The Constructed Objectivity of Mathematics and the Cognitive Subject[1]

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> «The problems of Mathematics are *not isolated problems in a vacuum*; there pulses in them the life of ideas which realize themselves *in concreto* through out human endeavours in our historical existence, yet forming an indissoluble whole transcend any particular science» [**Hermann Weyl**, 1949].

Introduction

This essay concerns the nature and the foundation of mathematical knowledge, broadly construed. The main idea is that mathematics is a human construction, but a very peculiar one, as it is grounded on forms of "invariance" and "conceptual stability" that single out the mathematical conceptualization from any other form of knowledge, and give unity to it. Yet, this very conceptualization is deeply rooted in our "acts of experience", as Weyl says, beginning with our presence in the world, first in space and time as living beings, up to the most complex attempts we make by language to give an account of it.

I will try to sketch the origin of some key steps in organizing perception and knowledge by "mathematical tools", as mathematics is one of the many practical and conceptual instruments by which we categorize, organise and "give a structure" to the world. It is conceived on the "interface" between us and the world, or, to put it in husserlian terminology, it is "designed" on that very "phenomenal veil" by which, simultaneously, the world presents itself to us and we give sense to it, while constituting our own "self".

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[1] To appear in "Epistemolgy of Physics and of Mathematics", M. Mugur-Schachter editor, Kluwer, 2001.

The mathematical structures are literally "drawn" on that veil and, as no other form of knowledge, stabilize it conceptually: geometric images and spaces, or the linguistic/algebraic structures of mathematics, set conceptual "contours" to relevant parts of the enormous amount of information that arrives upon us. Yet, this drawing is not arbitrary, as it is grounded on key regularities of the world or that we "see" in the world. That is, on these regularities that we forcibly single out by "reading" them according to our own search or projection of similar patterns, as living beings: symmetries, physical and biological symmetries, or the connectivity and continuity of space and time, for example.

Intersubjectivity and history add up to the early cognitive processes; they modify our forms of "intuition", including mathematical intuition, which is far from being stable in history. Indeed, mathematical intuition is constructed in a complex historical process, which begins with our biological evolution: the analysis of "intuition", not as a "magic" or inspeakable form of knowledge, but as a relevant part of human cognition, is one of the aims of this ongoing project. A project which can be named a "cognitive foundation of mathematics", as opposed to, or more exactly, complementing the metamathematical analysis of foundations largely developed in this century. Indeed, the foundational analysis of mathematics cannot be only a mathematical challenge, as proposed by Frege and Hilbert's mathematical logic: Hilbert's metamathematics uses mathematical methods and, by this, it became part of mathematics, the very discipline whose methods or whose whole it was supposed to found. Mathematical logic gave us an essential analysis of (logic/syntactic) proof-principles, and more it is giving: yet, we also need to go further and evidentiate "what is behind" these linguistic principles, their meaning as rooted in our practices of life. Persisting *only* on the proof-theoretic, thus mathematical, analysis of mathematics, would leave us in a cognitive deadlock, actually in a philosophical or even conceptual vicious circle: one cannot found mathematical methods and tools by mathematical methods and tools. For example, no mathematical methods and tools can prove their own "consistency", which is the metamathematical way to assure meaning to a mathematical theory. Sufficiently expressive, finitary theories, such as Arithmetic, have no *finitary* consistency proof; Set Theories which can represent a given infinity, need a larger one to be proven consistent. This is not a limitation to mathematics: in order to check the correctness of certain conceptual tools it is rather to be expected that one should "step back" from them and use different tools. Gödel's second incompleteness theorem proves exactly this: by Gödel's representation lemma, one can describe or encode the metatheory of arithmetic within arithmetic itself, which is thus part of the latter, and, then, prove that consistency is unprovable by that arithmetized metatheory. That is, the finitistic and mathematical metatheory of arithmetic is too weak to do the least job it was invented for, the proof of the consistency of Arithmetic, since it is given by the same finitary tools as Arithmetic itself.

And there begins the infinite regression of "relative consistency" results: if one wants to be sure that a given formal theory has a meaning (it is consistent), then one has to construct a stronger one and use it as metatheory (i.e. use induction over larger ordinal or, in Set Theory, assume the "existence" of larger cardinals). This is a non well-founded chain or infinite regression of theories, which shows that the foundational relation of mathematics to mathematical logic is conceptually non well-founded, when it is exclusive. That is, when metamathematics is fully mathematized or the foundational analysis is reduced to be only mathematical, one ends into a conceptual vicious circle. This is our main motivation to go further and *step outside mathematics* and try to ground it into the networks of our forms of knowledge and of possibly pre-conceptual experiences, as part of an investigation on human cognition.

