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SET-THEORETICAL MODELS OF λ-CALCULUS: THEORIES, EXPANSIONS, ISOMORPHISMS*

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0. Introduction

This paper mainly deals with the models for type free λ-calculus defined by Plotkin [18], Engeler [12] and Scott [22]. (See also [9] for a similar construction.)

Plotkin–Scott-Algebras (PSE-Algebras, in view of Engeler's approach) are built up in a very natural set theoretic way and provide a generalization of early ideas in Scott [20, 21]. Namely, the notion of application (interpreting formal application of λ-terms) generalizes the classical Myhill–Shepherdson–Rogers definition of application in $P\omega$, introduced to define Enumeration Operators (see [19, p. 143]). Abstraction is defined accordingly.

An interesting fact is that these definitions do not depend on codings of pairs and of finite sets, while the classical ones do. This doesn't affect the Recursion Theory one should be able to work out on PSE-Algebras (cf. [6, 16 §2, 21]), but does affect the model theory of **λ**-calculus (see [3] and Section 5). Moreover, for various reasons which should become clear in the next sections, these structures are very 'handy': it is easy to grasp the intuition on which the definitions rely and to modify them for the purpose of the model theory of **λ**-calculus we aim at.

Section 1.1 introduces **λ**-terms and CL-terms (terms of **λ**-calculus, $\boldsymbol{\lambda}\beta$, and of Combinatory Logic, CL) of various orders, corresponding to levels of functionality or number of **λ**-abstractions. Section 1.2 discusses the consequences in Combinatory Algebras of an early remark of Wadsworth (and Scott) on how to interpret the 'loss of information' which is implicit in performing combinatory reductions, as in any effective process.

Section 2 introduces PSE-Algebras and deals with the local analysis (according

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to the terminology in [2]) of Engeler's models. That is, a syntactical characterization is given of the true equalities in the free PSE-Algebras $\langle D_A, \cdot\rangle$ generated from a set of atoms A, with the 'canonical' interpretation of $\boldsymbol{\lambda}$-abstraction. Actually the partial order on these structures (i.e. set theoretic inclusion) matches perfectly well the very natural syntactical partial order over $\boldsymbol{\lambda}$-terms, given by inclusion of Boehm trees (the proofs are in Appendix B). This provides an algebraic characterication of $\boldsymbol{\lambda}$-terms possessing normal form.

Section 3 gives a semantical characterization of $\boldsymbol{\lambda}$-terms of any finite (and infinite) order, i.e., for $n \in \omega$, characterizes the class of terms such as $\lambda x_1 \cdots x_n.N$ according to n. In particular (closed) terms of order 0 are interpreted by the bottom element $\perp$ of the lattice-theoretic model considered and terms of order infinity by the top element. This is done in $\langle D_A, \cdot\rangle$, with a different interpretation of λ-abstraction.

Section 4 contains the main model-theoretic applications of this paper. Theorem 4.1 proves that, if α is an infinite cardinal, there exists a model of CL of cardinal α, where it is possible to give several interpretations of λ-abstraction, which yield different sets of true equations (i.e. a model of CL which yield models of $\lambda\beta$ with different theories; this is a strengthening of Los–Vaught theorem for λ-calculus). Theorem 4.6 gives a counter-part of Theorem 4.1: for any infinite cardinal α, there exists a non-extensional model of CL, which yields a unique model of $\lambda\beta$. These models will be called lambda-categorical. The proof is based on a Structureal Lemma for PSE-Algebras (Lemma 4.3).

Theorem 4.10 deals with a purely algebraic consequence of the previous results. As already mentioned, PSE-Algebras generalize application as defined for enumeration reducibility in $\langle P\omega, \cdot\rangle$. In fact, $\langle D_A, \cdot\rangle$ and $\langle P\omega, \cdot\rangle$ can be isomorphically embedded one into the other; but, using the previous local analysis, it is shown that for no A they are isomorphic (w.r.t. '$\cdot$').

An Intermezzo and Section 5 discuss extensionality and 'non well-founded' models.

In particular, by Theorem 1.12, the fixed point operator of λ-calculus, which gives the recursive definitions in the theory, is interpreted by Tarski's fixed point map in the models studied in Sections 2 and 3 (see Remark 3.9 for the generality of these models). This is not so in the case of structures which are not 'well founded', in the sense of Scott [22].

The notation is mainly from [2] and [17] unless explicitly defined (or elsewhere referred). Some acquaintance with Barendregt's book [2] is required.

Open problems are stated in several places and in the Conclusion.

Basic results:

Section 1: For (partly) known syntactic notions of order and tree of terms (BT and T), conditions are given on a Combinatory Algebra D, such that, for terms M, N,

$$M \text{ has proper order } 0 \;\Rightarrow\; D \models M = \perp.$$

If D yields a λ-model,

$$T(M) \subseteq T(N) \Rightarrow D \models M \leqslant N.$$

Section 2: Let $\langle D_A, \cdot, \boldsymbol{\lambda}\rangle$ be the canonical λ-expansion of the free PSE-Algebra $\langle D_A, \cdot\rangle$, over an arbitrary set A of generators, then

$$BT(M) \subseteq BT(N) \Leftrightarrow D_A \models M \subseteq N.$$

Then normal forms are semantically characterized.

Section 3: There is a non standard λ-expansion of $\langle D_A, \cdot\rangle$, say $\langle D_A, \cdot, \lambda^+\rangle$. If B is the largest element of D_A,

$$D_A^+ \models M = \bot \Leftrightarrow M \text{ is (closed) of order } 0,$$

$$D_A^+ \models M = B \Leftrightarrow M \text{ has order infinity.}$$

More generally, the order of λ-terms is semantically characterized. The theory of D_A is given by the above-mentioned trees T.

Section 4: PSE-Algebras with no atoms (i.e. any element is a set of pairs) are lambda-categorical. (Some Atomless PSE-Algebras are given, as quotient sets.) The canonical λ-expansion of $\langle D_A, \cdot\rangle$ is the least λ-expansion. By this, for any infinite cardinal there are (sub-) PSE-Algebras with several λ-expansions (and different equational theories, BT and T for example) as well as lambda-categorical ones. $\langle P\omega, \cdot\rangle \Rightarrow \langle D_A, \cdot\rangle$ (iso-embedded) and, for A countable, $\langle D_A, \cdot\rangle \Rightarrow \langle P\omega, \cdot\rangle$. But for no A they are isomorphic.

Section 5: There exist quotient PSE-Algebras where $Y_T \neq Y$. Though, for a suitable equivalence relation, $\langle P\omega, \cdot\rangle$ is isomorphic to a quotient atomless PSE-Algebra, $\langle D_A^0, \cdot\rangle$. Then $D_A^0 \models Y = Y_T$ (and $\langle P\omega, \cdot\rangle$ is lambda-categorical).

1. Approximation and application

1.1. Syntax

Combinatory Logic (CL) is the system whose terms are defined using just variables, two constants K and S and the formation rule

(MN) is a term, if M and N are terms.

MNP stands for $((MN)P)$. K and S satisfy

$$KMN = M, \qquad SPQR = PR(QR).$$

The '=' predicate satisfies the usual rules for equality (see e.g. [2, Section 7; 11].

λ-calculus $(\lambda\beta)$ is defined using just variables and the binding operator λ. The basic axiom schema is

(β) $\quad (\lambda x.M)N = [N/x]M,$

for N free for x in M, as usual.

'=' behaves similarly as in CL, including substitutivity (see e.g. [1, 2]).

Define now for terms in CL:

$$\lambda^* x.x \equiv SKK;\ \lambda^* x.M = KM, \text{ if } x \text{ is not free in } M;$$

$$\lambda^* x.PQ = S(\lambda^* x.P)(\lambda^* x.Q), \text{ otherwise.}$$

Strong Combinatory Logic (CLβ) is the extension of CL by the rule

$$(\xi) \qquad \frac{M = N}{\lambda^* x.M = \lambda^* x.N}$$

CLβ can be equivalently defined, as an equational theory, using the five Curry's axioms A_β (see [2, Section 7.3.6]).

CLβ and $\lambda\beta$ are necely related at a syntactical level. In particular one can go from λ-terms to CL-terms (and vice-versa) preserving provable equalities (see [2, Sections 7.1.4–7.3.1]). Barendregt's translations $(\ )_\lambda : \mathrm{CL} \to \lambda\beta$ and $(\ )_{\mathrm{CL}} : \lambda\beta \to \mathrm{CL}$, invertible up to provable equalities, are a tidy way for doing this.

The notion of order for a term informally corresponds to its 'functionality': a term of order 0, so to say, does not 'begin' with a $\lambda x \cdots$ (see Definition 1.2(i)) or, when applied to another term, does not 'act' on it. Formally, for terms in CL:

Definition 1.1. (i) A CL-term M is of *order* 0 ($M \in \mathrm{O}_0$) iff

$$\neg\exists\, N \in \{K, S, KU, SU, SVU\colon U, V \text{ CL-terms}\}\ \mathrm{CL}\beta \vdash M = N.$$

(ii) A CL-term is of *proper order* 0 ($M \in \mathrm{PO}_0$) iff $M \in \mathrm{O}_0$ and ($\neg\exists\vec{N}$ CL-terms $\mathrm{CL}\beta \vdash M = x\vec{N}$) (cf. [11, p. 145]; $\vec{N}, \vec{Q}, \ldots$ are finite vectors (sets) of terms, possibly empty).

Example. $x \in \mathrm{O}_0$; $SII(SII) \in \mathrm{PO}_0$, for $I = SKK$.

Working in $\lambda\beta$, it is easy to define terms of order n, for any $n \in \omega$, as well as terms of order infinity.

Definition 1.2. Let M be a λ-term. Then

(i) $M \in \mathrm{O}_n$ iff n is the largest such that $\exists N\, \lambda\beta \vdash M = \lambda x_1 \cdots x_n.N$.

(ii) $M \in \mathrm{O}_\infty$ iff $\forall n\, M \notin \mathrm{O}_n$

Example. $YK \in \mathrm{O}_\infty$, where Y is a 'fixed point operator'.

Proposition 1.3. (i) *Let M be a* CL-*term. Then*

(i.0) $M \in \mathrm{O}_0 \Leftrightarrow M_\lambda \in \mathrm{O}_0$;

(i.1) $M \in \mathrm{PO}_0 \Leftrightarrow M_\lambda \in \mathrm{O}_0$ and $\neg\exists\vec{N}\, \lambda\beta \vdash M_\lambda = x\vec{N}$.

(ii) *Let M be a λ-term. Then*

$$M \in \mathrm{O}_\infty \Leftrightarrow \forall n\, \exists m > n\, \exists N\, \lambda\beta \vdash M = \lambda x_1 \cdots x_m.N.$$

Proof. Easy. □

Clearly terms of order 0 are exactly the terms with no functionality: λ-terms in PO_0 are defined as in Definition 1.1(ii), using Definition 1.2(i) for O_0.

M is *solvable* iff $\exists n\ \exists y\ \exists \vec{N}\ \lambda\beta \vdash M = \lambda x_1 \cdots x_n.y\vec{N}$ (cf. [2]).

Lemma 1.4. *M is unsolvable iff*

(1) $M \in O_\infty$, *or*

(2) $\exists n \geqslant 0\ \exists N \in PO_0\ \lambda\beta \vdash M = \lambda x_1 \cdots x_n.N.$

Proof. $\Leftarrow$. By definition of head normal form [2, p. 41].

$\Rightarrow$. We prove $(M \notin O_\infty \Rightarrow (2))$, when M is unsolvable.

Let $M \notin O_\infty$. Assume that $n \geqslant 0$ is the largest such that $\lambda\beta \vdash M = \lambda x_1 \cdots x_n.N$. Then $\lambda\beta \nvdash N = x\vec{P}$, for M is unsolvable. Moreover $N \in O_0$ by maximality of n. Hence $N \in PO_0$. □

1.2. Semantics

Definition 1.5. Let '$\cdot$' be a binary operation (application) on a set D. Then $\langle D, \cdot\rangle$ is a *Combinatory Algebra* iff D contains elements $\boldsymbol{K}$ and $\boldsymbol{S}$ ($\boldsymbol{K} \neq \boldsymbol{S}$) satisfying:

$$\boldsymbol{K} \cdot d_0 \cdot d_1 = d_0,$$
$$\boldsymbol{S} \cdot d_0 \cdot d_1 \cdot d_2 = d_0 \cdot d_2(d_1 \cdot d_2) \qquad \text{for all } d_0, d_1, d_2 \in D.$$

Thus in a Combinatory Algebra $\langle D, \cdot\rangle$ one can interpret S and K of CL, by some $\boldsymbol{S}$ and $\boldsymbol{K}$. For each choice of $\boldsymbol{S}$ and $\boldsymbol{K}$ in D, one obtains an *expansion* $\langle D, \cdot, \boldsymbol{S}, \boldsymbol{K}\rangle \vDash$ CL, where CL contains S and K in the signature.

Definition 1.6. $\langle D, \cdot, \Psi_\lambda\rangle$ is a *Combinatory Model* iff setting $(D^n \to D) = \{f : D^n \to D : \exists d \in D\ \forall \vec{e} \in D^n\ f(e) = d.e_1 \cdots e_n\}$, the representable functions, one has:

(0) $\langle D, \cdot\rangle$ is a Combinatory Algebra.

(1) $\Psi_\lambda : (D \to D) \to D$ and $\Psi_\lambda(f).e = f(e)$.

(2) For $f \in (D^{n+1} \to D)$, $\lambda\vec{x} \in D^n.(\Psi_\lambda(\lambda y \in D.f(\vec{x}, y))) \in (D^n \to D)$, where $\lambda x \in D.(\cdots)$ is the function $d \vdash (\cdots)[d/x]$.

Combinatory Models correspond to Environmental Models, as defined in [13] or in [17]. Meyer's Combinatory Model Theorem proves the equivalence of this notion with his purely algebraic definition of Combinatory Model (see the following 'Discussion').

Let τ be an algebraic expression over D (see [2, p. 89]; i.e. τ is built up with variables, constants from D and '$\cdot$'). Then $\lambda x \in D.\tau$ is the function $d \vdash \cdot \tau[d/x]$. By combinatory completeness, i.e. by Definition 1.6(0), $\lambda x \in D.\tau \in (D \to D)$ possibly with parameters. We write $\boldsymbol{\lambda}x.\tau$ for $\Psi_{\boldsymbol{\lambda}}(\lambda x \in D.\tau)$. So, for $f \in (D \to D)$, $\lambda x.f(x)$ is the element of D which canonically represents the function f. Thus Definition 1.6(1) reads $(\boldsymbol{\lambda}x.f(x)) \cdot e = f(e)$, which better recalls the schema (β) of λ-calculus (cf. [2, ch. 5]). By a small abuse of language, we will also write $\boldsymbol{\lambda}d.f(d)$ for $\boldsymbol{\lambda}x.f(x)$ and consider $\boldsymbol{\lambda}$ as a map from $(D \to D)$ into D, writing $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ for $\langle D, \cdot, \Psi_{,\boldsymbol{\lambda}}\rangle$.

