REFLECTIONS ON FORMALISM AND REDUCTIONISM IN LOGIC AND COMPUTER SCIENCE

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Content:

This report contains a *preprint* (paper 1) and a *reprint* (paper 2). The first develops some epistemological views which were hinted in the second, in particular by stressing the need of a greater role of geometric insight and images in foundational studies and in approaches to cognition. The second paper is the "philosophical" part of a lecture in Type Theory, whose technical sections, omitted here, have been largely subsumed by subsequent publications (see references). The part reprinted below discusses more closely some historical remarks recalled in paper 1.

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Many logicians have now turned into applied mathematicians, whose role in Computer Science is increasingly acknowledged even in industrial environments. This fact is gradually changing our understanding of Mathematical Logic as well as its perspectives. In this note, I will try to sketch some philosophical consequences of this cultural and "sociological" change, largely influenced by Computer Science, by a critique of the role of formalism and reductionism in Logic and Computing.

In mathematics, we are mostly used to a dry and schematic style of presentation: numbered definitions and theorems scan the argument. This may add effectiveness and clarity, though nuances may be lost: the little space allowed here requires to stress effectiveness.

Themes - Three main aspects relating Computer Science and the logical foundation of mathematics will be mentioned below, namely

1. the growth of a pragmatic attitude in Logic
2. revitalization and limits of formalism and constructivism
3. the role of space and images.

1. **Pragmatism in Logic**

1.1 **Tool vs. foundation**. In Computer Science, Mathematical Logic is no longer viewed as a foundation, but as a tool. There is little interest in setting on firm grounds Cobol or Fortran, say, similarly as Logic aimed at founding Number Theory or Analysis by, possibly complete, axiomatic systems. The actual work is the invention of new programming languages and styles, or algorithms and architectures, by using tools borrowed from Mathematical Logic. This also may originate in attempts to base on clear grounds known constructs, but the ultimate result is usually a novel proposal for computing. Functional and Logic Programming are typical examples for this.

This "engineer's approach" in applied Logic is helping to change the philosophical perspective in pure Logic, as well.

1.2 **An analogy: the completeness of mechanical systems**. The interests in the foundation of mathematics, in this century, have been a complex blend of technical insights and philosophical views. We must acknowledge first that philosophical a priori, in Logic, may have had a stimulating role, in some cases. Yet ideologies and the blindness of many attitudes have been a major limitation for knowledge, an unusual phenomenon in scientific research. They have been the source of

- wrong conjectures (completeness, decidability, provability of consistency)
- false proofs, by topmost mathematicians (just to mention the main cases: an inductive proof of the consistency of Arithmetic, by Hilbert, refuted by Poincaré in 1904-05; a second attempt, based on a distinction of various levels of induction, debated by Hermann Weyl in the twenties).

The wrong directions taken by the prevailing formalist school may be understood as a continuation of a long lasting attitude in science and philosophy. On the shoulders of last centuries' giants, Newton and Laplace, typically, the positivist perspective believed in perfect and complete descriptions of the world by classical mechanics, namely by sufficiently expressive systems of partial differential equations. Similarly, in Logic, adequate axiomatic systems were supposed to describe completely Analysis or the whole of mathematics. Two levels of descriptions, both exhaustive, or complete in the sense of Logic: one of the world by mathematical equations, the other of mathematics by (finitely many) axioms.

Still, this posivistic vision, in Logic, was not compelled by the times. Hermann Weyl conjectured the incompleteness of number theory and the independence of the axiom of choice in "Das Kontinuum", in 1918. Poincaré rejected purely logical or linguistic descriptions as the only source for mathematics and stressed the role of geometric insight. As I will try to hint below, Poincaré's distinction between analytic work and the intuition of space as well as his approach to the foundation of mathematical knowledge may be today at the base of a renewed foundational work, similarly as his work on the three body problem is at the origin of contemporary mechanics (Lighthill[1986]).

1.3 **Foundation of mathematical knowledge**. Another mathematician should be quoted among those who did not except the reductionist and formalist attitudes, in the first part of this century: Federico Enriquez. Also in Enriquez's philosophical writings, the interest in the interconnections of knowledge and in its historical dynamics suggested more open philosophical perspectives. It may be fair to say that Poincaré, Weyl and Enriquez were interested in the foundation of mathematical knowledge more than in the technical foundation of mathematics.

The difference should be clear: by the first I mean the epistemology of mathematics and the understanding of it "as integral part of the human endeavor toward knowledge" (to put it in Weyl's words). Not a separated transcendence, isolated in a vacuum, but an abstraction emerging from our concrete experience of the real world of relations, symmetries, space and time (Poincaré[1902, 1905], Enriquez [1909], Weyl[1918, 1952]).

In contrast to this broad attitude, the purely technical, internal foundation, as pursued by formalistic and reductive programs in their various forms, somewhat reminds of a drawing of a book on a table and of the belief that the table really supports the book, while they are designed by us by the same technique and the same tools. This cognitive circularity is at the source of the negative results in Logic.

1.4 **The unity of formal systems**. The first step towards a more open attitude comes with the need for a variety of systems and the understanding of their interconnections. The laic attitude inspired by applications and the suggestions coming from geometric intuitions are at the origin of recent inventions of new systems of Logic, where unity is given not by a global, metaphysical, system, but by the possibility of moving from a system to another, by changes in the basic rules and by translations or connecting results. Indeed, Girard's focus on the structural rules in Girard[1987] and his seminal work in Girard[1992] are largely indebted to a pragmatic attitude that views Logic as part of (applied) mathematics, with no special "meta-status." In this perspective, geometric structures and applications suggest formal systems, guide toward relevant changes, propose comparisons, in the common mathematical style where connections and bridges preclude ideological closures within one specific frame. In this sense, its unity is a deep mathematical fact, as much as Klein's unified understanding of Geometry.

2. **Formalism and Constructivism in Computer Science (and their limits)**.

2.1 **Linguistic notations**. The volume in Combinatory Logic by Curry and Feys contains many pages on the renaming of bound variables and related matters (in set-theoretic terms, $\{x \mid P(x)\}$ is the same as $\{y \mid P(y)\}$). I believe that the foundational relevance of these pages, if any, may be summarized in about three lines. The formalist treatment is a typical example of purely symbolic manipulation, where meaning and structures are lost (see Curry's book on a formalist foundation of mathematics for an extremist's view in that direction). This opinion is shared by all working mathematicians who simply ignored the discussion and explain the problem to students in twenty seconds, on the blackboard. Still, it happened that variable binding is a crucial issue in computer programming. Thus, the discussion in Curry et al. has been largely and duly developed in functional programming and it is at the core of detailed treatments in implementations.

This example is just a small, but typical one, of the revitalization of formalism and constructivism due to Computer Science. It happens that computers proceed as our founding fathers of Logic described the parody of mathematics: linguistic definitions and formal deductions, with no meaning as a guidance. Meaningless, but effective constructions of programs, more than unifying insights and concepts. Mathematical invariants are lost, but denotations are very precise. This requires technically difficult insights into pure calculi of symbols and, sometimes, brand new mathematics. However, the branching of methods and results, due to translations and meaning, which are at the core of knowledge, may be lost within extremely hard, but closed, games of symbols. This is part of everyday's experience on the hacker's side even in theory of computing.