Clearly, also in the relation of cognition to mathematics there may be an (apparent) vicious circle. Jean Petitot in "The epistemology of physics versus a formalized epistemology", in this volume, raises the issues whether there may be a circularity in developing a cognitive approach to (the foundation of) mathematics, as suggested here, and, at the same time, a mathematical analysis of cognition, which he and many others are working out. I think that there is no danger of a conceptual circularity, in this case: indeed, a mathematical analysis of *some aspects* of human cognition is *possible*, exactly because mathematics is rooted in cognitive processes. The remarkable, yet not absolute, effectiveness of mathematics is due precisely to the fact that mathematics radicates in natural phenomena, from physical to biological ones, up to the «human endeavours towards knowledge,in our historical existence», as Weyl puts it. Moreover, in the case of cognition, these two dual approaches are actually *needed*, as the challenge is enormous (mathematics is very expressive) and only a variety of enriching or converging view points may help us in some further scientific understanding, including the use of mathematical tools. There is no conceptual vicious circle, though, as mathematics cannot describe alone all cognitive and historical knowledge processes: these must be analyzed also by the other forms of knowledge, in the methodological pluralism mentioned below. There is no entirely mathematized metatheory of human knowledge: *this* would lead to a vicious circle, as mathematics is one and a specific form of knowledge. It is instead the network of methods, of mutual understandings, interpretations and descriptions, by different conceptual tools that gives meaning to each theoretical description, in the interactive reflection of theories and interpretations that form our human and scientific understanding of the world. Mathematics is grounded in this network and its foundation may be analysed by appealing to various (all?) forms of knowledge, while using it to clarify *some* aspects of it: there is no circularity in this, but a search for the unity of knowledge via interactions, conceptual bridges, mutual enrichments.

1 - Methodological pluralism and the unity of knowledge

These "interacting" methods, cognition for mathematics and mathematics for cognition, are possible and needed, but not sufficient. I am a monist, as I do not believe in the distinction between body and soul, between concrete autonomous "objects" of reality and metaphysical ontologies, but I am not a *methodological* monist. That is, I do not believe that a unique method, the mathematical one for example, may provide the ultimate understanding of all phenomena, by a formal and complete reconstruction, or that "the essence of objectivity is mathematical". It may be so in physics, as mathematics has been largely designed over and with physical phenomena, in physical space and time; but this paradigm does not need to be transferable to other scientific analyses.

It should be clear that I do not doubt that, in the end, there is nothing else but "physical matter", whether we call it atoms, electrons or waves and strings or alike. However, I doubt that this (potential) reduction of "everything" to his material constituents implies that in other disciplines, beyond the one which proposed to us these notions, physics, must be reduced to the *same methodology*, the methodology of physics. Biological phenomena, say, do follow *also* "physical laws", but their biological behaviour cannot be *deduced* from the properties of physical matter: they lay at a different (further) phenomenal level. As suggested by [Petitot, 1995] in a different, but relevant, context, "dependence does not imply reducibity". Indeed, there are *qualitative differences* between the phenomena that we analyze in physics and those of life, as well as between individual, biological life and human intelligence or historical organisations. The emergence of one from the other results in apparent discontinuities, yet to be understood, as if material components, when they "get together", give rise to new phenomena or, at least, are better described by a different scientific method (even within physics it is so: "enough" particles, when they form a medium sized solid body, are better understood by classical mechanics than by Quantum Physics).

Mankind in its historical existence has been giving to itself a rich variety of forms of knowledge, exactly because the phenomenal world presents itself in different forms. It is too early, if ever possible, to pick up a method and propose it as absolute. Many of these forms of knowledge may or will be unified and reduced: the phenomenal descriptions of physics may be entirely reducible to (or handled by) mathematics for example, but, so far, that drawing of ours of the images and syntax of mathematics on the phenomenal veil, misses at least "colours", "nuances" and "intentions", which are at the core of life and human intelligence. There is no knowledge without intentionality for example (it even shows up in mathematical proofs, see below for references), without the guidance by feelings and passions, as "colours" and "nuances": we should not iterate Descartes' (and Kant's) error and introduce a primary break between rationality and the other forms of relation to the world we have. In particular, emotion and cognitive processes are not disjoint: intentionality sets the "direction" of analogies, say, or metaphors ("analogous to this", a "metaphor towards that" ...). Thus, emotion is not just a possible stimulus to knowledge, but, as interlocked to intentionality, contributes to constitute the content itself of analogies and metaphors, some key elements of human understanding and expression, even scientific one, as most scientific descriptions are "just" metaphors.

Our brain, indeed the simplest neural system, is first of all an "integrator", in the sense that it compares, integrates and synthesises very different perceptions and judgements. And so does our "historical brain", our human and collective form of intelligence. The objectivity and unity of knowledge is not given by a unique ultimate mathematization, but by the integration of a plurality of approaches, as a dialogue between different methodologies; it is given by setting bridges, by providing and relating common "supporting" points, here and there, as shared though spars, yet "communicating", foundations. The unity of scientific knowledge may be understood in terms of a "ring of disciplines", as beautifully described in [Bailly, 1991]. Or, as already stressed here, knowledge is a network which supports itself by a reflective equilibrium of conjectures and theories, by the exchanges and enrichments between essentially different approaches; this is its unity. In this network, mathematics stems out by its strong two-ways relation to physics, but physical methodology does not completely cover biological analysis, for example: the peculiar internal unity of living beings as well as their unique relation to their ecosystem have little to do with the mathematical singling out and setting of conceptual unities in microphysics, say. Living beings impose to us their unity and their living relations. Any form of "découpage", as a key to the physical description, "kills" the systemic unity of life, which constitutes its *essence* and forces itself throughout the phenomenal veil. On this veil, in biology, we are no longer free to draw contours and single out (ever changing) unities with our mathematical tools, or not as free as in physics (yesterday atoms, later electrons or muons, today strings or cords ...): the phenomenal donation of living beings has a qualitatively different, pre-existing, form of unity.

