Mathematics and the Biological Phenomena[1]

Giuseppe Longo

CNRS - Dept. de Mathematiques et Informatique
Ecole Normale Supérieure, Paris
http://www.dmi.ens.fr/users/longo

> *«The mathematician, who would address to other sciences, such as for example philosophy, and ask for definitions and deductions in the style of mathematics, is not wiser than the zoologist who would refute numbers because they are not living beings»* [**H. Weyl**, 1918]

Abstract

The first part of this paper highlights some key aspects of the differences in the use of mathematical tools in physics and in biology. Scientific knowledge is viewed as a network of interactions, more than as a hierachically organized structure where mathematics would display the essence of phenomena. The concept of "unity" in the biological phenomenon is then discussed. In the second part, a foundational issue in mathematics is revisited, following recent perspectives in the physiology of action. The relevance of the historical formation of mathematical concepts is also emphasized.

Part I: Reflections on Mathematics in Biology

Introduction: hierarchies of disciplines.

When hearing biologists about working methods in their discipline, one may often appreciate traces of the emotions of a scientific experience of great

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intensity and ... "acceleration", as the growth of this science, in the last few decades, the enriching or changes of paradigms has no analog, I believe, in other disciplines. As a mathematician, I never went through conflicts of scientific paradigms as strong as those experienced in biology: mathematics is remarkably stable since centuries. Or at least it appears to be so ... but it isn't: actual infinity, for instance, from differential calculus to Cantor, represented an extraordinary change in scientific paradigm. Yet, by listening to my colleagues at this conference, in particular, I perceive once more the importance and fascination of the changes biology went through, in a short amount of time: the extended use of tools from mathematics and physics, the novelty of Genetics, Molecular Biology, the progress in theories of evolution ... a crossing of methodologies. In some cases, methods of mathematical nature are (or have been) presented as "displaying the essence" of phenomena (see below).

I have always looked with caution, because of a kind of philosophical awareness, at the practice of lightly transferring working methods across disciplines, possibly parallel but still different, or at the attempt to cap one on another with pretences of unification. For instance, among some logicians the habit is still widespread of trying to legislate in mathematics (ah, ..., the "ultimate foundation" lies in this or that specific formal system ... the rigorous mathematician does not make use of "impredicative" definitions ... and he does not eat caviar when it is not meal-time, Brecht would say). Mathematical logic, on the contrary, is nowadays a *branch* of mathematics, though among the most beautiful and prolific, well beyond the foundational ambitions of its founding fathers. In short, Mathematical Logic does not explain, nor "found" mathematics, of which it is merely part. More precisely, mathematics *can not be deduced* from logical principles (there are theorems that prove this); similarly as chemistry *can not be deduced* from physics, nor biology from chemistry, as they are. Relationships among different disciplines or within the same one cannot be given as *hierarchical dominations*; instead they should contribute to improve the network of our knowledge by means of communication, connections and the highlighting of methodological differences as well as points of contact between one form of knowledge and another. This because, not only are there so many forms of comprehension (put another way, of "intelligences") in the world, but also because "emerging" phenomena, like life from inert matter or human thought from life, show "qualitative" differences which may depend on or, more precisely, may lead to different sorts of scientific methods to study them. The unity of knowledge is to be sought in *interconnections*, and not in ultimate logical or physical foundations, let alone in some "theory of everything" (physical-mathematical possibly, or, tomorrow, ... biological) which everything reduces to. To this aim, I will try to stress some of the crucial differences between the scientific method in biology and in mathematics, with a particular concern about what this has to do with the mathematisation of physics. First of all, I would say that there is a significant difference in the way we "look" at our objects of study or, also, at the way the objects present themselves to us. Many remarked on the difficulties and, perhaps, the inappropriateness of "giving a law", as general as it can be, once some biological facts have been observed. I think that one side of the problem lies in the ways the "objectivity" and "individuality" of the objects of knowledge comes to be in different settings, from biology to mathematics, to physics.

1. Example and Counterexample.

To begin with, talking to biologists, I started to understand the utterly different role of "examples" and "counterexamples" in biology with respect to mathematics, even when a biological theory is formulated. In mathematics, examples can only be "instances of explanation", sometimes they can provide clues to conjectures but they never ever build up to scientific knowledge; counterexamples are devastating: they demolish the theoretical proposal which is at the heart of mathematics, that is the *theorem*, the general law. This is not the case in biology, as far as I understand: examples, as resulting from an experiment typically, often seem to bear general and meaningful knowledge, they peacewise constitute a "*network of examples*", at the core of biological thoretizing. Counterexamples help to find the right contexts and experimental methodologies, plastically modify the network. This is not simply due to the fact that biology is an empirical science. The experimental physicist aims from the very beginning to the general, indeed mathematical law.

It is because of the meaning that examples can have in biology that I understand the possible reason why some refer to the analysis of life made by means of poetry: the description of Madame Bovary, thanks to Flaubert's poetry, rises to the general level and it has the status of true knowledge, not an intuitive (this is an abused term) one but a complex, deep and well thought out one, according to the talented novelist's own way. Starting from the particular, the reader has eventually gained a greater and very general knowledge of the human soul. Similarly, I have the impression that, every so often, the biologist need to "make theory", generalise, by means of examples; this is a difficult exercise especially when it does not concern a poetic description of human beings, but it has to do with contributing to scientific knowledge. It would be interesting to understand how in biology one enforces generality from examples, from the analysis of biological individuality, which is always presented as such and whether it is possible to foresee when this can be meaningful and mark the beginning of a theoretical proposal. And when is it that the "network of examples" becomes scientific knowledge and what does it mean to the biologist, in contrast with the experimental physicist, to "give the general law"? The sensitivity, which the biologist develops by means of his/her way of approaching problems and that is transmitted to the students, is a true "form of intelligence" which turns into scientific method. For the time being, I can only raise the question, as this point can be highlighted only by the experienced researcher in the field; yet, the dialogue with other scientific disciplines may be crucial.

