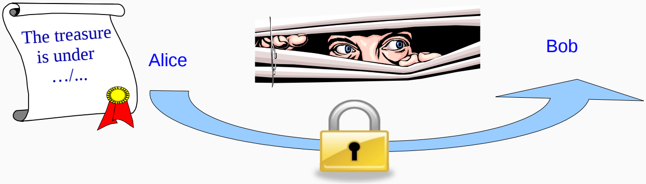

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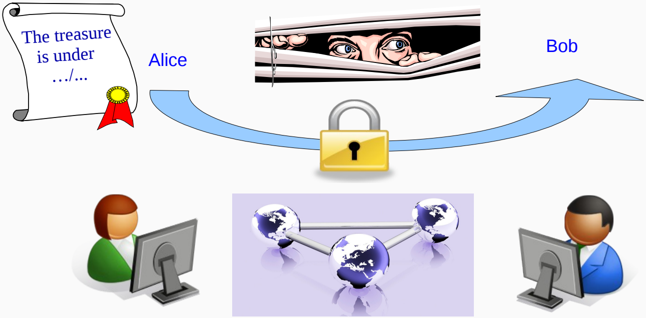

With the all-digital world, security needs are even stronger

Old Methods

Substitutions and permutations

Security relies on
the secrecy of the mechanism

Scytale - Permutation

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Alberti's disk
Mono-alphabetical Substitution

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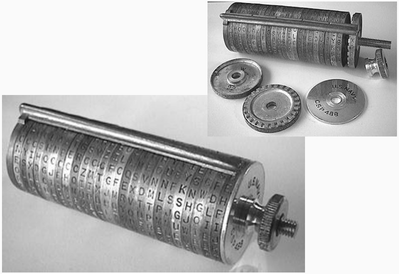

Wheel – M 94 (CSP 488)

Poly-alphabetical Substitution

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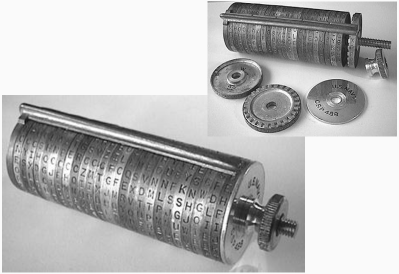

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Kerckhoffs' Principles (1)

La Cryptographie Militaire (1883)

Le système doit être matèriellement,
sinon mathématiquement, indéchiffrable

The system should be, if not theoretically unbreakable,
unbreakable in practice

$\longrightarrow$ If the security cannot be formally proven,
heuristics should provide some confidence.

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La clef doit pouvoir en être communiquée et retenue sans le secours de notes écrites, et être changée ou modifiée au gré des correspondants

The key should be rememberable without notes and should be easily changeable

$\longrightarrow$ The parameters specific to the users (the key) should be short

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Use of (Secret) Key

A shared information (secret key) between the sender and the receiver parameterizes the mechanism:

- Vigenère: each key letter tells the shift
- Enigma: connectors and rotors

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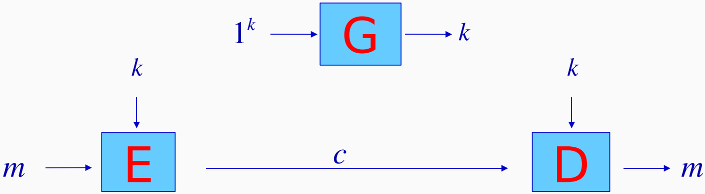
Security **looks** better: but broken (Alan Turing *et al.*)

Principles 2 and 3 define the concepts of symmetric cryptography:



Secrecy

It is impossible/hard to recover *m* from *c* only (without *k*)

Security

It is heuristic only: 1st principle

Symmetric Encryption

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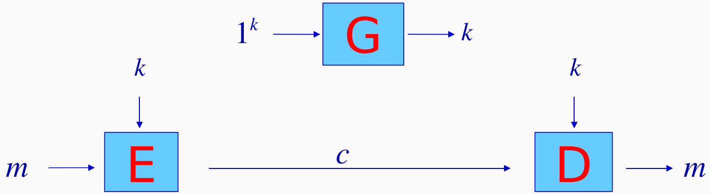

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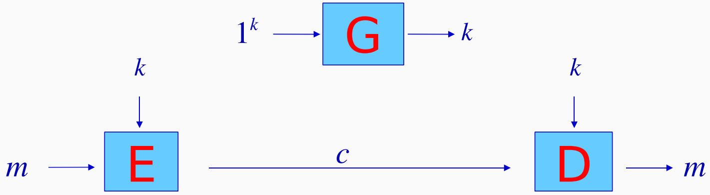

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Perfect Secrecy?

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Perfect Secrecy

The ciphertext does not reveal any (additional) information about the plaintext: no more than known before

- a priori information about the plaintext,
  defined by the distribution probability of the plaintext
- a posteriori information about the plaintext,
  defined by the distribution probability of the plaintext,
  given the ciphertext

Both distributions should be perfectly identical

One-Time Pad Encryption

Vernam's Cipher (1929)

- Encryption of $m \in \{0,1\}^n$ under the key $k \in \{0,1\}^n$:

$$
\begin{array}{rcl}
m = & \boxed{1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1} & \text{plaintext} \\
 & \oplus & \text{XOR (+ modulo 2)} \\
k = & \boxed{1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0} & \text{key = random mask} \\
 & = & \\
c = & \boxed{0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1} & \text{ciphertext}
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- Decryption of $c \in \{0,1\}^n$ under the key $k \in \{0,1\}^n$:

$$c \oplus k = (m \oplus k) \oplus k = m \oplus (k \oplus k) = m$$

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For any candidate $m \in \{0,1\}^n$, the key $k = c \oplus m$ would lead to c

$\Rightarrow$ no information about m is leaked with c!

Drawbacks

- The key must be as long as the plaintext
- This key must be used once only (one-time pad)

Theorem (Shannon – 1949)

To achieve perfect secrecy, A and B have to share a common string truly random and as long as the whole communication.

Thus, the above one-time pad technique is optimal…

Information Theory

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Perfect Secrecy vs. Practical Secrecy

- No information about the plaintext m is in the ciphertext c without the knowledge of the key k
  $\Rightarrow$ information theory
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Shannon also showed that combining appropriately permutations and substitutions can hide information: extracting information from the ciphertext is time consuming

Modern Symmetric Encryption: DES and AES

Combination of substitutions and permutations

DES (1977)
Data Encryption Standard

AES (2001)
Advanced Encryption Standard

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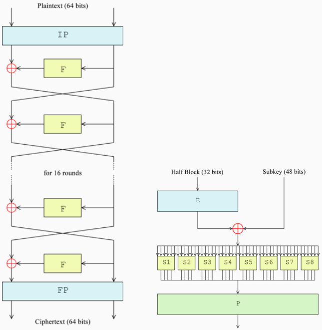

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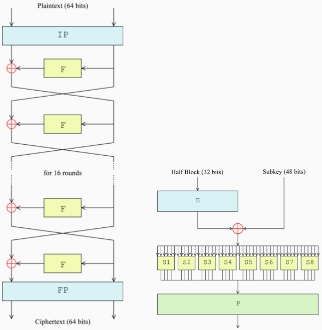

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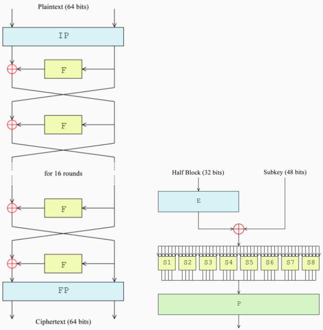

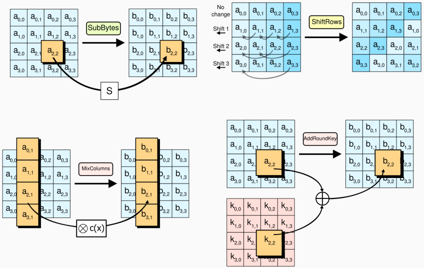

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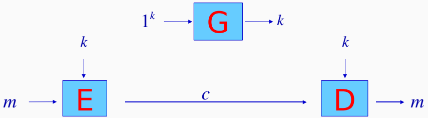
Symmetric Encryption – Secret Key Encryption

One secret key only shared by Alice and Bob: this is a common parameter for the encryption and the decryption algorithms

This secret key has a symmetric capability



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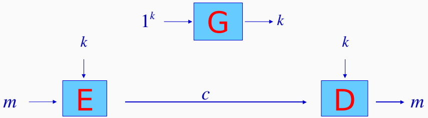

The secrecy of the key *k* guarantees the secrecy of communications but requires such a common secret key!

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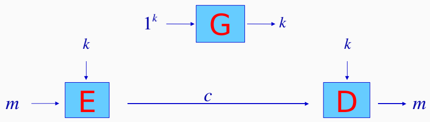

The secrecy of the key k guarantees the secrecy of communications but requires such a common secret key!

How can we establish such a common secret key?

Or, how to avoid it?

Asymmetric Encryption: Intuition

[Diffie-Hellman 1976]

Secrecy

- The recipient only should be able to open the message
- No requirement about the sender

Why would the sender need a secret key to encrypt a message?

