A Category-theoretic characterization of functional completeness(+).

Giuseppe Longo (*)
Computer Science Department,
Carnegie Mellon University

Eugenio Moggi
L.F.C.S.,
University of Edinburgh

and

Dipartimento di Informatica, Universita' di Pisa, Corso Italia 40, I-56100 Pisa

Abstract. Functional languages are based on the notion of application: programs may be applied to data or programs. By application one may define algebraic functions; and a programming language is functionally complete when any algebraic function $f(x_1,\ldots,x_n)$ is representable (i.e. there is a constant a such that $f(x_1,\ldots,x_n) = (\mathbf{a}\cdot x_1\cdot\ldots\cdot x_n)$. Combinatory Logic is the simplest type-free language which is functionally complete.

In a sound category-theoretic framework the constant **a** above may be considered as an "abstract gödel-number" for f, when gödel-numberings are generalized to "principal morphisms", in suitable categories. By this, models of Combinatory Logic are categorically characterized and their relation is given to lambda-calculus models within Cartesian Closed Categories.

Finally, the partial recursive functionals in any finite higher type are shown to yield models of Combinatory Logic.

(+) **Theoretical Computer Science**, 70 (2), 1990, pp.193-211. A preliminary version of this paper appeared in LNCS 176, Springer-Verlag.

(*) This author's work has been made possible, in part, by the generous hospitality of the Computer Science Dept. of Carnegie Mellon University, while teaching in this University during the academic year 1987/88.

§.1 **Introduction**

The Theory of Combinators or Combinatory Logic (**CL**) can be viewed as a prototype functional language (see Backus(1978)) or as a formal theory of functions. Category Theory provides a fruitful way for looking at the foundations of Mathematics and, nowadays, of Computer Science, since any "theory of programs" is surely better understood in terms of a "theory of functions", such as Category Theory, than by a "theory of sets". Objects in a category may be understood as the data types and morphisms as functions or, more generally, as transformations on types of data.

Connections between λ-calculus and Cartesian Closed Categories (CCC's, see §.3) have been widely explored by Lambek[1980], Scott[1980], Poignè (1984) and many others (see Lambek & P. Scott (1986) for surveys and references; Adachi(1983), Hayashi(1985), Curien(1986), Martini(1988) for further work). In particular, any CCC is a model of the typed λ–calculus.

In some CCC's objects may be related in several "interesting" ways. Scott(1980), Berry (1979), Obtulowicz (1982) and Koymans(1982) described the models of untyped λ-calculus as "reflexive" objects (see Barendregt (1984) or 3.9 below). Our main result characterizes those types, in a CCC, which are models of type-free Combinatory Logic (theor. 3.5).

Besides the results on CCC's and λ-calculus in the references above, there is at least one more reason for looking at CCC's when studying models of computations. If **C** is a CCC, then there exists an isomorphism

(1) $$\Lambda: \mathbf{C}(X\times Y,Z) \cong \mathbf{C}(X,Z^Y),$$

where X,Y,Z are objects in **C** and the exponent Z^Y represents $\mathbf{C}(Y,Z)$, the morphisms from Y to Z (see §.3 for details). By (1), for $f\in\mathbf{C}(X\times Y, Z)$, the function

(2) $$\Lambda(f) : x \longmapsto \lambda y.f(x,y)$$

is in $\mathbf{C}(X, Z^Y)$, i.e. it is a morphism in the category or, also, "it exists" in the intended universe. A partly informal connection to a simple fact in computability theory may help to understand our motivations and basic notions.

Let R be the recursive functions and $PR = \{\varphi_i\}_{i\in\omega}$ the partial recursive functions, where $\varphi : \omega \to PR$ is a given Gödel-numbering. Take any bijective and effective pairing $<,> : \omega \times \omega \to \omega$ and define, as usual,

$f : \omega^2 \to \omega$ is partial recursive iff, for $f'(<x,y>) = f(x,y)$, one has $f'\in PR$.

Note that a function $f: \omega^2 \to \omega$, which is recursive in each argument, doesn't need to be recursive in the sense above: take, say, g total non recursive and set $f(x,y) = g(\min\{x,y\})$.

Fact *$f : \omega^2 \to \omega$ is partial recursive iff $\exists s \in R\ \ \forall x, y\ \ \varphi_{s(x)}(y) = f(x,y)$.*

Proof By the s-m-n or iteration theorem. Δ

In other words, similarly as in (2), if $f : \omega^2 \to \omega$ is partial recursive, then the function

(2') $\quad x \mapsto \lambda y.f(x,y)\ \ (= \phi_{s(x)})$

has to be recursive, i.e. the index of $\lambda y.f(x,y)$ depends "uniformly effectively" on x or, also, s exists as a recursive function. In familiar CCC's for denotational semantics, say, where morphisms are continuous functions (see Scott (1976, 1982), Plotkin (1978)...), (1) requires that a function is continuous iff it is so in each argument in the "uniform" sense of (2).

Note also that the Fact above gives a characterization of acceptable Gödel-numberings of PR, as defined in Rogers(1967). Consider first a function $g: \omega \to PR$ and set $\Lambda^{-1}(g)(x,y) = g(x)(y)$. Observe that, by the Fact, $\Lambda^{-1}(g)$ partial recursive iff $\exists s \in R\ \ g = \phi \circ s$.

Corollary *$\psi : \omega \to PR$ is an acceptable Gödel-numbering iff $\Lambda^{-1}(\psi)$ is partial recursive and*
$\forall g : \omega \to PR\ \ (\Lambda^{-1}(g)$ is partial recursive $\Leftrightarrow\ \exists s \in R\ \ g = \psi \circ s)$.

By this, there is a natural category-theoretic generalization of the notion of (acceptable) Gödel- numbering:

1.1 **Definition** *Let X,Y be objects in a category* ***C****. A morphism $f \in \mathbf{C}(X,Y)$ is* ***principal*** *if*

$$\forall g \in \mathbf{C}(X,Y)\ \ \exists h \in \mathbf{C}(X, X)\ \ \ g = f \circ h\ .$$

As a matter of fact, consider the category **EN** of numbered sets whose objects are pairs $\underline{X} = (X,e_X)$, with $e_X : \omega \to X$ (onto) and morphisms defined by $f \in \mathbf{EN}(X,Y)$ iff $\exists f' \in R$ $f \circ e_X = e_Y \circ f'$. Then, for $e_\omega = id$ and any Gödel numbering φ of PR, i.e. for (ω, id) and (PR, φ) in **EN**, the corollary above becomes: $\psi \in \mathbf{EN}(\omega,PR)$ is an acceptable Gödel-numbering iff ψ is principal .

Principal morphisms have been introduced in Longo&Moggi (1984) (implicitly) and explicitly in Longo&Moggi (1984 a) for the purposes of higher type computability. The significance of this notion is confirmed in this paper by the fact that in CCC's principal morphisms, plus two simple conditions, characterize combinatory algebras, i.e. models of Combinatory Logic.

§.2 gives a categorical characterization of Combinatory Algebras in the general setting of Cartesian Categories, based on the notion of Kleene-universal morphism.

§.3 deals with that characterization within Cartesian Closed Categories. This relates our characterization to the understanding of models of the untyped λ–calculus within the framework of those for the typed one. Combinatory algebras and λ-models turn out to be tidily related in CCC's.

§.4 briefly presents examples of type-free models in the recursion theoretic hierarchy of partial functionals in higher types.

§.2 **Combinatory Algebras and Cartesian Categories**

In this section we discuss applicative structures (i.e. a set X with a binary operation $\cdot : X x X \to X$) and their relation to Cartesian Categories. In particular we first show how to construct a Cartesain Category out of an applicative structure and, then, under which circumstances one obtains a Combinatory Algebra within a Cartesian Category.