2. The "découpage".

I will, by contrast, try to single out how conceptualisations of mathematics, so independent from individual details and contexts, come to be. This process is like an element, a limiting and extremal one though, of an everyday practice that I shall call "découpage" (the *conceptual* "cutting out of figures", but appropriate for gesture and drawing too). This practice belongs to common sense and to different forms of knowledge, but it may be in physics or more precisely in physical-mathematical description that it finds its proper area of application[2]. Possibly, this will help us understanding the analogies as well as the methodological differences in the use of mathematics and in the

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[2] In fact I borrowed the term "découpage" from the thoughts on quantum physics of Mioara Mugur-Schacter (see [Mugur-Schacter,1993]), but I am giving it here a more simple and straightforward meaning. The meetings organised monthly by this former student of De Broglie have been for me an invaluable source for discussions, thanks to the presence of many physicists, mathematicians and philosophers like Francis Bailly, Jean Petitot, Hervé Barreau and many others (see the working groups in http://www.dmi.ens.fr/users/longo).

practice of "law making", and give us hints of where the difficulties might lie so that we can do better where possible.

When we look at things around us, the classification of what we see, the isolation of an entity, of an object, it follows almost always from our own choice. In other words, very rarely is an "object" presented to us with a ready made unit and identity. The process of singling out objects lies on the interface between us and the clues and signals that reach us from the world; this process is completely human in theory and in concept proposal, it is part of the relationship between human beings and the world.

Look around you, where is the unity of that rock, of that mountain? Of atoms and the solar system? A enormous mass of all sorts of information stands in front of us; we decide to say that here starts and there ends a certain unity, making a choice that is not merely subjective nor arbitrary, but built on the basis of a relationship with the world, which is phylogenetic, historical, cultural and scientific; we make up physical individuals, also by giving them names, in the intersubjective exchange. We pay attention to this or that outline or contour, we categorise, we classify colors and scents, we distinguish by means of names and gestures this object from that one: all this is built while we are acting in the world, *it is not already in the world*. It is the same kind of process that make us say: "we *learn* to see".

In reality, we are or become very good at singling out objects, especially when these are man-made: I know well what is the unity of this table or of this chair because I know how the carpenter made them, I have actually designed them. But it was, on the contrary, terribly hard to isolate the modern notion of atom or solar system. And immediately the physicist comes and tells us that, please, electrons and protons must be understood as traces of a new unity, the strings, chords or super-chords; that the solar system is filled with things and that the system itself is part of a bigger unity. We compose and recompose the information that reaches us through a very complex cultural dynamics: starting with the glance at the hill behind this house (by the way, where does it actually begin?), we, living and historic beings, propose delimitations and structures and so categorise the physical world. We do not do this arbitrarily, but grounded in actions, history, language as well as on some strong invariants which reach us from the world (symmetries, specific outlines, borders, when not ambiguous).

In particular, we use mathematics as an extremely precise drawing and language; in fact the most rigorous *language* we are able to think of. In the cases when language, drawing, and even mathematics may contribute to conceive and realise concretely a physical object, then we have a man-made one: the chair, the table, this computer of which, as I already said, we know well the structure and unity since it was devised by us. My cat, on the contrary, might be dubious about it: if by any chance he has developed any theory about it, he might think that my desk is made of four spikes sticking out from the parquet, of the same colour, on top of which a glass pane is laid. To him, who has not designed and never moved the desk, the unity probably lies in the "floor with spikes". Even the newborn baby or the blind person who suddenly acquires sight, learns to see, to "individuate" and isolate objects. He learns how to cutting them out from the background: the first step in the visual découpage.

It was a long route through evolution and history which led us to give unity to this desk or to the atom as physical objects: we have in fact conceived or moved or bought the former and we made an extensive use of mathematics for the latter.

Even everyday language is rich in "constructions of unity" that are not "already there": think of names of colours, so history dependent. Certainly the receptors of the retina have "pigments" sensitive to three primary colours (they have excitement peaks corresponding to the wavelengths of red, blue and green): these colours, as parameters in a three-dimensional space, allow the reconstruction of all possible wavelengths (but many other triples would do as well). Moreover, we are able to spot very minimal differences between colours. But it is a completely human and historic choice that of categorising colours, like separating yellow from purple, by giving, with a name first of all, "individuality" to this or that colour, marking borders in the "continuum" of wavelengths, between say "burnt sienna" and "red amaranth". In certain cultures one speaks only of black and white. It is said that in others, like the Inuit, there can be nearly ten different names for white and this allows them to make very fine distinctions with respect to different kinds of snow; other cultures add only red to black and white or, even if they have many different names for colours, do not conceptually isolate blue from green: ancient Greeks are an example of this. Every culture has its own categorisations of colours, of that continuum of light wavelengths that hits our retina. It would be interesting to know more about the role played, in the history of language, by those three primary colours that evolution has given us to reconstruct the others. They are as pivots upon which we build our mental categorisations and that probably drive our choices favouring some lengths over others, thus making non-arbitrary some of our categorisations. Because this is the point: scientific reconstructions of the world are *possible proposals* and yet they are *not arbitrary*. Thus, the foundational issue is in singling out the phenomenal "pivots" on which, along history, we built up our forms of knowledge.

Going back to the general point, thus, I do not support, by any means, the classic, positivist and neo-positivist theory which bases our relationship with the world on the assumption that there is an absolute objectivity in the "physical objects of medium size" (mind you, the typical examples of medium-size objects are always man-made ones), with clear cut outlines, limits and individuality while admitting the existence of a kind of discontinuity at the representation level of micro-physics and astrophysics: there, they say, we do in fact need some form of interpretation. Phenomena instead are always in between us and the world: perception does not begin and end with the senses while conscience is not autonomous and pre-structured. Consciousness and perceptions reciprocally influence each others formation. And they are built up in the "praxis, the common practice in and of the world, along our "forms of life", as intuited by philosophers (here I am thinking especially of what I have understood, or rather appreciated, of the late Husserl and Wittgenstein) and confirmed by many physiologists since Helmotz: «Perception is not representation: it is a simulated action projected onto the world» ... «perception is not so much a function of the intensity of an impulse as of its agreement of this with an hypothesis made by the brain » [Berthoz, 1997, p.61 and 147]. This "common practice" is at the heart of the very differentiated continuum of conceptual constructions: it ranges from the recognition of the common sense "objects of medium size" to the theoretical constructions of micro-physics[3].