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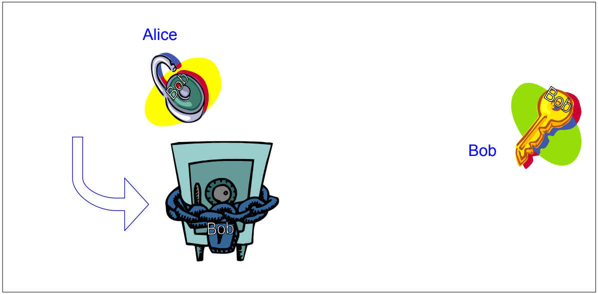

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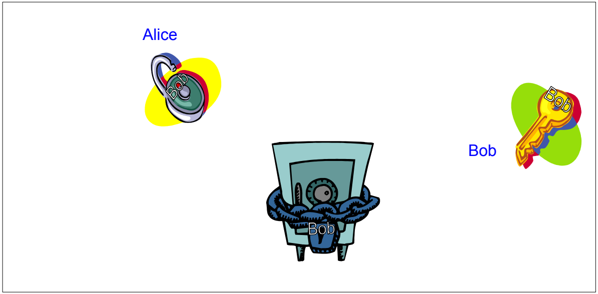

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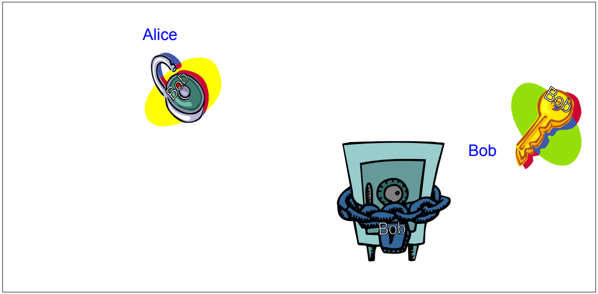

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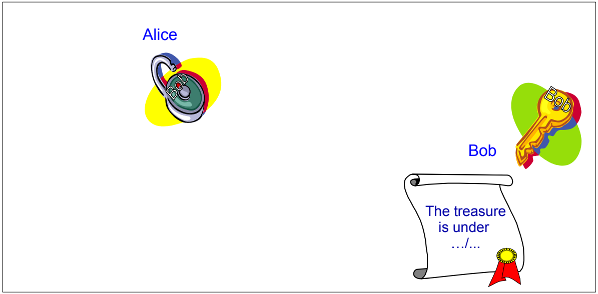

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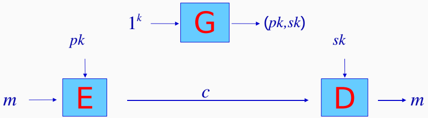
Public Key Cryptography – Diffie-Hellman (1976)

- Bob's public key is used by Alice as a parameter to encrypt a message to Bob
- Bob's private key is used by Bob as a parameter to decrypt ciphertexts

Asymmetric cryptography extends the 2nd principle:



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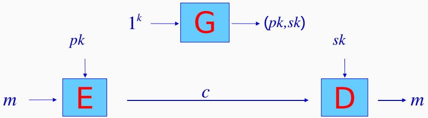

The secrecy of the private key *sk* guarantees the secrecy of communications

Provable Security

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Cryptography

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Definition

Computational Assumptions

Some Reductions

Basic Security Notions

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What is a Secure Cryptographic Scheme/Protocol?

- Symmetric encryption:
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- What does mean secrecy?
  $\rightarrow$ Security notions have to be formally defined
- How to guarantee above security claims for concrete schemes?
  $\rightarrow$ Provable security

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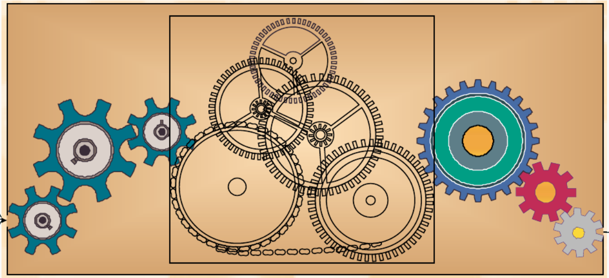

$\rightarrow$ solution

General Method

Computational Security Proofs

In order to prove the security of a cryptographic scheme/protocol, one needs

- a formal security model (security notions)
- acceptable computational assumptions (hard problems)
- a reduction: if one can break the security notions, then one can break the hard problem

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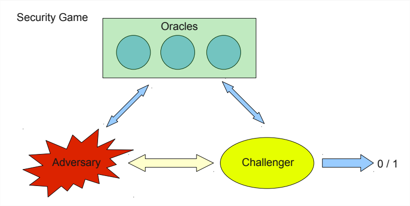

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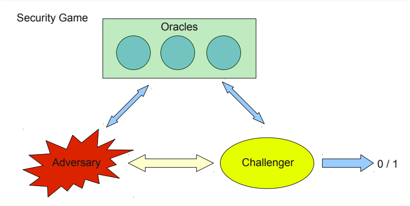
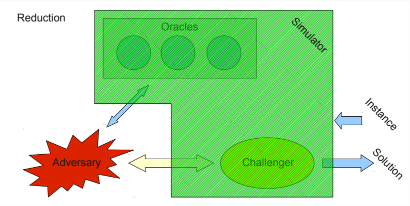
- acceptable computational assumptions (hard problems)
- a reduction: if one can break the security notions, then one can break the hard problem





Outline

Cryptography

Provable Security

Definition

Computational Assumptions

Some Reductions

Basic Security Notions

Conclusion

Integer Factoring
[Lenstra-Verheul 2000]

Integer Factoring

- Given $n = pq$
- Find p and q

Year	Required Complexity	n bitlength
before 2000	64	768
before 2010	80	1024
before 2020	112	2048
before 2030	128	3072
	192	7680
	256	15360

Note that the reduction may be lossy: extra bits are then required

Integer Factoring Records

Integer Factoring

- Given $n = pq$
- Find p and q

Digits	Date	Details
129	April 1994	Quadratic Sieve
130	April 1996	Algebraic Sieve
140	February 1999	
155	August 1999	512 bits
160	April 2003	
200	May 2005	
232	December 2009	768 bits

Integer Factoring Variants

RSA [Rivest-Shamir-Adleman 1978]

- Given $n = pq$, e and $y \in \mathbb{Z}_n^\star$
- Find x such that $y = x^e \bmod n$

Note that this problem is hard without the prime factors p and q, but becomes easy with them: if $d = e^{-1} \bmod \varphi(n)$, then $x = y^d \bmod n$

Flexible RSA [Baric-Pfitzmann and Fujisaki-Okamoto 1997]

- Given $n = pq$ and $y \in \mathbb{Z}_n^\star$
- Find x and $e > 1$ such that $y = x^e \bmod n$

Both problems are assumed as hard as integer factoring:
the prime factors are a trapdoor to find solutions

Integer Factoring Variants

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Discrete Logarithm

Discrete Logarithm Problem

- Given $\mathbb{G} = \langle g \rangle$ a cyclic group of order q, and $y \in \mathbb{G}$
- Find x such that $y = g^x$

Possible groups: $\mathbb{G} \in (\mathbb{Z}_p^\star, \times)$, or an elliptic curve

(Computational) Diffie Hellman Problem

- Given $\mathbb{G} = \langle g \rangle$ a cyclic group of order q, and $X = g^x$, $Y = g^y$
- Find $Z = g^{xy}$

The knowledge of x or y helps to solve this problem (trapdoor)

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Success Probabilities

For any computational problem P, we quantify the quality of an adversary $\mathcal{A}$ by its success probability in finding the solution:

$$\mathbf{Succ}^P(\mathcal{A}) = \Pr[\mathcal{A}(\text{instance}) \rightarrow \text{solution}].$$

We quantify the hardness of the problem by the success probability of the best adversary within time t: $\mathbf{Succ}(t) = \max_{|\mathcal{A}| \leq t}\{\mathbf{Succ}(\mathcal{A})\}$.

Note that the probability space can be restricted:
some inputs are fixed, and others only are randomly chosen.

Discrete Logarithm Problem

We usually fix the group $\mathbb{G} = \langle g \rangle$ of order q, and the generator g, but x is randomly chosen:

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Decisional Problem

(Decisional) Diffie Hellman Problem

- Given $\mathbb{G} = \langle g \rangle$ a cyclic group of order q, and $X = g^x$, $Y = g^y$, as well as a candidate $Z \in \mathbb{G}$
- Decide whether $Z = g^{xy}$

The adversary is called a distinguisher (outputs 1 bit).
A good distinguisher should behave in significantly different manners according to the input distribution:

$$\begin{aligned}\mathbf{Adv}_{\mathbb{G}}^{\mathsf{ddh}}(\mathcal{A}) = & \Pr[\mathcal{A}(X, Y, Z) = 1 | Z = g^{xy}] \\ & - \Pr[\mathcal{A}(X, Y, Z) = 1 | Z \stackrel{R}{\leftarrow} \mathbb{G}]\end{aligned}$$

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CDH $\leq$ DLP

Let $\mathcal{A}$ be an adversary against the **DLP** within time t, then we build an adversary $\mathcal{B}$ against the **CDH**: given X and Y, $\mathcal{B}$ runs $\mathcal{A}$ on X, that outputs x' (correct or not); then $\mathcal{B}$ outputs $Y^{x'}$.

The running time t' of $\mathcal{B}$ is the same as $\mathcal{A}$, plus one exponentiation:

$$\begin{aligned}\mathbf{Succ}_{\mathbb{G}}^{\mathbf{cdh}}(t') \geq \mathbf{Succ}_{\mathbb{G}}^{\mathbf{cdh}}(\mathcal{B}) &= \Pr[\mathcal{B}(X, Y) \to g^{xy} = Y^x] \\ &= \Pr[\mathcal{A}(X) \to x] = \mathbf{Succ}_{\mathbb{G}}^{\mathbf{dlp}}(\mathcal{A})\end{aligned}$$

Taking the maximum on the adversaries $\mathcal{A}$:

$$\mathbf{Succ}_{\mathbb{G}}^{\mathbf{cdh}}(t + \tau_{\mathsf{exp}}) \geq \mathbf{Succ}_{\mathbb{G}}^{\mathbf{dlp}}(t)$$

DDH $\leq$ CDH $\leq$ DLP

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DDH $\leq$ CDH

Let $\mathcal{A}$ be an adversary against the **CDH** within time t, we build an adversary $\mathcal{B}$ against the **DDH**: given X, Y and Z, $\mathcal{B}$ runs $\mathcal{A}$ on (X, Y), that outputs Z'; then $\mathcal{B}$ outputs 1 if $Z' = Z$ and 0 otherwise.