Given $\langle D, \cdot\rangle$, there may be several choices of $\boldsymbol{\lambda}$; each one provides a specific λ-*expansion* of $\langle D, \cdot\rangle$. Each Combinatory Model $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ naturally yields an expanded Combinatory Algebra: set $\mathbf{K} = \boldsymbol{\lambda}x.(\boldsymbol{\lambda}y(x))$, $\mathbf{S} = \boldsymbol{\lambda}xyz.xz(yz)$ (we omit '$\cdot$' in $d \cdot e$). Following [17], we call these expanded Combinatory Models λ-*Models*.

In view of Meyer's Lambda Model theorem, we shall ignore the distinction between these interchangeable notions of model of $\lambda\beta$ and use the phrase λ-model throughout the rest of this paper (cf. also [2] or [13]).

We write $D \models \cdots$ for $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ ($\langle D, \cdot, \mathbf{S}, \mathbf{K}\rangle) \models \cdots$, if there is no ambiguity.

In Section 4 the notion of lambda-categorical model will be introduced, that is of Combinatory Algebra with a unique λ-expansion. Some interesting models are lambda-categorical, some not. As a matter of fact, in a λ-model $\langle D, \cdot, \boldsymbol{\lambda}\rangle$, the key step in interpreting λ-terms is given by:

$$[\![\lambda x.M]\!]\sigma = \boldsymbol{\lambda}x.f(x)(=\boldsymbol{\lambda}(f)), \quad \text{where} \quad f(d) = [\![M]\!]\sigma_x^d$$

and f is in $(D \to D)$ by Definition 1.6(2) and by combinatory completeness (see [13] or [17] for details).

Discussion. (A) Apparently, Combinatory Algebras and λ-models differ only because of a choice function $\boldsymbol{\lambda}$ over the set of representable functions, see Definiton 1.5 and 1.6. Namely, given $\langle D, \cdot\rangle \vdash \mathrm{CL}$ and $f \in (D \to D)$, let $\boldsymbol{\lambda}$ choose an element, $\boldsymbol{\lambda}x.f(x)$, from the extensionality class of f, say $E_f = \{d : \forall e\, f(e) = d \cdot e\} \subseteq D$. Using $\boldsymbol{K}$ and $\boldsymbol{S}$, can't we always have also Definition 1.6(2) satisfied? The point is that, if so, the choice map $d \vdash \boldsymbol{\lambda}x.dx$ can be 'represented' in D. In fact by Definition 1.6(2) one has that $\boldsymbol{\lambda}xy.xy = \boldsymbol{\lambda}(\lambda x \in D.(\boldsymbol{\lambda}(\lambda y \in D.xy))) \in D$ and, for $d \in E_f$, $(\boldsymbol{\lambda}xy.xy)d = \boldsymbol{\lambda}y.dy = \boldsymbol{\lambda}y.f(y) = \boldsymbol{\lambda}(f)$. Using notation from [17], let's set ε for $\boldsymbol{\lambda}xy.xy$; then ε applied to $d \in E_f$ chooses the representative of f in E_f, i.e. $\varepsilon d = \boldsymbol{\lambda}(f)$ and, as it can be easily checked, one has

$$\forall e\; de = d'e \iff \varepsilon d = \varepsilon d'. \tag{1}$$

This is how one can have difficulties, trying to define a choice map $\boldsymbol{\lambda}$: in [4], for example, a Combinatory Algebra D is given (actually a λ-algebra, i.e. a model of strong Combinatory Logic, see [2]), where '$=$' is a Σ_2^0 complete predicate. It is then easy to understand why that Combinatory Algebra cannot be turned into a λ-model: because (1) would give a contradiction in the recursion theoretic hierarcy.

(B) Meyer [17] proves also the converse of the point above. Namely, if a Combinatory Algebra contains an ε satisfying (1) above and

$$\text{for all } d, e \in D, \quad \varepsilon de = de, \tag{2}$$

then $\langle D, \cdot, \varepsilon\rangle$ naturally yields a λ-model. Shortly, set $\boldsymbol{\lambda}(f)=\varepsilon d$ for $f\in(D\to D)$ and $d\in E_f$. then (2) gives Definition 1.6(1). As for Definition 1.6(2), if $f\in(D^{n+1}\to D)$ and $f(\vec{d}, e)=d_f e\vec{d}$, say, then by combinatory completeness $d_f e\vec{d}=d_f'\vec{d}e$, for some d_f'. Thus $\boldsymbol{\lambda}(\lambda e\in D.f(\vec{d}, e))=\varepsilon(d_f'\vec{d})$. By combinatory completeness again, $\varepsilon(d_f'\vec{d})=\varepsilon' d_f'\vec{d}$, for some ε'. Hence $\lambda\vec{d}\in D^n.\,\varepsilon(d_f'\vec{b})\in(D^n\to D)$. (Cf. Also [23] for a first-order approach.)

Definition 1.7. Two (expanded) Combinatory Algebra $\langle D_1, \cdot, (S_1, K_1)\rangle$ and $\langle D_2, x, (S_2, K_2)\rangle$ are *Equationally equivalent* iff $D_1\vDash M=N\Leftrightarrow D_2\vDash M=N$, for all CL-terms M, N.

As well known (β) or CL reductions entail a 'loss of information'. In $(\lambda x.M)N\to[N/x]M$, one knows 'where one goes, but not where one comes from'.

How can this be reflected in the semantics? Given a poset $\langle D, \leqslant\rangle$, let first say that $f: D\to D$ is *ω-continuous* iff, for any ω-chain $\{d_n\}_{n\in\omega}$, if $\bigsqcup d_n$ exists, then $f(\bigsqcup d_n)=\bigsqcup f(d_n)$.

Using ideas from Wadsworth's analysis of Scott's model D_∞, Wadsworth [27] (see also [2, 5]) define:

Definition 1.8. A Combinatory Algebra $\langle D, \cdot\rangle$ has *approximable application* iff

(i) $\langle D, \cdot, \leqslant\rangle$ is a poset, with least element $\perp$, such that '$\cdot$'$: D^2\to D$ is ω-continuous.

(ii) There exists a map $\mathrm{Seg}: Dx\omega\to D$ such that, for $d_n=\mathrm{Seg}(d, n)$, $\forall d, e\in D$ one has

(1) $d=\bigsqcup d_n$,
(2) $d_0=\perp$,
(3) $\perp e=\perp$,
(4) $d_{n+1}e\leqslant(de_n)_n$,
(5) $(d_n)_m=d_{\min\{n,m\}}$.

A way of understanding Definition 1.8 may be the following:
- d_n is d up to 'level n of information';
- applying no information, $\perp$, to something, one gets no information;
- if the operator has level $n+1$ of information, then it uses at most level n of information from the argument and provides at most a value with level n of information.

This has an immediate consequence for the semantics of the class of terms in CL where one can always perform reductions at the leftermost outermost level, i.e. for CL-terms in PO_0.

Theorem 1.9. *Let $\langle D, \cdot\rangle$ be a Combinatory Algebra with approximable application. Then*

$$M\in\mathrm{PO}_0\Rightarrow D\vDash M=\perp.$$

Proof. For the purpose of this proof, let's introduce a labeled CL, CL_0. The formation rules of CL-terms are extended by

$$M \in CL_0 \Rightarrow M^n \in CL_0 \quad \text{for all } n \in \omega;$$

the reduction rules are extended by

$$\begin{array}{lll}
(\text{Klab}) & (1) & K^{n+1}M \to (KM)^n, \\
 & (2) & (KM)^{n+1}N \to M^n, \\
(\text{Slab}) & (1) & S^{n+1}P \to (SP)^n, \\
 & (2) & (SP)^{n+1}Q \to (SPQ)^n, \\
 & (3) & (SPQ)^{n+1}R \to (PR^n(QR^n))^n, \\
(\text{Min}) & & (M^n)^m \to M^{\min(n,m)}.
\end{array}$$

$M \in CL_0$ is *completely labeled* iff each occurrence of S and K in M is labeled. Interpret CL_0-terms in D, by adding $[\![M^n]\!]_\sigma = ([\![M]\!]_\sigma)_n$, for all environment σ.

Claim 1. *Let $M \in PO_0$ and M^I a complete labeling of M. Then $CL_0 \vdash M^I \to N^0\vec{Q}$ for some N, $\vec{Q}$ in CL_0.*

In fact by definition of PO_0-terms, K and S (labeled) rules are always applicable at the 'head' of $M(M^I)$ and its contracts (in particular (Klab)(2) and (Slab)(3), up to label 0).

Claim 2. *If $CL_0 \vdash M \to N$, then $D \models M \leqslant N$.*

Use Definition 1.8(ii) and monotonicity of '$\cdot$'.
Let $M \in PO_0$. Then

$$\begin{array}{lll}
D \models M = \bigsqcup\{M^I : I \text{ complete labeling}\} & & \text{by Definition 1.8(i)–(ii)(1)} \\
\qquad\leqslant \bigsqcup\{N^0\vec{Q} : \forall I\, M^I \to N^0\vec{Q}\} & & \text{by Claims 1 and 2} \\
\qquad = \bot & & \text{by Definition 1.8(ii)(2) and (3).}
\end{array}$$

□

So much for Combinatory Algebras; Theorem 1.9 in full generality will be applied in Theorems 3.6 and 4.10.

In the next sections we will use two notions of 'tree of a λ-term'. For the notion of Böhm-tree of a λ-term M, $BT(M)$, we refer to [1] (or [2]). The partial order '$\subseteq$' on Böhm trees is the usual syntactic one: informally, put the always undefined element '$\bot$' at the bottom and then proceed inductively on the structure of the tree. Recall that $BT(M) = \bot$ iff M is unsolvable.

Definition 1.10. (Informal) Let $\Sigma = \{\lambda x_1 \cdots x_n.\bot : n \geqslant 0\} \cup \{T\} \cup \{\lambda x_1 \cdots x_n.y : n \geqslant 0\}$. Then the *Tree* of M, $T(M)$, is a Σ-labelled tree defined as follows.

$$
\begin{array}{ll}
T(M)=T & \text{if } M\in \mathrm{O}_\infty,\\
T(M)=\lambda x_1\cdots x_n.\bot & \text{if } M \text{ is unsolvable of order } n \text{ (see 1.4)},\\
T(M)=\lambda x_1\cdots x_n.y & \\
\qquad\qquad / \quad \backslash & \\
\quad T(M_1)\cdots T(M_p) &
\end{array}
$$

if M is solvable and has principal head normal form $\lambda x_1\cdots x_n.yM_1\cdots M_p$.

A Tree may be infinite: just mimic [2, p. 212] to give a formal definition. $T(M)$ is obtained from $\mathrm{BT}(M)$ 'displaying' the order of the unsolvable leaves. This can be done with the help of a Σ_2^0 oracle, writing his answers on leaves.

Definition 1.11. The set of Trees is partially ordered by

$T(M)\subseteq T(N)$ iff $T(N)$ is obtained from $T(M)$ by replacing $\bot$ in some leaves of $T(M)$ by Trees of λ-terms *or* by replacing some $\lambda x_1\cdots x_n.\bot$ by T.

Example.

$$
\begin{array}{c}
\lambda x.y\\
/ \quad \backslash\\
\lambda z.\bot \quad \lambda xy.\bot
\end{array}
\quad\subseteq\quad
\begin{array}{c}
\lambda x.y\\
/ \quad \backslash\\
\lambda zv.x \quad T\\
| \qquad\quad\\
z \qquad\quad
\end{array}
$$

($\bot$ in $\lambda z.\bot$ is replaced by $T(\lambda v.xz)$ and $\lambda xy.\bot$ by T.)

Levy [15] gives a partial order on terms, $\leqslant_L$, based on a notion of reduction and on the associated set of approximants. It is easy to prove that $M\leqslant_L N$ iff $T(M)\subseteq T(N)$.

Given a λ-model $\langle D,\cdot,\boldsymbol{\lambda}\rangle$, embed $(D\to D)$ with the pointwise partial order.

Theorem 1.12. *Let $\langle D,\cdot,\boldsymbol{\lambda}\rangle$ be a λ-model with approximable application. Assume also that $\boldsymbol{\lambda}:(D\to D)\to D$ is monotone. Then*

$$T(M)\subseteq T(N) \;\Rightarrow\; D\vDash M\leqslant N.$$

Proof. See Appendix A. $\square$

An easy consequence of Theorem 1.12 is that, in a λ-model D as in Theorem 1.12, all fixed point operators Y of $\lambda\beta$ coincide in D and they represent Tarski's fixed point map $Y_T(f)=\bigsqcup f^n(\bot)$. (See Appendix A for a proof: in Section 3 this is applied to a class of models.)

Definition 1.13. A λ-model $\langle D, \cdot, \lambda\rangle$ has *λ-approximable application* iff $\langle D, \cdot\rangle$ has approximable application and $\boldsymbol{\lambda} x.\bot = \bot$.

Proposition 1.14. *Let $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ be a λ-model with λ-approximable application. Then*

$$\mathrm{BT}(M) \subseteq \mathrm{BT}(N) \;\Rightarrow\; D \vDash M \leqslant N.$$

Proof. Forcing $\lambda x.\bot = \bot$, $T(M)$ collapses to $\mathrm{BT}(M)$: see Appendix A for details $\square$

2. Plotkin–Scott-Algebras and the local structure of Engeler's models

Definition 2.1. Let B be a non empty set such that

(o) $\beta \subseteq B \wedge b \in B \Leftrightarrow (\beta; b) \in B$

where β (as well as $\beta_1, \ldots, \gamma, \ldots$) range over finite sets. Define then

(i) '$\cdot$': $PB \times PB \to PB$ by $d \cdot e = \{b : \exists\, \beta \subseteq e\ (\beta; b) \in d\}$,

(ii) $\langle PB, \cdot\rangle$ is a *Plotkin–Scott-Algebra* (PSE-Algebra, in view of Engeler's approach).

Note that, by $\Leftarrow$ in (o), PB is closed under application '$\cdot$'. In Definition 2.4 a set B satisfying (o) is given (see also later).

Let D, E be topological spaces. Then $C(D, E)$ is the set of continuous functions from D to E.

Lemma 2.2. *Let B be as in Definition* 2.1. *Then*

(1) *$\langle PB, \subseteq\rangle$ is a complete algebraic lattice. The Scott topology on PB is given by the basis*

$$\{d \in PB : \beta \subseteq d\} \quad \textit{for } \beta \textit{ (finite) in } PB.$$

(2) *$f \in C(PB^n, PB)$ iff f is continuous in each argument iff*

$$f(\vec{d}) = \bigcup \{f(\vec{\beta}) : \vec{\beta} \subseteq \vec{d}\}.$$

(3) $(PB^n \to PB) = C(PB^n, PB)$.

Proof. (1) and (2) Routine.