2.2 **Denotational semantics**. Fortunately, though, even programming has been affected by meaning. In the last twenty and odd years, various approaches to the semantics of programming languages embedded programming into the broader universe of mathematics. Here is the main merit of the Scott-Strachey approach, as well as of the algebraic or other proposals, most of which are unifiable in the elegant frame of Category Theory (this is partly summarized in Asperti&Longo[1991]). In some cases, the meaning of formal systems for computing, over geometric or algebraic structures, suggested variants or extensions of existing languages. More often, obscure syntactic constructs, evident at most to the authors, have been clarified and, possibly, modified. As a matter of fact, in the last decade, computer manuals have slowly begun to be readable, as they are moving towards a more mathematical style, that is towards rigor, generality and meaning at once. We are not there yet, as most hackers think in terms of pure symbol pushing and are supported in this attitude by the formalist tradition in Logic. Many still do not appreciate from mathematics that the understanding and, thus, the design of a strictly constructive, but complicated system may also derive from highly non constructive, but conceptually simple, intellectual experiences.

2.3 **Resources and memory**. Brouwer, the founding father of intuitionism, explicitly considers human memory as "perfect and unlimited," for the purposes of his foundational proposal (Troelstra[1990]). This is implicitly at the base of the formalist approach as well. Indeed, computer memories are perfect and, by a faithful abstraction, unlimited. This has nothing to do with human memory, and mathematics is done by humans. One component of mathematical abstraction, as emerging from our "endeavour towards knowledge," may precisely derive from the need to organize language and space in least forms, for the purposes of memory saving. A "principle of minima" may partly guide our high level organization of concepts, Sambin[1990]. Moreover, imperfection of storage is an essential part of approximate recognition, of analogy building.

For the aim of founding mathematical knowledge, we need exactly to understand the emergence of abstraction, the formation of conceptual bridges, of methodological contaminations between different areas of mathematical thinking. There may be a need for psychology and neurophysiology in this approach. Good: the three mathematicians quoted in 1.3 have been often accused of "psychologism", of "wavering between different approaches", in their foundational remarks. The proposed "one-way" alternatives lead to the deadlock where formalism and reductionism brought us in understanding mathematics (and the world). Moreover, so much happened in this century in other areas of knowledge, that we should start to take them into account.

2.4 **Top down vs. bottom-up**. There is no doubt that formalism and reductionism have been at the base of Computer Science as it is today and of its amazing progress. In particular, top-down deductions and constructive procedures set the basis for the Turing and Von Neuman machines as well as for all currently designed languages, algorithms and architectures. Yet, there is a growing need to go beyond top-down descriptions of the world, even in Computer Science. The recent failures of strong Artificial Intelligence are the analogue of incompleteness and independence results in Logic: most phenomena in perception and reasoning escape the stepwise-deductive approach. Partly as a consequence of these failures, there is an increasing interest in bottom-up approaches. Relevant mathematics is being developed in the study of the way images, for example, organize space by singularities, or how the continuum becomes discrete and reassembles itself, in vision or general perception, in a way which leads from quantitative perception to qualitative understanding, Petitot[1992].

3. **Space and Images**

3.1 **Denotations and Geometry**. In the practice of mathematics, formal notations and meaning are hardly distinguished. Indeed, one may even have symbolic representations where, besides the geometric meaning, there exists a further connection to Geometry, at the notational level. Relevant examples of this are given by Feyman's and Penrose's calculi or by Girard's proof nets.

In Feyman's calculus, planar combinations of geometric figures allows computations representing subatomic phenomena. Penrose's extends familiar tensorial calculi over many dimensional vector spaces in a very powerful way: bidimensional connections between indexes explicitly use properties of the plane to develop computations. A more recent example may be found in Girard's Linear Logic. In this system, formal deductions are developed by drawing planar links between formulae in a proof tree. Proofs (and cuts) are carried on by an explicit use of the geometric representation, by modifying the links. As the proof-theoretic calculus is essentially complex, according to recent complexity results, the use of the geometric representations comes in as an essential tool for the formal computation.

In a sense, all these calculi derive from (physical) space or Geometry and, after an algebraic or syntactic description, end up in geometric representations, possibly unrelated to the original one. As a matter of fact, even Linear Logic originated in Geometry, as it was suggested by the distributive or linear maps over coherent spaces (Girard[1987]), and ends up into a Geometry of proofs.

3.2 **Geometric insight**. I would like to mention here the possible relations of the novel mathematical approaches to vision mentioned in 2.4, and similar ones in other forms of perception, to the wise blend of linguistic, or analytic, and geometric experiences required by the practice and the foundation of mathematics.

It should be clear that, in mathematics, synthetic explanations may provide an understanding and a foundation as relevant as stepwise reductionist descriptions. The drawing on a blackboard may give as much certainty as the search for least axioms for predicative Analysis. The point now is to understand what is behind the drawing, which intellectual experiences give to it so much expressiveness and certainty. The point is to turn this practice of human communication, by vision and geometric insight, into a fully or better understood part of knowledge. This is where our endeavor towards general knowledge cannot be separated from the foundation of mathematics. Mathematics is just a topmost human experience in language and space perception, unique as for generality and objectivity, but part of our relation to the world.

A few examples may be borrowed from neurophysiology (see Ninio[1991], Maffei[1992]). It seems that only human beings perform interpolations (vertices of a triangle or of a square seen as complete figures, sets of stars as a constellation...). Apparently this is done by minimal lines which complete incomplete images. We seem to interpolate by splines, when needed. More: there are neurons which recognise (send an impulse) only in presence of certain angles, or others which react only to horizontal or vertical lines. This is recomposed in intellectual constructions which are at the base of our everyday vision and of the so called optic illusions (which are just attempted reconstructions of images). And, why not, at the base of our geometric generalizations.

But how? How can we make this "composition of basic mental images" as part of a new foundation of mathematical knowledge, in the same way as formal, linguistic axioms, have been describing part of the analytic developments in mathematics?

I can only mention the problem, for the time being, and stress what is really missing, the possible source of incompleteness: the lack of Geometry and images in foundational studies. A modern rediscovery of these aspects may be at the core of an understanding of image recognition which goes together with an appreciation of geometric abstraction in mathematics.

In a sense we should enrich the insufficient attempts to deduce all of mathematics by linguistic axioms, by adding, at least, the knowledge we have today of space perception and of the process of image formation. This may help to focus the way in which mathematics emerges, surely by compositions of elementary components (the lines and triangles I mentioned before), but also by "synthesis" and reorganization of space, as mentioned in 2.4. In this, a renewed Artificial Intelligence, far away from the prevailing formalist one, may be a novel contribution of Computer Science to the foundation of mathematical knowledge, and conversely. The difficult point is to be able to move, in foundational studies and everyday's work, from local, quantitative and analytic approaches to global, qualitative and geometric perspectives and still preserve the crucial (informal) rigor of mathematics.

3.3 **The continuum and minima: more about reductionism**. In "Das Kontinuum" Weyl[1918], Hermann Weyl raises the issue of the continuum of Analysis vs. the continuum of time. The understanding of the latter is based on the simultaneous perception of past, present and future. In this irreducible phenomenological intuition of time it is not possible to isolate the temporal point, in contrast to the analytic description where this can be done by reduction to linguistic abstractions, that is to symbols and (derivable) properties. Weyl, a mathematician working also in relativity theory, expresses his unsatisfaction and raises a major point for mathematical knowledge: the convenient analytic unity of space and time does not correspond to our fundamental experience (see also Petitot[1992]). This problem has not been sufficiently studied since then, as we were mostly concerned by formalist reductions and the search for complete and (self-)consistent Set Theories, as a basis for Analysis. These formal theories have not been able to tell us anything even about the cardinality of an arbitrary set of reals (independence of the Continuum Hypothesis), let alone the profound mismatch between time and the analytic description of space, as given by the real line.