2.1 **Definition** *Let $A = (X,\cdot)$ be an applicative structure.*

(o) The set of ***monomials*** *over A is inductively defined by:*

$x,y....x_1,x_2$...(variables)...are monomials

$a,b....a_1,a_2$....(constants from X)... are monomials

MN is a monomial if M and N are monomials.

Substitution of constants for variables, i.e. $M[\underline{a}/\underline{x}]$, in monomials is defined by induction in the usual way. $M_1M_2...M_n$ stands for $(...(M_1M_2)...M_n)$.

(i) $f : X^n \to X$ is ***algebraic*** *if $f(\underline{a}) = M[\underline{a}/\underline{x}]$ for some monomial M and any $\underline{a} = (a_1,... ,a_n)\in X^n$ and $\underline{x}$ of length n . (That is, the set $\mathbf{P}^n(A) = \mathbf{P}[X^n,X]$ of algebraic functions of n-arguments is defined by the monomials over X, with at most n variables, modulo extensional equality.)*

(ii) $f : X^n \to X$ is ***representable*** *if $\exists a\in X\ \forall \underline{b}\in X^n\ f(\underline{b})= a\cdot b_1\cdot ...\cdot b_n$. That is, the set $RF^n(A)$ of representable functions of n-arguments is defined by*

$$RF^n(A)=\{f:X^n\to X \mid \exists a\in X \ \ \forall \underline{b}\in X^n \ f(\underline{b})= a\cdot b_1\cdot ...\cdot b_n \}$$

(If there is no ambiguity write P^n, RF^n for $P^n(A)$, $RF^n(A)$ and P, RF for P^1, RF^1).

(iii) A is ***functionally*** *(or combinatorially)* ***complete*** *if every algebraic function is representable, i.e. $\forall n\ \ P^n = RF^n$.*

2.2 **Theorem** (Curry-Schoenfinkel) *$A = (X,\cdot)$ is functionally complete iff $K = \lambda xy.x$ and $S = \lambda xyz.xz(yz)$ are representable.*

A **Combinatory Algebra** is a functionally complete applicative structure. It is **non trivial** when it contains at least two members. As wellknown, **Combinatory Logic,** the theory of combinatory algebras, is powerful enough to be a functional programming language for computing all partial recursive functions.

By using algebraic functions one may define a simple category over an arbitrary applicative structure.

2.3 **Definition** *Let $A = (X, \cdot)$ be an applicative structure. The category $\mathbf{P}_A$ of* ***polynomials over*** *A, has as*

***objects**: $X^n \in \mathbf{P}_A$, for all $n \in \omega$;*

***morphisms**: $f \in \mathbf{P}_A(X^n, X^m)$ iff*

$f : X^n \to X^m$ and $\forall i < m$ $pr^m{}_i \circ f \in P^n$, where $pr^m{}_i$ is the i-th projection.

(If there is no ambiguity write $\mathbf{P}(X^n, X^m)$ for $\mathbf{P}_A(X^n, X^m)$).

For example, $f(x,y) = (xb(xax),yxa)$ for $a,b \in X$, is in $\mathbf{P}(X^2,X^2)$. By substitution, one may easily show that morphisms are closed under composition; moreover, $pr^n{}_i \in \mathbf{P}(X^n,X) = \mathbf{P}^n$ and, thus, $\mathbf{P}_A$ is a category. If A is a combinatory algebra, then $\mathbf{P}_A$ may be considered as the category of representable morphism, by 2.2.

Recall now that a category **C** is **cartesian** if it has all finite products; i.e. if there is a terminal object T s.t. $\forall X \ \exists! f \in \mathbf{C}(X,T)$ and, for all objects X,Y, there is an object XxY, the product, with projections pr_1, pr_2 , and for all Z, an isomorphism <-,-> : $\mathbf{C}(Z,X) x \mathbf{C}(Z,Y) \to \mathbf{C}(Z,XxY)$ such that $pr_i \circ <f_1,f_2> = f_i$.

A category **C**, with a terminal object T , has **enough points** if

$$\forall a,b \in Ob_{\mathbf{C}} \ \ \forall f,g \in \mathbf{C}[a,b] \ \ \forall h \in \mathbf{C}[T,a] \ \ f \circ h = g \circ h \ \Rightarrow \ f = g .$$

This terminology derives from the idea that the arrows in **C**(T,a), when T is terminal, may be understood as the points of a . Note that most categories used in denotational semantics are categories of functions and, thus, have enough points.

2.4. **Remark** Let $A = (X,\cdot)$ be an applicative structure. It is easy to show that $\mathbf{P}_A$ is cartesian by using the projections pr_i 's above; indeed $X^n x X^m = X^{n+m}$ and X^0 is the terminal object. Moreover $\underline{P}_A$ has enough points.

2.5 **Definition** *Let **C** be a cartesian category. Then $u \in \mathbf{C}(XxY,Z)$ is* ***Kleene-***

universal *(K-universal) if* $\forall f\in \mathbf{C}(XxY,Z)$ $\exists s\in \mathbf{C}(X,X)$ $f = u\circ(sxid)$ *, i.e.*

$$
\begin{tikzcd}
X \arrow[d, dashed, "s"'] & & XxY \arrow[r, "f"] \arrow[d, dashed, "sxid"'] & Z \\
Y & & XxY \arrow[ur, "u"'] &
\end{tikzcd}
$$

2.6 **Remark** K-universality is a *weak* (co-)universality property, as no unicity of s is required. It has an obvious recursion theoretic meaning: K-universality generalizes the s-m-n (iteration) theorem, as pointed out in the Introduction.

2.7 **Definition** *Let* $\mathbf{C}$ *be cartesian and* $u\in \mathbf{C}(XxX,X)$. *Then* $u^{(n)}\in \mathbf{C}(XxX^n,X)$ *is inductively defined by:* $u^{(1)} = u$, $u^{(n+1)} = u^{(n)}\circ(uxid^n)$, *that is*

$u^{(n+1)}: XxXxX^n \xrightarrow{uxid^n} XxX^n \xrightarrow{u^{(n)}} X$, *for* $X^{n+1} = XxX^n$.

It is easy to observe that $u^{(n)}$ corresponds exactly to the application of $n+1$ arguments, from left to right, e.g. $u^{(2)}\circ\langle a,b,c\rangle = u\circ(uxid)\circ\langle a,b,c,\rangle = u\circ\langle u\circ\langle a.b\rangle,c\rangle$.

Recall now that a **retraction** $X < Y$ between objects X,Y in a category **C**, is a pair (i,j) s.t. $i\in C(X.Y)$, $j \in C(Y,X)$ and $j\circ i = id_X$. We then say that X is a **retract** of Y **via** (i,j).

2.8 **Lemma** *Let* $\mathbf{C}$ *be cartesian. Assume that, for some* U *in* $\mathbf{C}$, $UxU < U$ *and there is a K-universal* $u\in \mathbf{C}(UxU,U)$. *Then* $\forall n$ $u^{(n)}\in \mathbf{C}(UxU^n,U)$ *is K-universal.*

Proof By assumption, this is true for n = 1. Let $UxU < U$ via (i,j) and $f\in C(UxU^{n+1},U)$. Then, by the inductive hypotesis, for some $s^{(n)}\in C(U,U)$ the following diagram commutes

$$
(1) \qquad
\begin{tikzcd}
UxU^n \arrow[r, "jxid^n"] \arrow[d, "s^{(n)}xid^n"'] & UxUxU^n \arrow[r, "f"] & U \\
UxU^n \arrow[urr, "u^{(n)}"'] & &
\end{tikzcd}
$$

By assumption, for some $s\in C(U,U)$ one also has:

(2)
$$\begin{tikzcd} U x U \arrow[r, "i"] \arrow[d, "s x id"'] & U \arrow[r, "s^{(n)}"] & U \\ U x U \arrow[urr, "u"'] & & \end{tikzcd}$$

Then compute $\quad f = f \circ (j \text{ x } id^{n}) \circ (i \text{ x } id^{n})$

$= u^{(n)} \circ (s^{(n)} \text{ x } id^{n}) \circ (i \text{ x } id^{n})$ by (1)

$= u^{(n)} \circ (u \text{ x } id^{n}) \circ (s \text{ x } id^{n+1})$ by (2)

$= u^{(n+1)} \circ (s \text{ x } id^{n+1})$ Δ

We are now in the position to prove the main theorem of this section. Let's first express in suitable category-theoretic terms the notion of applicative structure.