(3) $\subseteq$ holds by the continuity of '$\cdot$'. As for $\supseteq$, note that $\{(\beta_1; (\beta_2; \ldots (\beta_n; b) \cdots) : b \in f(\beta_1, \ldots, \beta_n)\}$ represents $f \in C(PB^n, PB)$. $\square$

Theorem 2.3. *Let $\langle PB, \cdot\rangle$ be a PSE-Algebra. Define $\boldsymbol{\lambda} : C(PB, PB) \to PB$ by $\boldsymbol{\lambda}(f) = \boldsymbol{\lambda} x.f(x) = \{(\beta; b) : b \in f(\beta)\}$. Then $\langle PB, \cdot, \boldsymbol{\lambda}\rangle$ is a λ-model.*

Proof. Observe that $\boldsymbol{\lambda}$ is continuous, when $C(PB, PB)$ is given the Scott topology. Then the result easily follows by the lemma. $\square$

Of course, using Definition 1.6(2),

$$\boldsymbol{\lambda} x_1 \cdots x_n . f(x_1, \ldots, x_n) = \{(\beta_1; (\beta_2; \cdots (\beta_n; b) \cdots)) : b \in f(\beta_1, \ldots, \beta_n)\}.$$

Is there more than one way to turn a PSE-Algebra into a λ-model? This question will be answered in Sections 3 and 4. The equations PSE-Algebras may solve are discussed in Remark 3.9.

Definition 2.4. Let $A \neq \emptyset$. Define

$$B_0 = A, \qquad B_{n+1} = B_n \cup \{(\beta; b) : \beta \subseteq B_n \wedge b \in B_n\},$$
$$B = \bigcup B_n, \qquad D_A = PB.$$

(Recall that β ranges over finite sets. No element of A is denoted by $(\cdots; \cdots)$.)

Thus $\langle D_A, \cdot \rangle$ is the free PSE-Algebra generated from a set A of atoms. In this section (Theorem 2.8, proof in Appendix B) we syntactically characterize the set of true equations of λ-terms in the λ-expansion $\langle D_A, \cdot, \boldsymbol{\lambda} \rangle$, where $\boldsymbol{\lambda}$ is as in Theorem 2.3 and A is just a non empty set. $\langle D_A, \cdot, \boldsymbol{\lambda} \rangle$ has been defined in [12] (see also [18, 22]). A similar construction over a set of type symbols can be found in [9]. In Remark 5.8 it will be shown that also Scott's $P\omega$ model is (isomorphic to) a PSE-Algebra. But, by Theorem 4.8, for no A $\langle P\omega, \cdot \rangle \cong \langle D_A, \cdot \rangle$.

The intuition on which the construction of PSE-Algebras and the definition of '·' is based should be clear: $(\beta; b)$ is an 'elementary instruction' giving output b any time the input contains β. Thus $\forall\, d \in D_A$ $Ad = \emptyset$, since we assume A not to contain pairs such as $(\beta; b)$.

Note also that, by definition, in the case of D_A,

2.5. $\forall\, b \in B\ \exists\, \beta_n, \ldots, \beta_1\ \exists\, a \in A\ b = (\beta_n; \cdots (\beta_1; a) \cdots)$.

This makes D_A 'well founded' in the following sense: there is no infinite descending chain w.r.t. (the transitive closure of) the binary relation $<$ on B, where $b < (\beta; c) \Leftrightarrow b \in \beta \vee b = c \vee (c \in B \setminus A \wedge b < c)$.

The point is now to turn $\langle D_A, \cdot, \boldsymbol{\lambda} \rangle$ into a λ-model with λ-approximable application.

Definition 2.6. (i) (Simultaneous definition of $|\cdot|$ on B and on the finite parts of B, with range in ω.)

$$|b| = \begin{cases} 1 & \text{if } b \in A, \\ |\beta| + |c| & \text{if } b = (\beta; c), \end{cases}$$
$$|\beta| = \max\{|c| : c \in \beta\} + 1.$$

(ii) Let $d \in D_A$. Define $d_n = \{b \in d : |b| \leqslant n\}$. Clearly $|\beta|, |b| < |(\beta; b)|$.

Lemma 2.7. *$\langle D_A, \cdot, \boldsymbol{\lambda} \rangle$ has λ-approximable application.*

Proof. (Part 1: approximable application) We only check Definition 1.8(ii)(4), the rest is trivial.

$$d_{n+1}e=\{b:\exists\,\beta\subseteq e\ (\beta;b)\in d\wedge|\beta|+|b|\leq n+1\}$$
$$\subseteq\{b:\exists\,\beta\subseteq e_n\ (\beta;b)\in d\wedge|b|\leq n\}=(de_n)_n,$$

since $\forall\beta\,\forall b\,|\beta|,|b|\geq 1$ and $|\beta|\leq n$ implies $(\beta\subseteq e\Rightarrow\beta\subseteq e_n)$.

(Part 2: λ-approximable application) $\boldsymbol{\lambda}x.\emptyset=\{(\beta;b): b\in\emptyset\}=\emptyset$. □

Thus Proposition 1.14 applies and $\mathrm{BT}(M)\subseteq\mathrm{BT}(N)\Rightarrow D_A\vDash M\subseteq N$.

To prove the reverse implication one can use the classical Böhm-out technique à la Hyland. A revised version of it is in [5].[1] The point is to substitute Böhm's operator $C_p=\lambda x_0\cdots x_{p+1}.x_{p+1}x_0\cdots x_p$ by a C_p^- whose properties depend on the structure of D_A and such that Lemma 3.3 of [5] applies. The construction of such a C_p^- required 26 technical lemmas, in the case of Plotkin's $T\omega$. For D_A it turns out to be much simpler and it is shown in Appendix B.

Proposition 2.8. *Let $A\neq\emptyset$. Then*

$$\mathrm{BT}(M)\subseteq\mathrm{BT}(N)\ \Leftrightarrow\ \langle D_A,\cdot,\boldsymbol{\lambda}\rangle\vDash M\subseteq N.$$

Proof. By the proceeding remarks and Appendix B. □

Putting together Theorem 1.9 and Lemma 2.7 one has that M is unsolvable iff $\langle D_A,\cdot,\boldsymbol{\lambda}\rangle\vDash M=\emptyset$, thus $\langle D_A,\cdot,\boldsymbol{\lambda}\rangle$ is sensible in Barendregt's sense [2, p. 100].

We conclude with a simple characterization of λ-terms possessing normal form.

Let $(D_A)^0=\{[\![M]\!]: M\in\Lambda^0\}$ be the *interior* of $D_A.(D_A)^0$, as the set of objects interpreting closed λ-terms, can be algebraically characterized by taking $\mathbf{S}=\boldsymbol{\lambda}xyz.xz(yz)\in D_A$ and $\mathbf{K}=\boldsymbol{\lambda}xy.x\in D_A$ and closing with respect to '$\cdot$'.

Corollary 2.9. *Let $M\in\Lambda^0$. Then M has a normal form $\Leftrightarrow$ $\{d\in(D_A)^0: D_A\vDash d\subseteq M\}$ is finite and $[\![M]\!]$ is maximal in $(D_A)^0$.*

Proof. M has a normal forms iff $\mathrm{BT}(M)$ is finite and contains no Ω's. □

This fact is also true in the model $T\omega$; but the authors of [5] were too distracted by the hardware of $T\omega$, to point this out.

3. A semantical characterization of λ-terms of order n, for any $n\in\omega\cup\{\infty\}$

In this section we define a different λ-expansion of the applicative structure $\langle D_A,\cdot\rangle$ defined in Definition 2.4. Namely, for each $f\in(D_A\to D_A)$,

[1] Correction for [5, p. 316, def. 3.4(ii), line 2]: set $y_i=C_p^-\sigma_i$ instead of $\sigma_i=C_p^-y_i$, $0\leq i\leq n$.

$\lambda^+:(D_A \to D_A) \to D_A$ will choose a representative in the extensionality class of f, say $EC_f = \{d: \forall e\, f(e) = de\}$, different from $\boldsymbol{\lambda}(f)$.

Definition 3.1. Let $A \neq \emptyset$, $\langle D_A, \cdot \rangle$ as in Definition 2.4 and $\boldsymbol{\lambda}$ as in Theorem 2.3. Define $\lambda^+:(D_A \to D_A) \to D_A$ by

$$\lambda^+(f) = \lambda^+x.f(x) := \boldsymbol{\lambda}x.f(x) \cup A.$$

Note that for all $A \neq \emptyset$, the D_A's are objects of a Cartesian Closed Category (CPO's), with continuous maps as morphism. As already pointed out $(D_A \to D_A) = C(D_A, D_A)$: it is then easy to show that also $\boldsymbol{\lambda}$ and λ^+ are continuous maps. Moreover $C(D_A^n, D_A) \cong C(D_A, C(D_A^{n-1}, D_A))$. Thus

$$\lambda^+x_1 \cdots x_n.f(x_1, \ldots, x_n) = \lambda^+x_1(\lambda^+x_2 \cdots x_n.f(x_1, \ldots, x_n))$$

is well defined for all $f \in C(D_A^n, D_A)$. Lemmas 3.3, 3.4 and Corollary 3.5 show that $D_A^+ = \langle D_A, \cdot, \lambda^+ \rangle$ is a λ-model, for $A \neq \emptyset$ and D_A, $\cdot$, $\boldsymbol{\lambda}$ and λ^+ defined as in Definition 3.1.

Definition 3.2. Define $A_{(n)} = \lambda^+x_1 \cdots x_n.\emptyset$.

Lemma 3.3. (i) $A_{(0)} = \emptyset$,

(ii) $A_{(n+1)} = \lambda^+x.A_{(n)} = \boldsymbol{\lambda}x_1 \cdots x_n.A \cup \boldsymbol{\lambda}x_1 \cdots x_{n-1}.A \cup \cdots \cup A$,

(iii) $A_{(n)} \subset A_{(n+1)}$,

(iv) $\forall\, d \in D_A\ (A_{(n+1)})d = A_{(n)}$,

(v) $\lambda^+x_1 \cdots x_n.f(x_1, \ldots, x_n) = \boldsymbol{\lambda}x_1 \cdots x_n.f(x_1, \ldots, x_n) \cup A_n$, *for all* $f \in C(D_A^n, D_A)$,

(vi) $\lambda^+x_1 \cdots x_n.f(x_1, \ldots, x_n)\vec{d} = \lambda^+x_{p+1} \cdots x_n.f(d_1, \ldots, d_p, \ldots, x_n)$, *for all* $f \in C(D_A^n, D_A)$ *and* $\vec{d} = \{d_1, \ldots, d_p\}$, *with* $p \leq n$.

Proof. (i) Obvious; (ii) easy induction; (iii) by (ii).

(iv) by $Ad = \emptyset$, for all $d \in D_A$, and continuity of '$\cdot$' (recall that $\boldsymbol{\lambda}x.A = \{(\beta; b): b \in A\}$).

(v) induction, again; (vi) by (iv) and (v). □

Lemma 3.4. $\forall\, f \in (D_A \to D_A)$, $\lambda^+x.f(x)$ *is the largest element in* EC_f.

Proof. Let $d \in EC_f$. Clearly $d \cap A \subseteq \lambda^+x.f(x)$. Let $(\beta; b) \in d$. Then $b \in d\beta$, i.e. $b \in f(\beta)$ and we are done. □

Corollary 3.5. $D_A^+ = \langle D_A, \cdot, \lambda^+ \rangle$ *is a* λ*-model. Moreover it is the unique* λ*-expansion of* $\langle D_A, \cdot \rangle$ *satisfying* $\forall\, d \in D_A\ d \subseteq \lambda^+x.dx$.

Proof. The first part is by Lemma 3.3; note that if $\lambda\beta \vdash M = \lambda x_1 \cdots x_n.N$, then

$$\forall \sigma [\![M]\!]_\sigma^+ = \boldsymbol{\lambda}d_1 \cdots d_n.[\![N]\!]^+\sigma[\vec{d}/\vec{x}] \cup A_{(n)}, \quad \text{by Lemma 3.3(v).} \tag{1}$$

(Use 2.3 and what follows it.)

Assume now that $\langle D_1, \cdot, \lambda' \rangle$ is a λ-expansion such that $\forall d\, d \subseteq \lambda' x.dx$. Recall that

$$\forall\, d, e\ (d \cup A)e = de. \tag{2}$$

Then, for all $f \in (D_A \to D_A)$,

$$\begin{aligned} \boldsymbol{\lambda} y.f(y) \cup A &\subseteq \lambda' x.(\boldsymbol{\lambda} y.f(y) \cup A)x && \text{by assumption} \\ &= \lambda' x.(\boldsymbol{\lambda} y.f(y))x && \text{by (2)} \\ &= \lambda' x.f(x) && \text{by Theorem 2.3.} \end{aligned}$$

By Lemma 3.4, we are done. $\square$

Clearly $\langle D_A, \cdot \rangle$ possess at least $\mathrm{Card}(2^A)$ λ-expansions. This λ-model provides a semantical characterization of the λ-terms in PO_0, O_∞ and O_n, for all n.

Theorem 3.6. *Let M be a λ-term. Then*

$$M \in PO_0 \;\Leftrightarrow\; D_A^+ \vDash M = \emptyset.$$

Proof. $\Rightarrow$. This follows from Theorem 1.9. Notice that Theorem 1.9 depends only on the applicative structure of $\langle D_A, \cdot \rangle$, i.e. on $\langle D_A, \cdot \rangle$ as a Combinatory Algebra, not on the λ-expansions which may turn it into a λ-model.

$\Leftarrow$. Assume $M \notin PO_0$.

Case $\lambda\beta \vdash M = x\vec{Q}$, for some $Q_1, \ldots, Q_p$: Then, since $D_A^+ \vDash M = \emptyset \Leftrightarrow \forall \sigma\, [\![M]\!]_\sigma^+ = \emptyset$, $D_A^+ \nvDash M = \emptyset$, by taking $\sigma(x) = \boldsymbol{\lambda} x_1 \cdots x_p.A$.

Case $\lambda\beta \vdash M = \lambda x.N$, for some N: Then

$$\forall \sigma\, [\![M]\!]_\sigma^+ = \lambda^+ d.[\![N]\!]^+ \sigma_x^d = \boldsymbol{\lambda} d.[\![N]\!]^+ \sigma_x^d \cup A \neq \emptyset. \quad \square$$

Of course, Theorem 3.6.$\Leftarrow$ depends on the λ-expansion (cf. Proposition 2.8).

Theorem 3.7. *Let M be a λ-term. Then*

(i) $M \in O_n \Leftrightarrow (D_A^+ \vDash M \supseteq A_m \underset{(1)}{\Leftrightarrow} m \leq n)$,

(ii) $M \in O_\infty \Leftrightarrow D_A^+ \vDash M = B$.

Proof. (i). $\Rightarrow$. Assume $\lambda\beta \vdash M = \lambda x_1 \cdots x_n.N$, with $N \in O_0$. Then (1), in corollary 3.5, immediately gives $\underset{(1)}{\Leftarrow}$.

As for $\underset{(1)}{\Rightarrow}$, assume $m > n$.

Case $N \in PO_0$. By Theorem 3.6 and (1), $\forall \sigma\, [\![M]\!]_\sigma^+ = A_n$ (recall that $\boldsymbol{\lambda} x.\emptyset = \emptyset$), while $A_n \subset A_m$, for $m > n$.