The running time of $\mathcal{B}$ is the same as $\mathcal{A}$: $\mathbf{Adv}_{\mathbb{G}}^{\mathsf{ddh}}(t)$ is greater than

$$\begin{aligned}
\mathbf{Adv}_{\mathbb{G}}^{\mathsf{ddh}}(\mathcal{B}) &= \Pr[\mathcal{B} \to 1 | Z = g^{xy}] - \Pr[\mathcal{B} \to 1 | Z \stackrel{R}{\leftarrow} \mathbb{G}] \\
&= \Pr[\mathcal{A}(X, Y) \to Z | Z = g^{xy}] - \Pr[\mathcal{A}(X, Y) \to Z | Z \stackrel{R}{\leftarrow} \mathbb{G}] \\
&= \Pr[\mathcal{A}(X, Y) \to g^{xy}] - \Pr[\mathcal{A}(X, Y) \to Z | Z \stackrel{R}{\leftarrow} \mathbb{G}] \\
&= \mathbf{Succ}_{\mathbb{G}}^{\mathsf{cdh}}(\mathcal{A}) - 1/q
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Taking the maximum on the adversaries $\mathcal{A}$:

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Distribution Indistinguishability

Indistinguishabilities

Let $\mathcal{D}_0$ and $\mathcal{D}_1$, two distributions on a finite set X:

- $\mathcal{D}_0$ and $\mathcal{D}_1$ are perfectly indistinguishable if

$$\mathbf{Dist}(\mathcal{D}_0, \mathcal{D}_1) = \sum_{x \in X} \left| \Pr_{a \in \mathcal{D}_1}[a = x] - \Pr_{a \in \mathcal{D}_0}[a = x] \right| = 0$$

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Distribution Indistinguishability

Computational Indistinguishability

Let $\mathcal{D}_0$ and $\mathcal{D}_1$, two distributions on a finite set X,

- a distinguisher $\mathcal{A}$ between $\mathcal{D}_0$ and $\mathcal{D}_1$ is characterized by its advantage

$$\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) = \Pr_{a\in\mathcal{D}_1}[\mathcal{A}(a) = 1] - \Pr_{a\in\mathcal{D}_0}[\mathcal{A}(a) = 1]$$

- the computational indistinguishability of $\mathcal{D}_0$ and $\mathcal{D}_1$ is measured by

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$$\begin{aligned}\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) &= \Pr_{a \in \mathcal{D}_1}[\mathcal{A}(a) = 1] - \Pr_{a \in \mathcal{D}_0}[\mathcal{A}(a) = 1] \\ &= \Pr[a \in \mathcal{D}_1 : \mathcal{A}(a) = 1] - \Pr[a \in \mathcal{D}_0 : \mathcal{A}(a) = 1] \\ &= 2 \times \Pr[a \in \mathcal{D}_b : \mathcal{A}(a) = b] - 1\end{aligned}$$

Computational Indistinguishability

$$\begin{aligned}\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) &= \Pr_{a\in\mathcal{D}_1}[\mathcal{A}(a)=1] - \Pr_{a\in\mathcal{D}_0}[\mathcal{A}(a)=1] \\ &= \Pr[a\in\mathcal{D}_1 : \mathcal{A}(a)=1] - \Pr[a\in\mathcal{D}_0 : \mathcal{A}(a)=1]\end{aligned}$$

Equivalent Notation

Let $\mathcal{D}_0$ and $\mathcal{D}_1$, two distributions on a finite set X,

$$\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) = 2 \times \Pr[a\in\mathcal{D}_b : \mathcal{A}(a)=b] - 1$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) = \Pr_{a\in\mathcal{D}_0}[\mathcal{A}(a)=1] - \Pr_{a\in\mathcal{D}_1}[\mathcal{A}(a)=1]$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\begin{aligned}
\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) &= \Pr_{a\in\mathcal{D}_0}[\mathcal{A}(a)=1] - \Pr_{a\in\mathcal{D}_1}[\mathcal{A}(a)=1] \\
&= \sum_{x\in X} \begin{pmatrix} \Pr_{a\in\mathcal{D}_0}[\mathcal{A}(a)=1 \wedge a=x] \\ -\Pr_{a\in\mathcal{D}_1}[\mathcal{A}(a)=1 \wedge a=x] \end{pmatrix}
\end{aligned}$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) \;=\; \sum_{x\in X}\begin{pmatrix} \Pr_{a\in\mathcal{D}_0}[\mathcal{A}(a)=1 \wedge a=x] \\ -\Pr_{a\in\mathcal{D}_1}[\mathcal{A}(a)=1 \wedge a=x]\end{pmatrix}$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\begin{aligned}
\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) &= \sum_{x \in X} \begin{pmatrix} \Pr_{a \in \mathcal{D}_0}[\mathcal{A}(a) = 1 \wedge a = x] \\ -\Pr_{a \in \mathcal{D}_1}[\mathcal{A}(a) = 1 \wedge a = x] \end{pmatrix} \\
&= \sum_{x \in X} \begin{pmatrix} \Pr_{a \in \mathcal{D}_0}[\mathcal{A}(x) = 1 \wedge a = x] \\ -\Pr_{a \in \mathcal{D}_1}[\mathcal{A}(x) = 1 \wedge a = x] \end{pmatrix}
\end{aligned}$$

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For any distinguisher $\mathcal{A}$, we have

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For any distinguisher $\mathcal{A}$, we have

$$\begin{aligned}\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) &= \sum_{x\in X}\begin{pmatrix}\Pr_{a\in\mathcal{D}_0}[\mathcal{A}(x)=1\wedge a=x]\\ -\Pr_{a\in\mathcal{D}_1}[\mathcal{A}(x)=1\wedge a=x]\end{pmatrix}\\ &= \sum_{x\in X}\Pr[\mathcal{A}(x)=1]\times\begin{pmatrix}\Pr_{a\in\mathcal{D}_0}[a=x]\\ -\Pr_{a\in\mathcal{D}_1}[a=x]\end{pmatrix}\end{aligned}$$

$a=x$ and $\mathcal{A}(x)=1$ are independent events

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) = \sum_{x \in X} \Pr[\mathcal{A}(x) = 1] \times \begin{pmatrix} \Pr_{a \in \mathcal{D}_0}[a = x] \\ -\Pr_{a \in \mathcal{D}_1}[a = x] \end{pmatrix}$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\begin{aligned}
\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) &= \sum_{x \in X} \Pr[\mathcal{A}(x) = 1] \times \begin{pmatrix} \Pr_{a \in \mathcal{D}_0}[a = x] \\ - \Pr_{a \in \mathcal{D}_1}[a = x] \end{pmatrix} \\
&\leq \sum_{x \in X} |\Pr[\mathcal{A}(x) = 1]| \times \begin{vmatrix} \Pr_{a \in \mathcal{D}_0}[a = x] \\ - \Pr_{a \in \mathcal{D}_1}[a = x] \end{vmatrix}
\end{aligned}$$

A better analysis could be done here

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) \leq \sum_{x \in X} |\Pr[\mathcal{A}(x) = 1]| \times \left| \begin{array}{c} \Pr_{a \in \mathcal{D}_0}[a = x] \\ -\Pr_{a \in \mathcal{D}_1}[a = x] \end{array} \right|$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\begin{aligned}
\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) &\leq \sum_{x \in X} |\Pr[\mathcal{A}(x) = 1]| \times \left| \begin{array}{c} \Pr_{a \in \mathcal{D}_0}[a = x] \\ - \Pr_{a \in \mathcal{D}_1}[a = x] \end{array} \right| \\
&\leq \sum_{x \in X} \left| \Pr_{a \in \mathcal{D}_0}[a = x] - \Pr_{a \in \mathcal{D}_1}[a = x] \right|
\end{aligned}$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) \;\leq\; \sum_{x \in X} \left| \Pr_{a \in \mathcal{D}_0}[a = x] - \Pr_{a \in \mathcal{D}_1}[a = x] \right|$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\begin{aligned}\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) &\leq \sum_{x\in X} \left| \Pr_{a\in\mathcal{D}_0}[a = x] - \Pr_{a\in\mathcal{D}_1}[a = x] \right| \\ &\leq \mathbf{Dist}(\mathcal{D}_0, \mathcal{D}_1)\end{aligned}$$

Relations between Indistinguishability Notions

For any distinguisher $\mathcal{A}$, we have

$$\mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(\mathcal{A}) \leq \textbf{Dist}(\mathcal{D}_0,\mathcal{D}_1)$$

Theorem

$\textbf{Dist}(\mathcal{D}_0,\mathcal{D}_1)$ *is the best advantage any adversary could get, even within an unbounded time.*

$$\forall t, \quad \mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(t) \leq \textbf{Dist}(\mathcal{D}_0,\mathcal{D}_1).$$

With a better analysis, we can even get

$$\forall t, \quad \mathbf{Adv}^{\mathcal{D}_0,\mathcal{D}_1}(t) \leq \frac{1}{2} \cdot \textbf{Dist}(\mathcal{D}_0,\mathcal{D}_1).$$

Hybrid Technique

Let us consider the distributions $\mathcal{D}_A$ and $\mathcal{D}_B$:

$$\mathcal{D}_A = (g^x, g^{y_1}, g^{xy_1}, \ldots, g^{y_n}, g^{xy_n}) \subseteq \mathbb{G}^{2n+1}$$
$$\mathcal{D}_B = (g^x, g^{y_1}, g^{z_1}, \ldots, g^{y_n}, g^{z_n}) \subseteq \mathbb{G}^{2n+1}$$

$$\mathbf{Adv}^{\mathcal{D}_A, \mathcal{D}_B}(t)?$$

We define the hybrid distribution

$$\mathcal{D}_i = (g^x, g^{y_1}, g^{xy_1}, \ldots, g^{y_i}, g^{xy_i}, g^{y_{i+1}}, g^{z_{i+1}}, \ldots, g^{y_n}, g^{z_n})$$

$$\mathcal{D}_0 = \mathcal{D}_B \qquad \mathcal{D}_n = \mathcal{D}_A.$$

Hybrid Technique

Let us consider the distributions $\mathcal{D}_A$ and $\mathcal{D}_B$:

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$$\mathcal{D}_B = (g^x, g^{y_1}, g^{z_1}, \ldots, g^{y_n}, g^{z_n}) \subseteq \mathbb{G}^{2n+1}$$

$$\mathbf{Adv}^{\mathcal{D}_A, \mathcal{D}_B}(t)?$$

We define the hybrid distribution

$$\mathcal{D}_i = (g^x, g^{y_1}, g^{xy_1}, \ldots, g^{y_i}, g^{xy_i}, g^{y_{i+1}}, g^{z_{i+1}}, \ldots, g^{y_n}, g^{z_n})$$

$$\mathcal{D}_0 = \mathcal{D}_B \qquad \mathcal{D}_n = \mathcal{D}_A.$$

Hybrid Technique

Let us consider the distributions $\mathcal{D}_A$ and $\mathcal{D}_B$:

$$\mathcal{D}_A = (g^x, g^{y_1}, g^{xy_1}, \ldots, g^{y_n}, g^{xy_n}) \subseteq \mathbb{G}^{2n+1}$$
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$$\mathbf{Adv}^{\mathcal{D}_A,\mathcal{D}_B}(t)?$$

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$$\mathcal{D}_0 = \mathcal{D}_B \qquad \mathcal{D}_n = \mathcal{D}_A.$$

Hybrid Technique

Let $\mathcal{A}$ be an adversary within time t, against $\mathcal{D}_A$ vs. $\mathcal{D}_B$.

Given a **DDH** input (X, Y, Z), we generate the hybrid instance:

$$\mathcal{I}_i = (X, g^{y_1}, X^{y_1}, \ldots, g^{y_{i-1}}, X^{y_{i-1}}, Y, Z, g^{y_{i+1}}, g^{z_{i+1}}, \ldots, g^{y_n}, g^{z_n})$$

Note that

$$\left.\begin{array}{l} \bullet \text{ if } Z = g^{xy}, \text{ then } \mathcal{I} \in \mathcal{D}_i \\ \bullet \text{ if } Z \xleftarrow{R} \mathbb{G}, \text{ then } \mathcal{I} \in \mathcal{D}_{i-1} \end{array}\right\} \quad \begin{array}{l} \mathbf{Adv}^{\mathcal{D}_i, \mathcal{D}_{i-1}}(\mathcal{A}) \leq \mathbf{Adv}^{\mathsf{ddh}}_{\mathbb{G}}(t') \\ \text{where } t' \leq t + 2(n-1)\tau_{\mathsf{exp}} \end{array}$$

$$\mathbf{Adv}^{\mathcal{D}_A, \mathcal{D}_B}(\mathcal{A}) \quad = \quad \mathbf{Adv}^{\mathcal{D}_n, \mathcal{D}_0}(\mathcal{A})$$

Hybrid Technique

Let $\mathcal{A}$ be an adversary within time t, against $\mathcal{D}_A$ vs. $\mathcal{D}_B$.

Given a **DDH** input (X, Y, Z), we generate the hybrid instance:

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$$\begin{aligned} \mathbf{Adv}^{\mathcal{D}_A,\mathcal{D}_B}(\mathcal{A}) &= \mathbf{Adv}^{\mathcal{D}_n,\mathcal{D}_0}(\mathcal{A}) \\ &= \left| \Pr_{\mathcal{D}_0}[\mathcal{A} \to 1] - \Pr_{\mathcal{D}_n}[\mathcal{A} \to 1] \right| \end{aligned}$$

Hybrid Technique

Let $\mathcal{A}$ be an adversary within time t, against $\mathcal{D}_A$ vs. $\mathcal{D}_B$.

Given a **DDH** input (X, Y, Z), we generate the hybrid instance:

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Hybrid Technique

Let $\mathcal{A}$ be an adversary within time t, against $\mathcal{D}_A$ vs. $\mathcal{D}_B$.

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$$\mathbf{Adv}^{\mathcal{D}_A, \mathcal{D}_B}(\mathcal{A}) = \sum_{i=1}^{n} \left| \Pr_{\mathcal{D}_{i-1}}[\mathcal{A} \to 1] - \Pr_{\mathcal{D}_i}[\mathcal{A} \to 1] \right|$$

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Hybrid Technique

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Hybrid Technique

Let $\mathcal{A}$ be an adversary within time t, against $\mathcal{D}_A$ vs. $\mathcal{D}_B$.

Given a **DDH** input (X, Y, Z), we generate the hybrid instance:

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$$\mathbf{Adv}^{\mathcal{D}_A,\mathcal{D}_B}(\mathcal{A}) \leq \sum_{i=1}^{n} \mathbf{Adv}^{\mathbf{ddh}}_{\mathbb{G}}(t')$$

Hybrid Technique

Let $\mathcal{A}$ be an adversary within time t, against $\mathcal{D}_A$ vs. $\mathcal{D}_B$.

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$$\begin{aligned} \mathbf{Adv}^{\mathcal{D}_A, \mathcal{D}_B}(\mathcal{A}) &\leq \sum_{i=1}^{n} \mathbf{Adv}^{\mathsf{ddh}}_{\mathbb{G}}(t') \\ &\leq n \times \mathbf{Adv}^{\mathsf{ddh}}_{\mathbb{G}}(t') \end{aligned}$$

Hybrid Technique

Let $\mathcal{A}$ be an adversary within time t, against $\mathcal{D}_A$ vs. $\mathcal{D}_B$.

Given a **DDH** input (X, Y, Z), we generate the hybrid instance:

$$\mathcal{I}_i = (X, g^{y_1}, X^{y_1}, \ldots, g^{y_{i-1}}, X^{y_{i-1}}, Y, Z, g^{y_{i+1}}, g^{z_{i+1}}, \ldots, g^{y_n}, g^{z_n})$$

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$$\mathbf{Adv}^{\mathcal{D}_A, \mathcal{D}_B}(\mathcal{A}) \quad \leq \quad n \times \mathbf{Adv}^{\mathbf{ddh}}_{\mathbb{G}}(t')$$

Let $\mathcal{A}$ be an adversary within time t, against $\mathcal{D}_A$ vs. $\mathcal{D}_B$.

Given a **DDH** input (X, Y, Z), we generate the hybrid instance:

$$\mathcal{I}_i = (X, g^{y_1}, X^{y_1}, \ldots, g^{y_{i-1}}, X^{y_{i-1}}, Y, Z, g^{y_{i+1}}, g^{z_{i+1}}, \ldots, g^{y_n}, g^{z_n})$$

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$$\mathbf{Adv}^{\mathcal{D}_A, \mathcal{D}_B}(\mathcal{A}) \quad \leq \quad n \times \mathbf{Adv}_{\mathbb{G}}^{\textbf{ddh}}(t')$$

Theorem

$$\forall t, \qquad \mathbf{Adv}^{\mathcal{D}_A, \mathcal{D}_B}(t) \leq n \times \mathbf{Adv}_{\mathbb{G}}^{\textbf{ddh}}(t + 2(n-1)\tau_{\text{exp}})$$

Basic Security Notions

Outline

Cryptography

Provable Security

Basic Security Notions

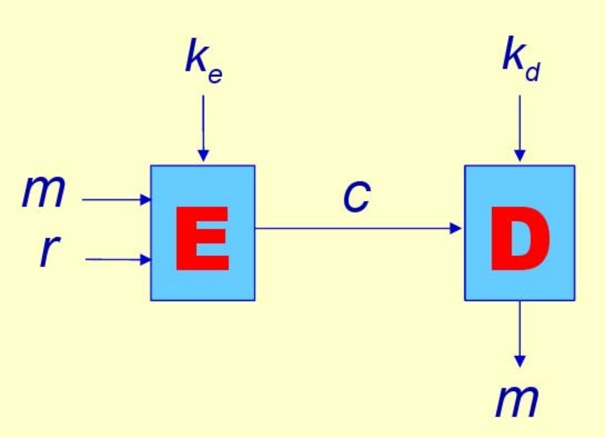
Public-Key Encryption

Variants of Indistinguishability

Signatures

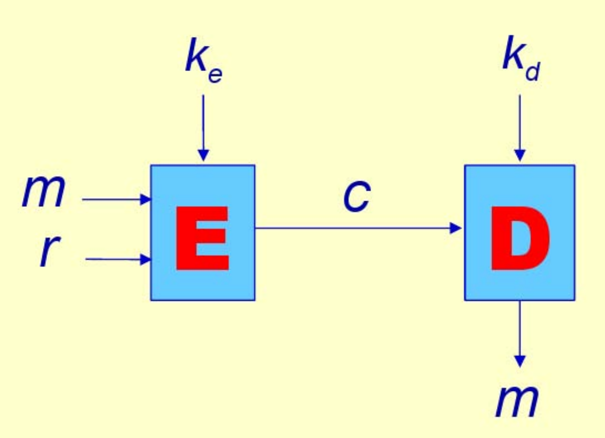
Conclusion

Public-Key Encryption



Goal: Privacy/Secrecy of the plaintext

Public-Key Encryption



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OW − CPA

One-Wayness

For a public-key encryption scheme $\mathcal{S} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$, without the secrete key *sk*, it should be computationally impossible to recover the plaintext *m* from the ciphertext *c*:

$$\mathbf{Succ}_{\mathcal{S}}^{\mathsf{ow}}(\mathcal{A}) = \Pr[(sk, pk) \leftarrow \mathcal{K}(); m \stackrel{R}{\leftarrow} \mathcal{M}; c = \mathcal{E}_{pk}(m) : \mathcal{A}(pk, c) \rightarrow m]$$

should be negligible.