2.9 **Definition** *Let* $\mathbf{C}$ *be a cartesian category,* T *its terminal object and* U *an object in* $\mathbf{C}$*, with* $T < U$ *and* $u \in \mathbf{C}(U x U, U)$*. The* ***applicative structure associated to u****,* $\underline{A}(u)$*, is given by* $\underline{A}(u) = (\mathbf{C}(T,U), \cdot)$*, where* $a \cdot b = u \circ \langle a,b \rangle$*.*

Note that, in a category $\mathbf{C}$ with a terminal object T, $T < U$ simply generalizes the set-theoretic notion that U is "not empty".

Remark. We say that $g \in \mathbf{C}(U,U)$ **induces** $f : \underline{A}(u) \to \underline{A}(u)$ if $f(h) = g \circ h$ for all $h \in \mathbf{C}(T,U)$. Then, it is straightforward to prove that all algebraic functions defined by a polynomial in n variables with constants in $\underline{A}(u)$, are induced by morphisms in $\mathbf{C}(U^{n},U)$. One only has to "intepret" variables as projections, constants as the maps from U^{n} to U factoring through the terminal object and argue by induction on the structure of the "algebraic term" defining the function. For example, for $f(x_1,x_3) = (x_3 \cdot x_1) \cdot a$ write

$u \circ \langle pr_3, pr_1 \rangle : U^3 \to U x U \to U$, which is $x_3 \cdot x_1$, and then

$f = u \circ \langle u \circ \langle pr_3, pr_1 \rangle, a \circ ! \rangle : U^3 \to U x U \to U$, where $! : U \to T$.

2.10 **Theorem** *Let* $\mathbf{C}$ *be a cartesian category. Assume that for some object* U*, one has* $T < U$*,* $U x U < U$ *and there exists a K-universal* $u \in \mathbf{C}(U \text{ x } U, U)$*. Then* $\underline{A}(u)$ *is a combinatory algebra.*

Proof Let $T < U$ via (i,j). Then, by 2.8, $\forall n$ $\forall f \in \mathbf{C}(U^{n},U)$ $\exists s \in \mathbf{C}(U,U)$ the following diagram commutes, with $[f] = s \circ i$:

$$\begin{tikzcd}
T\mathrm{x}U^{n} \arrow[r, "i\mathrm{x}id^{n}"] \arrow[dr, "{[f]\mathrm{x}id^{n}}"'] & U\mathrm{x}U^{n} \arrow[r, "pr_2"] \arrow[d, "s\mathrm{x}id^{n}"] & U^{n} \arrow[r, "f"] & U \\
& U\mathrm{x}U^{n} \arrow[urr, "u^{(n)}"'] & &
\end{tikzcd}$$

Thus $u^{(n)}\circ[f]\times id^n = f\circ pr_2\circ i\times id^n = f$. Since $u^{(n)}$ is the application, from left to right, of its $n+1$ arguments, $[f]$ represents , in the sense of definition 2.1(ii), the function from $A(u)^n$ to $A(u)$ induced by f . By 2.2, we only needed to consider $f\in\mathbf{C}(U^2,U)$ and $g\in\mathbf{C}(U^3,U)$ such that $f(x_1,x_2) = x_1$ and $g(x_1,x_2,x_3) = x_1\cdot x_3\cdot(x_2\cdot x_3)$, as, then, $[f]$ and $[g]$ would represent K and S, respectively. For this purpose, just take $f = pr^2{}_1\in\mathbf{C}(U^2,U)$ and $g = u\circ<u\circ<pr^3{}_1,pr^3{}_3>,u\circ<pr^3{}_2,pr^3{}_3>>\in\mathbf{C}(U^3,U)$. Δ

For the reader's convenience, suppose, say, that **C** is a category of sets and let's compute explicitly $[pr^2{}_1]$, i.e. the element K in U . By the diagram,

$$u^2\circ(s\circ i_T\times id^2) = pr^2{}_1 .$$

As morphisms are just ordinary functions and we may identify the set U with the set of points $\mathbf{C}(T,U)$, for $T = \{0\}$, the diagram gives

$$\forall a_1,a_2\in U \quad u^2(s\circ i_T(0), a_1,a_2) = pr^2{}_1(a_1,a_2) = a_1 .$$

Thus, by the definition of application, for $K = s\circ i_T(0)$, one has

$$K\cdot a_1\cdot a_2 = u^2\circ(s\circ i_T(0), a_1,a_2) = a_1.$$

It is very simple to prove a converse of 2.10, in the sense of 2.11: it just formally says the wellknown fact that the (η expansion of) the identity is the universal function in Combinatory Logic (λ-calculus). Indeed, the next theorem proves this and, moreover, it shows that, by applying the construction in 2.9 to a combinatory algebra, one gets back to the given combinatory algebra. In the sequel, we will often use an informal lambda-notation for algebraic functions, e.g. $\lambda xy.x$ for $pr^2{}_1$.

2.11 **Theorem** *Let $A = (X,\cdot)$ be a combinatory algebra and $\boldsymbol{P}_A$ be the category of polynomials over A . Then $T < X$, $X\mathrm{x}X < X$ in $\boldsymbol{P}_A$ and, for $u(x,y) = x\cdot y$, $u\in\boldsymbol{P}(X^2,X)$ is K-universal in the category $\boldsymbol{P}_A$. Moreover $\underline{A}(u) = A$.*

Proof $T < X$ trivially holds, for $X\neq\emptyset$. Let now $c,c_1,c_2\in X$ represent $\lambda xyz.zxy$, $\lambda xy.x$, $\lambda xy.y$, respectively, in the sense of 2.1(ii). As usual, c is the element which codes pairs in λ-calculus, while c_1, c_2 will be used to define projections. Thus, for $[x,y] = cxy$ and $p_i(x) = xc_i$, one has $[-,-]\in P(X^2,X)$, $p_i\in P(X,X)$ and $X\mathrm{x}X < X$ via $([-,-],<p_1,p_2>)$.

Note that XxX exists by 2.4 . Finally, assume that $f \in P(X^2,X)$ and that $a \in X$ represents f. Then $f = u \circ ((\lambda x.ax) \times id)$ and, hence, u is K-universal.

It is easy to check from the definition that $\underline{A}(u) = A$, up to isomorphisms. Δ

2.12 **Corollary** *Let* $A = (X,\cdot)$ *be an applicative structure. Then A is a combinatory algebra iff, in* $\boldsymbol{P}_A$*, one has* $T < X$, $XxX < X$ *and, for* $u(x,y) = x\cdot y$*, u is K-universal.*

Proof $(\Rightarrow)$ by 2.11; $(\Leftarrow)$ by 2.10. Δ

2.13 **Remark** If **C** has enough points and $u : U \times U \to U$ is K-universal, then the category $\mathbf{P}_{\underline{A}(u)}$, over A(u), is a full sub-cartesian category of **C**. The proof easily follows from the previous observations, 2.10 and the assumption that **C** has enough points.