2.1 Découpage in Physics.

The way of isolating "objects" by conceptual constructions and, with them, invariants and laws is a central feature of the relationship between mathematics and physics, when we want to acknowledge the key role of mathematics as the language of physics. Mathematics helps us to "découper"

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[3] With respect to this, René Thom takes an "intermediate" position. He underlines the role of the "meaningful forms": kinds of sensitive domains which emerge from the background, thanks to borders; these are real structural discontinuities [Thom, 1988; this idea was developed in several papers by Jean Petitot, along the lines of Husserl's].

(cut out) with great care and precision physical objects, by producing conceptual abstractions, often very complex ones, that do not arise arbitrarily but are instead built according to the different ways our intelligence is composed and built in respect to the world. This process is particularly evident in micro-physics. Wave-particle duality is proposed, cut out from the background, given a name, a mathematical description and unity; physicists are so good at playing this fruitful game, that every month propose new names for new "objects" and tell us that those of the previous months did not have any real unity after all, or did not even exist: "atoms" are cut down into new "atoms". Moreover in micro-physics, the choice of measure instruments is already a way of pre-conceiving objects: the découpage of this or that entity is made up thanks to that choice too and it is, afterward, developed in the most daring theoretical constructions, as with quantum physics, whose correlation to the world, the proposed découpage, needs continuous verification, revision and updating.

One can think, as an older example in physics, of the ancient Greeks with their imaginative categorisation of the elements: the concepts of "fire", "water" or "ground". To single out "primary elements" they devised non-existant unities of matter, though not unreasonable ones. Now think again about the découpage of quantum physics which proposes to "destroy" individuals, the particles, which *disappear* in the "quantum field", *in the interaction*. Nothing is left, nothing is studied further but for (weak) interaction. The concept of interaction is the proposed mathematical unity.

Physical theories that are proposed nearly always have some sort of legitimacy: the ways in which the elements of the Greek physics were combined or the Galileo-Newton's law of gravitation allow a first, non-magical, glance at the world. Certainly a glance in needs of revision. Usually such glances are given by means of a mathematical formalisation in one of the many possible ways. The "universal" laws, mathematical ones, are the formal expressions of these conceptual cuts. In physics, it should be clear by now, one can cut out from reality "almost" as one wishes, with theoretical proposals that are not arbitrary, because they are rich in history and knowledge, and are driven by our relationship with the world and by its "making resistance". I say "making resistance" because there exist things outside of our brains, which we bump into, which send or bounce back signals to us, which impose certain outlines or borders and numerous regularities; for instance the symmetries in crystals and in reflections of light ... . The point is how we go about describing them, using mathematics in particular, and how our self develops in its relationship with the world, while trying to grasp and describe it.

In summary, the conceptual act of "découpage" and of proposing new individualities or unities is typical of physics and the usage of mathematics in it. Mathematics, in fact, is nothing else than pure découpage, in particular, it is the theoretical découpage of physiscs, or better *is nothing else than the work on the découpage*. Mathematics is like ... "the smile without the cat" (it is not by chance that Lewis Carroll was a mathematician). Once the cat is gone, one works only on the smile: one squares it (a fundamental concept for smiles), transforms it, observes that its negative square root has an unforeseen geometrical significance ... . In fact I am recalling something that should be clearly understood since long, and that to an Egyptian of 800 B.C. I would explain by telling him that doing mathematics is like studying his "rectangular piece of land", about which he knows many things, but "without land": one is left with just the rectangle, with all its properties, which we establish by means of laws, independently from any actual piece of land. Only laws are left. This was the crucial observation of Greek mathematics: not only does one give a general rule to build right-angled triangles or circles but goes up to the level of giving "laws", such as that expressed by Pythagoras' Theorem or by the relationship between the circumference and radius in a circle. These laws *do not depend* on any specific triangle or circle. One reaches the invariance not only of shape, but also of theorems, of *proofs*.

Starting then from everyday practice, découpage is a "gesture", a line drawn on a sheet, but overall a conceptual operation for which mathematics is the principal instrument, the tip of the iceberg. For those who wish to identify human intelligence with the inclination toward the stability of the "découpages" drawn, toward the "mathematical law", possibly described with a finite number of linguistic symbols, this explicit tip of the iceberg of a particular practice becomes the top of a hierarchical mountain, the only possible one, with the bright beauty of the abstract axioms of Arithmetics. But this blurs the idea of *individuality* of the objects under study, when these are alive, or better, it hides points which it is better be aware of, in order to devise new methods and instruments. Since this individuality of the living beings, in contrast to inert matter, is at the core of the biologist's analysis.

2.2 Découpage in Biology.

With respect to the variety of forms of (mathematical) "découpage" in physics, this more or less free reconstruction of possible "borders", we do not have the same kind of relationship with living things, nor does a cat. Independently from us, the dog or the living cell have precise borders. They have an almost aggressive unity that if wounded can be lost, by losing life that is the glue that keeps everything together. "Object-ness" in biology is different matter: the biologist is, we are all, surrounded by living individuals whose unity is there; it is manifest or, at least, is crucial in the analysis of phenomena. It is not one of the theoretical tasks of the biologist to isolate or invent "individuals", such as quarks or positrons. This may have some role in the analysis of organisms, as they are compound objects, or in studying symbiosis phenomena, clusters of life ... . Or, in genetics one may "create" new individuals, or variants of existing ones, but the "glue" of life will set up a new autonomous unity which does not need to depend entirely on the conceptual construction of the "creator". And even in these limit cases the biologist does not perform, a priori, any "découpage" of the world in order to isolate "entities" and "objects" in the same way as the physicist can do.

The quantum physicist, as I was saying, does not see anything but for some macroscopic clue on some measure instrument, some tick-tick of a machine that he built more or less arbitrarily or, better, following a theoretical hypothesis whose inspiration comes often from completely different experiences. Then he says that there, there is a neutrino, here an electron, a wave-particle which ... jumps from one orbit to another. Using mathematics he cuts out these realities and then makes theories entirely built on the mathematical structure that, boldly, he forces on the world. And he does very well to work this way; try and try again, we now know very many things about universe in physics.