Case $N \equiv x_i\vec{Q}$, for some $\vec{Q}$ and $i \leq n$. Take $b = (\beta_1; \cdots (\underset{i\text{th}}{0}\,; \cdots (\beta_m; a) \cdots))$ for some $\vec{\beta}$ in D_A and $a \in A$. Clearly $b \in A_m \setminus A_n$; thus $b \in \boldsymbol{\lambda} d_1 \cdots d_n.d_i(⟦\vec{Q}⟧^+\sigma[\vec{d}/\vec{x}])$. Then by the definition of $\boldsymbol{\lambda}$, one has $a \in \emptyset(⟦\vec{Q}⟧\sigma[\vec{\beta}/\vec{x}]) = \emptyset$.

Case $N \equiv y\vec{Q}$, for some $\vec{Q}$ and $y \neq x_i$, $i \leq n$. Take $\sigma(y) = \emptyset$, then $⟦M⟧^+_\sigma = A_n \subset A_m$, for $m > n$, by (1) in Corollary 3.5. A contradiction.

So far for (i). $\Rightarrow$.

$\Leftarrow$. Assume $M \notin \mathrm{O}_n$.

Case $M \in \mathrm{O}_p$, with $p \neq n$: Then by $\Rightarrow$, we are done.

Case $M \in \mathrm{O}_\infty$: Then, since $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$,

$$\forall p\, \forall \sigma\, ⟦M⟧^+_\sigma \supseteq A_p, \quad \text{by Proposition 1.3(ii) and (1).} \tag{3}$$

This contradicts $\underset{(1)}{\Leftrightarrow}$, again.

(ii)

Claim. $B = \bigcup A_{(n)}$.

Clearly $\bigcup A_{(n)} \subseteq B$. Conversely, if $b \in B = \bigcup B_n$ (see Definition 2.1), $b = (\beta_1; \cdots (\beta_p; a))$ for some $\vec{\beta}$ in D_A, $a \in A$ (by 2.5). By Lemma 3.3(ii) and Theorem 2.3, $A_{(p+1)} \supseteq \boldsymbol{\lambda} x_1 \cdots x_p.A = \{(\beta_1; \cdots (\beta_p; a)) : a \in A, \vec{\beta} \text{ in } D_A\}$.

Thus,

$$B \subseteq \bigcup A_{(n)}. \tag{4}$$

Now, assume $M \in \mathrm{O}_\infty$, then

$$\forall \sigma\, ⟦M⟧^+_\sigma = B, \quad \text{by (3) and (4).}$$

Conversely, $D^+_A \vDash M = B$ implies

$$\forall n\, M \notin \mathrm{O}_n, \quad \text{by (i) and (4).} \quad \square$$

In view of (4), let's write $A_\infty = B$.

Corollary 3.8. *Let* M *be a* λ-*term. Then*

$$M \textit{ is unsolvable} \Leftrightarrow \exists\, n \in \omega \cup \{\infty\}\; D^+_A \vDash M = A_{(n)}.$$

Proof. $\Rightarrow$. By Lemma 1.4 we have two cases.

Case $M \in \mathrm{O}_\infty$: Then $D^+_A \vDash M = A_{(\infty)}$, by Theorem 3.7(ii).

Case $\lambda\beta \vdash M = \lambda x_1 \cdots x_n.N$, for some $n \in \omega$ and $N \in \mathrm{PO}_0$. Then

$$\forall \sigma\, ⟦M⟧^+_\sigma = \boldsymbol{\lambda}\vec{x}.\emptyset \cup A_{(n)} = A_{(n)} \quad \text{by (1) and Theorem 3.6.}$$

$\Leftarrow$. Assume $\lambda\beta \vdash M = \lambda x_1 \cdots x_m.yQ_1 \cdots Q_p$, for some $m \in \omega$, y and $\vec{Q}$ (i.e. assume that M is solvable). Assume also that $D^+_A \vDash M = A_{(n)}$.

By Theorem 3.7(i),

$$D^+_A \vDash M = A_{(n)} \;\Rightarrow\; n \leq m.$$

Thus $n \leq m$.

Case $y \neq x_i$, $\forall\, i \leq m$: Take $\sigma(y) = \boldsymbol{\lambda} x_1 \cdots x_p.A$. Then

$$\begin{aligned}[\![M]\!]^+_\sigma &= \lambda^+ x_1 \cdots x_m.A \\ &\neq A_{(n)}, \quad \text{by Definition 3.2 and Lemma 3.3(iv).}\end{aligned}$$

A contradiction.

Case $y = x_i$, for some $i \leq m$: Take $\beta_i^a = \{(\emptyset; \cdots (\emptyset; a) \cdots)\}$ of length p, for some $a \in A$, and $b^a = (\beta_1; \cdots (\beta_i^a; \cdots (\beta_m; a) \cdots)$, for some $\vec{\beta}$. Then $b^a \notin A_{(n)}$, by Lemma 3.3(ii) and $n \leq m$. Nonetheless

$$\forall \sigma\, b^a \in \{(\beta_1; \cdots (\beta_m; b) \cdots)/b \in \beta_i\, [\![\vec{Q}]\!]^+ \sigma[\vec{\beta}/\vec{x}]\} \subseteq [\![M]\!]^+_\sigma$$

since $\beta_i^a d_1 \cdots d_p = \{a\}$. A contradiction, again. □

By Theorem 3.7, the witness n of the RHS of Corollary 3.8 is unique and it is the order of the unsolvable term M.

Note that D_A^+ provides a semantical characterization of unsolvable terms, with their functionality. Moreover the functionality of solvable terms is also characterized, by Theorem 3.7(i), though it never occurs that a solvable and an unsolvable term are equated. Finally, by the montonicity of λ^+, Theorem 1.12 applies; thus

$$T(M) \subseteq T(N) \;\Rightarrow\; D_A^+ \models M \subseteq N.$$

The author believes that this model is 'very sensible' although such a definition wouldn't fit Barendregt's (cf. [2, p. 100]).

With some patience, one should also be able to work out the following fact:

$$T(M) = T(N) \;\Leftrightarrow\; D_A^+ \models M = N.$$

Actually, we claim that the technique used in Appendix B gives also the following: let A have at least two elements and $a_0 \in A$, define $\lambda^1 x.f(x) = \boldsymbol{\lambda} x.f(x) \cup (A \setminus \{a_0\})$, then

$$T(N) \subseteq T(M) \;\Leftrightarrow\; \langle D_A, \cdot, \lambda^1 \rangle \models M \subseteq N.$$

Remark 3.9. How do the models defined in Sections 2 and 3 relate to the set-theoretic construction in Scott [22]?

Given $A \neq \emptyset$, set $A^* = \bigcup_n A^n$, where $A^0 = \{\langle\ \rangle\}$, $A^n = \{f : f : \{0, \ldots, n-1\} \to A\} = \{\langle a_1, \ldots, a_n \rangle : a_1, \ldots, a_n \in A\}$ and $\langle\ \rangle$ is the empty tuple (Scott [22, p. 229]).

'Zermelo's least solution' of the equation $C = C^*$ (cf. Scott [22, p. 234]), would be

$$B_Z = \bigcap \{C : \emptyset \in C \wedge (a_1, \ldots, a_n \in C \Rightarrow \langle a_1, \ldots, a_n \rangle \in C)\},$$

that is, B_Z is the least set containing $\emptyset$ and closed under formation of finite tuples. The λ-model is given by $X.Y = \{b : \exists n\, \exists a_1, \ldots, \exists a_n \in Y\ \langle b, a_1, \ldots, a_n \rangle \in X\}$, for $X, Y \subseteq B_Z$ and $\lambda x.f(x) = \{\langle\ \rangle\} \cup \{\langle b, a_1, \ldots, a_m \rangle : b \in f(\{a_1, \ldots, a_m\})\}$, for f continuous over the parts of B_Z (see Scott [22, pp. 229–234]).

Define B as in Definition 2.4, starting with $A = \{\emptyset\}$, and $' : B_Z \to B$ by $\langle\ \rangle' = \emptyset$,

$\langle b, a_1, \ldots, a_n\rangle' = (\beta, b')$, where $\beta \subseteq B$ contains exactly (and without repetitions) $a'_1, \ldots, a'_n$. This is well defined, as a map, since $\langle a, b, c\rangle$ is not given as $\langle a, \langle b, c\rangle\rangle$ (it is not a Kuratowski tuple, say). In particular $\langle b\rangle' = (\emptyset; b')$, $\langle b, \langle\ \rangle\rangle' = (\{\emptyset\}; b')$ and $\langle a, \langle b, c\rangle\rangle' = (\{\langle b, c\rangle'\}; a') \neq (\{b', c'\}; a') = \langle a, b, c\rangle'$.

Let B'_Z be B_Z modulo the following relation: $x \sim z$ iff $x' = z'$. Thus two $n+1$-tuples are equivalent iff the last n elements are the same up to a permutation (inductively over the 'depth' of tuples). Note that working modulo '$\sim$' does not affect the model, i.e. does not affect application and abstraction as defined above. Clearly B'_Z and B are isomorphic. Thus Scott's model construction (i.e. the 'full' λ defined on p. 233), modulo '$\sim$', is isomorphic to $\langle D_A, \cdot, \lambda^+\rangle$, where $A = \{\emptyset\}$.

More generally, given a set A of 'atoms', i.e. non-tuples, consider Scott's equation $C = A \cup C^*$ (Scott [22, p. 244]). Then

$$B_S = \bigcap \{C: A \subseteq C \wedge \emptyset \in C \wedge (a_1, \ldots, a_n \in C \Rightarrow \langle a_1, \ldots, a_n\rangle \in C)\}$$

is the least solution of $C = A \cup C^*$. Up to the equivalence '$\sim$' above, over tuples, B_S is the same as the set B defined in Definition 2.4, starting with $A_0 = A \cup \{\emptyset\}$. This generalization turns out to be essential for the purposes of Theorem 4.1.

Intermezzo (on extensionality). When dealing with applicative structures, by 'extensional applicative structure', we mean a pair $\langle D, \cdot\rangle$ such that

(E) $\quad \forall\, d, e \in D\, (\forall\, c \in D\; dc = ec \Rightarrow d = e).$

Dana Scott in discussions with Roger Hindley, Henk Barendregt and the author pointed out that the right notion of extensionality for structures, which yield λ-models, is the satisfaction of rule (ξ), see Section 1, in the sense described in, say, [13], [17] or [2]. As a matter of fact, the correct meaning of the word is that a function is uniquely determined by its argument-value correspondence. This is exactly what (ξ) takes care of.

However, one may be interested in applicative structures which may not yield λ-models. Still they may satisfy (E) above. That is, they may contain just one representative for each representable function. Some interesting domains satisfying (E), which do not need to be λ-models, are characterized in [10].

In view of the general discussion on applicative structures carried out in Section 1, we keep the slightly improper use of (E) for extensionality and consider (ξ) as a weak extensionality property, following [1] and [2]. Note that λ-models are exactly Combinatory Algebras satisfying (ξ). In case of λ-models, (E) corresponds to the axiom schema (η) or rule (ζ). The following discussion is divided into two parts. Both are concerned with a technique for constructing extensional λ-models due to Scott. Remark (i), in Part I, answers a question raised by Scott, while (ii) states a conjecture and an argument which makes it plausible. Part II presents a construction which will be used in Sections 4 and 5.

Part I: None of the models studied so far is extensional; namely, in general, EC_f contains more than one element. Throwing away some elements, can we turn

$\langle D_A, \cdot\rangle$ into an extensional λ-model? There doesn't seem to be an elementary direct way for such a construction, starting from a λ-expansion of $\langle D_A, \cdot\rangle$ (see Definition 4.4 for an indirect argument).

Scott [21] (see also [22]) presents an elegant technique to construct an extensional substructure of the λ-model $P\omega$. This technique applies to 'almost' (see later) any λ-model satisfying $(\eta^-)\lambda x.x \leq \lambda xy.xy$, which is a c.p.o.

Scott's argument is the following: Let

$$I_0 := I \equiv \lambda x.x \qquad I_{n+1} \equiv \lambda xy.I_n(x(I_n y)).$$

Set $d_{(n)} = [\![I_n]\!] \in P\omega$ and $d_\infty = \bigcup d_{(n)}$. Then $E\omega = \{d_\infty e \colon e \in P\omega\} \subseteq P\omega$ is an extensional λ-model (see [22]).

Remark. (i) Scott [22, p. 251] points out that d_∞ is the least solution of

$$d = d_{(0)} \cup (\boldsymbol{\lambda} xy.d(x(dy))),$$

and remarks that d_∞ doesn't seem to be the interpretation of any closed λ-term. And in fact it is not. By induction, one has

$$\forall n\, \lambda\beta \vdash I_n = \lambda x_0 \cdots x_n.x_0(I_{n-1}x_1)(I_{n-2}x_2)\cdots x_n. \tag{1}$$

Now $d_\infty \neq \emptyset$; thus, if $[\![M]\!] = d_\infty$, M is solvable, say $\lambda\beta \vdash M = \lambda x_1 \cdots x_p.x_j\vec{Q}$. Then one can derive a contradiction from $\forall n\, d_n \subseteq [\![M]\!]$ and (1), by applying them to the right $\vec{C}$ in $P\omega$, depending on j.

(ii) Let $(B, \leq_n)$ be the set of Böhm-like trees partially ordered by (possibly infinite) η-expansions (see [2, p. 230]). It is then easy to show (use cofinality of chains) that $\bigsqcup BT(I_n)$ is the 'Nakajima-tree' of $\lambda x.x$ (see [2, p. 51]). *Conjecture*: $E\omega$ is equationally equivalent to Scott's inverse limit D_∞. (Added in proof: Karst Koymans recently proved this conjecture.)

Part II: Scott's (η^-) property seems to be a fairly natural property for a λ-model $\langle D, \cdot, \lambda'\rangle$ which is a poset. It says that λ' chooses the largest element in EC_f, for each $f \in (D \to D)$. That is exactly what λ^+ does in the case of D_A^+. Nonetheless the technique of Part I doesn't apply to D_A^+ (thus to no other λ-expansion of $\langle D_A, \cdot\rangle$). In fact, by Theorem 3.7(i) and (4) in proof of theorem 3.7, one has $d_\infty = B$. Therefore

$$\{d_\infty e \colon e \in D_A\} = \{B\},$$

since $\forall\, e \in D_A\ Be = B$, i.e. the extensional substructure collapses to a singleton.

How to force (η^-) into Engeler's construction and still obtain an interesting d_∞?

Given $\lambda' \neq \lambda^+$, $\forall d\, d \subseteq \lambda' x.dx$ is false because of those d's containing elements of A, which do not act as 'instructions' (see Corollary 3.5). Thus, what one can do, with $a \in A$, is to force $a = (\{a\}; a)$ (or, also, $a = (\emptyset; a)$, see Remark 5.8), i.e. force a to act. That is, set $a \simeq (\{a\}; a)$ and consider $\tilde{B} = B/{\simeq}$, $\tilde{D}_A = \langle P\tilde{B}, \cdot\rangle$.[1] (It is easy to define hereditarely $\simeq$ on B, see the Remark below.)