§.3 From universal to principal morphisms.

In this section we look at K-universal morphisms in the context of CCC's. This framework will motivate the informal connections we hinted between classical notions in recursion theory and our categorical generalizations, as K-universal will relate to principal morphisms similarly as universal functions relate to gödel numberings. Moreover, these categories provide a widely used frame for typed calculi.

A Cartesian Closed Categories (CCC) **C** is a Cartesian Category such that for all objects Y, Z there is an object Z^Y, which "represents" **C**(Y,Z) . More precisely, for all X, there exists an isomorphism Λ: **C**(XxY,Z) $\cong$ **C**(X,Z^Y) such that :

$$\begin{tikzcd} XxY \arrow[r, "h"] \arrow[d, "\Lambda(h)xid"'] & Z \\ Z^{Y}xY \arrow[ur, "eval"'] & \end{tikzcd}$$

The advantage of dealing with CCC's is that for any object X one may consider its "function space" X^X, as an object. As a matter of fact, functional completeness, in its various form of increasing strength (combinatory algebras, λ-algebras, λ-models (see later)), expresses some sort of privileged relation between an object in a category and its "function space".

To see this in a unified way, let us first recall the definition of a category which has been widely used for the semantics of types over models of λ-calculus (see Scott (1976), Hindley (1983)). It actually originated in Proof Theory, by work of Kreisel (see also Girard(1972), Troelstra(1973)) and is now widely used in the semantics of Intuitionistic Logic and higher order calculi (Hyland(1982/1987), Longo&Moggi(1988) and many others).

3.1 **Definition** *Let $A = (X,\cdot)$ be an applicative structure. Define then:*
(1) The category $\boldsymbol{PER_A}$ *of* ***partial equivalence relations*** *given by:*
objects: *$R \in \boldsymbol{PER_A}$ iff R is an equivalence relation on a subset X_R of X, i.e.*
$X_R = dom\ R = rangeR$.
morphisms: *for $R \in \boldsymbol{PER_A}$ let $\pi_R\{n\} = \{m \mid nRm\}$; then*
$f \in PER(R,S)$ iff $\exists g \in \boldsymbol{P}(X,X)$ $f \circ \pi_R = \pi_S \circ g$ on X_R, i.e. the following diagram commutes:

$$\begin{tikzcd}
X_R \arrow[r, dashed, "g\lceil X_R"] \arrow[d, "\pi_R"'] & X_S \arrow[d, "\pi_S"] \\
X_R/R \arrow[r, dashed, "f"'] & X_S/S
\end{tikzcd}$$

(2) The category $\boldsymbol{ER_A}$ *of (total)* ***equivalence relations*** *given as above by using equivalence relations on X (i.e. $X_R = X$ in (i)).*

$\mathbf{PER_A}$ and $\mathbf{ER_A}$ are indeed categories. As for $\mathbf{P_A}$ we write $(\mathbf{P})\mathbf{ER}(R,S)$ for $(\mathbf{P})\mathbf{ER_A}(R,S)$, when unambiguous.

3.2 **Proposition** *Let $A = (X,\cdot)$ be an applicative structure. Then one as:*
(i) If $T < X$ and $XxX < X$ in $\boldsymbol{P_A}$, then $\boldsymbol{ER_A}$ and $\boldsymbol{PER_A}$ are CC's with enough points. Moreover, $\boldsymbol{P_A}$ and $\boldsymbol{ER_A}$ are full sub-CC's of $\boldsymbol{PER_A}$.
(ii) If A is a combinatory algebra, then $\boldsymbol{PER_A}$ is a CCC.

Proof (i) Let $XxX < X$ via $([-,-],<p_1,p_2>)$. Then RxS may be defined componentwise, by

$$a(R \times s)b \quad \text{iff} \quad (p_1(a))R(p_1(b)) \ \underline{\text{and}} \ (p_2(a))S(p_2(b)).$$

This turns $\mathbf{PER_A}$ into a CC.
Observe now that $\forall n$ $X^n < X$ in $\mathbf{P_A}$ by $X^0 = T < X$ and by iterating $XxX < X$. Thus $\mathbf{P_A}$ may be faithfully embedded in $\mathbf{PER_A}$, by associating to X^n the identity relation, id_n, restricted to the image of X^n in X. Moreover, $\mathbf{P_A}$ is full in $\mathbf{PER_A}$, since $\mathbf{P}(X^n,X^m) \cong \mathbf{PER}(id_n, id_m)$, as sets, by the following isomorphism (take $m = 1$, for the sake of simplicity). Let $g \in \mathbf{P}[X^n,X]$, then, for $x = [x_1,...,x_n]$, define $G(g) \in \mathbf{PER}[id_n, id]$ by $G(g)(x) = g(p_1(x),...,p_n(x)) = g(x_1,...,x_n)$. $G(g)$ is computed, in the sense of 3.1, by

$g \circ \langle p_1, \ldots, p_n \rangle : X \to X^n \to X$. G is an isomorphism, whose inverse is given as follows: if $h \in \mathbf{PER}[id_n, id]$ is computed by $h' \in \mathbf{P}[X,X]$, then $G^{-1}(h) = h' \circ [-, \ldots, -] : X^n \to X \to X$. By definition, $\mathbf{ER}_A$ is a full sub-CC of $\mathbf{PER}_A$, and both categories have enough points (the reader may prove it by exercise, observing that $\mathbf{P}(X,X)$ contains the constant functions).

(ii) The object S^R in $\mathbf{PER}_A$, which represents $\mathbf{PER}(R,S)$, is defined as follows:

$$aS^Rb \quad \text{iff} \quad \forall x,y \in X \; (xRy \Rightarrow (ax)S(by)).$$

Recall now that, by assumption, each function in $P(X,X)$ is representable. The rest is easy. Δ

3.3 **Remark** Note that all what is needed in 3.2(ii) in order to prove that $\mathbf{PER}_A$ is a CCC, is that each function in $\mathbf{P}(X,X)$ and, hence, in $\mathbf{PER}(R,S)$ for each R,S, has representatives in X (see §.4 for an application of this).

In CCC's K-universal morphisms and principal morphisms, which were defined in 1.1, are tidely related.

3.4 **Proposition** *Let* $\boldsymbol{C}$ *be a CCC and* Λ *the isomorphism* $\boldsymbol{C}(X\mathrm{x}Y, Z) \cong \boldsymbol{C}(X, Z^Y)$. *Then* $u \in \boldsymbol{C}(X\mathrm{x}Y,Z)$ *is K-universal iff* $\Lambda(u) \in \boldsymbol{C}(X,Z^Y)$ *is principal.*

Proof The isomorphism Λ implies, by definition, the equivalence of the following diagrams:

$$\begin{tikzcd} X\times Y \arrow[r, "f"] \arrow[d, "s\times id"'] & Z \\ X\times Y \arrow[ur, "u"'] & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} X \arrow[r, "\Lambda(f)"] \arrow[d, "s"'] & Z^Y \\ X \arrow[ur, "\Lambda(u)"'] & \end{tikzcd}$$

And we are done. Δ

By 3.2 and 3.4 one may then easily restate 2.10 and 2.11 in terms of CCC's and principal morphisms.

3.5 **Proposition** *(i) Let* $\boldsymbol{C}$ *be a CCC. Assume that, for some* U *in* $\boldsymbol{C}$, $T < U$, $U\mathrm{x}U < U$ *and there exists a principal* $p \in \boldsymbol{C}(U,U^U)$. *Then* $\underline{A}(\Lambda^{-1}(p))$, *defined as in 2.9, is a combinatory algebra.*

(ii) Let $A = (X,\cdot)$ *be a combinatory algebra. Then, in the CCC* $\boldsymbol{PER}_A$, *one has* $T < X$, $X\mathrm{x}X < X$ *and, for* $u(x,y) = xy$, $\Lambda(u) \in \boldsymbol{PER}(X,X^X)$ *is principal. Moreover*

$\underline{A}(u) = A$.