Chosen-Plaintext Attacks

In the public-key setting, the adversary has access to the encryption key (the public key), and thus can encrypt any plaintext of its choice: chosen-plaintext attack

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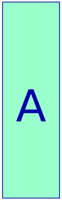
$\mathbf{OW} - \mathbf{CPA}$ Security Game



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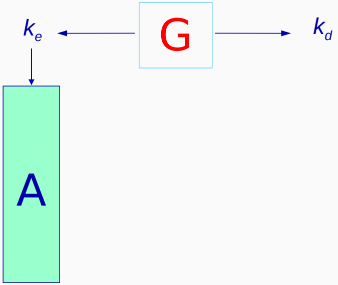
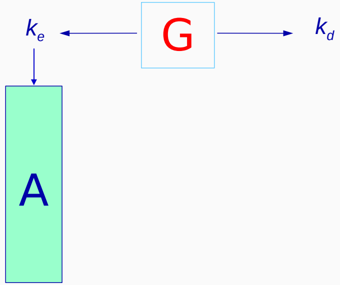


OW − CPA Security Game

m^* random
r^* random



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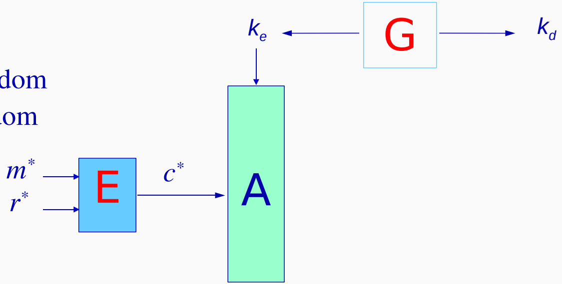

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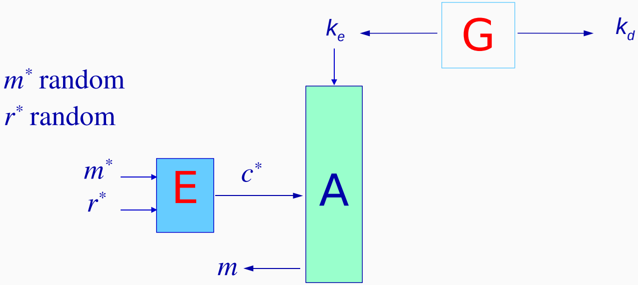

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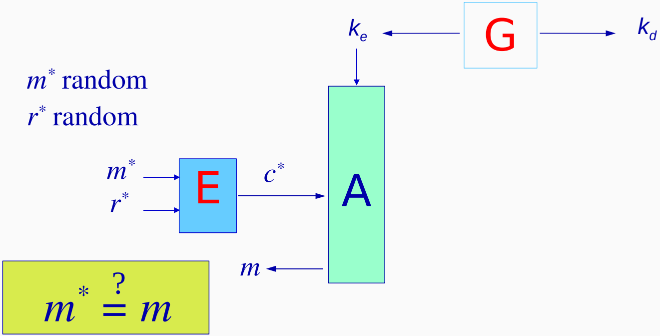

ElGamal Encryption

The ElGamal encryption scheme $\mathcal{EG}$ is defined,
in a group $\mathbb{G} = \langle g \rangle$ of order q

- $\mathcal{K}(\mathbb{G}, g, q)$: $x \stackrel{R}{\leftarrow} \mathbb{Z}_q$, and $sk \leftarrow x$ and $pk \leftarrow y = g^x$
- $\mathcal{E}_{pk}(m)$: $r \stackrel{R}{\leftarrow} \mathbb{Z}_q$, $c_1 \leftarrow g^r$ and $c_2 \leftarrow y^r \times m = pk^r \times m$.
  Then, the ciphertext is $c = (c_1, c_2)$
- $\mathcal{D}_{sk}(c)$ outputs $c_2/c_1^x = c_2/c_1^{sk}$

Theorem (ElGamal is $\mathrm{OW-CPA}$)

$$\mathbf{Succ}_{\mathcal{EG}}^{\mathsf{ow-cpa}}(t) \leq \mathbf{Succ}_{\mathbb{G}}^{\mathbf{cdh}}(t)$$

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Let $\mathcal{A}$ be an adversary against $\mathcal{EG}$, we build an adversary $\mathcal{B}$ against **CDH**: let us be given a **CDH** instance (X, Y)

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  $= \Pr[c_2/m = c_2/m^*] = \Pr[c_2/m = \mathbf{CDH}(X, Y)] \leq \mathbf{Succ}_{\mathbb{G}}^{\mathbf{cdh}}(t)$

Is $\mathrm{OW}-\mathrm{CPA}$ Enough?

For a yes/no answer or sell/buy order,
one bit of information may be enough for the adversary!

How to model that no bit of information leaks?

Semantic Security [Goldwasser-Micali 1984]

For any predicate f, $\mathcal{E}(m)$ does not help to guess $f(m)$, with better probability than $f(m')$ (for a random but private m'): in the game

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then,

$$\mathbf{Adv}^{\text{sem}}_{\mathcal{S}}(\mathcal{A}) = \left|\Pr[p = f(m)] - \Pr[p = f(m')]\right|.$$

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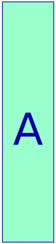

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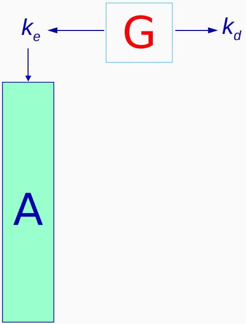

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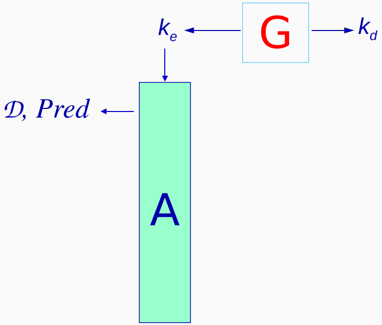
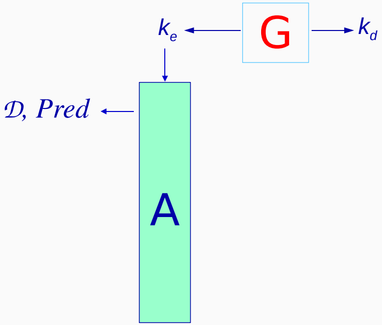


$m^*, m' \leftarrow \mathcal{D}$

r random



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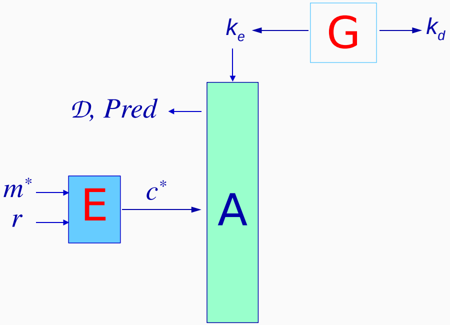

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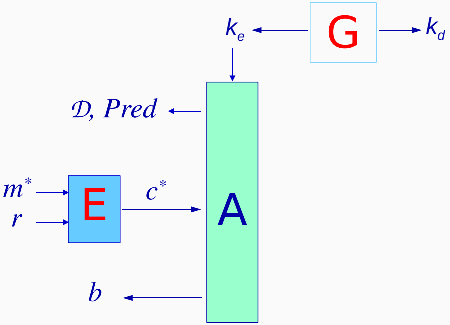

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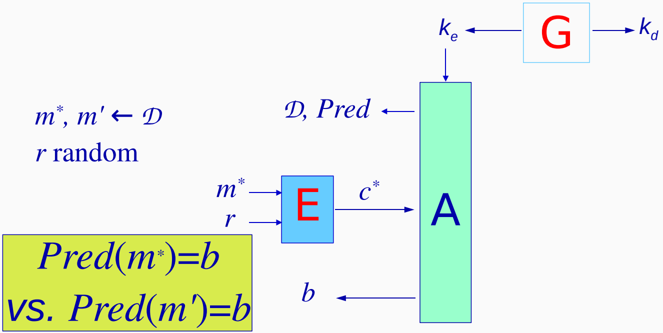


Another equivalent formulation (if efficiently computable predicate):

IND − CPA

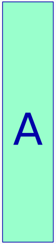
After having chosen two plaintexts m_0 and m_1, upon receiving the encryption of m_b (for a random bit b), it should be hard to guess which message has been encrypted: in the game

$$(sk, pk) \leftarrow \mathcal{K}(); (m_0, m_1, \text{state}) \leftarrow \mathcal{A}(pk);$$
$$b \stackrel{R}{\leftarrow} \{0,1\}; c = \mathcal{E}_{pk}(m_b); b' \leftarrow \mathcal{A}(\text{state}, c)$$

then,

$$\begin{aligned} \mathbf{Adv}^{\text{ind-cpa}}_{\mathcal{S}}(\mathcal{A}) &= \left|\Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0]\right| \\ &= \left|2 \times \Pr[b' = b] - 1\right| \end{aligned}$$

IND − CPA Security Game



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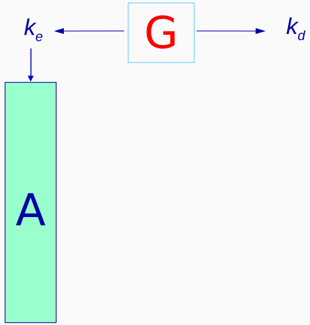

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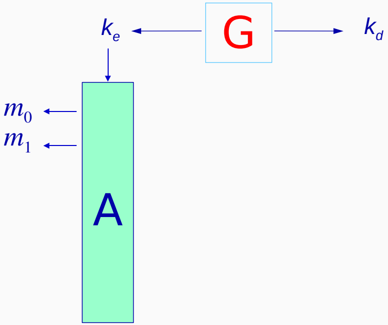