This need of ours to "fill up the gaps", possibly by continuity, may probably go together with the principles of minima, mentioned in 2.3 and 3.2 as a possible description of some aspects of abstraction (memory optimization and the formation of images, respectively). These principles are usually very complex in mathematics and, when referring to them, we depart from reductionism. Yet another relevant mathematical experience then, to be added to the continuum of time, which seems to escape reductionism. Reductions are surely a relevant part of scientific explanations, however they are far from proposing complete methodologies or providing the only possible foundation of knowledge.

In conclusion, we need to focus on alternative approaches to formalism and reductionism both in applied as well as in theoretical approaches to cognition. In 2.4 and 3.2, the role is mentioned of current inverse paradigms with respect to the prevailing top-down, deductive formalizations: bottom-up descriptions, for example, which may give a complementary account of perception and conceptual abstraction. What really matters now is to extend, not to keep reducing our tools. Our rational paradigms must be made to comprehend the mathematical, indeed human, intuition of space and time. In other words, we need to lower the amount of magic and mystery in these forms of intuition, and bring them into the light of an expanded rationality.

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SOME ASPECTS OF IMPREDICATIVITY

Notes on Weyl's Philosophy of Mathematics and on todays Type Theory

Part I (*)

Giuseppe Longo

"The problems of mathematics are not isolated problems in a vacuum; there pulses in them the life of ideas which realize themselves in concreto through out human endeavors in our historical existence, but forming an indissoluble whole transcend any particular science"
Hermann Weyl, 1944.

(*) First part of a lecture delivered at the **Logic Colloquium 87**, European Meeting of the ASL, and written while teaching in the Computer Science Dept. of Carnegie Mellon University, during the academic year 1987/88. The generous hospitality and the exceptional facilities of C.M.U. were of a major help for this work.
(The second, more technical, part of this lecture has been largely supersed by Longo&Moggi (**Mathematical Structures in Computer Sciences**, 1 (2), 1991), under ftp as omegaSetModel.ps.gz)

1. Logic in Mathematics and in Computer Science.

There is a distinction which we feel a need to stress when talking (or writing) for an audience of Mathematicians working in Logic. It concerns the different perspectives in which Logic is viewed in Computer Science and in Mathematics. In the aims of the founders and in most of the current research areas of Logic within Mathematics, Mathematical Logic was and is meant to provide a "foundation" and a "justification" for all or parts of mathematics as an established discipline. Since Frege and, even more, since Hilbert, Proof Theory has tried to base mathematical reasoning on clear grounds, Model Theory displayed the ambiguities of denotation and meaning and the two disciplines together enriched our understanding of mathematics as well as justified many of its constructions. Sometimes (not often though) results of independent mathematical interest have been obtained, as in the application of Model Theory to Algebra; moreover, some areas, such as Model Theory and Recursion Theory, have become independent branches of mathematics whose growth goes beyond their original foundational perspective. However, these have never been the main aims of Logic in Mathematics. The actual scientific relevance of Logic, as a mathematical discipline, has been its success in founding deductive reasoning, in understanding, say, the fewest rational tools required to obtain results in a specific area, in clarifying notions such as consistency, categoricity or relative conservativity for mathematical theories.

This is not so in Computer Science, where Mathematical Logic is mostly used as a ***tool****, not as a foundation.* Or, at most, it has had a mixed role: foundational and "practical". Let us try to explain this. There is no doubt that some existing aspects of Computer Science have been set on clearer grounds by approaches indebted to Logic. The paradigmatic example is the birth of denotational semantics of programming languages. The Scott-Strachey approach has first of all given a foundation to programming constructs already in use at the time. However, the subsequent success of the topic, broadly construed, is mostly due to use that computer scientists have made of the denotational approach in the design new languages and software. There are plenty of examples - - from Edinburgh ML to work in compiler design to the current research in polymorphism in functional languages. Various forms of modularity, for example, are nowadays suggested by work in Type Theories and their mathematical meaning. In these cases, results in Logic, in particular in lambda-calculus and its semantics, were not used as a foundation, in the usual sense of Logic, but as guidelines for new ideas and applications. The same happened with Logic Programming, where rather old results in Logic (Herbrand's theorem essentially) were brought to the limelight as core programming styles. Thus Mathematical Logic in Computer Science is mostly viewed as one of the possible mathematical tools, perhaps the main one, for applied work. Its foundational role, which also must be considered, is restricted to conceptual clarification or "local foundation", in the sense suggested by Feferman for some aspects of Logic in Mathematics, instead of the global foundation pursued by the founding fathers of Logic. Of course, the two aspects, "tool" and "local foundation", can't always be distinguished, as a relevant use of a logical framework often provides some sort of foundation for the intended application.

It is clear that this difference in perspective deeply affects the philosophical attitude of researchers in Logic according to whether they consider themselves as pure mathematicians, possibly working at foundational problems, or applied mathematicians interested in Computer Science. The later perspective is ours.

In the sequel we will be discussing "explanations" of certain impredicative theories, while we will not try to "justify" them. This is in accordance with the attitude just mentioned. By **explanation** we essentially mean "explanation by translation", in the sense that new or obscure mathematical constructions are better understood as they are translated into structures which are "already known" or are defined by essentially different techniques. This will not lay foundations for nor justify those constructs, where by **justification** we mean the reduction to "safer" grounds or an ultimate foundation based on "unshakable certainties", in Brouwer's words [1923,p.492]. The same aim as Brouwer's was shared by the founders of proof theory.

However, we believe that there is no sharp boundary between explanation and foundation, in a modern sense. The coherence among different explanations, say, or the texture of relations among different insights and intuitions does provide a possible, but never ultimate, foundation.

1.1 Why Weyl's philosophy of Mathematics?

As applied mathematicians, we could avoid the issue of foundations and just discuss, as we claimed, explanations which provide understanding of specific problems or suggest tools for specific answers to questions raised by the practice of computing. *However, in this context, we would like to justify not the mathematics we will be dealing with, as, we believe, there is no ultimate justification, but the methodological attitude which is leading our work.* Our attempt will be developped in this part of the paper, mostly following Hermann Weyl's philosophical perspective in Mathematics. At the same time, with reference to the aim of this talk, we will review Poincaré's and Weyl's understanding to the informal notion of "impredicative definition". The technical part, Part II, is indeed dedicated to the semantics of impredicative Type Theory and may be read independently of Part I (the reader should go to Longo[89] or Asperti&Longo[91] for part II or its recent developments).

The reader may wonder why we should refer to Weyl in the philosophical part of a lecture on impredicative systems, since Weyl's main technical contribution to Logic is the proposal for a predicative foundation of Analysis (see §.4). The point is that, following Poincare', Weyl gave a precise notion of predicative (and thus impredicative) definition, see §.4. Moreover, and this is more relevant here, his proposal, as we will argue, is just one aspect of Weyl's foundational perspective. His very broad scientific experience led him to explore and appreciate, over the years, several approaches to the foundation of Mathematics, sometimes borrowing ideas from different viewpoints. The actual unity of Weyl's thought may be found in his overall philosophy of mathematics and of scientific knowledge, a matter he treated in several writings from 1910 to 1952, the time of his retirement from the Institute for Advanced Studies, in Princeton. In our view, Weyl's perspective, by embedding mathematics into the real world of Physics and into the "human endeavors of our historical existence", suggests, among other things, the open minded attitude and the attention to applications, which are so relevant in an applied discipline such as Logic in Computer Science.