[1] By a different argument, Scott derives the same observation (see remark on p. 251 of [22]).

$\tilde{D}_A$ is no longer well founded and there is no way of turing $\langle \tilde{D}_A, \cdot \rangle$ into a Combinatory Algebra with approximable application: this is an immediate consequence of Theorems 1.9 and 5.6 (see later) which gives a semantical characterization of closed λ-terms of order 0, different from Theorem 3.6; namely,

$$M \in O_0 \Leftrightarrow \tilde{D}_A \models M = A, \quad \text{for } M \text{ closed.}$$

Notice that, in $\tilde{D}_A$, $\forall b \in \tilde{B}(\emptyset; b) \notin d_\infty = \bigcup d_{(n)}$, by (1) in the remark. Thus $d_\infty \emptyset = \emptyset$; since one also has $d_\infty B = B$, then Scott's technique applies and the extensional substructure is not trivial.

In this model, Tarski's fixed point operator is not λ-definable (see Section 5).

Remark. Clearly not any equivalence relation $\sim$ on B turns $P\tilde{B}$ into a non trivial applicative structure. (*Hints*: Take a map of B into B, whose range contains $B \setminus A$. Define hereditarely on the pairs $(\beta; b)$ in B the least equivalence relation $\sim$ such that, for $c \in B$, $c \sim f(c)$. Then $\langle P\tilde{B}, \cdot \rangle$ is a (non trivial) PSE-Algebra. (This generalizes the technique used above, where f_a and $f_\emptyset$ are such that $f_a(a) = (\{a\}; a)$ and $f_\emptyset(a) = (\emptyset; a)$.) We claim that, given B and $\sim$ on B, if $\langle P\tilde{B}, \cdot \rangle$ is a well defined PSE-Algebra, then $\sim$ may be defined as above. (See the Conclusion for a further discussion.)

4. Expansions in any cardinal. Lambda-categorical models. D_A and $P\omega$

The theory $\lambda\beta$ is not complete, in the sense that there exist (closed) terms M, N such that $\lambda\beta \nvdash M = N$ and $\lambda\beta + \{M = N\}$ is consistent. Moreover $\lambda\beta$ does not possess finite models. Then, by Łos–Vaught theorem, $\lambda\beta$ possesses, in any infinite cardinal, non 'elementarely equivalent models' (see [17] for a first-order characterization of λ-model).

The local analysis of $\langle D_A, \cdot, \boldsymbol{\lambda} \rangle$ and of $\langle D_A, \cdot, \lambda^+ \rangle$ gives a stronger fact: in the case of $\lambda\beta$, we can obtain, in any cardinal, non equationally equivalent λ-models over the same applicative structure (see Theorem 4.1). Theorem 4.6 proves a counterpart of Theorem 4.1.

We first state an obvious generalization of Theorem 3.6. Namely, instead of using the specific λ-expansion D_A^+ of $\langle D_A, \cdot \rangle$, given by λ^+ as defined in Definition 3.1, we could use any λ-expansion $D'_A = \langle D_A, \cdot, \lambda' \rangle$, defined by a choice map λ' over the representable functions, such that $\lambda'(f) = \boldsymbol{\lambda}(f) \cup d'$, where $d' \subseteq A$ and $d' \neq \emptyset$. Any such a D'_A trivially gives the same characterization of λ-terms in PO_0 as in Theorem 3.6.

Theorem 4.1. *For any infinite-cardinal α, there exists a Combinatory Algebra $\langle D, \cdot \rangle$ such that*

(1) $\mathrm{card}(D) = \alpha$,

(2) $\langle D, \cdot \rangle$ has λ-expansions which yield non equationally equivalent λ-models.

Two arguments for Theorem 4.1 follow. The first was suggested by the referee and directly uses Lowenheim–Skolem theorem, Proposition 2.8 and Theorem 3.6. The original proof (proof II) to some extent works out the details of proof I and more closely investigates the structure of definable submodels and their relation to models. As a consequence, say, it suggested Remark 4.2.

Let A be α. Define $B \subseteq A$ and $\langle D_A, \cdot\rangle$ as in Definition 2.4. By the construction, $\mathrm{card}(B) = \alpha$ and $\mathrm{card}(D_A) = 2^\alpha$.

Proof I. Following Meyer [17] or Scott [23], formalize λ-calculus (and its models), in a first-order way. In particular an expanded combinatory algebra $\langle D, \cdot, \mathbf{S}, \mathbf{K}\rangle$, i.e. a model of CL with a given interpretation of S and K, is a λ-model iff, for $\boldsymbol{\varepsilon} = \mathbf{S}(\mathbf{KI})$ (where $I \equiv SKK$), Meyer's axioms (1) and (2) in the Discussion of p. 7 are satisfied (cf. also Remark 5.8, where axiom (3) is a stability condition, i.e. $\boldsymbol{\varepsilon\varepsilon} = \boldsymbol{\varepsilon}$ iff $\boldsymbol{\varepsilon} = \boldsymbol{\lambda} xy.xy$).

Consider now the first-order theory of $\langle D_A, \cdot\rangle$. Let $\mathbf{K} = \boldsymbol{\lambda} xy.x$, $K^+ = \lambda^+ xy \cdot x$ and similarly for $\mathbf{S}$ and S^+. By Theorem 2.3 and Corollary 3.5, both the corresponding $\boldsymbol{\varepsilon}$ and ε^+ satisfy Meyer's axioms and give different λ-expansions of $\langle D_A, \cdot\rangle$ (actually $\boldsymbol{\lambda}$ and λ^+).

By Lowenheim–Skolem theorem, there exists a substructure $\langle D, \cdot\rangle$ of $\langle D_A, \cdot\rangle$ such that:

(i) $\mathbf{K}, \mathbf{S}, K^+, S^+ \in D \subseteq D_A$,

(ii) $\mathrm{card}(D) = \alpha$,

(iii) $\langle D_A, \cdot\rangle$ is an elementary extension of $\langle D, \cdot\rangle$.

Thus, by Proposition 2.8, $\langle D, \cdot, \mathbf{K}, \mathbf{S}\rangle \vDash M = N$ iff $\mathrm{BT}(M) = \mathrm{BT}(N)$. Choose now a term in PO_0, say $SII(SII)$. Then by Theorem 3.6, $\langle D, \cdot, K^+, S^+\rangle \vDash M = SII(SII)$ iff $M \in \mathrm{PO}_0$. Hence the theories of these expansions of $\langle D, \cdot\rangle$ differ.

Proof II. (1) Let L be a 'rich enough' first-order language L for Set Theory with two sorts of variables, one of which ranges only over finite subsets of B (in other words, formalize what we have been doing so far, using $\beta, \gamma, \ldots$ and $b, c, \ldots$). Since $A \subseteq B$ contains atoms, and, for the purposes of the proof, we need them as 'non tuples', express $(\beta; b)$ in such a way that no element of A is denoted by $(\cdots;\cdots)$.

Let $D_A^C = \{d : d \subseteq B$ and d is definable over $B\} \subseteq D_A$, i.e. the elements of D_A^C are defined by formulas of L using constants from B. By definition, D_A^C is closed under '$\cdot$'. Clearly $\mathbf{K} = \{(\beta; (\gamma; b)) : b \in \beta\}$ and $\mathbf{S} = \{(\beta; (\gamma; (\delta, b))) : b \in \beta\delta(\gamma\delta)\}$ are in D_A^C. Thus $\langle D_A^C, \cdot\rangle$ is a Combinatory Algebra and card $(D_A^C) = \alpha$.

(2) Let $(D_A^C \to D_A^C)$ be the set of representable functions over D_A^C (see Definition 1.6). For $f \in (D_A^C \to D_A^C)$ define $\bar{f} \in C(D_A, D_A)$ by $\bar{f}(d) = \bigcup \{f(\beta) : \beta \subseteq d \ \beta$ finite$\}$. Since any finite $\beta \subseteq B$ is in D_A^C and D_A is a complete algebraic lattice, $\bar{f}$ is well defined and is the unique continuous extension of f with respect to Scott's topology over D_A.

Let $\boldsymbol{\lambda}$ be as in Theorem 2.3 and let d' be a definable (possibly empty) subset of A. Define a λ-model $\langle D_A, \cdot, \lambda'\rangle$ (i.e. a λ-expansion of $\langle D_A, \cdot\rangle$) by $\lambda'(f) = \boldsymbol{\lambda}(f) \cup d'$. Set then $\lambda'_C(f) = \lambda'(\bar{f})$, for $f \in (D^C_A \to D^C_A)$. We need to show that $\lambda'_C x.f(x) = \lambda'_C(f) \in D^C_A$, in order to prove that $\langle D^C_A, \cdot, \lambda'_C\rangle$ is a λ-model; the rest is trivial.

Let $d_f \in D^C_A$ be a representative for f. Then d_f represents also $\bar{f}$: by continuity of '$\cdot$', one has $\bar{f}(d) = \bigcup_{\beta \subseteq d} d_f\beta = d_f d$, for all $d \in D_A$.

Then $\lambda'_C(f) = \lambda'(\bar{f}) = (\{(\beta, b)\colon b \in d_f\beta\} \cup d') \in D^C_A$ and $\langle D^C_A, \cdot, \lambda'_C\rangle$ is clearly a λ-model.

Set now $\mathbf{K}' = \lambda' x.(\lambda' y.(\mathbf{K}xy))$ and $\mathbf{S}' = \lambda' x.(\lambda' y.(\lambda' z.(\mathbf{S}xyz)))$: then $\mathbf{K}' = \mathbf{K} \cup \{(\beta; b)\colon b \in d'\} \cup d'$, similarly for $\mathbf{S}'$. Thus $\mathbf{K}'$ and $\mathbf{S}'$ are in D^C_A and interpret the λ-terms $K \equiv \lambda xy.x$ and $S \equiv \lambda xyz.xz(yz)$ in $\langle D^C_A, \cdot, \lambda'_C\rangle$. By definition of λ'_C, they also interpret K and S in $\langle D^C_A, \cdot, \lambda'_C\rangle$. Then, if M is a closed term, one has $[\![M]\!]_\sigma = [\![M]\!]^C_\sigma$, where $[\![\cdot\cdot\cdot]\!]^C_\sigma$ is the interpretation in D^C_A: just write M using only S and K's. Thus $\langle D^C_A, \cdot, \lambda'_C\rangle \models M = N$ iff $\langle D_A, \cdot, \lambda'\rangle \models M = N$, for closed M, N. Theorems 2.8 and 3.6 (see above) give the result. $\square$

Remark 4.2. Scott's approach to the construction of models of λ-calculus has been a topological one (cf. Scott [20]). By giving the Scott topology to some lattices, the set of continuous functions could be isomorphically embedded into the lattices themselves. The definable models D^C_A above, with the induced topology, do not have this property. As a matter of fact, let D^C_A be as in Theorem 4.1 (proof II). If $\mathrm{card}(D^C_A) = \alpha$, then the continuous functions on D^C_A are more than α. To prove this, assume that the language L has a constant symbol for B. Then consider in D^C_A the sublattice G_A, whose least and largest element are $\emptyset$ and B and containing the singletons of elements of B as incomparable elements (with respect to $\subseteq$). Give to G_A the induced topology. Clearly, for A is α, there are 2^α continuous maps of G_A into itself, i.e. $\mathrm{card}(C(G_A, G_A)) = 2^\alpha$. Since G_A is trivially a continuous lattice, G_A is an injective space (cf. Scott [20]). Thus any $f \in C(G_A, G_A)$ can be extended to $f^* \in C(D^C_A, G_A) \subseteq C(D^C_A, D^C_A)$.

This is why the model has been defined using just the representable functions, which are much less than the continuous ones, but still enough.

Question. Is there a topology on D^C_A such that the representable functions are exactly the continuous ones?

(We claim that there is no such a topology.)

Theorem 4.1 proves that, for any infinite cardinal, there exists a subalgebra of a PSE-Algebra (a *sub-PSE-Algebra*, say) with several λ-expansions, yielding different theories.

Of course, if the cardinal is 2^α, for some infinite α, one could take a full PSE-Algebra.

A PSE-Algebra may or may not contain (elements with) atoms; D_A, as defined in Definition 2.4, does contain atoms. They have been used in the proof of Theorem 4.1.

By an abuse of language, we say that a PSE-Algebra is *atomless* if all its elements are atomless, i.e. do not contain atoms. For example, $\tilde{B}$, defined at the end of the Intermezzo, is such that $\tilde{D}_A$ $(=P\tilde{B})$ is atomless, by the identification $a = (\{a\}; a)$ (see also Remark 5.8).

Given B as in Definition 2.1, let's say that $d \in PB$ is *saturated* iff

$$((\beta; b) \in d \wedge \beta \subseteq \gamma \Rightarrow (\gamma; b) \in d).$$

The following lemma gives some general information on the embeddings of $C(PB, PB)$ into PB, i.e. on λ-expansions. Claim 1, say, proves that, in order to satisfy Definition 1.6(1)–(2), they must be continuous. Thus, in a PSE-Algebra, Definition 1.6(1)–(2), are equivalent to Definition 1.6(1) and the continuity of the embedding.

Lemma 4.3 (Main Structural Lemma). *Let $\langle PB, \cdot \rangle$ be a PSE-Algebra. Assume that $\langle PB, \cdot, \lambda' \rangle$ is a λ-model. Then $\lambda'(f)$ is saturated, for all $f \in C(PB, PB)$.*

Proof. Recall first that $(PB \to PB)$, the set of representable functions (cf. Definition 1.6), coincide with $C(PB, PB)$: in fact any continuous f is represented at least by $\boldsymbol{\lambda}(f) = \{(\beta; b): b \in f(\beta)\}$, the canonical representative of f. By the assumption, λ' satisfies Definition 1.6(1)–(2). Using Definition 1.6(2), set $\varepsilon' = \lambda' xy.xy$ (cf. the Discussion following Definition 1.6). As in that Discussion, one then has: $\varepsilon' d = \lambda' x.dx = \lambda'(f)$ iff $d \in EC_f$ (i.e. iff d represents f).

Claim 1. *λ' is continuous (and hence monotone).*

Since $\boldsymbol{\lambda}(f) \in EC_f$, then $\lambda'(f) = \varepsilon' \boldsymbol{\lambda}(f)$. Use then the continuity of $\boldsymbol{\lambda}$ and of '$\cdot$'.

For the sake of simplicity, set now $\phi(d) = \lambda x \in PB.dx$; that is $\phi : PB \to C(PB, PB)$ gives the function represented by d. By definition, $\lambda'(\phi(d)) = \lambda' x.dx$ and $\phi(\lambda'(f)) = f$. Since '$\cdot$' is monotone, then ϕ also is monotone.

Claim 2. $\alpha = \{(\beta; b)\} \Rightarrow \alpha \subseteq \lambda'(\phi(\alpha))$ $(= \lambda' x.\alpha x)$.