As summarized in Barendregt (1983/4), models of the purely equational theory of λ-calculus, λ-algebras, are characterized as reflexive objects in a CCC, i.e. as U such that $U^U < U$.

λ-algebras, originally called pseudo-λ–models in Hindley & Longo (1980), are exactly the models of Combinatory Logic with β-equality (see Hindley & Longo (1980) or Hindley & Seldin (1986)). Moreover, one exactly has the "first order" models of λ-calculus, λ-models (see 3.6 below), if **C** has enough point (Barendregt (1984)). We conclude this section by a very simple proof of the latter characterization, based on the work done on principal morphisms.

3.6 **Definition** (Meyer(1982)) $(X, \cdot, k, s, \varepsilon)$ is a **λ-model** if it satisfies the following axioms:

(k)	$\forall x,y$	$kxy = x$
(s)	$\forall x,y,z$	$sxyz = xz(yz)$
(ε_1)	$\forall x,y$	$\varepsilon xy = xy$
(ε_2)	$\forall x,y$	$(\forall z\ xz = yz) \Rightarrow \varepsilon x = \varepsilon y$
(ε_3)		$\varepsilon\varepsilon = \varepsilon$

In Hindley & Longo (1980) and Scott(1980) equivalent characterizations to 3.6 are given.

Note that there may be several choices of k,s and ε , in an applicative structure $(X,\cdot)$ which yields a λ-models (see Longo (1983)). Given ε as in (ε_1), (ε_2) and (ε_3), though, $(X, \cdot, \varepsilon)$ uniquely determine the interpretation $[-]$ of the λ-terms. Thus $[\lambda xy.x]_\xi$ and $[\lambda xyz.xz(yz)]_\xi$, for any environment ξ, provide the "intended" k and s in X, even though there may be other $\underline{k}$ and $\underline{s}$ with the same behaviour (see Meyer (1982), Barendregt (1984) for a discussion; in Longo (1983) interesting λ-models with several ε's are discussed as well as λ-models with a unique ε; in particular the classical $(P\omega,\cdot)$ turns out to have just one ε (Bruce & Longo (1984))).

The obvious syntactic definition of λ-model is easily related to the definition in 3.6 (see Hindley & Longo (1980), Meyer (1982), Barendregt (1984)).

3.7 **Lemma** *Let* ***C*** *be a category. Then*

(i) If $X < Y$ via (i,j), then j is principal;

(ii) If $X < Y$ and $f \in \boldsymbol{C}(Y,X)$ is principal, then there exists $g \in \boldsymbol{C}(X,Y)$ such that $X < Y$ via (g,f) .

Proof (i) Just notice that the following diagram commutes, for all $h \in \mathbf{C}(Y,X)$, where $X < Y$ via (i,j):

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "h"'] & X \\ X \arrow[r, "i"] & Y \arrow[u, "j"'] \end{tikzcd}$$

Thus $h = j \circ (i \circ h)$.

(ii) Since f is principal, $\exists s \in \mathbf{C}(X,X)$ $j = f \circ s$. Then, for $g = s \circ i$, one has $f \circ g = j \circ i = id_X$. As a diagram:

$$\begin{tikzcd} Y \arrow[r, "i", no head] \arrow[dr, "g"'] & X \arrow[r, "j"] \arrow[d, "s"] & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

Let M be a lambda-term in which x and y may occur. Following Barendregt(1984), we write $[[M]]_{xy}$ for the intended map from UxU to U.

3.8 **Theorem** *Let* $\mathbf{C}$ *be a CCC. Assume that, for some* U *in* $\mathbf{C}$, $U^U < U$ *via* (ψ,ϕ). *Then* $T < U$, $UxU < U$ *and* ϕ *is principal.*

Proof Clearly, ϕ is principal by 3.7(i) and $T < U$, for $T < U^U < U$. We only need to show that $UxU < U$. Following the interpretation of λ–terms in Barendregt[1984], set

$$[-,-] = [[\lambda z.zxy]]_{xy} : UxU \longrightarrow U$$
$$p_i = [[x(\lambda x_1 x_2.x_i)]]_x : U \longrightarrow U .$$

By using the Λ and eval, one may easily show that these are all morphisms in $\mathbf{C}$. Conversely, let $pr_1, pr_2 \in \mathbf{C}(UxU,U)$ be the projections in $\mathbf{C}$, which is a CCC. Note that from $U^U < U$ one may deduce $(U^U)^U < U^U$, via (ψ', ϕ'), say. Now, $\Lambda(pr_1) \in \mathbf{C}(U,U^U)$ and $\Lambda(\Lambda(pr_i)) = \Lambda^2(pr_i) \in \mathbf{C}(T,U^U)^U) \cong (U^U)^U$. Define then

$$p_i = \lambda x.x(\psi(\psi'(\Lambda^2(pr_i))) \in \mathbf{C}(U,U).$$

Thus $p_i \circ [-, -] = [[(\lambda z.zx_1x_2)(\lambda x_1x_2.x_i)]]_{x1x2} = [[x_i]]_{x1x2} = pr_i : UxU \longrightarrow U$ and $([-,-],<p_1,p_2>)$ gives the required retraction. Δ

Note that in 3.8, from $U^U < U$, we directly derived the pairing and projection functions which are given in combinatory algebras (see 2.11). Objects U such that $U^U < U$, are usually called reflexive.

In the next corollary, for the sake of simplicity, we identify points $a : T \to X$ of X in $\mathbf{PER}_A$ and elements of X, and write, say, $\psi(a) \in X$ instead of $\psi \circ a : T \to X^X \to X$ or even $\psi \circ \phi \in X^X$ instead of $\Lambda(\psi \circ \phi) : T \to X^X$.

3.9 **Corollary** *(i) Let* $\mathbf{C}$ *be a CCC with enough points. Assume that* $\mathbf{C}$ *has a reflexive object* U*, i.e.* $U^U < U$ *via some* (ψ, ϕ)*. Then, for* $\varepsilon = \psi(\psi \circ \phi)$*,* $(\underline{A}(\Lambda^{-1}(\phi)), \varepsilon)$ *is a* λ*-model.*

(ii) Let $A = (X, \cdot, \varepsilon)$ *be a* λ*-model. Then, in the CCC* $\mathbf{PER}_A$*, there exist* $\psi \in \mathbf{PER}(X^X, X)$ *and* $\phi \in \mathbf{PER}(X, X^X)$ *such that* $X^X < X$ *via* (ψ, ϕ)*. Moreover,* $(X, \cdot) \cong \underline{A}(\Lambda^{-1}(\phi))$*, and* $\varepsilon = \psi(\psi \circ \phi)$*.*

Proof (i) By 3.8, $\phi \in \mathbf{C}(U, U^U)$ is principal, $T < U$ and $U x U < U$. Thus, for $a \cdot b = eval \circ \phi x id \circ <a,b>$ ($= \phi(a)(b)$, for short) and for some K and S, $(U, \cdot, K, S)$ is a combinatory algebra, i.e. (k) and (s) in 3.6 hold. Set now $\varepsilon = \psi(\psi \circ \phi)$. Then one has

(ε_1) $\varepsilon xy = \psi(\psi \circ \phi)xy = \psi(\phi(x))y = xy$

(ε_2) $\forall z \; xz = yz$ $\Rightarrow$ $\phi(x) = \phi(y)$, for $\mathbf{C}$ has enough points

$\Rightarrow$ $\varepsilon x = \psi(\psi \circ \phi)x = \psi(\phi(x)) = \psi(\phi(y)) = \varepsilon y$

(ε_3) $\varepsilon\varepsilon = \psi(\psi \circ \phi)(\psi(\psi \circ \phi)) = \psi(\phi(\psi(\psi \circ \phi))) = \varepsilon$

(ii) Let $f \in \mathbf{PER}(X, X)$ and $a \in X$ be a representative for f. Define the $\psi(f) = \varepsilon a$. By (ε_2), ψ is well defined. Recall also that X^X corresponds to the collection of representable functions, hence $\psi \in \mathrm{PER}(X^X, X)$.