But to biologists, even cells are visible, perhaps only through a microscope, and they have discussed them, building on their blatant individualities, for many years now. It is not possible for them to proceed like the physicist and if they try to do so it immediately leads to oversimplifications: as soon as one cuts along a plane of biological reality in a mathematician's fashion, one cuts out living individualities. If one isolate an individual from its ecosystem, as if the latter was an irrelevant element of "friction", one definitely compromises the individual itself: the ecosystem, in fact, is far from playing a role analogous to that of friction for a falling body in physics, since it is the ecosystem itself which contributes to form individuals and species.

To isolate a law, in a physico-mathematical fashion, is like cutting out an aspect of reality, choosing a plane into which one decides to work that "cuts through" the unity of biological individuals. So they are killed and lose what is essential to them, the life, which grounds their unity.

In physics one observes falling bodies, isolates them singularly, and then proposes a general law of the phenomenon, regardless of their individuality and of friction. "Si defalcano le bagaglie" (Remove that which is superfluous), said Galileo. It is a great idea and the talent of the physicist is in being able to choose correctly what is "superfluous". Even so, it is like cutting a three-dimensional space with a two-dimensional plane: the section contains only a part of the information. But the physicist is happy enough with that and decides that for the problem he is considering, this section is what matters more in order to be able to abstract away the general mathematical law. Carrying on with the previous example, if the problem is that of falling bodies, then the unity and individuality of those bodies and their characteristics with respect to friction are uninteresting and distracting. If a body is falling in the absence of friction (a very plausible mathematical and physical hypothesis: especially nowadays that we know how to recreate almost perfectly empty space, or travel to it) and receives a push, it reacts *only* with its weight. Mathematics describes perfectly the parametrized curve of its trajectory, independently of whether it is a stone, a feather or a cat which is falling. On the contrary, if one stimulates a live neuron, a cell or a cat, as biological individuals, with a toothpick, a stick, or a discharge of electricity, it will react not with respect to a single parameter, but with *its entire unity as a living being*. A mathematical découpage with a beautiful, synthetic and effective law, such as universal gravitation is, obtained on a carefully chosen mathematical plane, would not have the same significance as before. This is one of the reasons why the conceptual categorisation that biologists do is qualitatively different: the unity of living beings does not allow the same kind of conceptual cuts.

In other words, I believe that the mathematical practice of "découpage", as described above for physics, is one of the greatest difficulties which the biologists encounter when trying to use mathematics: they cannot "dissect" a cat with laws and concepts that are perfectly stable and completely independent from life. One cannot break its unity as a living being and study the abstract properties of the smile. Well, in fact one could do it, but one would do something different from biology or, at most, one could capture only part of the biological analysis. The challenge is on how to go further in the dialogue between these essentially different disciplines, in a way that the *methods* of *both* are enriched: mathematics, in spite of the formalist and platonist myth, is so "plastic" and capable of novelties that we should not limit ourselves to the passive transfer of methodologies and tools.

3. Morphogenesis.

The analysis of morphogenesis is one of the earliest, most relevant (and elegant) examples of application of mathematics to biology. Some early mathematical observations on morphogenesis go back to Galileo: the physical resistance of limbs to strain is not linear. The section of an elephant leg is proportionally much bigger than that of an ant, yet an elephant cannot lift fifteen times its weight as the ant does! During the first half of this century, a major Italian mathematician, Vito Volterra, and D'Arcy Thompson in England studied biology as a sort of mechanics: different forms which are transformed continuously from one into the other; topological deformations which allow movement from one kind of fish to another; embryos which are moulded while growing, as the flow of lava in a valley. Briefly, they ingeniously searched for general rules of stability and physical-mathematical equilibrium, around which to organise living things. Alan Turing, in the early '50s, and especially René Thom later contributed to these efforts. Moreover, the latter has proposed some very original mathematical instruments to analyse the morphogenesis of living beings. Moving away from the *quantitative* analysis of equilibrium and towards the *qualitative* one of the interplay between stability and singularities ("catastrophes"), Thom managed to simplify considerably the number of variables one needs to examine in any single problem. Indeed, "add more variables" is the standard answer of the mathematician to the complaints of the biologist, who finds the proposed model too "poor". Thom reversed this attitude and invented new mathematical tools for a more qualitative analysis. The original inspiration was still from physics, but it was remarkably adapted to biology [Thom, 1972]. Thus one can better appreciate the global dynamics and the evolution of the studied phenomena, in particular the evolution of life. For instance, qualitatively speaking, one can describe the development of a jellyfish as analogous to that of a drop of milk falling in a glass of water. In Thom's approach, the global behavior of a phenomenon is more relevant than the number and precise value of the variables involved: he steps back from the wishful thinking of having precise solutions, but he gains very suggestive descriptions. In doing this, Thom drifts away also from any formalist and logicist paradigm for the foundations of mathematics (for this reason, in a different context, the Apologie du Logos [Thom,1990], he even talks of a "logical delirium") and places geometry, space, the continuum back at the center of mathematics and at the heart of its foundations. I wonder then, why many biologists are so puzzled when confronted with such beautiful mathematics?

It seems to me that using these methods one can satisfactorily study *physical* properties of bones or biological membranes, such as the effects of weight or of the superficial tension or of viscosity. These are properties of *physical objects* which by chance may sometimes also be alive. But their being alive is not at all essential: a plastic bag full of paraffin or a drop of viscous liquid would take the same shape if thrown into water. The method helps biology but is not internal to it: one better realises this if one wants to make progress. When doing this kind of analysis, the fact that legs or membranes are alive is secondary. What matters are the *materials* these things are made of. Materials, whose nature will certainly contribute to moulding life, bones for an elephant, or shape for a jellyfish, but they do not have much to do with aspects that characterise life and its evolution, along the reproduction of species (and their evolving characteristics). One has performed a physical découpage; it is like having "flayed" a living being to observe its membranes and bones as physical objects. To get back to the unity of it, Thom embraces a Lamarckian finalism, for which it is function which shapes or plastically moulds an organ, and explicitly refers to Aristotle and Bergson.