For

$$\begin{aligned}
&(\lambda' x.\alpha x)\beta = \alpha\beta = \{b\} \\
&\quad \Rightarrow \exists\, \beta' \subseteq \beta\ (\beta'; b) \in \lambda' x.\alpha x \\
&\quad \Rightarrow b \in (\lambda' x.\alpha x)\beta' = \alpha\beta'. \\
&\quad \Rightarrow \beta' = \beta, \quad \text{otherwise } \alpha\beta' = \emptyset \\
&\quad \Rightarrow (\beta; b) \in \lambda' x.\alpha x\ (= \lambda'(\phi(\alpha))).
\end{aligned}$$

Let now $f \in C(PB, PB)$ and $(\beta; b) \in \lambda'(f)$. From $\beta \subseteq \gamma$ we have to deduce that $(\gamma; b) \in \lambda'(f)$. Set then $\alpha_0 = \{(\beta; b)\}$, $\alpha_1 = \{(\gamma; b)\}$ and $e = \{(\delta; b): \beta \subseteq \delta\}$. Clearly $\phi(\alpha_0) = \phi(e)$.

Then one has

$$\begin{array}{ll} \alpha_1 \subseteq \lambda'(\phi(e)) & \text{by Claim 2} \\ \quad \subseteq \lambda'(\phi(e)) & \text{by } \alpha_1 \subseteq e \text{ and monotonicity of } \phi \text{ and } \lambda' \\ \quad = \lambda'(\phi(\alpha_0)) & \text{by } \phi(\alpha_0) = \phi(e) \\ \quad \subseteq \lambda'(f) & \text{by } \alpha_0 \subseteq \lambda'(f)\text{, that is by } \phi(\alpha_0) \subseteq \phi(\lambda'(f)) = f. \end{array}$$

Thus $(\gamma; b) \in \lambda'(f)$. □

Definition 4.4. A Combinatory Algebra is *lambda-categorical* if it has a unique λ-expansion (that is the applicative structure uniquely determines the λ-model).

Theorem 4.5. (i) *Let $\langle PB, \cdot \rangle$ be a PSE-Algebra and $f \in C(PB, PB)$. Then EC_f, the extensionality class of f, contains a unique atomless and saturated element.*
(ii) *Any atomless PSE-Algebra is lambda-categorical.*

Proof. (i) Let $d \in EC_f$ be saturated. Take $e \in EC_f$ and $(\beta; b) \in e$. Then $b \in e\beta = d\beta$, that is $\exists\, \beta' \subseteq \beta\ (\beta'; b) \in d$. By saturation, $(\beta; b) \in d$.
(ii) By Lemma 4.3 and (i). □

Recall that $\tilde{D}_A$ is atomless. By the Remark at the end of the Intermezzo, it is easy to define more atomless PSE-Algebras. By Theorem 4.5, in order to obtain several λ-expansions in a PSE-Algebras, one needs atoms, namely objects with no 'functional behaviour'. But, still, they do some work: their use may affect the theory. As a matter of fact, the sub-PSE-Algebras given in Theorem 4.1 have as many λ-expansions as their cardinality. Of course, if the cardinal is large enough, most of them will yield the same theory (i.e. will be equationally equivalent), for there are only 2^ω extensions of pure λ-calculus. The canonical map, **λ**, is the smallest one giving a λ-expansion (whereas λ^+ is the largest, cf. Lemma 3.4); it is the unique map, satisfying Definition 1.6, whose range contains only atomless elements.

Any extensional Combinatory Algebra is trivially lambda-categorical.

Theorem 4.6. *For any infinite cardinal α, there exists a non extensional sub-PSE-Algebra $\langle D, \cdot \rangle$, with $\mathrm{card}(D) = \alpha$, which is lambda-categorical.*

Proof. Let A be α. Take $\tilde{B}$ and $\tilde{D}_A$ $(= P\tilde{B})$ as in the Intermezzo. $\langle \tilde{D}_A, \cdot \rangle$ is a Combinatory Algebra of cardinal 2^α. By Theorem 4.5(ii), it has a unique λ-expansion: the canonical one, **λ**.

Clearly $\langle \tilde{D}_A, \cdot \rangle$ is not extensional: just observe, say, that $\{(\beta; b)\}$ and $\{(\beta; b), (\gamma; b)\}$, for $\beta \subseteq \gamma$, represent the same function.

Then argue as in Theorem 4.1 (Proof I) to define a lambda-categorical sub-PSE-Algebra of cardinal α. □

A lambda-categorical Combinatory Algebra yields a unique theory, as λ-model. But it may have several K's and S's, giving different theories, as expanded Combinatory Algebras (i.e. as models of CL, cf. Definition 1.5).

For example, define λ^- over $\tilde{D}_A$ as in Definition B.5 (Appendix B) for D_A, using $\boldsymbol{\lambda}$ instead of λ^0. By Lemma 4.3 and Definition 4.4, λ^- doesn't give a λ-expansion of $\langle \tilde{D}_A, \cdot\rangle$ (even not of $\langle D_A, \cdot\rangle$(!)). Set $K^- = \lambda^- xy.x$ $(=\{(\{b\}; \delta; b): b \in \tilde{B}\})$ and $S^- = \lambda^- xyz.xz(yz)$ $(= \cdots$ exercise $\cdots)$. By Lemma B.6(1), $\langle \tilde{D}_A, \cdot, K^-, S^-\rangle \models$ CL.

Question. Are $\langle \tilde{D}_A, \cdot, \mathbf{K}, \mathbf{S}\rangle$ and $\langle \tilde{D}_A, \cdot, K^-, S^-\rangle$ equationally equivalent?

The theories of the λ-models in Theorem 4.6 will be discussed in Section 5.

Another application of the previous results (namely Theorem 1.9 and Lemma 2.7) relates D_A and $P\omega$. In particular the isomorphisms between $\langle D_A, \cdot\rangle$ and $\langle P\omega, \cdot\rangle$ as applicative structures.

'$\cdot$' over $P\omega$ is defined as for Enumeration Reducibility (see [19, p. 146; 2, p. 469]). That is, for codings of the finite sets $\{E_n\}_{n\in\omega}$ and of pairs $(\,,\,)$, $CG = \{m: \exists\, E_n \subseteq G\ (n, m) \in C\}$. By $\langle D, x\rangle \Rightarrow \langle D', \cdot\rangle$ we mean that D can be isomorphically embedded into D', w.r.t. 'x' and '$\cdot$'.

Proposition 4.7. *Let* $A \neq \emptyset$. *Then one has*

(i) $\langle P\omega, \cdot\rangle \Rightarrow \langle D_A, \cdot\rangle$

(ii) *If A is countable, then* $\langle D_A, \cdot\rangle \Rightarrow \langle P\omega, \cdot\rangle$.

Proof. (*Notation*: $(\,,\,)$ and $\{E_n\}_{n\in\omega}$, finite sets, are as in [19]; in particular $E_0 = \emptyset$, $E_1 = \{0\}$. Set also $\#E_n = n$).

Notice first that

(0) $\forall n \in \omega \setminus \{0\}\ \exists! k\ \exists! n_1 \cdots \exists! n_k\ n = (n_1; \cdots (n_k, 0) \cdots) \wedge n_k \neq 0.$

(i) Define (simultaneously), for some $a \in A$,

$$[\cdot]: \omega \to D_A,$$
$$h: \{E_n\}_{n\in\omega} \to \{\beta: \beta \subseteq B \text{ finite}\},$$
$$\text{first}: [\omega] \to B$$

by

$$[0] = \{a, (\emptyset; a), (\emptyset; (\emptyset; a)), \ldots\},$$
$$\text{first}([0]) = a;$$

Let $n = (n_1, (n_2, \ldots (n_k, 0) \cdots), n_k \neq 0$: then set

$$[n] = \{(\beta'_{n_1}; (\beta'_{n_2}; \cdots (\beta'_{n_k}; b) \cdots): b \in [0]\}$$

where, for $E_n = \{m_1, \ldots, m_q\}$,

$$\beta'_n := h(E_n) = \{\text{first}([m_1]), \ldots, \text{first}([m_q])\} \quad (\text{with } h(\emptyset) = \emptyset)$$

and, for $p=(p_1,(p_2,\ldots(p_r,0)\neq 0$,

$$\text{first}([p])=(\beta'_{p_1};(\beta'_{p_2};\cdots(\beta'_{p_r};a)\cdots).$$

Finally define $f:P\omega\to D_A$ by

$$f(C)=\bigcup\{[n]:n\in C\}.$$

Claim. (1) $(\beta'_m;b)\in[(n,p)]\Leftrightarrow m=n\wedge b\in[p]$.
(2) $E_n\subseteq C\in P\omega\leftrightarrow\beta'_n\subseteq f(C)$.

Part (1) easily follows by the definitions (note that it holds also for $(n,p)=0$, i.e. $n=0\wedge p=0$). As for (2), notice that $\beta'_n\subseteq f(E_n)$. Clearly f is injective.

Compute now

$$\begin{aligned}f(C)f(G)&=\{b:\exists\beta\subseteq f(G)(\beta;b)\in f(C)\}\\&=\{b:\exists n(\beta'_n\subseteq f(G)\wedge\exists p(\beta'_n;b)\in[(n,p)]\wedge(n,p)\in C)\}\quad\text{by (1)}\\&=\{b:\exists\, E_n\subseteq G\ \exists p\ b\in[p]\wedge(n,p)\in C\}\quad\text{by (2), (1)}\\&=\bigcup\{[p]:\exists\, E_n\subseteq G\ (n,p)\in C\}=f(CG).\end{aligned}$$

(ii) Define, for $A=\{a_0,a_1,a_2,\ldots\}$

$$\text{map}:B\to\omega\quad(\text{notation: }\mathbf{b}=\text{map}(b))$$

$$g:\{\beta:\beta\subseteq B(\text{finite})\}\to\omega$$

by

$$\mathbf{a}_n=(1,n),\qquad \boldsymbol{\beta}\to\mathbf{b}=(g(\beta),\mathbf{b})$$

where, for $\beta=\{b_1,\ldots,b_q\}$,

$$g(\beta)=\#\{\mathbf{b}_1,\ldots,\mathbf{b}_q\}.$$

Define

$$f':D_A\to P\omega\quad\text{by}\quad f'(d)=\{\mathbf{b}:b\in d\}.$$

Claim. (1) map *is injective*; g *is injective,*
(2) $\forall\, d\in D_A\ 0\notin f'(d)$;
(3) f' *is injective.*

As for (1)+(2), define $|\ |:B\to\omega$ as in Definition 2.6. The proof easily follows by (combined) induction on $|b|$ (and $|\beta|$). As for (3), it is an obvious consequence of (1). Then compute

$$\begin{aligned}f'(d)f'(e)&=\{q:\exists\, E_n\subseteq f'(e)\,(n,q)\in f'(d)\}\\&=\{\mathbf{b}:\exists\beta\, E_{g(\beta)}\subseteq f'(e)\,(g(\beta),\mathbf{b})\in f'(d)\}\end{aligned}$$

by (2), since $E_1=\{0\}$ (so it is not the case that $(n,q)=\mathbf{a}$ for $a\in A$),

$$\begin{aligned}&=\{\mathbf{b}:\exists\,\beta\subseteq e\,(\beta;b)\in d\},\quad\text{by (1)}\\&=f'(de).\quad\square\end{aligned}$$

The embeddings in the proof of Proposition 4.7 do not preserve S and K. In [12] it is shown that for any applicative structure $\langle A, x\rangle$, one has

$$\langle A, x\rangle \Rightarrow \langle D_A, \cdot\rangle.$$

Corollary 4.8. *Let $A \neq \emptyset$. Then for any countable $\langle E, x\rangle$, one has*

$$\langle E, x\rangle \Rightarrow \langle D_A, \cdot\rangle.$$

Proof. Just use $\langle E, x\rangle \Rightarrow \langle D_E, \cdot\rangle \Rightarrow \langle P\omega, \cdot\rangle \Rightarrow \langle D_A, \cdot\rangle$. $\square$

In particular $\langle E, x\rangle$ may be a countable extensional Combinatory Algebra.

Discussion 4.9. $P\omega$ is not (as) well founded (as D_A; see Definition 2.6). Namely, there is no way to mimic the definition of $|\ |$ given for D_A (see 2.5 and Definition 2.6) on $P\omega$. If, in view of (0) in Proposition 4.7, one sets $|n| = k$ for $n \neq 0$, then $|\ |$ cannot be extended to 0 in a way to have always $|n|, |m| < |(n, m)|$. In fact $0 = (0, 0) = (0, (0, 0)) \cdots$ (and this is the only 'bad guy', but any coding must have at least one $\cdots$). Thus under the standard coding (but this can be generalized):

$$\forall C \in P\omega \quad \{0\}C = \{0\}.$$

This is what we have been taking care of in Proposition 4.7(i). Clearly non well-founded codings in even a stronger sense would make the result false (see [3] for strongly non well-founded codings). Note that the proof of Proposition 4.7 does not depend on properties of the 'standard' codings, other than their almost well foundedness (this discussion continues in Remark 5.8, where a change in the definition of the set D_A, say D_A^0 gives $\langle P\omega, \cdot\rangle \cong \langle D_A^0, \cdot\rangle$.

Theorem 2.8 and Hyland's result [2, 19.1.9] show that $\langle P\omega, \cdot\rangle$ and $\langle D_A, \cdot\rangle$ can be turned into equationally equivalent λ-models. Proposition 4.7 tells us about isomorphic embeddings. Nonetheless in no case $\langle P\omega, \cdot\rangle$ and $\langle D_A, \cdot\rangle$ can be made isomorphic.

Theorem 4.10. $\forall A\ \langle P\omega, \cdot\rangle \not\cong \langle D_A, \cdot\rangle$.

Proof. We first need a few remarks.

$P\omega$ **Claims.** (1) $\forall C \in P\omega\ \emptyset C = \emptyset$ *and* $\{0\}C = \{0\}$;

(2) $\forall E_n \neq \emptyset\ \exists \vec{C}\ E_n\vec{C} = \{0\}$;

(3) $\forall C \in P\omega$ (*C infinite* $\Rightarrow \forall h\ \exists k > h\ \exists\ G_1, \ldots, G_k\ C\vec{G} \neq \emptyset$).

(1) is obvious. As for (2), take the largest k such that $m_k \neq 0 \wedge (m_1, (m_2, \ldots (m_k, 0) \cdots) \in E_n$. Then $E_nE_{m_1} \cdots E_{m_k} = \{0\}$. (3) is trivial.

D_A **Claims.** (4) $\forall d \in D_A$ ($\exists e\ de = d \Rightarrow d = \emptyset \vee d$ *infinite*);

(5) $\forall d \in D_A$ ($\exists e\ de$ *infinite* $\Rightarrow d$ *infinite*);

(6) $\forall \beta \in D_A$, *finite*, $\exists h\ \forall k > h\ \forall d_1, \ldots, d_k\ \beta\vec{d} = \emptyset$.