As for ϕ, define $\phi(a) = \lambda x.ax$ for any $a \in X$. Clearly $\phi \in \mathrm{PER}(X, X^X)$. Compute then

$\phi(\psi(f))$ $= \lambda x.\varepsilon ax$, if a represents f

$= f$, by (ε_1).

Thus $X^X < X$, via (ψ, ϕ).

Finally, $(X, \cdot) \cong \underline{A}(\Lambda^{-1}(\phi))$, since $\phi(a)(b) = ab$. Moreover $\psi(\phi(a)) = \varepsilon a$ and, hence, ε represents $\psi \circ \phi$. Thus

$\psi(\psi \circ \phi)$ $= \varepsilon\varepsilon$, by definition of ψ

$= \varepsilon$, by (ε_3) . Δ

3.10 **Corollary** *Let* $\mathbf{C}$ *be a CCC with enough points. Assume that, for some* U *in* $\mathbf{C}$*,* $U^U < U$ *via* (i,j)*. If* $u \in \mathbf{C}(UxU, U)$ *is K-universal, then also* $\underline{A}(u)$ *can be turned into a* λ*-model.*

Proof By 3.7 (ii) and 3.9(i). Δ

We conclude this section by summarizing the connections between models of the type-free λ-calculus and categories obtained so far. This provides a unified framework for the topic.

3.11 **Theorem** *Let* $\boldsymbol{C}$ *be a CCC with enough points and* A *an object of* $\boldsymbol{C}$. *Then*

(1) $A^A \cong A \;\Rightarrow\; A$ *is a model of* $\lambda\beta\eta$

(2) $A^A < A \;\Rightarrow\; A$ *is a model of* $\lambda\beta$

(3) $\exists p \in \boldsymbol{C}[A, A^A]$ *principal*, $T < A$ *and* $A \times A < A \;\Rightarrow\; A$ *is a model of CL.*

Conversely,

(1) A *is a model of* $\lambda\beta\eta \;\Rightarrow\; A^A \cong A$ *in* $\boldsymbol{PER}_A$

(2) A *is a model of* $\lambda\beta \;\Rightarrow\; A^A < A$ *in* $\boldsymbol{PER}_A$.

(3) A *is a model of CL* $\Rightarrow\; \exists p \in \boldsymbol{PER}[A, A^A]$ *principal*, $T < A$ *and* $A \times A < A$ *in* $\boldsymbol{PER}_A$.

Remark In Scott(1980), Koymans (1982) and Barendregt (1983/4) the Karoubi-Scott envelope $\mathbf{C}_A$ is used for characterizing λ-algebras and λ–models. λ-algebras are given as in 3.11(2), except that it is not required that the frame category $\mathbf{C}$ has enough points; conversely, if $(A,\cdot)$ is a λ-algebra $\mathbf{C}_A$ does not need to have enough points. (If $(A,\cdot)$ is a λ-model, then $\mathbf{C}_A$ has enough points and is a full subCCC of $\mathbf{PER}_A$, see Scott(1976)).

The construction of $\mathbf{C}_A$, though, does not work on combinatory algebras, the case treated in this paper, as it may be easily checked by inspecting the proofs of 4.12-13 in Barendregt (1983) or 5.5.11-13 in Barendregt (1984). Note also that by interpreting types as objects in $\mathbf{PER}_A$, the ordinary interpretation of λ-terms in a λ–model A is tidily related, by a completeness results, to formal type assignement (see Hindley (1983)). This is not so when using $\mathbf{C}_A$.

Finally, there is a further motivation for the interest of the $\mathbf{PER}_A$ construction on (possibly partial, see §.4) combinatory algebras: $\mathbf{PER}_A$ is equivalent to the category $\mathbf{M}_A$ of the **modest** sets, a subcategory of Hyland's Efffective topos, relativized to A. These "realizability" toposes provide interesting models of Intuitionistic ZF. By this and by relevant properties of $\mathbf{PER}_A$, viewed as $\mathbf{M}_A$, namely its "small completeness", $\mathbf{PER}_A$ yields also models for higher order calculi (see Hyland (1987), Longo & Moggi (1988) for results and more references).

§.4 Remarks on higher type objects as models for type-free calculi

The aim of this informal section is to provide some examples of categories and objects with the properties mentioned so far and discuss connections to Higher Type Recursion Theory, a topic which actually originated this work. In particular, it will be hinted how to construct lots of λ–models in recursion theoretic hierarchies of functions and functionals. The observation is that at any finite higher type (higher than 1) one has model of the type-free λ-calculus.

4.1 **Lemma** *Let* **C** *be a CCC and* T *its terminal object. Then for all* $X,Y \in Ob_{\mathbf{C}}$ *, if* $T < Y$ *, then* $X < X^Y$ *.*

Proof The retraction (i,j) is given by $i = \Lambda(pr_1) \in \mathbf{C}(X, X^Y)$ and $j = eval \circ \langle id,t \rangle \in \mathbf{C}(X^Y, X)$ for some fixed $t \in \mathbf{C}(X^Y, Y)$. Indeed,

$$
\begin{aligned}
j \circ i &= eval \circ \langle id,t \rangle \circ \Lambda(pr_1) \\
&= eval \circ (\Lambda(pr_1) x id) \circ \langle id,t \rangle \\
&= pr_1 \circ \langle id,t \rangle \\
&= id
\end{aligned}
$$

Δ

4.2 **Definition** *Let Tp , the* ***Curry types*** *over an atomic type 1, be the least set such that* $1 \in Tp$ *and* $\sigma,\tau \in Tp \Rightarrow \sigma \to \tau \in Tp$ *. Then , for* X *in a CCC* **C***, set*

$$X^1 = X \text{ , and, for } A = X^\tau \text{ and } B = X^\sigma \text{, } X^{\sigma \to \tau} = B^A.$$

Recall that in a CCC an object U is reflexive if $U^U < U$.

4.3 **Proposition** *Let* U *be reflexive in a CCC* **C***. Then, for* $\{U^\sigma\}_{\sigma \in Tp}$ *as in 4.2, one has* $\forall \sigma, \tau \in Tp \;\; U^\sigma < U^\tau$ *in* **C***.*

Proof Assume, by induction on the structure of types, that $U < U^\sigma$ and $U < U^\tau$. Then $U^U < U^{\sigma \to \tau}$. Similarly, $U^{\sigma \to \tau} < U^U < U$, by induction and the reflexivity of U . Clearly, $U < U^U$, as $T < U$. Thus, $\forall \gamma,\mu \in Tp \;\; U^\gamma < U < U^\mu$ in **C** . Δ

As already mentioned, the notion of "principal morphism" and its properties were suggested by some work in higher type recursion theory in Longo&Moggi(1984). In that paper, the authors investigated the higher type partial recursive functionals, namely the type structure $\{PR^\sigma\}_{\sigma \in Tp}$ over the set PR of the partial recursive functions. Within $\{PR^\sigma\}_{\sigma \in Tp}$ one can reconstruct the Kleene-Kreisel countable and continuous functionals of Higher Type Recursion Theory (see Ershov(1976) or Longo&Moggi(1984)). $\{PR^\sigma\}_{\sigma \in Tp}$ may be defined as in 4.2, in the category **CCD** below. We need first to recall a few definitions.