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$b \in \{0,1\}$
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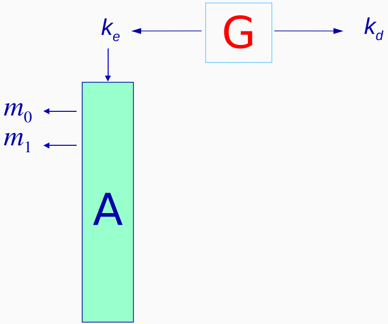

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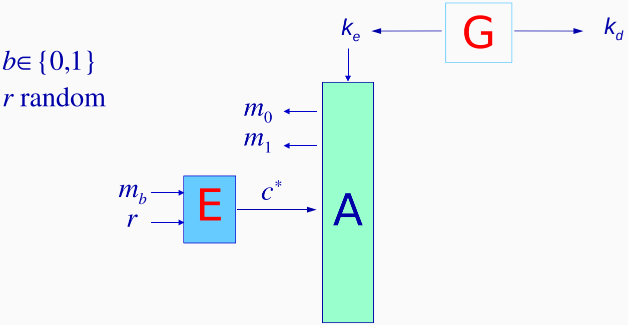

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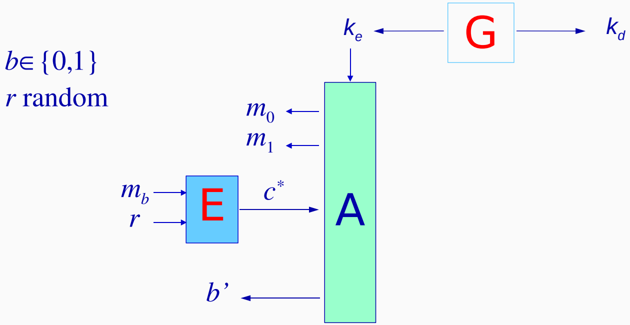

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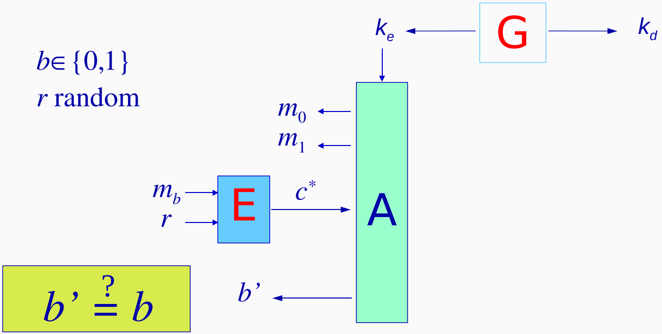

Indistinguishability implies Semantic Security

Let $\mathcal{A}$ be an adversary within time t against semantic security, we build an adversary $\mathcal{B}$ against indistinguishability:

- $\mathcal{B}$ runs $\mathcal{A}$ to get $\mathcal{D}$ and a predicate $\mathcal{P}$;
  it gets $m_0, m_1 \stackrel{R}{\leftarrow} \mathcal{D}$, and outputs them;
- the challenger encrypts m_b in c
- $\mathcal{B}$ runs $\mathcal{A}$, to get the guess p of $\mathcal{A}$ about the predicate $\mathcal{P}$ on the plaintext in c;
  - if $\mathcal{P}(m_0) = \mathcal{P}(m_1)$, $\mathcal{B}$ outputs a random bit b';
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Note that (if diff denotes the event that $\mathcal{P}(m) \neq \mathcal{P}(m'))$

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- $\mathcal{B}$ runs $\mathcal{A}$, to get the guess p of $\mathcal{A}$ about the predicate $\mathcal{P}$ on the plaintext in c;
  - If $\mathcal{P}(m_0) = \mathcal{P}(m_1)$, $\mathcal{B}$ outputs a random bit b',
  - otherwise it outputs b' such that $\mathcal{P}(m_{b'}) = p$.

Note that (if diff denotes the event that $\mathcal{P}(m) \neq \mathcal{P}(m')$)

$$\mathbf{Adv}^{\mathsf{sem}}(\mathcal{A}) = \left|\Pr[p = \mathcal{P}(m) | c = \mathcal{E}(m)] - \Pr[p = \mathcal{P}(m') | c = \mathcal{E}(m)]\right|$$

Indistinguishability implies Semantic Security

Let $\mathcal{A}$ be an adversary within time t against semantic security, we build an adversary $\mathcal{B}$ against indistinguishability:

- $\mathcal{B}$ runs $\mathcal{A}$ to get $\mathcal{D}$ and a predicate $\mathcal{P}$;
  it gets $m_0, m_1 \stackrel{R}{\leftarrow} \mathcal{D}$, and outputs them;
- the challenger encrypts m_b in c
- $\mathcal{B}$ runs $\mathcal{A}$, to get the guess p of $\mathcal{A}$ about the predicate $\mathcal{P}$ on the plaintext in c;
  - If $\mathcal{P}(m_0) = \mathcal{P}(m_1)$, $\mathcal{B}$ outputs a random bit b',
  - otherwise it outputs b' such that $\mathcal{P}(m_{b'}) = p$.

Note that (if diff denotes the event that $\mathcal{P}(m) \neq \mathcal{P}(m')$)

$$\begin{aligned}
\mathbf{Adv}^{\mathsf{sem}}(\mathcal{A}) &= \left|\Pr[p = \mathcal{P}(m) | c = \mathcal{E}(m)] - \Pr[p = \mathcal{P}(m') | c = \mathcal{E}(m)]\right| \\
&= \left|\begin{array}{l} \Pr[p = \mathcal{P}(m) | c = \mathcal{E}(m) \wedge \mathsf{diff}] \\ - \Pr[p = \mathcal{P}(m') | c = \mathcal{E}(m) \wedge \mathsf{diff}] \end{array}\right| \times \Pr[\mathsf{diff}]
\end{aligned}$$

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$$\mathbf{Adv}^{\mathsf{ind}}(\mathcal{B}) = \left|\Pr[b' = 1|b = 1] - \Pr[b' = 1|b = 0]\right|$$

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If diff denotes the event that $\mathcal{P}(m_0) \neq \mathcal{P}(m_1)$

$$
\begin{aligned}
\mathbf{Adv}^{\mathsf{ind}}(\mathcal{B}) &= \left| \Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0] \right| \\
&= \left| \begin{array}{l} \Pr[b' = 1 | b = 1 \wedge \mathsf{diff}] \\ - \Pr[b' = 1 | b = 0 \wedge \mathsf{diff}] \end{array} \right| \times \Pr[\mathsf{diff}]
\end{aligned}
$$

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&= \left| \begin{array}{l} \Pr[\mathcal{P}(m_1) = p | c = \mathcal{E}(m_1) \wedge \mathsf{diff}] \\ - \Pr[\mathcal{P}(m_1) = p | c = \mathcal{E}(m_0) \wedge \mathsf{diff}] \end{array} \right| \times \Pr[\mathsf{diff}]
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$$

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$$
\begin{array}{rcl}
\mathbf{Adv}^{\text{ind}}(\mathcal{B}) & = & \left|\Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0]\right| \\
& = & \left| \begin{array}{l} \Pr[b' = 1 | b = 1 \wedge \text{diff}] \\ - \Pr[b' = 1 | b = 0 \wedge \text{diff}] \end{array} \right| \times \Pr[\text{diff}] \\
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& = & \mathbf{Adv}^{\text{sem}}(\mathcal{A}) \leq \mathbf{Adv}^{\text{ind}}(t')
\end{array}
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&= \left|\begin{array}{l} \Pr[\mathcal{P}(m_1) = p|c = \mathcal{E}(m_1) \wedge \mathsf{diff}] \\ - \Pr[\mathcal{P}(m_1) = p|c = \mathcal{E}(m_0) \wedge \mathsf{diff}] \end{array}\right| \times \Pr[\mathsf{diff}] \\
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&= \mathbf{Adv}^{\mathsf{sem}}(\mathcal{A}) \leq \mathbf{Adv}^{\mathsf{ind}}(t')
\end{aligned}$$

The running time t' of $\mathcal{B}$ = one execution of $\mathcal{A}$ (time t), two sampling from $\mathcal{D}$ (time τ_D), two evaluations of the predicate $\mathcal{P}$ (time $\tau_{\mathcal{P}}$)

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&= \left| \begin{array}{l} \Pr[\mathcal{P}(m_1) = p | c = \mathcal{E}(m_1) \wedge \mathsf{diff}] \\ - \Pr[\mathcal{P}(m_0) = p | c = \mathcal{E}(m_1) \wedge \mathsf{diff}] \end{array} \right| \times \Pr[\mathsf{diff}] \\
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\end{aligned}$$

The running time t' of $\mathcal{B}$ = one execution of $\mathcal{A}$ (time t), two sampling from $\mathcal{D}$ (time τ_D), two evaluations of the predicate $\mathcal{P}$ (time $\tau_{\mathcal{P}}$)

$$\mathbf{Adv}^{\mathsf{sem}}(t) \leq \mathbf{Adv}^{\mathsf{ind}}(t + 2\tau_{\mathcal{D}} + 2\tau_{\mathcal{P}})$$

Semantic Security implies Indistinguishability

Let $\mathcal{A}$ be an adversary within time t against indistinguishability, we build an adversary $\mathcal{B}$ against semantic security:

- $\mathcal{B}$ runs $\mathcal{A}$ to get (m_0, m_1);
  it sets $\mathcal{D} = \{m_0, m_1\}$, and $\mathcal{P}(m) = (m \stackrel{?}{=} m_1)$;
- the challenger chooses $m, m' \stackrel{R}{\leftarrow} \mathcal{D}$, and encrypts m in c
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$$\mathbf{Adv}^{\mathsf{sem}}(\mathcal{B}) \quad = \quad \left|\Pr[p = \mathcal{P}(m)] - \Pr[p = \mathcal{P}(m')]\right|$$