To prove (4), assume that $d \neq \emptyset$ is finite and take the largest n such that $(\beta_1; \cdots (\beta_n; a) \cdots) = b \in d$ for some $\vec{\beta}$, a. Then $\forall e\ b \notin de$. (5) and (6) are proved similarly.

Assume now that $f : P\omega \to D_A$ is an isomorphism, for some $A \neq \emptyset$. Let $\mathbf{K}$, $\mathbf{S}$, $\mathbf{I}$ interpret K, S, I in $P\omega$. Clearly $\langle D_A, \cdot, f(\mathbf{K}), f(\mathbf{S})\rangle$ is a Combinatory Algebra, namely it is a particular expansion of $\langle D_A, \cdot\rangle$, with interpretation, say, $[\![\ ]\!]^f_\sigma : \mathrm{CL} \to D_A$. $\langle D_A, \cdot\rangle$ has approximable application (see Definition 2.6, part 1), and this depends only on the properties of $\langle D_A, \cdot\rangle$ as applicative structure. Thus Theorem 1.9 applies and

$$\begin{aligned} \emptyset &= [\![SII(SII)]\!]^f && \text{by 1.9} \\ &= f(\mathbf{SII}(\mathbf{SII})) && \text{by def. of } [\![\ ]\!]^f \text{ and } f \\ &= f([\![SII(SII)]\!]) && \\ &= f(\emptyset) && \text{by } P\omega \models SII(SII) = \emptyset \ [2, \text{ch. } 19.1]. \end{aligned}$$

By assumption f is injective, thus, by (1) and (4), $f\{0\})$ is infinite. Finally, observe that

$$\forall\, C \neq \emptyset \quad f(C) \text{ is infinite.}$$

In fact: if C is finite, then use (2), the fact that $f(\{0\})$ is infinite and (5). If C is infinite, then use (3) and (6). This concludes the proof. □

5. Non-well-founded models

The notation is as at the end of the Intermezzo, where $\tilde{D}_A = \langle \tilde{D}_A, \cdot\rangle$ was defined. In particular recall that, in $\tilde{D}_a$, $a = (\{a\}; a).\boldsymbol{\lambda}$ is as in Theorem 2.3.

The motivation for defining $\tilde{D}_A$ are given in the Intermezzo. Its properties will be proved by applying Theorem 1.9 to an elementary substructure.

Lemma 5.1. *In $\tilde{D}_A$ one has:*

(i) $Ad = A \cap d$.

(ii) $a \in d \wedge a \in e \Rightarrow a \in de$, *for* $a \in A$.

(iii) $d \subseteq \boldsymbol{\lambda} x.dx$

(iv) $\langle \tilde{D}_A, \cdot, \boldsymbol{\lambda}\rangle$ *is a* λ-*model.*

Proof. By the definitions. □

Given $a \in A$, let c_a be the constant symbol for $\{a\}$ and $\Lambda(c_a)$ the set of λ-terms built up using also c_a, where $[\![c_a]\!] = \{a\}$.

Lemma 5.2. *Let $\sigma : \mathrm{var} \to \tilde{D}_A$ be constantly equal to $A \in \tilde{D}_A$. Then*

$$\forall\, a \in A \quad \forall\, M \in \Lambda(c_a) \quad a \in [\![M]\!]\sigma.$$

Proof. (By induction.) If $M \equiv c_a$ or $M \equiv x$, we are done. If $M \equiv PQ$, use 5.1(ii) and the induction. If $M \equiv \lambda x.N$, then $[\![M]\!]\sigma = \{(\beta; b): b \in [\![N]\!]\sigma_x^\beta\}$. Notice now that $N[c_a/x] \in \Lambda(c_a)$; then $a \in [\![N]\!]\sigma_x^{\{a\}} = [\![N[c_a/x]]\!]\sigma$, by the induction hypothesis, so $a = (\{a\}, a) \in [\![M]\!]\sigma$. □

Lemma 5.3. *Let σ be as in Lemma 5.2. Then one has for any $M \in \Lambda$:*

(i) $A \subseteq [\![M]\!]\sigma$;
(ii) $A = [\![M]\!]\sigma \Rightarrow M \in \mathrm{O}_0$;
(iii) $M \in \Lambda^0 \Rightarrow \tilde{D}_A \vDash A \subseteq M$.

Proof. (i) If $M \in \Lambda$, then $\forall a\, M \in \Lambda(c_a)$.

(ii) If $\lambda\beta \vdash M = \lambda x.N$, for some N, then $(\gamma; b) \in [\![M]\!]\sigma \wedge \gamma \subseteq \beta \Rightarrow (\beta; b) \in [\![M]\!]\sigma$. Thus $A \subseteq \tilde{B}$ would be saturated, while $(\{a\}; a) \in A$ and, say, $(\{a, (\emptyset, a)\}; a) \notin A$.

(iii) By (i). □

Definition 5.4. $S_A = \{d: A \subseteq d\} \subseteq \tilde{D}_A$.

By Lemma 5.1(ii), S_A is closed w.r.t. '$\cdot$'. Moreover, by Lemma 5.3(iii), $(\tilde{D}_A)^0 = \{[\![M]\!]: M \in \Lambda^0\} \subseteq S_A$; thus $\mathbf{K}, \mathbf{S} \in S_A$ and $\langle S_A, \cdot\rangle$ is a Combinatory Algebra. Consider S_A embedded with the induced topology, where $\tilde{D}_A$ is given the Scott's topology.

Lemma 5.5. $\langle S_A, \cdot\rangle$ *has approximable application.*

Proof. Set $\perp = A$, then S_A is a poset satisfying Definition 1.8(i). Define then $(\ )_n$ as follows.

If $b \in \tilde{B} = B/\simeq$, then b, as equivalence class in B, contains a shortest element (in B), say sh(b): this element is obtained by 'collapsing' all $(\{a\}; a)$ to a. Let $|\ |: B \to \omega$ be as in 2.5. Define, for $d \in S_A$, $d_n = \{b \in d: |\mathrm{sh}(b)| \leq n+1\}$.

Then (1), (2), (3), (5) of Definition 1.8 trivially hold. As for (4):

$$\begin{aligned} d_{n+1}e &= \{b: \exists\, \beta \subseteq e\ (\beta; b) \in d \wedge |\mathrm{sh}(\beta; b)| \leq n+2\} \\ &= \{b: \exists\, \beta \subseteq e\ (\beta; b) \in d \wedge (|\mathrm{sh}(\beta)| + |\mathrm{sh}(b)| \leq n+2)\} \\ &\subseteq \{b: \exists\, \beta \subseteq e_n\ (\beta; b) \in d \wedge |\mathrm{sh}(b)| \leq n+1\} = (de_n)_n \end{aligned}$$

by the definitions of $|\ |$ and $(\ )_n$. □

Theorem 5.6. *Let $M \in \Lambda^0$. Then $\tilde{D}_A \vDash M = A \Leftrightarrow M \in \mathrm{O}_0$.*

Proof. Consider $\langle S_A, \cdot, \mathbf{K}, \mathbf{S}\rangle$, $\mathbf{K}, \mathbf{S} \in S_A \subseteq \tilde{D}_A$; then, for $M \in \Lambda^0$, $S_A \vDash d = M$ iff $\tilde{D}_A \vDash d = M$, i.e. the interpretations of $M \in \Lambda^0$ in S_A and $\tilde{D}_A$ coincide. By Lemma 5.5, 1.9 and 5.3(ii) give the result. □

Consider now Tarski's fixed point operator $Y_T(f) = \bigsqcup f^n(\perp)$. As for any continuous map from $C(\tilde{D}_A, \tilde{D}_A)$ into $\tilde{D}_A$, we may say that Y_T is λ-definable in case $Y_T \circ \phi$ is so, for $\phi(d) = \lambda x \in \tilde{D}_A.dx$. That is the λ-definability of Y_T amounts to say that $Y_T \circ \phi \in C(\tilde{D}_A, \tilde{D}_A)$ is represented by the interpretation of some closed λ-term.

Proposition 5.7. (i) *Tarski's fixed point operator Y_T over $\tilde{D}_A$ is not λ-definable.*

(ii) *$\langle S_A, \cdot\rangle$ can be turned into a λ-model. Moreover, in S_A, Y_T is λ-defined by the interpretation of Curry's paradoxical combinator Y.*

Proof. (i) Just note that $Y_T(\lambda x \in \tilde{D}_A, x) = Y_T \circ \phi(\boldsymbol{\lambda} x.x) = \emptyset$ and use Lemma 5.3(iii) (or Theorem 5.6).

(ii) We first show that any continuous function over S_A is representable. For $f \in C(S_A, S_A)$ define $\tilde{f}: \tilde{D}_a \to \tilde{D}_A$ by $\tilde{f}(d) = f(d \cup A)$. Clearly $\tilde{f} \in C(\tilde{D}_A, \tilde{D}_A)$ and $\tilde{f} \upharpoonright S_A = f$. Moreover $\boldsymbol{\lambda} x.\tilde{f}(x) \in S_A$, since $a \in \tilde{f}(\{a\}) = f(A)$, for all $a \in A$. Thus $f \in C(S_A, S_A)$ iff $f \in (S_A \to S_A)$. Set now $\boldsymbol{\lambda}'(f) = \boldsymbol{\lambda} x.\tilde{f}(x)$, for $f \in (S_A \to S_A)$. Then

$$\forall\, d \in S_A\ \boldsymbol{\lambda}'(f)d = \{b : \exists\, \beta \subseteq d\ b \in \tilde{f}(\beta) = f(\beta \cup A))\}$$
$$= \bigcup_{\beta \subseteq d} f(\beta \cup A) = f(d)$$

by the definition of the induced topology over S_A. The rest is easy.

Finally, $\boldsymbol{\lambda}'$ is monotone; then Theorem 1.12 applies, for $\langle S_A, \cdot, \boldsymbol{\lambda}'\rangle$ is a λ-model with approximable application by Lemma 5.5. Thus

$$T(M) \subseteq T(N) \Rightarrow S_A \models M \subseteq N. \tag{0}$$

By the Remark after 1.12 (and Appendix A), we are done. □

Remark 5.8. $P\omega$ is (isomorphic to) an atomless PSE-Algebra.

Just take $A = \{a\}$ and set $a \simeq (\emptyset; a)$. Then, for $B' = B/\simeq$ and $D^0_A = PB'$, $\langle P\omega, \cdot\rangle \simeq \langle D^0_A, \cdot\rangle$. The isomorphism follows by the proof of $\langle P\omega, \cdot\rangle \Rightarrow \langle D_A, \cdot\rangle$ given in Proposition 4.7 (or by an easy set-theoretic argument).

By [17], any Combinatory Algebra containing an ε such that

(1) $\varepsilon de = de$,

(2) $\forall e (de = d'e \Rightarrow \varepsilon d = \varepsilon d')$,

(3) $\varepsilon\varepsilon = \varepsilon$,

can be turned into a λ-model, by setting $\boldsymbol{\lambda}(f) = \varepsilon d$, for $d \in EC_f$. (Note that $\varepsilon = \boldsymbol{\lambda} xy.xy$.) Conversely, from $\langle D, \cdot, \boldsymbol{\lambda}\rangle = \lambda\beta$ one may define $\varepsilon = \boldsymbol{\lambda} xy.xy$. (Note that $\boldsymbol{\lambda}(f) = \varepsilon d$, for $d \in EC_f$.) (This has already been discussed before Definition 1.7.)

Thus $\langle D, \cdot\rangle$ is lambda-categorical iff such an $\varepsilon \in D$ is unique. By this, isomorphisms of applicative structures preserve lambda-categoricity. Now $\langle D^0_A, \cdot\rangle$ is lambda-categorical, by Definition 4.4, since it is an atomless PSE-Algebra. Thus also $\langle P\omega, \cdot\rangle$ is lambda-categorical. Of course, this gives another proof of Theorem 4.10. (For a direct proof of lambda-categoricity of $\langle P\omega, \cdot\rangle$, see [9].)

Finally, notice that (0) above does not depend on the cardinality of A. In view of the proof of Theorem 5.6, we claim that the lambda-categorical models, given in Theorem 4.6, have all the same theory, independently of the cardinal (i.e. they are all equationally equivalent),

6. Conclusion

The basic view point in this paper has been the analysis of theories of PSE-Algebras. But, more than this, we have been looking at applications of this study. Thus λ-expansions have been studied and related to the local analysis of models or used for the semantical characterization of interesting classes of terms. Moreover the results in Sections 2 and 3 were applied in the lambda-categoricity and cardinality theorems of Section 4.

Similarly, Section 5 gave some results on the connections between non well-foundeness, substructures and true equalities. This was done using quotient PSE-Algebras.

In our views two kinds of questions are naturally raised by this work.

(1) Given a PSE-Algebra how can one characterize quotient sets which are again PSE-Algebras? That is, generalize the technique used in Sections 4 and 5 (some hints are given in the Remark at end of the Intermezzo). More: are there general results relating equivalence relations on PSE-Algebras and the theories of their quotient sets? In Section 5 a non well-founded quotient PSE-Algebra gives $Y \neq Y_T$; another turns out to be isomorphic to $P\omega$, thus $Y = Y_T$ holds in it.

Several results on quotient sets for similar structures (namely, filter domains) are given in [10]. Still, filter λ-models are built over more 'structured' bases (theories of type assignment). Thus, this kind of results are more general (and, perhaps, more difficult) in the set theoretic framweork of PSE-Algebras.

(2) PSE-Algebras solve some equations (cf. Remark 3.9). Can one carry on a general category theoretic study of the theories of solutions of domain equations (in the sense of Scott)?

It is not clear at all whether category theoretic notions may characterize theories. Take for example Scott's basic equation $D = D \to D$. The inverse limit solution, D_∞, to this equation, in the category of cpo's, seems to yield different theories according to the projections one uses (the 'canonical' ones or Park's, see [2, ch. 18]). As a matter of fact, the Approximation Theorem holds only when the canonical projections are used.

Moreover an equation may have lambda-categorical solutions and/or non lambda-categorical ones, which in turn may yield different theories.

Appendix A: Proofs of Theorem 1.12 and Proposition 1.14

As for Theorem 1.12, one can use a variant for trees of the Approximation Theorem à la Scott–Wadsworth–Hyland (see [2, 19.1.8]).

Since, in the models we work at, we do not require that $\boldsymbol{\lambda}x.\perp = \perp$, consider the extension $\lambda\beta\Omega_1$ of $\lambda\beta$ obtained by adding the constant Ω to the formation rules of the λ-terms, and the axioms $\Omega M \to M$, only.

Extend the definition of Tree to terms of $\lambda\beta\Omega_1$ by $T(\lambda x_1 \cdots x_n.\Omega\vec{M}) = \lambda x_1 \cdots x_n.\perp$, for $0 \leqslant n$, $\vec{M}$ possibly empty.