2. Objectivity and independence of formalism

The idea of an "ultimate foundation" is, of course, a key aspect of Mathematical Logic since its early days. For Frege or, even more, Hilbert this meant the description of techniques of thinking as a safe calculus of signs, with no semantic ambiguities. With reference to Geometry, the paradigm of axiomatizable mathematics for centuries, "it must be possible to replace in all geometric statements the words point, line, plane by table, chair, mug", in Hilbert's words, as quoted in Weyl[1985, edited,p.14]. The certainty could then be reached by proving, for the calculus, results of consistency, categoricity, decidability or conservative extension (relative to some core consistent theory). The independence of meaning goes together, for Hilbert, with the independence from contextual worlds of any kind: "...mathematics is a presuppositionless science. To found it I do not need God, as does Kronecker, or the assumption of a special faculty of our understanding attuned to the principle of mathematical induction, as does Poincaré, or the primal intuition of Brouwer, or finally, as do Russell and Whitehead, axioms of infinity, reducibility, or completeness, which in fact are actual, contentual assumptions that cannot be compensated for by the consistency proofs" (Hilbert[1927,p.479]). A similar deep sense of the formal autonomy of mathematics may be found in the many lectures that Hilbert delivered in the twenties, in a sometimes harsh polemic against Brouwer. In particular, in disagreement with Brouwer 's view on existence proofs, Hilbert claims, in several places, that the interest of a proof of existence resides exactly in the elimination of individual constructions and in that different constructions are subsumed under a general idea, independent of specific structures. Hilbert's strong stand towards the independence of mathematics is absolutely fascinating and clearly summarizes the basic perspective of modern mathematics and its sense of generality. By this, Hilbert "... succeeded in saving classical mathematics by *a radical reinterpretation of its meaning* without reducing its inventory, namely by ..... transforming it in principle from a system of intuitive results into a game with formulas that proceeds according to fixed rules" (Weyl[1927,p.483]). Indeed, Weyl acknowledges "... the immense significance and the scope of this step of Hilbert's, which evidently was made necessary by the pressure of the circumstances" (Weyl[1927,p.483]). Hilbert's "bold enterprise can claim one merit: it has disclosed to us the highly complicated and ticklish structure of Mathematics, its maze of back connections, which result in circles of which it cannot be gathered, at first glance whether they might not lead to blatant contradictions... but what bearing does it have on cognition, since its formulas admittedly have no material meaning by virtue of which they could express intuitive truth ?" Weyl[1949,p.61].

Thus Weyl suggests that one should derive our understanding of mathematics also from entirely different perspectives.

3. Predicative and non-predicative.

Weyl's own partial commitment to intuitionism, at Hilbert's annoyance, spans the twenties ("I now give up my own attempt and join Brouwer ", Weyl [quoted in van Heijenoort, p.481]). The roots of this change of perspective in a disciple of Hilbert, may be found in the main foundational writing of Weyl's, i.e. in **Das Kontinuum**, 1918. As a working mathematician, Weyl cares about the actual expressive power of mathematical tools. He is very unsatisfied though with the "...crude and superficial amalgamation of formalism and empiricism... still so successful among mathematicians" (Weyl[1918, preface]). He is as well aware of the shaky sands (see later) on which the structure of classical mathematics is built, not long before revealed by the paradoxes.

Weyl's way to get by the foundational problem, together with Poincaré's thought, is the beginning of the contemporary **definitionist** approach to Mathematics.

Poincaré blames circularities for all troubles in Mathematics, in particular when the object to be defined is used in the property which defines it. In these cases "... by enlarging the collection of sets considered in the definition, one changes the set being defined".... "From this we draw a distinction between two types of classifications...: the **predicative** classification which cannot be disordered by the introduction of new elements; the **non-predicative** classifications in which the introduction of new elements necessitates constant modification" (Poincaré [1913,p.47]).

Weyl takes up Poincaré 's viewpoint and gives a more precise notion of predicativity. First he points out that impredicative definitions do not need to be paradoxical, but rather they are implicitly circular and hence improper (Weyl[1918], Feferman[1986]). Then he stresses that impredicativity is a *second order* notion as it typically applies in the definition of sets which are impredicatively given when "quantified variables may range on a set which includes the definiendum", Weyl[1918,I.6]. That is a set b is defined in an impredicative way if given by

(1) $$b = \{ x \mid \forall y \in A. P(x,y) \}$$

where b may be an element of A.

The discussion of impredicative definitions in the second order case is motivated by

Poincaré and Weyl's interest in the foundation of Analysis and, hence, in second order Arithmetic (see also Kreisel[1960], Feferman[1968 through 1987]). Thus the need to talk of sets of numbers, provided that this is done in the safe stepwise manner of a predicative, definitionist approach: "... objects which cannot be defined in a finite number of words....are mere nothingness" (Poincaré [1913,p.60]).

On these grounds Weyl sets the basis for the modern work in predicative analysis, which has been widely developed by Feferman, Kreisel and other authors in Proof Theory. The crucial impredicative notion in Analysis is that of least upper bound (or greatest lower bound). Both are given by intersection or union (i.e. by universal or existential quantification) with the characteristic in (1), since the real number being defined, as a Dedekind cut, may be an element of the set over which one takes the intersection or union that defines it. That is, for the greatest lower bound,

$$\text{g.l.b.}(A) = \cap\{ r \mid r\in A \} ,$$

where g.l.b.(A) may be in A .

In **Das Kontinuum**, Weyl proposes to consider the totality of the natural numbers and induction on them as sufficiently known and safe concepts; then he uses explicit and predicative definitions of subsets and functions, within the frame of Classical Logic, as well as definable sequences of reals, instead of sets, in order to avoid impredicativity. Weyl's hinted project has been widely developped in Feferman[1987].

3.1 More circularities

At this stage, are we really free of the dangerous vortex of circularities? Observe that even the collection ω of natural numbers, if defined by comprehension, is given impredicatively, following Frege and Dedekind :

$$x\in\omega \;\Leftrightarrow\; \forall Y\, (\forall y\, (y\in Y \Rightarrow y+1\in Y) \Rightarrow (0\in Y \Rightarrow x\in Y))$$

Thus also inductive definitions turns out to be impredicatively given, classically. A set defined inductively by a formula A , say, is the intersection of all the sets which satisfy A . As a matter of fact, Kreisel[1960,p.388] suggests that there is no convincing purely classical argument "...which gives a predicative character to the principle of inductive definitions".

Poincaré and Weyl get by this problem by considering ω and induction as the irreducible working tools for Mathematics. This approach is very close to the intuitionistic perspective, where the stepwise generation of the sequence of numbers is the core mathematical intuition (except that definitionists consider ω as a totality).

Observe that ω and induction are treated in a second order fashion, as the quantifications above are over sets. In other words, they rely on full second order comprehension for sets, which is usually given impredicatively. It is time, though, to discuss whether the circularity at the core of impredicative definitions really appears only at higher order.

Consider for example, a set S of natural numbers, and define m as the least number in S. Of course m is in S and, hence, if one understands this definition classically, the totality S , which is used in the definition of m , contains m itself. To put it otherwise, S is not known as long as we do not get m too. The definitionist approach or some mild constructivism save us from this: S is known if defined in a (finitistic) language or one may compute m by inspection of the sequence of numbers. Or, also, as some definitionists would say, m is impredicatively given only if the *only* way to define it is via S .