Nevertheless it seems to me that central to the formation of living beings are phenomena like "redundancy" (of differentiation of species or of genetics, among others) and "latent potentials" (in the evolution of species), along reproduction and, thus, evolution, as very well described in [Gould, 1989]. As for latent potentials, consider, for instance, how the ears of birds and mammals seem to have been formed. Gould explains that certain reptiles had developed, in contrast with others, a double articulation for the jaw; the upper part of it became "transformed" later in the ear bones of the species which originated from this transformation. This theory seems to be confirmed by the available fossil documentation. The potential in this transformation is *latent* as it is almost impossible to predict its future development; and even with hindsight, it is very hard to understand this evolutive phenomenon in terms of any kind of physical optimisation or within a finalistic scheme.

This understanding, and similarly for the redundancy of life in evolution, has nothing to do with concepts of optimality and with the minimality principles which have driven the physical-mathematical analysis of equilibrium and stability, from Galileo to Thom. Moreover, in biology the idea of causality itself seems to have a different meaning from that in physics. This is another difficult problem, for the treatment of which, choosing among the many papers on the subject, I refer to the reflections of a physicist, [Bailly, 1991, 1994].

Clearly, mathematical analyses such as those of morphogenesis can teach us many things (and, as Thom showed, can help to create innovative mathematics). Even so, unity of a physical kind, built around concepts of stability or geodetics, as proposed by Thom or as examined by others around notions such as that of a "well of attraction", as it is for clouds around a tornado, when used to analyse mathematically the unity of a living being, can only lead to partial results. They may possibly describe a posteriori certain phenomena but they are not able to capture the differentiations, the different evolution steps, rarely optimal but often "contingent" and at most compatible, that are taken. They cannot account for the responses that living beings give to external stimuli by means of their plurality of levels (mental, biochemical, physical, ...) and because the contexts within which living beings dwell are themselves alive and do not simply bear a geometrical structure.

Moving briefly to another proposed formal description of living systems, cybernetics, even the cybernetic system, a machine that we can build, has its own unity (and following Wiener, there have been other interesting proposals, of a mathematical kind, to understand the biological unity of which we are talking). Nevertheless, in this case the mathematical découpage is easy: we have designed the machine. Moreover it is also easy to violate its physical unity; one might for instance add a lamp, a small new bit or propose some mechanical updating. On the contrary, when, by means of genetic manipulation, one substitutes a leg for an antenna of a drosophila fly as they are genetically "homologue" organs, one is driven by the tracks of the unity of the fly and of its species (only genetically homologue organs allow us to do these monstrosities). One cannot break this unity, but one can put it together in a different way. Homology is a difficult biological concept extensively treated by Prochiantz in his book. It is probably a key element of the qualitative difference of living beings and, I think, of great interest for the mathematician unwilling to boldly transfer methods from physics to biology.

To conclude, it seems to me that as far as the conceptual act of découpage is concerned, biology as a science is placed at the opposite end to mathematics, especially when the latter is regarded in its relationships with physics: in mathematics there are only invariants, well "singled out". It is the land of conceptual stability; in mathematics, one is constantly looking for laws, the most general possible. Mathematics is a peculiar "conceptual essence" that has been distilled from physics and its space. More precisely, it is a "possible essence", as one may propose, say, different mathematical descriptions of the phenomenal continua (standard or non-standard analyses), or different descriptions of space, the euclidean and non-euclidean geometries. Mathematics is made of invariant structures, of propositions which are *completely independent from individualities*, with their contingent and contextual characteristics that are so relevant in living phenomena. As I was saying, it is perhaps here that there lies one of the difficulties in the mathematisation of biological facts or, more simply, to the biologist's "law-making" in a physical-mathematical style; this is one of the issues to deal with and it is a very basic one.

I believe that as long as one thinks that mathematics has to be "grounded" on the most rigid axiomatics, possibly minimal, mainly centered around logic and language or that the way to meet mathematics with biology is always to go via physics, these difficulties will not be resolved. They will also stay as long as *this* mathematics remains the "paradigm" of human rationality. It should be pointed out instead that it is human rationality, which is so rich and has many different facets, that *enters* mathematics. For this reason the "vitalist" intuitions, the sense of time flowing for instance, impossible to freeze by the points of analysis (see [Weyl, 1918] or [Thom,1992]), that of the evolution of living beings and the sense of unity as something different from the union of the parts, felt by certain philosophers such as Nietzsche or Bergson, these need to be reintegrated into scientific analysis, to enrich a notion of rationality so impoverished by formalism, positivism and physicalist reduction; and we must question what is the right method in order to go further. Thom has started doing this, but the physical-mathematical paradigm holds sway and it is one of the obstacles to an otherwise fruitful dialogue between him and biologists, with their particular "intelligence" about living beings.

4. The biological phenomenon.

I think that the problem underlying the issues I tried to discuss, is huge and much greater than the scope of a single paper or a single conference. Modern scientific knowledge, beginning with Galileo, Descartes, Newton and, for its philosophy, Kant, was built thanks to a crucial distinction: physics studies the *phenomenon*, mainly via its master device, mathematics, but not the *essence* of the matter ("matter in itself" is not knowable, stated Kant). On the one hand, there is an "essentialist ontology", which we are not interested in and leave to medieval scholastic philosophers, as well as to those of my colleague mathematicians who are Platonists (and for which mathematics is *the essence* of the world, independent and predating it; nature simply moulds herself by adapting to the perfect geometric shapes of mathematics). On the other hand there is the scientific analysis of natural phenomena.

At the risk of repeating myself, I wish to stress again that the *essence* of a body which is falling is not important: stone, feather or cat, the gravitational law, the physical phenomenon, is always one and the same. In physics, the "phenomenal veil" is like a cloth where one draws out the object of study: that very veil that is constructed in, *which actually is*, the interface between us and the world, as it is constructed by our action in the world as living beings. And mathematics supplies precisely the conceptual structure which best specifies the phenomenon level, at least in physics: it is the texture of that cloth, and it organises and selects the forms drawn on it.