Note that $\lambda\beta\Omega_1 \vdash M = N \Rightarrow T(M) = T(N)$. Define then the set of *Tree Approximate normal forms* by

$$TA(M) = \{P\colon T(P) \subseteq T(M) \text{ and } P \text{ is in } \beta\Omega_1\text{-n.f.}\}.$$

The basic Lemmas A.1 and A.2 go through as for the Approximation Theorem for Böhm-trees. Namely, define first a labelled $\lambda\Omega_1$-calculus ($\lambda\beta\Omega_1^N$). That is extend the formation rules of λ-terms by allowing labels in ω over terms. Take as axioms the axioms in [2, 14.1.4], *except* $\lambda x.\Omega \to \Omega$. ([1, 7.18–7.19], except for $\lambda x.\Omega^P \to \Omega$, best fits our approach.) By the same argument as in [1, 7.23], one has:

Lemma A.1. *Each completely labelled term has a normal form.*

If M is labelled, set $T(M) = T(|M|)$, where $|M|$ is obtained from M leaving out all the labels.

Given a λ-model with approximable application interpret labelled terms by $[\![\Omega]\!]_\sigma = \perp$, $[\![M^n]\!]_\sigma = ([\![M]\!]_\sigma)_n$.

Lemma A.2. *Let M, Q be labelled terms, in $\lambda\beta\Omega_1^N$. Assume that $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ has approximable application. Then, for $M \to Q$, one has:*

(i) $D \vDash M \leqslant Q$.

(ii) $T(Q) \subseteq T(M)$.

(The *proof* is as in [2, 19.1.6], just notice that $d_0 = \perp$ of Definition 1.8 implies the validity of $(\lambda x.M)^0 N = ([\Omega/x]M)^0$ in D.)

Definition A.3. Let $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ be a λ-model with approximable application. Then $\boldsymbol{\lambda}$ is $\perp$*-monotone* iff for any algebraic expression r over D and $n \in \omega$, one has

$$\boldsymbol{\lambda}(\lambda d_1 \cdots d_n \in D.\perp) \leqslant \boldsymbol{\lambda}(\lambda d_1 \cdots d_n \in D.r)$$

(or, equivalently, $D \vDash \lambda x_1 \cdots x_n.\Omega \leqslant \lambda x_1 \cdots x_n.M$, for all M in $\lambda\beta\Omega_1$ and $n \in \omega$).

Lemma A.4. *Let $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ be as in Definition* A.3. *Assume that $\boldsymbol{\lambda}$ is monotone or that D has λ-approximable application. Then $\boldsymbol{\lambda}$ is $\perp$-monotone.*

Proof. By an easy induction or trivial. $\square$

Lemma A.5. *Let* $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ *be a* λ*-model with approximable application and* $\perp$*-montone* $\boldsymbol{\lambda}$. *Then*

$$N \in TA(M) \Rightarrow D \models N \subseteq M.$$

Proof. By assumption $T(N)$ is finite and the Ω's in N and the $\perp$'s in $T(N)$ are in a one–one correspondence. Thus M is obtained from N (up to $\beta\Omega_1$-equality) by replacing some Ω's in N by other terms or some $\lambda x_1 \cdots x_n.\Omega$ by some $\lambda x_1 \cdots x_m.Q$, for $m \geq n$ (see Proposition 1.3(ii)).

$\perp$-monotonicity of $\boldsymbol{\lambda}$ and the monotonicity of '$\cdot$' give the result. $\square$

Approximation Theorem A.6 (for Trees). *Let* $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ *be as in Lemma* A.5. *Then*

$$D \models M = \bigsqcup \{N \colon N \in TA(M)\}.$$

Proof.

$$\begin{aligned} D \models M &= \{M^I \colon I \text{ complete labelling}\} && \text{by Definition 1.8} \\ &\leq \{Q \colon |Q| \in TA(M)\} && \text{by Lemmas A.1 and A.2(i)} \\ &\leq \{N \colon N \in TA(M)\} && \text{by } D \models Q \leq |Q| \\ &\leq M && \text{by Lemma A.5.} \quad \square \end{aligned}$$

Theorem 1.12 now follows from Lemma A.4 and Theorem A.6, by the same argument as in [2, 19.1.9, 19.1.11] applied to truncated Trees.

Moreover, let Y be a fixed point operator in $\lambda\beta$, then $TA(Y) = \{\lambda y.y^n\Omega : n \in \mathbb{N}\}$, by Theorem A.6. Therefore, for $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ as in Lemma A.5, one has

$$D \models Y = \bigsqcup \lambda y.y^n\Omega.$$

Thus, if Y_T is Tarski's fixed point operator over $\langle D, \cdot\rangle$, $[\![Y]\!]_\sigma d = \bigsqcup d^n \perp = Y_T(d)$ and $\boldsymbol{\lambda}(Y_T) = \boldsymbol{\lambda} d.\, Y_T(d) = \boldsymbol{\lambda} d.[\![Y]\!]_\sigma d = [\![Y]\!]_\sigma$, since Y is not in O_0 (i.e. begins with $\lambda y \cdots$). Note that, if D is as in Lemma A.5, then

$$M \in \mathrm{O}_\infty \Rightarrow D \models M = \bigsqcup_n \lambda x_1 \cdots x_n.\perp.$$

As for the *proof of Proposition* 1.14, let $\lambda\beta\Omega$ be the extension of $\lambda\beta\Omega_1$ obtained by adding

$$\lambda x.\Omega \to \Omega.$$

For $\mathrm{BT}(M)$ = 'Böhm tree of M', let

$$A(M) = \{N \colon \mathrm{BT}(N) \subseteq \mathrm{BT}(M) \text{ and } N \text{ is in } \beta\Omega\text{-n.f.}\}.$$

Given a $\lambda\Omega$-term N, let N^* be obtained from N, performing also $\lambda x.\Omega \to \Omega$, if any such reduction is possible. Then one has

$$N \in TA(M) \;\Leftrightarrow\; N^* \in A(M). \tag{1}$$

Assume now that $\langle D, \cdot, \boldsymbol{\lambda}\rangle$ is a λ-model with λ-approximable application. Then, if N is in $\beta\Omega_1$-n.f.,

$$D \vDash N^* = N \quad \text{by } N^* \in TA(N), \text{ Lemma A.5 and } \boldsymbol{\lambda}x.\bot = \bot. \tag{2}$$

Finally

$$\begin{aligned} D \vDash M &= \bigsqcup\{Q: Q \in TA(M)\} && \text{by Lemma A.4 and Theorem A.6}\\ &= \bigsqcup\{Q^*: Q \in TA(M)\} && \text{by (2)}\\ &= \bigsqcup\{N: N \in A(M)\} && \text{by (1).} \end{aligned}$$

This is the Approximation Theorem for Böhm trees: Proposition 1.14 then follows as in [2, 19.1.11].

Appendix B: $D_{\mathrm{A}} \vDash M \subseteq N \Rightarrow \mathrm{BT}(M) \subseteq \mathrm{BT}(N)$

This appendix completes the proof of Lemma 2.7, thus the notation is as in Section 2.

Lemma B.1. (i) *Let* $f \in (D_{\mathrm{A}} \to D_{\mathrm{A}})$. *Then* $\lambda x.f(x)$ *is saturated* (*i.e.* $(\beta; b) \in \boldsymbol{\lambda}x.f(x) \wedge \beta \subseteq \gamma \Rightarrow (\gamma; b) \in \boldsymbol{\lambda}x.f(x)$).

Let $\mathbf{A} = B \setminus A$. Then

(ii) $(d \cap \mathbf{A})e = de$,

(iii) $d \subseteq \mathbf{A} \Leftrightarrow d \subseteq \boldsymbol{\lambda}x.dx$.

Proof. (i) By monotonicity of f. (ii) By definition. (iii)$\Rightarrow$. $(\beta; b) \in d \Rightarrow b \in d\beta \Rightarrow (\beta; d) \in \boldsymbol{\lambda}x.dx$.

$\Leftarrow \boldsymbol{\lambda}x.dx$ does not contain element of A. □

Note that (ii) and (iii) hold just because one can distinguish between elements of A and elements of $\mathbf{A}$. Fix now $a_0 \in A \neq \emptyset$.

Definition B.2. Let $f \in (D_{\mathrm{A}} \to D_{\mathrm{A}})$. Define

$$\lambda^0 x.f(x) = \boldsymbol{\lambda}x.f(x) \cup \{a_0\}.$$

(*Notation.* For $f \in C(D_{\mathrm{A}}^n, D_{\mathrm{A}})$, set

$$\lambda^0 x_1 \cdots x_n.f(x_1, \ldots, x_n) = \lambda^0 x_1(\lambda^0 x_2 \cdots x_n.f(x_1, \ldots, x_n));$$

by the continuity of $\cup$ and $\boldsymbol{\lambda}$, this is a good definition.)

Remark B.3. By definition

$$\boldsymbol{\lambda}x_1 \cdots x_n.f(x_1, \ldots, x_n) = \{\beta_1; \cdots (\beta_n; b) \cdots): b \in f(\beta_1, \ldots, \beta_n)\},$$

while $\lambda^0 x_1 \cdots x_n.f(x_1, \ldots, x_n)$ contains also a_0 and all elements of the type $(\beta_1; a_0), \ldots, (\beta_1; \cdots (\beta_{n-1}; a_0)$ for arbitrary β's.

Lemma B.4. *Let $f \in C(D^n_A, D_A)$. Then*

(1) $0 \leq i \leq n \Rightarrow (\lambda^0 x_1 \cdots x_n.f(x_1, \ldots, x_n))d_1 \cdots d_i$
$$= \lambda^0 x_{i+1} \cdots x_n.f(d_1, \ldots, d_i, x_{i+1}, \ldots, x_n)$$
(2) $0 \leq p < n \Rightarrow a_0 \in (\lambda^0 x_1 \cdots x_n.f(x_1, \ldots, x_n))d_1 \cdots d_p$.

Proof. Easy. □

Notation. Given $P \subseteq D^n_A$, $\beta_1, \ldots, \beta_n$ are *minimal such* (mins) $P(\beta_1, \ldots, \beta_n)$ iff
(i) $P(\beta_1, \ldots, \beta_n)$ holds, and
(ii) if $\gamma_1 \subseteq \beta_1, \ldots, \gamma_n \subseteq \beta_n$ and $\vec{\gamma} \neq \vec{\beta}$
then $\neg P(\gamma_1, \ldots, \gamma_n)$.

Definition B.5. Let $f \in C(D^n_A, D_A)$. Define

$$\lambda^- x_1 \cdots x_n.f(x_1, \ldots, x_n) = \{b \in \lambda^0 x_1 \cdots x_n.f(x_1, \ldots, x_n): \exists c\ \exists \vec{\beta}\ b = (\beta_1; \cdots (\beta_n; c) \cdots) \Rightarrow \vec{\beta} \text{ mins } c \in f(\vec{\beta})\}$$

Lemma B.6. *Let $f \in C(D^n_A, D_A)$. Then*
(1) $(\lambda^- x_1 \cdots x_n.f(x_1, \ldots, x_n))d_1 \cdots d_n = f(d_1, \ldots, d_n)$.
(2) *If* $\forall d_1, \ldots, d_{n-1} \exists d_n\, f(d_1, \ldots, d_{n-1}, d_n) \neq \emptyset$, *then* $(0 \leq p < n \Rightarrow (\lambda^- x_1 \cdots x_n.f(x_1, \ldots, x_n))d_1 \cdots d_p$ *contains a_0 and it is not saturated (cf. Lemma* B.1)).

Proof. (1) By definition and Lemma B.4(1).
(2) Let's write $F^-_n := \lambda^- x_1 \cdots x_n.f(x_1, \ldots, x_n)$. Then

$$a_0 \in F^-_n d_1 \cdots d_p = \{c : \exists\ \vec{\beta} \subseteq \vec{d}\ (\beta_1; \cdots (\beta_p; c) \cdots) \in F^-_n\},$$

since $p < n$ and by Lemma B.4(2) and the definition of F^-_n. As for 'non saturation', notice first that

$$\begin{aligned} f(e_1, \ldots, e_n) &= F^-_n e_1 \cdots e_n, \quad \text{by (1)} \\ &= \{b : \exists\ \vec{\beta} \subseteq \vec{e}\ (\beta_1; \cdots (\beta_n; b) \cdots) \in F^-_n\}; \end{aligned}$$

and hence, by the assumption on f,

$$\forall\ p < n\ \forall\ e_1, \ldots, e_p\ \exists\ \beta_1 \subseteq e_1, \ldots, \exists\ \beta_p \subseteq e_p \quad \exists\ \beta_{p+1}, \ldots, \exists \beta_n\ \exists b\ (\beta_1; \cdots (\beta_n; b) \cdots) \in F^-_n.$$

Recall now that by definition of F^-_n, these $\beta, \ldots, \beta_n$ are 'minimal such', thus, in particular, $\forall\ \gamma \supset \beta_{p+1}\ (\gamma; (\beta_{p+2}; \cdots (\beta_n; b) \cdots) \notin F^-_n d_1 \cdots d_p$. □

Definition B.7. Let $C_p^- \equiv \lambda^- x_0 \cdots x_{p+1}.x_{p+1}x_0 \cdots x_p$.
Let $\Lambda(D_A)$ the set of λ-terms built up using also constants (symbols) from D_A.

Proposition B.8. (I) $D_A \models C_p^- x_0 \cdots x_{p+1} = x_{p+1}x_0 \cdots x_p$.

(ii) *Let* $\sigma_1, \ldots, \sigma_m$ *and* $\tau_1, \ldots, \tau_q$ *be terms in* $\Lambda(D_A)$. *If* $n \neq t$ *and* $m, q < p$, *then*

$$D_A \not\models \lambda x_1 \cdots x_n.C_p^- \sigma_1 \cdots \sigma_m \subseteq \lambda x_1 \cdots x_t.C_p^- \tau_1 \cdots \tau_q.$$

Proof. (i) by Lemma B.6(1).

(ii) Clearly $f(d_0, \ldots, d_{p+1}) = d_{p+1}d_0 \cdots d_p$ satisfies the conditions on f in Lemma B.6(2).

Assume $D_A \models \lambda x_1 \cdots x_n.C_p^- \sigma_1 \cdots \sigma_m \subseteq \lambda x_1 \cdots x_t.C_p^- \tau_1 \cdots \tau_q$.

Case $n < t$. Apply both LHS and RHS to $x_1, \ldots, x_n$. Then

$$D_A \models C_p^- \sigma_1 \cdots \sigma_m \subseteq \lambda x_{n+1} \cdots x_t.C_p^- \tau_1 \cdots \tau_q.$$

This is impossible since the LHS contains a_0 by Lemma B.6(2), while the RHS doesn't.

Case $t < n$. Apply both LHS and RHS to $x_1, \ldots, x_t$. *Then*

$$D_A \models \lambda x_{t+1} \cdots x_n.C_p^- \sigma_1 \cdots \sigma_m \subseteq C_1^- \tau_1 \cdots \tau_q.$$

This is impossible since the RHS is not saturated by Lemma B.6(2), while the LHS is saturated by Lemma B.1(i). □

Proposition B.8 is the C-lemma, Lemma 3.3 of [5]. Thus the rest of the proof of Proposition 2.8 is exactly as in that paper.

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