However, the first order circularities are not always so simple to solve. Consider "...the standard ("intended") interpretation of intuitionistic implication. This interpretation, when applied to iterated implications, has the same degree of impredicativity as full comprehension itself in the sense that being a proof of such an implication is defined by a formula containing quantifiers over all proofs of (arbitrary) logical complexity" Troelstra[1973,p.8] (a viewpoint confirmed in a personal communication). Kreisel[1968,p.154-5] shares the same views "....Heyting's ... implication certainly does not refer to any list of possible antecedents. It simply assumes that we know what a proof is". All proofs, not just the proofs of the antecedent. Indeed, a proof of $A \rightarrow B$ in Heyting's sense, is defined as a computation which outputs a proof of B for any proof of A , as input. But the proofs of A or B are not better known than the proofs of $A \rightarrow B$, as they may refer to, or contain as a subproof, proofs of $A \rightarrow B$. For example, one may have obtained b in B from c in $A \rightarrow B$ and a in A , i.e. $b = c(a)$.

This shows a circularity in the heart of a rather safe approach to foundation. Even though Weyl, because of his attention to Analysis, was explicitly referring only to second order impredicativity, the same circularity that he and Poincaré describe arises here (see the quotation from Poincaré[1913] above).

Another impredicative first order definition will be crucial later, when discussing the semantics of Girard's system F . Consider the following extension of Curry's Combinatory Logic, where terms are defined as always (variables, constants K, S, δ and application)

Definition. Combinatory Logic with a **delta** rule (C.L.δ) is given by

$\forall x,y \;\; Kxy = x$

$\forall x,y,z \;\; Sxyz = xz(yz)$

$\forall x \;\; \delta xx = x$.

We claim that it is sound to say that the definition of δ is impredicative here. Indeed, the δ axiom is equivalent to

$$M = N \;\Rightarrow\; \delta MN = M \quad \text{, for arbitrary terms } M, N .$$

Thus δ, by definition, internalizes "=", or checks equality in the system described by K, S, and δ. But δ itself contributes to define "=", or the definition of δ refers to "=", which we are in the process of defining. This violates Poincare's restriction above.

An inspection of Klop's proof in Barendregt[1984] may give a feeling of where impredicativity comes in: an infinite B hm-tree is reduced to different trees, with no common reduct, by δ , once that the entire tree is known to δ .

(Note that, in contrast to Church's delta, one does not ask for M and N to be in normal form. As a matter of fact, C.L.δ is provably not Church Rosser, by a result in Klop[1980]. It is consistent, though, by a trivial model, where impredicativity is lost: just interpret δ by K . One may wonder if there is any general theorem to be proved here about impredicatively given reduction systems and the Church Rosser property.)

A further understanding of the impredicative nature of C.L.δ is proposed in Part II of the original version of this note, Longo[89] (see also Asperti&Longo[91;ch.12] for an update). By using what may be roughly considered an extension of it, an interpretation of impredicative second order Type Theory is given.

4 The rock and the sand.

"With this essay, we do not intend to erect, in the spirit of formalism, a beautiful, but fake, wooden frame around the solid rock upon which rests the building of Analysis, for the purpose of then fooling the reader -- and ultimately ourselves -- that we have thus laid the true foundation. Here we claim instead that an essential part of this structure is built on sand. I believe I can replace this shifting ground with a trustworthy foundation; this will not support everything that we now hold secure; I will sacrifice the rest as I see no other solution." (Weyl[1918,preface]).

With this motivation Weyl proposes his definitionist approach to Analysis. This is based on a further critique of Hilbert's program. If we could "...decide the truth or falsity of every geometric assertion (either specific or general) by methodically applying a deductive technique (in a finite number of steps), then mathematics would be *trivialized*, at least in principle" (Weyl[1918; I.3]). Weyl's awareness of the limitations of formalism is so strong (and his mathematical intuition so deep) that, at the reader's surprise, a few lines below, he conjectures that there may be number theoretic assertions independent of the axioms of Arithmetic (in 1918!). (Indeed, he suggests, as an example, the assertion that, for reals r and s , $r < s$ iff there exists a rational q such that $r < q < s$. There may be cases where ".. neither the existence nor the non-existence of such a rational is a consequence of the axioms of Arithmetic". Can we say anything more specific about this, now that we also know of mathematical independence results such as Paris-Harrington's ?). Two sections later, Weyl conjectures that "...there is no reason to believe that any infinite set must contain a countable set". This is equivalent to hinting the independence of the axiom of choice.

This insight of Weyl's into mathematical structures seems scarcely influenced, either positively or negatively, by the predicativist approach he is proposing. It is more related to an "objective" understanding of mathematical definitions, in the sense below, and to his practical work.

" In our conception, the passage from the property to the set... simply means to impose an *objective* point of view instead of one which is purely logical; that is we consider as prevailing the coincidence of objects (in *extenso*, as logicians say) - ascertainable only by means of knowledge of them - instead of logical equivalence" (Weyl[1918,I.4]). Thus, *even though we are far from a platonist attitude, the conceptual independence of mathematical structures from specific formal denotation is the reason for the autonomy of mathematics from Logic.*

And now comes the aspect that makes Weyl such an open scientific personality. Just as for Poincaré, Weyl's proposal for a predicative foundation of Analysis does not rule his positive work in Mathematics. This emerges from both authors' work (see Goldfarb[1986] or Browder[1985] for Poincaré; much less has been said about Weyl and we can only refer to our experience in a one year long seminar on Weyl, in Pisa, in 1986/7, where mathematicians and physicists from various areas, Procesi, Catanese, Barendregt, Tonietti, Rosa-Clot and others surveyed his main contributions. The reader may consult Chandarasekharan[1986]. It is unfortunate, though, that the latter volume ignores Weyl's contribution to Logic).

However, Weyl's overall philosophical perspective in the foundation of mathematics related to his main technical contributions, if one looks beyond his specific, though relevant, proposal for a predicativist Analysis. Following Hilbert, Weyl stresses the role of "creative definitions" and "ideal" elements: limits points or "imaginary elements in geometry... ideals numbers in number theory... are among the most fruitful examples of this method of ideal elements" (Weyl[1949,p.9]) "... [which is] the most typical aspect of mathematical thinking" (Weyl[1918,I.4]). For example, "... affine geometry... presupposes the fully formed concept of real number -- into which the entire analysis of continuity is thrown" (Weyl[1949,p.69]). On the other hand, Weyl aims at a blend of "... *theoretical constructions* .. bound only by... consistency; whose organ is creative imagination [of ideal objects]" and "... *knowledge or insight...* which furnishes truth, its organ is "seeing" in the widest sense... Intuitive truth, though not the ultimate criterion, will certainly not be irrelevant here" (Weyl[1949,p.61]). But the intuitive insight of the working mathematician cannot be limited to Brouwer's intuition: "...mathematics with Brouwer gains its highest intuitive clarity.... However, in advancing to higher and more general theories the inapplicability of the simple laws of classical logic results in an almost unbearable awkwardness" (Weyl[1949,p.54].

An example may explain what kind of intuition Weyl is referring to. In Weyl[1918], the other major theme is the discussion of the geometric and the physical continuum. As a disciple of Husserl, Weyl adheres to a phenomenological understanding of time "as the form of pure consciousness" (Weyl[1949,p.36]) . The phenomenological perception of the passing moment of physical time is irreducible, in Weyl's thought, to the analytic description of the real numbers in Mathematics, since the ongoing intuition of past, present and future, as a continuum, is extraneous to logical principles and to any formalization by sets and points. "... the continuum of our intuition and the conceptual framework of mathematics are so much distinct worlds, that we must reject any attempt to have them coincide. However, the abstract schemes of mathematics are needed to make possible an exact science of the domains of objects where the notion of continuum intervenes" (Weyl[1918,II.6]).

In other words, *not all of what is interesting or that we can "grasp" of the real world is mathematically describable. Weyl is aware of this, raises the issue, stresses the uncertainties and... keeps working, with a variety of tools.* "... large parts of modern mathematical research are based on a dexterous blending of axiomatic and constructive procedures. One should be content to note their mutual interlocking.." and resist adopting "...one of these views as the genuine primordial way of mathematical thinking to which the other merely plays a subservient role" (Weyl[1985,edited,p.38]).