On the contrary it seems to me that in biology, the unity of the living being "tears" this veil, on which by means of mathematics we have worked out modern physics; it seems also that it does not need to conform to selective projections and the very unity appears almost the "essence" of the living being, forcing a nearly "ontological" analysis: with much care and modesty we are tempted to reread Saint Thomas ... . Or, at least, a revision of the concept of "biological phenomenon" is forced upon us. It is perhaps because of this notion, or rather of this "scientific practice", that the beautiful paradigm of modern science, based on mathematical-physics, is again questioned: *we cannot isolate phenomena from their "essence", understood as the unity of the living being.*

Mathematics however is *one possible construction*: it is not "already there". It materialises in the interface between us and the world, the world that we want to study. It is possible that it must lose some of the features which define it, perhaps some of its conceptual stability, but it is not impossible that in doing so it will become possible to offer even to biology some new tools. At any rate, it is mandatory to go into the kitchen with biologists to analyse their method and logic, not just the *facts* which we want to "mathematise", and to be ready again to question our principles and the rules of the game, in particular as they have been moulded together the growth of another discipline, physics.

By this, I am not stating that life can *by no means* be reduced to the physical world, and that we cannot pursue the scientific analysis of the emergence of living beings and of thought from matter: I think that all of us who are monists see that particles or whatever collect into atoms, these atoms into molecules, the molecules into living cells, and these into "thinking" organisms. The scientific analysis of all this is what we are aiming for. What I am saying here is that the *paradigms* of *biological knowledge* cannot be reduced to those of *physical knowledge*, at least as they are presently understood with their ancient and deep philosophical and historical ways.

We can also detected a sign of this in the multiplicity of our forms of understanding of the world, which reflect in the variety of scientific disciplines. In fact these do not arise randomly; they are not arbitrary. The world appears to us in aggregations diversified by quality; from inorganic matter to life, up to thinking, as we understand it in the dialogue among humans and in history; there are qualitative jumps, emerging entities that need to be caught. At the interface with these different aspects of the world we build interpretations and different kind of descriptions, which can grow and divide into different kinds of knowledge. The unity is in the connections, in mutual influences, hidden passages from one to the other that need to be closely analysed; unity is not necessarily found in reduction to just one form of knowledge, even though this might perhaps arise at a later stage, but in "two ways" communicating paths, in comparing methodologies.

To conclude, the relationship between mathematics and biology is a very difficult problem, and a very interesting one too: it is possible to deal with it by means of old slogans, according to which "in any science there is as much of proper science as there is mathematics in it" (Kant, Metaphysical Principles of the Natural Sciences, Preface), as well as importing into biology some techniques, possibly bearing the stamp of physics, already consolidated in mathematics; it is however also possible to try to discuss, building bridges in both directions, not only from mathematics to biology, but also embed mathematics into the world, and particularly into life; going, in a certain sense, in the opposite direction: from biology to mathematics, to a kind of mathematics which needs to be rebuilt.

I wish to recall then that a great part of really innovative mathematics - think for instance of the infinitesimal analysis of Newton, Leibniz and Lagrange - has been designed around new applications, such as in this case the physics of movement: well beyond the tools of the existing geometry and algebra, physical observation forced a new mathematics. The use of the infinite as an actual limit and of the continuum, in making mathematical calculation (the differential calculus), started an immense conceptual and foundational revolution, based on an original outlook at physics and space, that uses the actual infinite to understand the finite (movement around us). Even in the limited space of the new problems raised by biology, if a technical dialogue is possible, it requires a radical change or a completely new synthesis of paradigms, which will need not be inspired by physics. My bet is that a questioning of the foundations of mathematics can contribute to this aim.

Part II Naturalizing Mathematics

Introduction: brain plasticity and action.

When attempting to analyse the constitution of mathematical thought, as an epistemological undertaking and therefore as an analysis of a "knowledge process", one should set this conceptual construction of ours in the context of other forms of knowledge, as «the problems of mathematics are not isolated in a vacuum» as Weyl stresses [Weyl, 1927].

The challenge is in the "singling out" of its specific abstract nature, independent or transcending any specific form of knowledge; yet the constitution of mathematical invariants is just one of the integrated aspects of our scientific endeavor. Reflections ongoing in Cognition and Biology may be part of such an enterprise, as they may help us to begin to understand how life and intersubjective exchange managed to form thought, when the latter is seen as an evolving continuum (although continuity does not imply differentiability) leading up through evolution and history to human thought - a component of life and history seen for what it is, both contingent and historical.

The high standard of "objectivity" of mathematics, w.r.t. other forms of knowledge may be partly ascribed to its being grounded in common experiences, directly related to our living being, as thought is rooted in life and parallels it. Intersubjective exchange, through history, adds on top of this to reach the peculiar conceptual stability and invariance which is proper to some aspects of human thought, such as mathematics. But one should also be bold enough to recall that life (and history) are necessary to thinking: a Lapalissian truth, of which certain developments in idealism and formalism have caused us to lose sight. In short, instead of starting from absolute forms of knowledge, "universal laws of thought", possibly independent of actual, living and historical being, one should try to reverse the paradigm and re-construct them by a close look at their dynamical constitution throughout our action and presence in the world.

An important element of any "naturalistic" analysis of thought is the consideration of the relationships between individuation in ontogenesis, cerebral plasticity and the evolution of the species. The importance of this remark, which we owe to biology, is tremendous, since it allows us to assign a biological foundation to the hypothesis that our self is formed in relation to the world about us and that it becomes "other", i.e. it differenciates and specifies, through collective and individual development in life and history. We are never "ready formed" to live: thanks to our cerebral plasticity we define ourselves within and in synchrony with our milieu. In particular, the constituting process of thought is part of the adaptive relationship between organism and milieu, it emerges from our praxis in the world.

And, it all begins with ... early forms of life, because there is a continuity in the complex development of the nervous system: we all have common ancestors who contributed to the phylogenetic heritage of a primitive map of the brain. In contrast to inanimate object, any form of life possesses an embryonic form of "thinking": the first intentional action is in the amoebae which "chooses" a direction to move, as simple and purely chemical this reflex may be; the squid, escaping towards the closest but sufficiently large hide, while the predator is made blind by its ink, is making an early "geometric reconstruction" or "evaluation" of space. Evolution of thought, in all its aspects, parallels the evolution of life [Prochiantz, 1997].