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$$\begin{aligned} \mathbf{Adv}^{\mathsf{sem}}(\mathcal{B}) &= \left|\Pr[p = \mathcal{P}(m)] - \Pr[p = \mathcal{P}(m')]\right| \\ &= \left|\Pr[m = m_p] - \Pr[m' = m_p]\right| \end{aligned}$$

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&= \left|\Pr[m = m_{b'}] - \Pr[m' = m_{b'}]\right|
\end{aligned}$$

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&= \left|\Pr[m = m_{b'}] - \Pr[m' = m_{b'}]\right| \\
\mathbf{Adv}^{\text{ind}}(\mathcal{A}) &= \left|\Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0]\right| \\
&\quad \text{where } m = m_b
\end{aligned}$$

Semantic Security implies Indistinguishability

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Semantic Security implies Indistinguishability

$$\begin{aligned}\mathbf{Adv}^{\mathsf{sem}}(\mathcal{B}) &= \left|\Pr[m = m_{b'}] - \Pr[m' = m_{b'}]\right| \\ &= |\Pr[m_b = m_{b'}] - \Pr[m_d = m_{b'}]| \\ &\quad \text{where } m = m_b \text{ and } m' = m_d\end{aligned}$$

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&= \left|\Pr[b = b'] - \Pr[d = b']\right|
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&\quad \text{where } m = m_b \text{ and } m' = m_d \\
&= \left|\Pr[b = b'] - \Pr[d = b']\right| \\
&= \left|\Pr[b = b'] - 1/2\right|
\end{aligned}$$

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&= \Big|\Pr[b = b'] - \Pr[d = b']\Big| \\
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&= \mathbf{Adv}^{\mathsf{ind}}(\mathcal{A})/2 \leq \mathbf{Adv}^{\mathsf{sem}}(t')
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The running time t' of $\mathcal{B}$ = one execution of $\mathcal{A}$ (time t)

$$\mathbf{Adv}^{\mathsf{ind}}(t) \leq 2 \times \mathbf{Adv}^{\mathsf{sem}}(t)$$

ElGamal Encryption

The ElGamal encryption scheme $\mathcal{EG}$ is defined,
in a group $\mathbb{G} = \langle g \rangle$ of order q

- $\mathcal{K}(\mathbb{G}, g, q)$: $x \overset{R}{\leftarrow} \mathbb{Z}_q$, and $sk \leftarrow x$ and $pk \leftarrow y = g^x$
- $\mathcal{E}_{pk}(m)$: $r \overset{R}{\leftarrow} \mathbb{Z}_q$, $c_1 \leftarrow g^r$ and $c_2 \leftarrow y^r \times m = pk^r \times m$.
  Then, the ciphertext is $c = (c_1, c_2)$
- $\mathcal{D}_{sk}(c)$ outputs $c_2 / c_1^x = c_2 / c_1^{sk}$

Theorem (ElGamal is $\mathrm{IND-CPA}$)

$$\mathbf{Adv}_{\mathcal{EG}}^{\mathsf{ind-cpa}}(t) \leq 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t)$$

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$$\mathbf{Adv}_{\mathcal{EG}}^{\mathsf{ind-cpa}}(t) \leq 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t)$$

ElGamal is $\mathrm{IND}-\mathrm{CPA}$: Proof

Let $\mathcal{A}$ be an adversary against $\mathcal{EG}$, we build an adversary $\mathcal{B}$ against **DDH**: let us be given a **DDH** instance (X, Y, Z)

- $\mathcal{A}$ gets $pk \leftarrow y = g^x$ from $\mathcal{K}$, and outputs (m_0, m_1)

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- $\mathcal{A}$ gets $pk \leftarrow y = g^x$ from $\mathcal{K}$, and outputs (m_0, m_1)
- The challenger chooses $r^* \stackrel{R}{\leftarrow} \mathbb{Z}_q^\star$ and sets $c_1 \leftarrow g^{r^*}$

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- The challenger chooses $b \stackrel{R}{\leftarrow} \{0, 1\}$, sets $c_2 \leftarrow y^{r^*} \times m_b$, and sends $c = (c_1, c_2)$

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- The challenger chooses $b \stackrel{R}{\leftarrow} \{0,1\}$, sets $c_2 \leftarrow y^{r^*} \times m_b$, and sends $c = (c_1, c_2)$
- $\mathcal{A}$ receives $c \leftarrow (c_1, c_2)$, and outputs b'

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- The challenger chooses $b \stackrel{R}{\leftarrow} \{0, 1\}$, sets $c_2 \leftarrow y^{r^*} \times m_b$, and sends $c = (c_1, c_2)$
- $\mathcal{A}$ receives $c \leftarrow (c_1, c_2)$, and outputs b'
- $2 \times \Pr[b' = b] - 1 = \mathbf{Adv}_{\mathcal{EG}}^{\mathrm{ind-cpa}}(\mathcal{A})$

ElGamal is $\mathbf{IND-CPA}$: Proof

Let $\mathcal{A}$ be an adversary against $\mathcal{EG}$, we build an adversary $\mathcal{B}$ against **DDH**: let us be given a **DDH** instance (X, Y, Z)

- $\mathcal{A}$ gets $pk \leftarrow X$ from $\mathcal{B}$, and outputs (m_0, m_1)
- The challenger chooses $r^* \stackrel{R}{\leftarrow} \mathbb{Z}_q^\star$ and sets $c_1 \leftarrow g^{r^*}$
- The challenger chooses $b \stackrel{R}{\leftarrow} \{0, 1\}$, sets $c_2 \leftarrow y^{r^*} \times m_b$, and sends $c = (c_1, c_2)$
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- $\mathcal{B}$ sets $c_1 \leftarrow Y$
- The challenger chooses $b \stackrel{R}{\leftarrow} \{0, 1\}$, sets $c_2 \leftarrow y^{r^*} \times m_b$, and sends $c = (c_1, c_2)$
- $\mathcal{A}$ receives $c \leftarrow (c_1, c_2)$, and outputs b'
- $2 \times \Pr[b' = b] - 1 = \mathbf{Adv}^{\mathsf{ind-cpa}}_{\mathcal{EG}}(\mathcal{A})$

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- $\mathcal{B}$ sets $c_1 \leftarrow Y$
- $\mathcal{B}$ chooses $b \stackrel{R}{\leftarrow} \{0, 1\}$, sets $c_2 \leftarrow Z \times m_b$,
  and sends $c = (c_1, c_2)$
- $\mathcal{A}$ receives $c \leftarrow (c_1, c_2)$, and outputs b'
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- $\mathcal{B}$ sets $c_1 \leftarrow Y$
- $\mathcal{B}$ chooses $b \stackrel{R}{\leftarrow} \{0, 1\}$, sets $c_2 \leftarrow Z \times m_b$, and sends $c = (c_1, c_2)$
- $\mathcal{B}$ receives b' from $\mathcal{A}$ and outputs $d = (b' = b)$
- $2 \times \Pr[b' = b] - 1 = \mathbf{Adv}^{\text{ind}-\text{cpa}}_{\mathcal{EG}}(\mathcal{A})$

ElGamal is $\mathrm{IND-CPA}$: Proof

Let $\mathcal{A}$ be an adversary against $\mathcal{EG}$, we build an adversary $\mathcal{B}$ against **DDH**: let us be given a **DDH** instance (X, Y, Z)

- $\mathcal{A}$ gets $pk \leftarrow X$ from $\mathcal{B}$, and outputs (m_0, m_1)
- $\mathcal{B}$ sets $c_1 \leftarrow Y$
- $\mathcal{B}$ chooses $b \stackrel{R}{\leftarrow} \{0, 1\}$, sets $c_2 \leftarrow Z \times m_b$, and sends $c = (c_1, c_2)$
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  $= 0$, otherwise

ElGamal is $\mathrm{IND}-\mathrm{CPA}$: Proof

As a consequence,

- $|2 \times \Pr[b' = b | Z = \mathbf{CDH}(X, Y)] - 1| = \mathbf{Adv}_{\mathcal{EG}}^{\mathsf{ind-cpa}}(\mathcal{A})$
- $\left|2 \times \Pr[b' = b | Z \stackrel{R}{\leftarrow} \mathbb{G}] - 1\right| = 0$

$$\begin{aligned} \mathbf{Adv}_{\mathcal{EG}}^{\mathsf{ind-cpa}}(\mathcal{A}) &= 2 \times \left| \begin{array}{l} \Pr[d = 1 | Z = \mathbf{CDH}(X, Y)] \\ - \Pr[d = 1 | Z \stackrel{R}{\leftarrow} \mathbb{G}] \end{array} \right| \\ &= 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(\mathcal{B}) \leq 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t) \end{aligned}$$

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As a consequence,

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$\mathcal{RSA}$ Encryption [Rivest-Shamir-Adleman 1978]

$\mathcal{RSA}$ Encryption

The RSA encryption scheme $\mathcal{RSA}$ is defined by

- $\mathcal{K}(1^k)$: p and q two random k-bit prime integers, and an exponent e (possibly fixed, or not): $sk \leftarrow d = e^{-1} \bmod \varphi(n)$ and $pk \leftarrow (n, e)$
- $\mathcal{E}_{pk}(m)$: the ciphertext is $c = m^e \bmod n$
- $\mathcal{D}_{sk}(c)$: the plaintext is $m = c^d \bmod n$

Theorem ($\mathcal{RSA}$ is $\mathrm{OW-CPA}$, but…)

A deterministic encryption scheme cannot be $\mathrm{IND-CPA}$

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Outline

Cryptography

Provable Security

Basic Security Notions

- Public-Key Encryption
- Variants of Indistinguishability
- Signatures

Conclusion

Indistinguishability vs. Find-then-Guess

[Bellare-Desai-Jokipii-Rogaway 1997]