The working mathematician has to be able to use axiomatic or constructive methods as well as the intuition, in the sense above, with its real and historical roots. For example, in Weyl's opinion, there are (at least) two different notions of function, both relevant, for different purposes. The functions which express the dependence on time, a continuum; the functions which originate from arithmetical operations (Weyl[1918,I.8]).

This openness of Weyl's, who was sometimes "accused" of eclecticism, is surely due to the variety of his contributions (in Geometry, Algebra, mathematical Physics... see Chand.[1986]). Thus, his permanent reference to the physical world and to the "human endeavors in our historical existence" provides the background and ultimate motivation of Mathematics, as we will argue in §.5. Moreover, because of his broad interests, Weyl was used to borrowing, or inventing, the most suitable tools for each specific purpose of knowledge. However, Weyl was not an applied mathematician. He was more an "inspired" mathematician (as suggested by Tonietti[1981]), as he mostly aimed at pure knowledge, at mathematical elegance and at a unified understanding while his ideas are constantly drawn from applications and lead by references to the real world: the phenomenological time of Physics, the patterns of symmetries in nature, in art... all brought together by the "reasonableness of history" (§.5). An attitude similar to Weyl's, we claim, may also help in our work, as logicians, in Computer Science.

4.1 Impredicative Type Theory and its semantics.

Let's try to illustrate, by an example, what we mean by a "similar attitude". The example refers to the topic of the technical sections in Part II of the full version of this paper, Longo[89] (one may consult Asperti&Longo[91] for an updated presentation).

However, the sketchy presentation here only uses a few concepts from type theory, which are recalled in place. If these seem too unfamiliar to the reader, he/she may directly go to the next section.

Second order lambda calculus, and its impredicative theory of types, originated in Proof Theory and intuitionistic Logic by the work of Girard[1972] (and Troelstra[1973]). The calculus was brought to the limelight when computer scientists, like Reynolds, Liskov or Burstall, Lampson or Cardelli and others suggested or made use, in actual language design, of expressive forms of modularity which are tidily represented in Impredicative Type Theory (**ITT**). Indeed, the programming languages they proposed support "polymorphism" in ways more or less directly inspired by ITT: roughly, programs may be applied to types and, by this, output a new program which is an (automatically) instantiated version of the original program, by the input type. These programs have type $\forall t \in Tp.A$, where Tp represents the collection of types. Here comes the circularity of impredicative definitions, since $\forall t \in Tp.A$ is a type, i.e. is in Tp, and its defination uses a quatification over Tp itself (see §.3). The most recent language, Quest in Cardelli[1988], is an explicit extension of Girard's system F.

Of course this raised and still raises lots of questions. One clear point is that at Digital, say, (but not only there) there are people interested in implementing a languages, such as Quest and its derivatives, with the "obscure", but powerful features of impredicative definitions. It is almost a "concrete reality", as it runs on hardware. As logicians, we may help at its understanding and, perhaps, at its growth. Even consistency issues are still open, as Quest properly extends F by many facilities of functional programming. In particular, it allows recursive definitions and has notions of records, subtypes, "powertypes" and inheritance whose definition requires extra rules and axioms. A relative "consistency result" is by all means relevant, even if one had to use highly non-constructive tools: on the average they are much more reliable than the actual correcteness of programs, in whatever language! (A model for the core of Quest is given in Cardelli&Longo[91]). More generally in Computer Science, the "mathematical semantics", i.e. the translation into mathematical structures designed by essentially different tools, adds understanding, since complicated, though effective, constructions may be better displayed to our minds by possibly non-constructive, but intellectually simple methods. The interpretation enriches our overall knowledge by establishing connections, unifying and relating notions, allowing the use of methods from one area to another and, most of all, suggests extensions and changes of the syntax one is given to interpret. This is exactly what happened with several functional languages, whose design grew with their denotational understanding (e.g. Edinburgh ML).

In the case in discussion, the models in Part II, which are based on early work of Girard and Troelstra, happen to interpret extensions of systems F in other directions, namely interpret Coquand and Huet Calculus of Constructions (see Ehrhard[1988]). This may suggest extensions obtained by putting together the various features. Of course, further expressiveness would pose more semantic challenges and would bring us as close as conceivable to "the Tarpean rock" of the paradoxes. But these semantic challenges may provide informative connections with other areas or even suggest brand new Mathematics.

This is the most recent story of the semantic investigation of Type Theory: since Moggi's suggestion for a categorical understanding of Girard-Troelstra models, by hinting the "small completeness" of a category of "sets", several papers and discussions raised issues in Topos Theory and shed more light on topos theoretic models of Intuitionistic Set-Theory (**IZF**; see Pitts[1987], Hyland[1987] and Asperti&Longo[91] for detailed work and further references).

At this point, one may wonder how constructive are the tools used in this kind of work. As the reader may see in the above references, the definition of the models requires the use of powerset operation and second order impredicative comprehension. On the other hand, Hyland's Effective Topos, **Eff**, which provides the categorical frame, is a model of IZF and uses a key intuitionistic fact; namely, the computability of all functions, as the so called "Church Thesis" (a precise statement in the system) is valid in the topos. However, it realizes principles that go beyond Intuitionism, namely, "Markov Principle" and the "Uniformity Principle" (see the Discussion at the end of this note).

In short, several aspects of Mathematical Logic get together by the challenge coming from practical issues, since the current questions raised in the semantics of Type Theory originate in functional programming (also Moggi's hint was given as an answer to a computer language question). Moreover, tools from a variety of perspectives are needed to understand them better, with no philosophical preclusion.

Of course, an entirely different activity may also be relevant. For example, some may try to find "safer" tools, definitionist, say, or strictly constructive tools, for the same purposes, and carry on a reductionist analysis of the languages. This may turn out to be as relevant as the work done in the thirties in computability, since we may all gain a further insight and new languages with different programming facilities may be suggested. Indeed, programming is difficult, thus the more viewpoints we have the better we may suggest how to program, as each approach may answer different questions and solve different problems. However a point should be clear, in view of the introductory distinction we made on the use of Logic in mathematics and in Computer Science. In the foundation of mathematics a strong philosophical commitment may motivate the researcher's work, may give it intellectual unity and, hence, contribute to the foundational aim. This is clear from the history of Logic, where even the personal fights (Poincaré, Hilbert, Brouwer were not always friendly at each other) stimulated the discussion. But, when using Mathematical Logic as a tool, as in most work in Computer Science, an "a priori" philosophical commitment doesn't need to be a stimulus, and it may just result in an intellectual and practical limitation. What we need are explanations, by informative translations or interpretations, say, and unified frameworks, which may suggest new ideas, not ultimate foundations.

5. Symbolic constructions and the reasonableness of history

" To fulfill the demand of objectivity we construct an image of the world in symbols" (Weyl[1949,p.77]). For example, " symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection", it is also a key structural property in the natural world. We will then "... generalize this concept gradually... through the process of mathematical abstraction along the road that will finally lead us to a mathematical idea of great generality" (Weyl[1952,p.5-6]).

Along this road, the demand for objectivity causes a permanent tension between intuitive descriptions (of symmetry, of time...) and the need to eliminate aesthetic appreciations and subjective understanding of the physical world. However, complete "ego-extinction" is an impossible task, as Weyl knows from his experience in the mathematics of relativity.