Riemann, Helmotz, Mach, Poincaré, Enriques, and H. Weyl have attempted to develop theses upon which this approach rests. It was in movement, in the sensori-motor system, in the phenomenon of sight, in our life, in the context of history, that they researched in quest of these "acts of experience" (to put it with Weyl) which are the basis upon which we may analyse the foundations and origins of mathematics (see the works cited below for these authors, as well as [Boi, 1995] for a broad historical-phylosophical survey).

Many philosophers and logicians have nonetheless explained to us (and to them), and continue to reiterate at the present time, that one must not confuse *foundation* with *genesis*. In fact, this is a fundamental distinction, proposed by Frege, among others, and a distinction out of which was born Hilbert's formal Theory of Proof, one of the most beautiful branches of modern mathematics, and its daughter, Computer Science. With the aim of founding human knowledge (and/or mathematics) independently of "human being", i.e. with no analysis of the concrete "knowledge process" and genesis, we were lead to reify rationality in fantastic machines: a remarkable fall-out. However, in the field of the philosophy of knowledge, the insistence upon maintaining this distinction is nowadays only a limitation of the analysis of our forms of knowledge, for it is precisely where the frontiers of genesis and foundation get together, that we may locate the origin of this wonderful conceptual structure we call mathematics. By isolating certain fundamental or invariant elements of the practice of proof (the principles of proof), Proof Theory can help us to take some first steps towards connecting foundations and genesis, although subsequently one must enrich this connection by making a cognitive analysis of those very conceptual invariants which the connection reveals, and which constitute a very distinctive mixture of our language and our relationship with the sensed environment (for example, connections between formal induction and order in mental spaces; between geometric construction principles and the space of senses ...).

We are therefore talking about intersubjectivity and language, an essential tool of human thought, as well as about touch, simian and human caresses, smiling, dialogue between minds, dialogue through history, in all its forms. For we must take great care not to fall into the trap of making "all things biology", by isolating the living being and its brain from the rest of the world and the brains of the others: by this we would pass from the Scylla of the "linguistic turn" in philosophy of knowledge (largely due to Frege), which focused the entire analysis into logic and language, as absolute, human independent realities, to the Caribdi of biologist reductionism.

On the contrary, brain plasticity is the "bridge" between human life and its contexts, up to human history as locus of intersubjective, explicit constructions. And this is crucial from the cognitive point of view. Thought is not a deposit banked in the brain, but rather lives and moves in dialogue with the environment and through history. Thought is generated by praxis in the world; yet, neurobiologists can identify traces of it in the nervous system, thanks especially to observation of sensori-motor functions, at the core of brain plasticity, but these traces "acquire sense" only in contexts and history, in intersubjective activity.

This last specific issue about cerebral plasticity, i.e. the role of sensori-motor functions, is often stressed in analyses made by neurobiologists, especially those who are interested in cognition (Edelman, Prochiantz, Maffei ... see the bibliography). Plasticity is observable essentially wherever "feedback activity" occurs, that is to say, in cerebral activities, when sensation is the stimulus for an action which in turn gives rise to a new sensation. See for example the increased structure, number and connections in the cortical neurons associated with the fingers of a violinist's left hand, dancers or pianists who can "think with hands and feet as well as with their brains", as Prochiantz puts it, in the sense in which these activities in the world mould the brain and create in the trained artist a new kind of unity of mind and body. See, more generally, the idea of "the brain as a complex development of the reflex pathway", and the differentiated structure of the fore-brain, formed in response to the need to link sensorimotor reflexes to the sensory modes.

One of the great challenges, amongst these analyses of the constitution of "self" in ontogenesis, is the analysis of the "interaction" between the primitive map of the brain, as "memory" of the species, and adaptation by individuation, based on the ontogenetic plasticity of the brain, which retains traces of individual experience and enables a faculty of "individual memory". This is what is at stake here, and an inevitable point of challenge through which any "naturalisation" of thought must pass, as long as one knows that one must link specifically biological studies to the many different analyses of human conceptualisation and mental activities (origin and foundation of language, of mathematics, of madness...). In short, biology is helping us to understand the systemic unity of senses, the constituting of thought and environment.

By this, it is only in this dialogue between biology and the disciplines which reveal the foundations and the dynamic (both historical and individual) of the various forms of knowledge and concept-construction, like mathematics, that one can hope to conduct an epistemological analysis which has its starting-point within the subject (as Mathematical Logic has) and yet connects the subject with the world. Let us try to examine just one aspect of this.

1. The concept of unidimensional mathematical line.

> «Oceans... mountains.... rivers ... are not swallowed into me by seeing, but the images of them only ... are preserved in my memory»; as for knowledge of geometry, though, «I do not remember images of it, but knowledge itself» ... «Memory contains the reasons and innumerable laws of numbers and dimensions; yet, ... I have seen the lines drawn by an architect, even as small as the thread of a spider's web; but the mathematical lines are not the images of those dimensions which my eye of flesh shewed into me. Everyone knows them ... within himself, within memory» [**Augustinus**, 401; lib. X, c.VIII - XII].

In the rest of this lecture, I will try to discuss an example of "cognitive foundation" of a mathematical concept. It is just one single point, but the analysis of conceptual geneses may only proceed, as for now, by cautious case analyses. And "possible stories": as all scientific knowledge, this approach does not provide "absolute knowledge" or "unshakable certainty", to refer to the words of the fathers of Mathematical Logic as a foundation, but possible theories that may be falsified, locally verified, revised or gradually expanded.

The questioning I want to deal with is beautifully raised by Saint Augustine in the quotation above: how can we have the concept of "line with no tickness", essential to mathematics, while in no way these lines belong to the world? Saint Augustine's answer is not ours, as it is based on his original (and remarkable) platonism, which embeds ontologies into memory[4]; yet this very relevance he gives to memory may be seen as a peculiar cognitive insight.

1.1 Trajectories and anticipation.

Certain regularities of physics, chemistry and life, like symmetries, light reflections, minimal trajectories which the world around us imposes on us, all "influence" the growth of living structures by our activities as beings in the world, and later contribute to our conceptual constructs. In other words, the pre-conceptual experiences make possible (and contribute to the meaning of) our subsequent conceptualizations. In particular, they make the possible universes of mathematics non-arbitrary: as I was saying earlier, epistemological analysis must start from these "objective" facts of the phenomenal world about us and extend towards the phylogenetic, the ontogenetic, and lastly, the intersubjective domains. The objectivity and effectiveness of mathematics is in its rooting in the world and in our action in it.