FtG – CPA

- The challenger flips a bit b
- The challenger runs the key generation algorithm $(sk, pk) \leftarrow \mathcal{K}()$
- The adversary receives the public key pk, and chooses 2 messages m_0 and m_1
- The challenger returns the encryption c of m_b under pk
- The adversary outputs its guess b' on the bit b

$$\mathbf{Adv}_{\mathcal{S}}^{\mathsf{ftg-cpa}}(\mathcal{A}) = \mathbf{Adv}_{\mathcal{S}}^{\mathsf{ind-cpa}}(\mathcal{A}) = \left|2 \times \Pr[b' = b] - 1\right|$$

Note: the adversary has access to the following oracle, only once:
$\mathrm{LR}_b(m_0, m_1)$: outputs the encryption of m_b under pk

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- The challenger flips a bit b
- The challenger runs the key generation algorithm $(sk, pk) \leftarrow \mathcal{K}()$
- The adversary receives the public key pk, and chooses 2 messages m_0 and m_1 Find stage
- The challenger returns the encryption c of m_b under pk
- The adversary outputs its guess b' on the bit b

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- The challenger returns the encryption c of m_b under pk
- The adversary outputs its guess b' on the bit b Guess stage

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Left-or-Right Indistinguishability

[Bellare-Desai-Jokipii-Rogaway 1997]

LoR – CPA

- The challenger flips a bit b
- The challenger runs the key generation algorithm $(sk, pk) \leftarrow \mathcal{K}()$
- The adversary receives the public key pk, and asks LR on any pair (m_0, m_1) of its choice
- The challenger answers using LR_b
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$$\mathbf{Adv}_{\mathcal{S}}^{\mathsf{lor-cpa}}(\mathcal{A}) = \left|2 \times \Pr[b' = b] - 1\right|$$

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Find-then-Guess vs. Left-or-Right

Theorem (FtG $\overset{n}{\sim}$ LoR)

$$\forall t, \quad \mathbf{Adv}_{\mathcal{S}}^{\mathsf{ftg-cpa}}(t) \leq \mathbf{Adv}_{\mathcal{S}}^{\mathsf{lor-cpa}}(t)$$
$$\forall t, \quad \mathbf{Adv}_{\mathcal{S}}^{\mathsf{lor-cpa}}(t) \leq n \times \mathbf{Adv}_{\mathcal{S}}^{\mathsf{ftg-cpa}}(t)$$

where n is the number of LR queries

LoR $\Rightarrow$ FtG is clear

FtG $\Rightarrow$ LoR: hybrid distribution of the sequence of bits b

- The Left distribution is $(0, 0, \ldots, 0) \in \{0, 1\}^n$, for the LR queries
- The Right distribution is $(1, 1, \ldots, 1) \in \{0, 1\}^n$, for the LR queries
- Hybrid distribution: $\mathcal{D}_i = (0, \ldots, 0, 1, \ldots, 1) = 0^i 1^{n-i} \in \{0, 1\}^n$

$$\mathbf{Dist}(\mathcal{D}_0, \mathcal{D}_n) = \mathbf{Adv}_{\mathcal{S}}^{\mathsf{lor-cpa}}(\mathcal{A}) \quad \mathbf{Dist}(\mathcal{D}_i, \mathcal{D}_{i+1}) \leq \mathbf{Adv}_{\mathcal{S}}^{\mathsf{ftg-cpa}}(t)$$

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$$\begin{aligned} \forall t, \quad \mathbf{Adv}_{\mathcal{S}}^{\text{ftg}-\text{cpa}}(t) &\leq \mathbf{Adv}_{\mathcal{S}}^{\text{lor}-\text{cpa}}(t) \\ \forall t, \quad \mathbf{Adv}_{\mathcal{S}}^{\text{lor}-\text{cpa}}(t) &\leq n \times \mathbf{Adv}_{\mathcal{S}}^{\text{ftg}-\text{cpa}}(t) \end{aligned}$$

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Find-then-Guess vs. Left-or-Right

Theorem (FtG $\overset{n}{\sim}$ LoR)

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Real-or-Random Indistinguishability

[Bellare-Desai-Jokipii-Rogaway 1997]

RoR − CPA

- The challenger flips a bit b
- The challenger runs the key generation algorithm $(sk, pk) \leftarrow \mathcal{K}()$
- The adversary receives the public key pk, and asks RR on any message m of its choice
- The challenger answers using RR_b:
  - if $b = 0$, the RR_0 encrypts m Real
  - if $b = 1$, the RR_1 encrypts a random message Random
- The adversary outputs its guess b' on the bit b

$$\mathbf{Adv}_{\mathcal{S}}^{\mathrm{ror-cpa}}(\mathcal{A}) = \left|2 \times \Pr[b' = b] - 1\right|$$

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Left-or-Right vs. Real-Random

Theorem (LoR $\sim$ RoR)

$$\forall t, \quad \mathbf{Adv}_{\mathcal{S}}^{\mathsf{ror-cpa}}(t) \leq \mathbf{Adv}_{\mathcal{S}}^{\mathsf{lor-cpa}}(t)$$
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LoR $\Rightarrow$ RoR is clear (using $m_0 = m$ and $m_1 \stackrel{R}{\leftarrow} \mathcal{M}$)

RoR $\Rightarrow$ LoR: $\mathcal{B}$ flips a bit d, and uses m_d for the RR oracle, then forwards $\mathcal{A}$'s answer

$$\Pr[d \leftarrow \mathcal{B} | \mathsf{Real}] = \Pr[d \leftarrow \mathcal{A}] \quad \Pr[d \leftarrow \mathcal{B} | \mathsf{Random}] = 1/2$$

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Outline

Cryptography

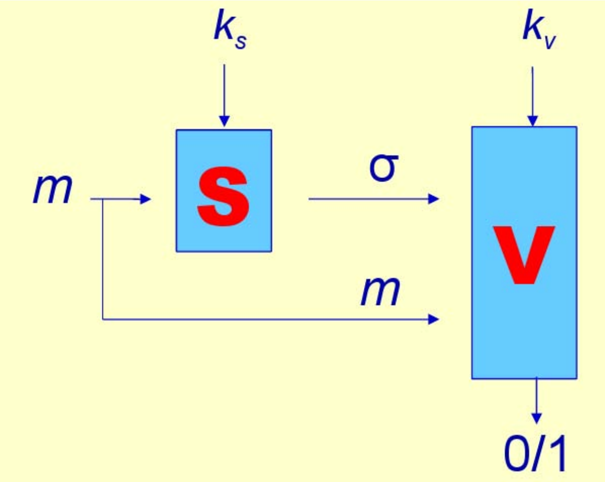
Provable Security

Basic Security Notions

- Public-Key Encryption
- Variants of Indistinguishability
- Signatures

Conclusion

Signature



Goal: Authentication of the sender

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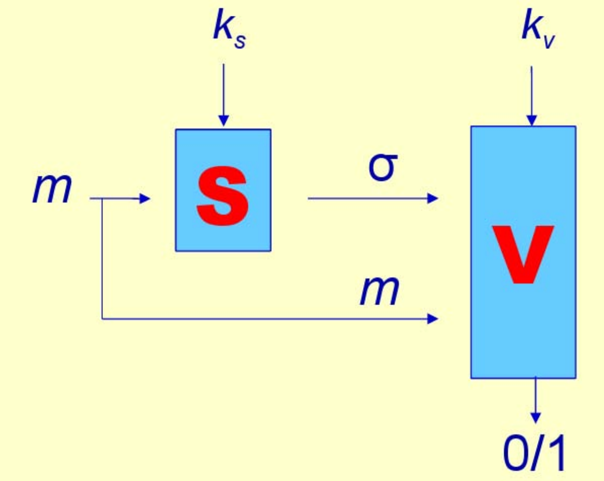

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EUF − NMA

Existential Unforgeability

For a signature scheme $\mathcal{SG} = (\mathcal{K}, \mathcal{S}, \mathcal{V})$, without the secrete key *sk*, it should be computationally impossible to generate a valid pair (m, σ):

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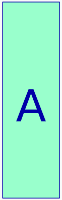

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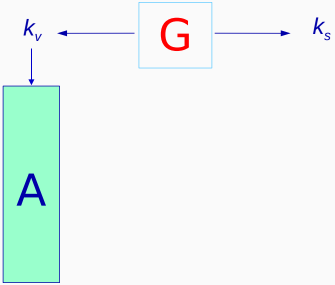

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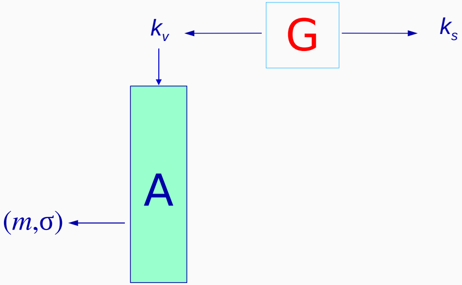

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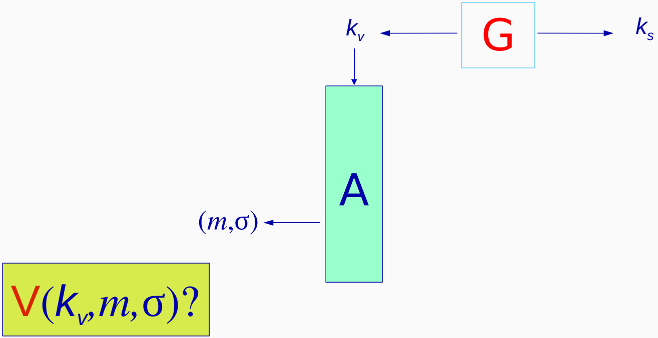

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The RSA signature scheme $\mathcal{RSA}$ is defined by

- $\mathcal{K}(1^k)$: p and q two random k-bit prime integers, and an exponent v (possibly fixed, or not): $sk \leftarrow s = v^{-1} \bmod \varphi(n)$ and $pk \leftarrow (n, v)$
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