We believe that in his last book, **Symmetry**, 1952, one finds the core of Weyl's perspective. His view emphasizes the dramatic growth of mathematics as "... a proud tree which... raises its... branches into the thin air, but which at the same time sucks its strength through thousands of roots from the earth of intuitions and real representations" (quoted in Weyl[1985,editor's note]). These roots provide the robust foundation of Mathematics, by branching into two main directions: the reference to the physical world and our intellectual history.

Symmetry is based on examples from art, chemistry, crystallography, and physics which gradually lead to the difficult mathematics of the classification of finite groups. Weyl, for example, takes the reader from the symmetry of ornaments in art, to the symmetries of crystals, describing Leonardo's naÂve work on the groups of orthogonal transformations, based on observation. He points out that Leonardo had a complete list of the orthogonally inequivalent finite groups of orthogonal transformations. Then he shows how the complexities and surprises of the classification of finite groups are derivable by non-trivial tools from analytic geometry and a blend of metric and lattice structures. This abstraction though is reached by a continuous interaction between the formalism introduced and its actual meaning.

There is probably an implicit reference, in this work, to Galileo, an author widely studied by Weyl's master of philosophy, Husserl. The approach is dual, though, with respect to Galileo's. It is not the book of nature which is written by God in the language of mathematics and that we just read, as Galileo thought, but mathematics is written by us in our interaction with nature. **Symmetry** is a unified reading of the real world and human art, from which mathematical concepts emerge. When reading **Symmetry** one has the feeling that *mathematics is created by or, simply, is the common aspect of a variety of* *concrete (spacial) as well as linguistic experiences. It is designed by these in the same way as the space of relativity is created by the presence of objects.*

The approach, though, is far from flatly empirical, as we understand it: *our creative imagination interacts with the real world and tends to the design of an autonomous world of concepts. This world is autonomous from reality in that it has a further justification: the history of our language, a key tool for abstraction, and our history as human beings.* This is why, in his critique of Hilbert, Weyl may say: " what compels us to take as a basis precisely the particular axiom system developed by Hilbert? Consistency is indeed a necessary but not a sufficient condition for this. For the time being we probably cannot answer this question except by asserting our belief in the *reasonableness of history* which brought these structures forth in a living process of intellectual development -- although, to be sure, the bearers of this development, dazzled as they were by what they took for self-evidence, did not realize how arbitrary and bold their construction was." (Weyl[1927,p.484]). We conclude by observing that this very viewpoint, which Weyl consistently assumed through out his life (and which we share), is expressed even more clearly twenty years later, in Weyl[1949,p.62]: "But perhaps this question can be answered by pointing toward the essentially historical nature of that life of the mind of which my own existence is an integral but not autonomous part. It is light and darkness, contingency and necessity, bondage and freedom, and it cannot be expected that a symbolic construction of the world in some final form can ever be detached from it."

Discussion (This discussion refers to Part II of Longo[89], which is missing here. However, the general lines may be clear to the reader, just by knowing that **Per** is the set of partial equivalence relations over ω, the natural numbers, or the quotients over subsets of ω, and that **PER** is the associated category with computable maps as morphisms; see Asperti&Longo[91] for details).

By the Girard-Troelstra interpretation, we have taken back the impredicativity of Type Theory to the more familiar impredicative definition of a set as intersection: the set defined may be a member of the set over which we take the intersection (indeed, $\cap_{R\in\mathbf{Per}}F(R) \in \mathbf{Per}$). This is the circularity which was discussed in part I when considering inductive definitions, greatest lower bounds of reals etc... It relies on taking a powerset of a (countably) infinite set.

As for the rest, it should be clear the the role of **Eff** and intuitionistic logic, here, is more than essential, as explained in depth, among others, in Pitts[1987]. As a matter of fact, **Eff** is a model of IZF (informally, in the same sense as the topos **Set** is a model of classical ZF). The crucial fact is that in **Eff** there are very few morphisms, as they are all computable, or realized by elements of ω via Kleene's application. This corresponds to the validity in **Eff** of "Church Thesis", i.e. that "internally" any function from numbers to numbers is computable.

However, **Eff** realizes several principles which do not strictly belong to intuitionistic Logic. For example, the Uniformity Principle (UP) and Markov Principle (MP). Let Φ be a formula of IZF. Then

(UP) $\forall R \in \mathbf{Per}\ \exists a \in \omega\ \ \Phi(a,R) \ \Rightarrow\ \exists a \in \omega\ \ \forall R \in \mathbf{Per}\ \Phi(a,R)$

(MP) $\forall R \in \mathbf{Per}\ \ (\forall a \in \omega\ (a\ R\ a) \vee \sim(a\ R\ a)) \wedge \sim\sim\exists a \in \omega\ (a\ R\ a) \ \Rightarrow\ \exists a \in \omega\ (a\ R\ a)$.

As a matter of fact, both the constructive virtues of **Eff** and its less constructive ones are fully used. The computability of its morphisms, i.e. "Church Thesis", appears all the time and makes $[\Pi_{a \in A} G(a)]_{\mathbf{PER}}$, the internal indexed product in the model, much "smaller" than $\Pi_{a \in A} G(a)$, the classical set-theoretic product. The crucial isomorphisms of $\cap_{R \in \mathbf{Per}} F(R)$ and $[\Pi_{R \in \mathbf{Per}} F(R)]_{\mathbf{PER}}$ (see Longo&Moggi[1991] for details) uses (UP), which intuitionists do not generally view as a constructive principle. Observe, though, that (UP) is equivalent to the contraposit of König's lemma -- ...in a brown finitely branching infinite tree, if *for any* branch *there exists* a node where the branch switches to green, then *there exists* a level such that *any* branch is green... -- . Thus, it is classicaly equivalent to König's lemma, a rather convincing (and accepted) proof method. **Eff** gives meaning to (UP), under certain circumstances, also in a constructive framework.

As for Markov Principle, (MP), which russian constructivists like, it shows up at another point. The understanding of the recursive definitions of data types has been a relevant success of denotational semantics. Since the early work of Scott, they became much more familiar to computer scientists and more widely used in programming. The idea is that a recursive definition of a type of data yields an equation which needs to be solved over some structure, in the same way one gives meaning to "$x^2+8 = x$" by finding a solution for it over, say, the structure of the the complex numbers.

In **Eff** one may find solutions to all relevant domain equations as follows. Consider a constructive version of Scott domains (essentially, the computable substructures of the effectively given domains in Scott[1982]). When taking continuous and computable maps as morphisms, they form a Cartesian Closed Category, which can be fully and faithfully embedded in **Eff** (when **Eff** is constructed out of Kleene's $\underline{\omega}$). In this way the limit constructions, needed to solve the equations, can be carried on within **Eff**. The embedding, though, requires, in an essential way, Markov's Principle (see Rosolini[1986,th] or Longo[1988]).

The relevance of the "effective" set-theoretic environment, provided by **Eff**, is becoming clear in applications. In Cardelli&Longo[1991], for example, naîve, but complete, set-theoretic interpretation may be given to several programming language constructs: being a subtype is interpreted by "being a subset", a record by the obvious list of indexed sets etc... The constructive nature of the model and its internal Logic do the rest and let these notions be inherited at higher types and higher order very smoothly.

In a sense, **Eff** and its Logic provide a formalized example of the "dexterous blending of axiomatic and constructive procedures" Weyl was referring to as the practice of Mathematics. We are more than "content to note their mutual interlocking", also because we have, in this case, a clear understanding of what is being used and when and we are far from the "superficial mixture of formalism and sensism", which Weyl was blaming among mathematicians and which would be of little help in the applications of Logic to Computer Science.