A subset D of a poset (X,≤) is **pairwise consistent** if any pair of elements in D has an upper bound in X . (X,≤) is **pair-coherent** if any pairwise consistent subset has a l.u.b.. It is an easy consequence of Plotkin(1978;theor.11), that P, the partial number theoretic functions, is reflexive in the category of ω-algebraic pair-coherent cpo's. Indeed, the work in

Plotkin(1978) can be "effectivized". Denote by X_o the collection of the compact elements of a ω-algebraic pair-coherent cpo $(X,\leq)$.

4.4 **Definition.** *Let* $\underline{X} = (X,X_o ,e_o,\leq)$ *be an ω-algebraic pair-coherent cpo and* $e_o :\omega\to X_o$ *(bijective). Set* $e_o(n)\Uparrow e_o(m)$ *for "*$e_o(n)$ *and* $e_o(m)$ *have an upper bound". Then* $\underline{X}$ *is* ***effectively given*** *if*

(1) $e_o(n)\Uparrow e_o(m)$ *is a decidable predicate*

(2) $\exists g\in R\ \forall n,m\ (e_o(n)\Uparrow e_o(m) \Rightarrow e_o (g(n,m)) = sup\{e_o(n), e_o(m)\})$.

It is easy to show that, if $\underline{X}$ to $\underline{Y}$ are effectively given, then the space of continuous functions from $\underline{X}$ to $\underline{Y}$ is also effectively given. Indeed, the corresponding category, with continuous functions as morphism, is cartesian closed.

Recall now that ideals are downward closed directed subset of a poset $(X,\leq)$. The idea now is to take, within an effectively given $(X,X_o,e_o,\leq)$, only the l.u.b of those ideals of X_o , which are indexed over a recursively enumerable set. This is done in analogy with the partial recursive functions (or the r.e. sets), as they are exactly the partial functions approximated by r.e. collections of functions with finite domain (or by r.e. coleections of finete sets). By this, one may call **computable** elements the l.u.b of the r.e. indexed ideals.

4.5 **Definition** ***If*** $\underline{X} = (X,X_o ,e_o,\leq)$ *is an effectively given ω-algebraic pair-coherent cpo, then* X_c *is the corresponding* ***constructive and pair-coherent domain (ccd)*** *, i.e. the subposet of computable elements.*

Note that this definition makes sense, since X_c is dense in X , w.r.t. the Scott topology, and, by the ideal completion of X_c, one recovers exactly X (to within isomorphism).

It is easy to show that the category **CCD** whose objects are ccd's and morphisms the continuous and computable functions (computable as elements of the function spaces) is cartesian closed. By an easy constructive variant of Plotkin's theorem (use computable retractions) one has that PR is reflexive in **CCD**.

Consider now the type structure $\{PR^\sigma\}_{\sigma\in Tp}$ constructed over PR in **CCD**. These are the higher type partial recursive functionals. By 4.3,

$\{PR^\sigma\}_{\sigma\in Tp}$ *yield a (type-free) λ-model in any finite type, as* $PR^{\sigma\to\sigma} < PR^\sigma$.

CCD tidely relates by theorem 4.6 below, to categories previously defined, provided the a minor generalization is made.

So far we have only been dealing with **total** applicative structures, i.e. where "·" is everywhere defined, as combinatory algebras are total structures. There exist, though, interesting **partial** applicative structures: for example Kleene's K = $(\omega,\cdot)$, where $n\cdot m = \phi_n(m)$ for some acceptable gödel numbering $\phi : \omega \to PR$ of the partial recursive functions.

Given a partial applicative structure $B = (X,\cdot)$, one may define the categories $\mathbf{P}_B$, $\mathbf{ER}_B$ and $\mathbf{PER}_B$ as in 2.3 and 3.1, with a minor caution. Since we deal here with categories of total morphisms, we consider only total polynomial in each $\mathbf{P}(X^n,X^m)$; in particular in $\mathbf{P}(X,X)$, when defining $\mathbf{ER}(R,S)$ in 3.1. As for $\mathbf{PER}(R,S)$, each $f\in\mathbf{PER}(R,S)$ must be "computed", in the sense of 3.1, by a (possibly partial) $g\in\mathbf{P}(X,X)$ which must be total on domR , though. (Partial morphisms are a different, long, story, see Longo & Moggi (1984a), Asperti & Longo (1988), Rosolini & Robinson (1988), Curien & Obtulowcitz (1988), Moggi (1986) for categories with partial morphisms). If $XxX < X$ in $\mathbf{P}_B$, then 3.2(i) applies similarly and $\mathbf{P}_B$ and $\mathbf{ER}_B$ are full sub-CC of the CC $\mathbf{PER}_B$. Moreover, if B is a partial combinatory algebra, then $\mathbf{PER}_B$ is a CCC by 3.2(ii) and 3.3. The remaining results carry on similarly.

Consider now Kleene's K = $(\omega,\cdot)$. Clearly, K is a partial combinatory algebra, as it contains (indices for) partial k and s .

4.6 **Theorem** (Generalized Myhill-Shepherdson Th.) ***CCD** is equivalent, as category, to a full subCCC of* $\mathbf{ER}_K$.

(For a proof see Ershov(1974), Giannini&Longo(1984), Rosolini(1986), Berger(1986), where the theorem is given for a slightly bigger category than **CCD**)

It should be clear why Kleene's K is only a partial combinatory algebra and not a total one. If ω^ω represents $\mathbf{PER}(\omega,\omega) = \mathbf{P}(\omega,\omega)$, then there is no principal $p\in\mathbf{PER}(\omega,\omega^\omega)$, as there is no gödel-numbering or effective enumeration of R .

Note that ω may be turned into an object $\omega^\perp$ of **CCD** in a natural way by adding a least, bottom, element $\perp$ which interprets undefined computations (see Longo & Moggi (1984a) or Asperti & Longo (1988-9) for the "bottom functor" in CCC's with partial morphisms). $\mathbf{CCD}(\omega^\perp,\omega^\perp)$, then, coincides with PR plus the everywhere constant functions on $\omega^\perp$. From any acceptable gödel-numbering of PR it is easy to construct a principal $p\in\mathbf{CCD}(\omega^\perp,PR)$; however, $\omega^\perp\times\omega^\perp < \omega^\perp$ in **CCD** fails, by a simple continuity argument. Thus also $(\omega^\perp,\cdot)$ does not yield a combinatory algebra. By $PR^{\sigma\to\sigma} < PR^\sigma$ in **CCD** and 3.9, though, λ-models may be found at any finite higher type.

Note that p above or gödel-numberings are principal morphisms which cannot be turned into retractions, by the latter observation plus 3.8(1) or by observing that, given a function, there is not uniform effective choice of one of its indices. More examples could be given by taking combinatory algebras which cannot be turned into λ-models. In Barendregt &

Koymans (1980) an example is given by using the term model of Combinatory Logic, i.e. by a "model" constructed by purely syntactic tools. (It is surprising that no one has been constructed so far a "proper" combinatory algebra, independently of the syntax, while we have plenty of λ-models).

Since the theory of Combinators appears to be the "minimal" functional language, the categorical characterization we gave, entirely independent of the sintax, should give an insight into the actual mathematical semantics of functional abstraction and application.

Aknoledgements A few discussions, in Pisa, with Dana Scott made us understand the category- theoretic significance of a notion we had. Roger Hindley suggested several improvements and changes in the manuscript.