The analysis in [Viviani, 1991] of the mathematical relationship between curve and speed of movement of the hand is a fine example of this. The gesture we make is not arbitrary, it follows regularities, it follows lines of minimal variation of acceleration. These lines can even be found in art, in forms drawn with "minimal jerk" (a jerk is the derivative of an acceleration): a difficult mathematical curve is accomplished by or instilled into the gesture. In this, there is a very close interaction between physical space with its regularities and the way the body functions. The shape is made thanks to the presence of physical objects in the world, including our bodies, and our bodies are moulded, shape themselves in evolution, by their action in the world, with its geodesics and its symmetries, as pointed out in morphogenetic analysis (see above). The mathematical description of these shapes is subsequently our bid to represent them; it is not an arbitrary description, but it is far from being unique, as non-euclidean geometries or the non-standard continua teach us. What gives it its objectivity is the existence of these regularities in the world, which impose themselves upon our existence and which will underlie any good (coherent, even if "non-standard") representation of the world,

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[4] For Saint Agustine, we experience the absolute in memory: « [numbers and dimensions]... have an absolute existence {valde sunt}» ... and God « ... surely lives in our memory {Tu habitas certe in ea [memoria]} , ... as I find Thee in it, by remembering» [Augustinus, 401; lib. X, c. XXIV-XXV].

constructed in intersubjectivity[5].

We are not, however, passive in the face of these regularities. Trajectories for example, which are at the heart of our actions, are not passively followed by our bodies: physiologists explain to us that we anticipate the movement to be made. In particular, in ocular pursuit, one looks ahead of the target, and in so doing, we make ready the trajectory to follow, if for example we are to capture a prey: «in other words, we move towards where we are looking, and not the opposite.» ... «The brain is not made up of simple systems which transform sensory signals into motor commands: it is made up of closed circuits. Action modifies perception at source.» [Berthoz, 1997; p.201-221]. We are actually constantly interrogating the world about us, consulting data-receptors, according to needs and intentions; we regulate the sensitivity of our receptors, anticipate their feed-back data, compare it with all other data in combination, on the basis of an internal simulation-representation, usually analogical, of the expected consequences of action.

I think that our conceptual constructs, especially mathematical ones, are not only generated by these phenomena, but also obey the same paradigm: we act in the world, and then we propose predictions in the form of explicit representations, which are not arbitrary, since they are rich in memory (memory which distills invariants out of the mass of data, but also that memory which is shared with others, historical memory, see [Longo, 1995]; the role of memory is the great intuition of Saint Augustine).

Then, upon these "predictions" we construct metaphors by analogy, which link a mathematical structure and/or a method of working to another (or even are at the core of the mathematical proposal, see [Lakoff&Nunez, 1997] for the mathematical metaphors about phenomenal continua). Language and logic, the latter the locus of explicit and conscious conceptual invariance and methodological stability, both play a huge part in the last fragment of this constituting interaction, but they are not its ultimate foundation, much less its only foundation.

1.2 Genesis and foundation: from trajectories to shapes.

For the time being, I have attempted here (and in some of the quoted writings) to sketch out the possible role of movement and geodetics or optimal lines, of symmetries in the constitution of invariants, as products of action and

[5] In [Longo, 1997] the role of symmetries in the various geometries is discussed.

multisensorial experiences of the world: jointly to memory, multisensorial experiences are at the core of constituting of invariants. Perhaps even new machines could be derived from this analysis: current machine design increasingly departs from its formalist origin (consider, say, "hybrid systems" - as a blend of digital and analog computing, or the geometric analysis of concurrent, distributed and asynchronous computing).

And I stressed the role of trajectories, as pre-conceptual experiences. But, these trajectories are primarily predictions, as one learns from the physiology of action [Petit, 1997]; "acts of experience", not of our individual lives, but rich in the history of evolution: their experience is also in our philogenetic memory.

One must understand, in order to grasp this change of perspective, that geometry is not, or is no longer, especially since the great debate of the last century, a "science of figures", but a "science of space" (Riemann). In fact it is a science of "movement in space" (Poincaré). Now one can more precisely wonder in what sense geometry could be a "perceptual abstraction", or what this "invisible thing which underlies the visible" actually is (to paraphrase Merleau-Ponty). I believe that nowadays, thanks to the breakthroughs in biology and cognition, one can get to consider geometry to be a science of "action and prediction of movement" within space: segment, curve and circle are neither "abstract forms" of material objects, nor "ideal figures" (what does this mean?), but rather predictions of trajectories. The point is that *prediction is already an abstraction*. This is the main thesis outlined in [Longo, 1997]: *the trajectory predicted or anticipated by the gaze is abstract, as much as it is the memory of it.*

By this thesis I am seeking to respond, though remaining a monist, to the questions raised by Saint Augustine about the foundations of mathematics, quoted in the prologue to this section. In other words, the act of predicting, anticipating a trajectory, grounded on memory, this act of a living being, constitutes the ancient and pre-human embryo of human geometric abstraction, given through history in language and drawings. It is the cognitive origin of these lines which we can conceive of as having no thickness, since they are pure directions, and of these curves, perfectly smooth and optimal, since they are pure trajectories, or rather predictions of trajectories. *We can conceive unidemensional lines because of these experiences; the very mathematical construction makes sense to us, because we understand it in reference to the pre-conceptual construction.*

This act of experience is also there, not only in praxis, but also in the phylogenetic memory of our relation to space (certain visual neurons are activated for "directions" in space). We will use this perhaps also to form figures by interpolation, to follow or reconstruct contours, by integrating the visual perceptions of them, on our way towards that geometry of forms which is a part of the geometry of space. Language will add its contribution, by attaining the objectivity of shared constructs, but so will the act of drawing, with the look which anticipates the gesture of the hand. History, too, will contribute to the constitution of invariance and conceptual stability necessary to mathematical construction, thanks also to the great variety of practical experiences and of linguistic and formal descriptions, including the variety of notations, for numbers, or in mathematics in general, or the many mathematical descriptions of space: mathematical invariance is grounded in pre-conceptual experiences and in independence from specific explicit historical constructions.

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