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APPENDIX: from the JOURNAL OF SYMBOLIC LOGIC, 53,3, 1993.

REVIEW OF:

Solomon Feferman: " Weyl vindicated: **Das Kontinuum** 70 years later "

The review of this interesting paper is divided into two parts, as the paper itself is split into a philosopical or historical part and a technical one. I will first hint the technical aspects, which are very relevant for predicativity and set on firm grounds an established area of investigation, and later discuss the informal preliminaries (and the title of the paper!). These present a rather partial view of Weyl's contribution to the foundation of Mathematics, a view which I do not share, because of both the historical and the foundational perspective proposed by the author.

The core of the approach presented beginning with section 4, is the notion of "definability". In the "definitionist" approach, which is essentially the Feferman's view, given (possibly) for granted the totality of the collection of the natural numbers, the other mathematical notions must be defined in some sort of "stratified" way. Stratification may mean entirely typed definitions following Russell, or, more weakly, "predicative" definitions, in the sense made clear by Poincare' and Weyl. In short, << ... we draw a distinction between two types of classifications...: the predicative classification which cannot be disordered by the introduction of new elements; the non-predicative classifications in which the introduction of new elements necessitates constant modification>> (Poincaré [1913, p.47]). As the author is interested, in the end, in the foundation of Analysis, impredicativity is understood as a second order notion as it typically applies in the definition of sets. And a set is impredicatively given when <<...quantified variables may range on a set which includes the definiendum...>>, Weyl[1918, I.6]. That is, a set b is defined in an impredicative way if it is given by

(1) $b = \{ x \mid \text{for all } y \text{ in } A, P(x,y) \}$

where b itself may be an element of A.

As a matter of fact, the author refers to Weyl's ideas in his 1910 paper and in the 1918 book with a remarkable philological attention. The core of the technical part reconstructs with great care Weyl's hint towards a predicative Analysis. Indeed, Weyl's book is extremely incomplete and vague in this proposal, while rich of mathematically deep observations. The author takes up Weyl's largely informal remarks step by step and turns them into a clear and complete approach to predicative mathematics. In particular, section 5 carefully describes iteration and induction, and clarifies the connections between the various different forms of Weyl's "principle of iteration", by a careful formalization of Weyl's informal hints. Section 6 faithfully rewrites in modern formalism Weyl's axiom system: the structure and role of explicit definitions and comprehension axioms is clearly presented and this contributes, in an essential way, to the understanding of a few very obscure pages of Weyl's book. Section 7 compares Weyl's approach, as formalized in the previous section, to extant approaches to predicative Analysis. This sections, in particular, refers to the author's (and others', such as Kreisel's) work in predicative systems, since the early 60-ties (see references). Weyl's approach turns out to be "equivalent" to sufficiently large fractions of (formally) rather expressive systems. By this, one may say that Weyl's aim of setting on clear grounds large part of Analysis is achieved. Indeed, by the extension to "flexible types" in section 8, the author updates even further the previous approach to the current debate on higher order type systems. An interesting result of "conservative extension" of the system w.r.t. to PA is also stated. In section 9, a few arguments are hinted towards the actual expressiveness of the predicative systems proposed, for the purposes of applicable Analysis, to Physics, tipically. Recent and unpublished work by the author and Simpson reinforces the thesis just sketched in this section, by a technically deep insight into various area of "predicative" mathematics. However, it is known that Lebesgue measures, least upper and largest lower bounds escape predicativity. Indeed, "the continuum" goes beyond it.

Let's now go back to a "critique" of the historical and epistemological perspective presented by the author in the first part. For this purpose, I will also borrow a few ideas from the first part of Longo[1989]. First of all, in my view, the main relevance of Weyl's "The Continuum" is not due to the informal hints towards predicative Analysis, but to the critique of the formalist approach to Analysis and to a deep and original discussion on the relation between our intuition of the physical continuum and its mathematical formalizations. In my opinion, instead, the paper "Weyl vindicated: ..." (and its title), suggests a rather partial view of this broad, deep, wonderful thinker and mathematician. Of course, the author's work towards the clarification of Weyl's specific proposal is an interesting and a relevant contribution to predicative Analysis, as I tried to say. The danger is that the reader may classify and reduce by this Weyl's contribution to the foundation of mathematics as one of the many (attempted) "reductionist" formalizations. As a matter of fact, Weyl's very broad scientific experience led him to explore and appreciate, over the years, several approaches to the foundation of Mathematics, sometimes wavering between different viewpoints. The actual unity of Weyl's epistemological view may be found in his overall philosophy of Mathematics and of scientific knowledge, a matter he treated in several writings from 1910 to 1952, the time of his retirement from the Institute for Advanced Studies, in Princeton. As an important aspect of this view, one should stress a crucial critique to Hilbert's approach: Weyl keeps stressing, in several writings, that what really matters in (meta)mathematics is the relevance of axiom systems, in their broad connections to the structures under investigation and the physical reality; consistency is a necessary, but far less relevant condition. A further critique of Hilbert's program is clearly expressed in Weyl's book. If we could <<...decide the truth or falsity of every geometric assertion (either specific or general) by methodically applying a deductive technique (in a finite number of steps), then Mathematics would be trivialized, at least in principle...>> (Weyl[1918; I.3]). Weyl's awareness of the limitations of formalism is so strong (and his mathematical intuition so deep) that, at the reader's surprise, a few lines below, he conjectures that there may be number theoretic assertions independent of the axioms of Arithmetic (in 1918!). (Indeed, he suggests, as an example, the assertion that, for reals r and s, $r < s$ iff there exists a rational q such that $r < q < s$. There may be cases where <<... neither the existence nor the non-existence of such a rational is a consequence of the axioms of Arithmetic>>. Can we say anything more specific about this, now that we also know of mathematical independence results such as Paris-Harrington's ? This would really vindicate Weyl, whose views on this matter greatly annoyed his former professor and major academic authority, David Hilbert). Two sections later, Weyl conjectures that <<...there is no reason to believe that any infinite set must contain a countable set>>. This is a very early hint in the right direction for the independence of the axiom of choice (!).

This insight of Weyl's into mathematical structures seems scarcely influenced, either positively or negatively, by the predicativist approach he is proposing. It is more related to an "objective" understanding of mathematical definitions and to his practical work. What "objective" or "independent from specific formalizations" means here is a delicate issue, for which one may consult Weyl[1918] (or Longo[1989;1991]).

Indeed, what really interests Weyl is the understanding of mathematics as part of our human endeavour towards knowledge, in particular of the physical world. Weyl stresses the inadequacies of the mathematical formalization with respect to a crucial aspect of our physical experience (see chapter II, §.6): our intuition of the continuity of space and time (Weyl greatly contributed to the mathematics of Einstein's relativity). In his view, the phenomenal experience of time, as past, present and future, is unrelated to the mathematical treatment of the real numbers. Time cannot be decomposed in points. Present lasts continuously, it is <<something ever new which endures and changes in consciousness>>. In our perception of time, <<an individual point is non-independent... it exists only as a point of transition.. it cannot be exhibited in any way ... only an *approximate*, never an exact determination of it is possible>> (again, chapter II, §.6). Even the use of limit points or ideal constructions, the essence of mathematics according to Weyl, do not help us sufficiently in grasping the <<irreducible>> perception of the continuum (references are made for this to Husserl and Bergson).

The depth and philosophical difficulties of Weyl's chapter II do not allow us to go any further into this here. I believe though that there is a strong need to revisit these aspects of Weyl's reflexion. In particular today, in view of the increasing interests in "theories of knowledge" as part of broadly construed attempts to reconsider our understanding and (possibly mathematical) description of the world.

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