References

Adachi T. (1983) "A categorical characterization of lambda-calculus models" Dept. of Info. Sci. Tokyo Inst. Tech., n. C-49.

Asperti A., Longo G. (1988) "Categories of partial morphisms and the relation between type-structures" Lecture delivered at the **Semester on theory of Computation**, Banach Mathematical Center, Warsaw, December 1985 (preliminary version: CAAP '86, LNCS 214, Spriger-Verlag).

Asperti A., Longo G. (1989) "**Categories for denotational semantics:** category theory for the working computer scientist", book, in preperation.

Backus J. (1978) "Can Programming be liberated of the Von Neumann style? A functional style and its algebra of programs" **CACM** 21, 8 (613-641).

Barendregt H. (1983) "Lambda-calculus and its models" in Proceedings of **Logic Colloqium 82** (Lolli, Longo, Marcja eds), North-Holland.

Barendregt, H. (1984), **The lambda calculus; its syntax and semantics**, Revised and expanded edition, North Holland.

Barendregt, H.P., Koymans, K., (1980) **Comparing some classes of lambda calculus models,** in: Hindley and Seldin (1980), pp. 287-302.

Berger, U. [1986], **Berechenbarkeit in Hoheren Typen Nach Einem Ansatz Von Ju.L. Ersov**, dissertation Munchen.

Berry G. (1979) "Some Syntactic and Categorial Constructions of lambda-calculus models" INRIA, Valbonne.

Bruce K., Longo G. (1984) "On Combinatory Algebras and their expansions," **Theor. Comp. Sci. 1-2, 31,** (31-40).

Curien P.-L. (1986) "Categorical Combinators" **Info.&Co.** (to appear).

Curien P.L. (1986) **Categorical Combinators and Functional Programming**, Pitman.

Curien, P-L., Obtulowicz, A., (1988) "Partiality, Cartesian Closedness and Toposes", **Info.&Co.** (to appear).

Ershov, Ju.L. (1973) "Theorie der Numerierungen I", **Zeitschr, f. math. Logik und Grundl. d. Math.**, Bd 19.

Ershov, Ju.L. (1975) "Th. d. Numm. II", **Zeitschr, f. math. Logik und Grundl. d. Math.**, Bd 21.

Ershov Yu. L. (1976) "Model C of the partial continuous functionals," **Logic Colloquium 76** (Gandy, Hyland eds.) North Holland, 1977.

Giannini P., Longo G. (1984) "Effectively given domains and lambda calculus semantics," **Info.&Co.** 62, 1 (36-63).

Girard, J. (1972) "Interpretation fonctionelle et elimination des coupure dans l'arithmetic d'ordre superieur," These de Doctorat d'Etat, Paris.

Hayashi S. (1985) "Adjunction of semifunctors: categorical structures in non-extensional lambda-calculus" **Theor. Comp. Sci.** 4

Hindley, R. (1983) "The completeness theorem for typing lambda-terms," **Theor. Comp. Sci.** 22 (1-18).

Hindley R., Longo G. (1980) "Lambda-calculus models and extensionality," **Zeit. Math. Logik Grund. Math.** n. 2, vol. 26 (289-310).

Hindley R., Seldin, J. (eds.) (1980) **To H.B. Curry: Essays in Combinatory Logic, Lambda calculus and formalism**, Academic Press.

Hindley R., Seldin J. (1986) **Introduction to Combinators and Lambda-Calculus**, London Mathematical Society.

Hyland M. (1982) "The effective Topos," *in* **The Brouwer Symposium**, (Troelstra, Van Dalen eds.) North-Holland.

Hyland M. (1987) "A small complete category" Lecture delivered at the Conference **Church's Thesis after 50 years**, Zeiss (NL), June 1986 (**Ann. Pure Appl. Logic**, to appear).

Koymans K. (1982) "Models of the lambda calculus", **Info.&Co.** 52, pp. 306-332.

Lambek J. (1980) "From lamba-calculus to cartesian closed categories," *in* Hindley andSeldin[1980], (pp. 375-402).

Lambek J., Scott P.J. (1986) **Introduction to higher order Categorial Logic**, Cambridge University Press.

Longo, G. (1983) "Set-Theoretical Models of Lambda-calculus: Theories, Expansions, Isomorphisms," **Annals Pure Applied Logic** vol. 24, (153-188).

Longo, G. (1984) "Limits, higher type computability and type free languages," **Math. Found. Comp. Sci.** , Prague (Chytil, Koubek eds.), LNCS 176 , Springer-Verlag, 1984 (pp. 96-114).

Longo G.(1987) "On Church's Formal Theory of functions and functionals," Lecture delivered at the Conference **Church's Thesis after 50 years**, Zeiss (NL), June 1986 (**Ann. Pure Appl. Logic**, to appear).

Longo G., Moggi E. (1984) "The Hereditary Partial Recursive Functionals and Recursion Theory in higher types," **J. Symb. Logic**, vol. 49, 4 (pp. 1319-1332).

Longo G., Moggi E. (1984a) "Cartesian Closed Categories of Enumerations and effective Type Structures," **Symposium on Semantics of Data Types** (Khan, MacQueen, Plotkin eds.), LNCS 173, Springer-Verlag, (pp. 235-247).

Longo G., Moggi E. (1984b) "Gödel numberings, principal morphisms, combinatory algebras" *Prelim. version of this paper* **Math. Found. Comp. Sci.**, Prague 1984 (Chytil, Koubek eds.) LNCS 176 , Springer-Verlag, 1984 (pp. 397-406)).

Longo G., Moggi E. (1988) "Constructive Natural Deduction and its Modest Interpretation," Report CMU-CS-88-131, to appear in "**Semantics of natural and programming languages**" Stanford, March 1987 (**M.I.T. Press**, Meseguer ed.).

MacLane S. (1971) **Categories for the working mathematician** Springer-Verlag.

Martini S. (1988) "Modelli non estensionali del polimorfismo in programmazione funzionale" Tesi di Dottorato, Pisa.

Meyer, A. R. (1982) "What is a model of the lambda calculus?", **Information and Control** 52, (pp.87-122).

Moggi E.(1986) "Categories of partial morphisms and the partial lambda-calculus" Proc. Workshop on **Category Theory and Comp. Sci.**, Guildford, **LNCS 240**.

Moggi E.[1988] "The partial lambda-calculus" Ph. D. Thesis, Edinburgh.

Obtulowicz A. , Wiweger A. (1982) "Categorial, Functorial and algebraic aspects of the type-free lambda-calculus" **Universal Algebra and Applications** Banach Center Publications vol. 9, (399-422) **PWN**-Polish Scientific Publishers, Warszawa.

Plotkin G. (1978) "T^ω as a universal domain," **J. Comp. Syst. Sci.**, 17 (pp. 209-236).

Poigne' A. (1985) "On specification Theories and Models with higher types", **Info.&Co.** 6, (pp. 1-46).

Robinson E., Rosolini P. [1988] "Categories of partial maps" **Info&Co.**, to appear.

Rogers H. (1967) **Theory of recusive functions and effective computability**, MacGraw Hill.

Scott D. (1976) "Data types as lattices," **SIAM Journal of Computing**, 5 (pp. 522-587).

Scott D. (1980) "Relating theories of the lambda-calculus," *in* Hindley/Seldin(1980).

Scott, D. (1982) "Some ordered sets in Computer Science," in **Ordered Sets** (Rival Ed.), Reidel.

Smyth M. (1977) "Effectively Given Domains", **Theoret. Comput. Sci. 5,** pp 255-272.

Troelstra A. (1973) **Metamathematical investigation of Intuitionistic Arithmetic and Analysis**. **LNM 344** , Springer-Verlag, Berlin.