As for comparing models, in physics the mathematical model is usually more rich than the phenomenon: in microphysics, say, a few sparks in a machine may lead to a complex theory of particles. In biology, the mathematical model is always poorer than the phenomenon, with its unbreakable living unity: it is only a "conceptual section" of this unity. Moreover, biology, as many little mathematized disciplines, deals with "examples" and "counterexamples" in a way which largely differs from the (physico-)mathematical practice: an example may constitute knowledge, it is not just an instance of a general "theorem"; a counterexample need not demolish a theory, but may suggest variations. Stability and invariance, repeatability of phenomena, as well as the absolute generality of a law are not at the core of life. Each living being is unique; more than elementary interactions, biology analizes individual totalities; contexts and ecosystems affect the repeatability of each experiment; more than stability and invariance, what really matters in biology is variation, non-isotropy, diversity; in biology, finalism contributes to explanation, on top of "physical" causality (living beings "want to" survive, to improve or maintain their metabolism: see

[Bailly, 1991] and [Longo, 1998] for references and further analogies and differences with physics).

The successes of mathematics in physics should not leave us blind with respect to the peculiar phenomenology of life. The enterprise of applying mathematics to the sciences of life is particularly challenging, as it cannot be a simple "transfer" of techniques, but the methods should complement each other with no philosophical ultimate priority; on the contrary, we should try to make more explicit the "philosophical peculiarity" of the biological analysis, which may suggest novel mathematical tools, as it often happened in the historical relation between mathematics and physiscs. Thus, beyond using established tools from mathematical physics, say, as specific mathematical structures, one should also try to establish the interaction at the level of "conceptual invariants", as they are defined below. This are more "plastic", as I will try to argue, more subject to further mathematical specification, in particular in interaction with essentially different methodologies, as those of biology.

2 - On the genesis of conceptual invariants.

Invariance and stability are key components of mathematics. I would even dare to define mathematics, among our attempts to describe the world, by structuring it, as the locus of conceptual invariance and stability. We will try to analyze invariance and stability at the level of "concepts" (this section) and at the level of "mathematical structures" (§. 4). Husserl's philosophy, (see also [Petitot, 1995] and more references in this volume), provides a guideline for our approach, as mathematics "transcendental objectivity" is the result of a path or genesis through "immanent cognitive and historical acts" (see [Husserl, 1936]).

2.1 - Integers

The concept of integer number is perfectly stable: it is invariant w.r.t. notations and meaning as no other practical concept. It is (remarkably) stable through history. Indeed, an embryo of counting is surely pre-human.

There is large evidence today that the rat, the monkey and the human baby (at four and a half months!) distinguish between sets of 1 or 2 or even 3 objects, independently of the nature of the objects being enumerated (see [Dehaene, 1998]). That is, these various mammals exhibit similar reactions when they perceive two sounds, two flashes, or two dots on a screen. Pairs of objects or threes are recognised as such, even if they are in motion, distorted or continually changing: this shows that it is not a pattern that it is recognized. Evolution then appears to have constructed neurones which react to the number of events occurring in the environment, no matter what the nature of these events, providing there are a small number of them: two or three, rarely four (p. 37 of [Dehaene, 1998; french version], experiments conducted by R. Thompson; se also the references to K. Wynn's work.) For successful survival, it seems necessary to be able to recognise and compare at least a few samples of food, objects or useful markers, even when 'secondary' changes are affecting the perception of these; and this became part of our phylogenetic memory, as a common pre-conceptual experience.

Here we have an example of early "mathematical" invariance, a key concept in mathematics, to which history has given depth and complexity. The experience of a variety of notations is part of the path leading to the explicit concept of "number", *as independent from reference and notations*. A far from obvious path: all early counting notations, as pointed out in [Dehaene, 1998], are identical up to 3 (I, II, III; also our Arabic 2 and 3 themselves are two or three connected horizontal lines, probably for writing convenience). Yet, the early sumerians numbers used to differ, beyond 3, according to the counted set (the 5 of 5 sheeps was written differently from that of 5 houses ...); the late sumerians and the Egyptians made the fantastic step to get to a notation independent from meaning.

In short, I claim that the ancient appreciation of the independence of "counting" from examples and events, up to 3, as a shared praxis of many species, is *behind*, it is the primary ground on which lies the "foundation" of the concept of number, as an invariant: from the phylogenetic memory of this pre-conceptual invariant, to the practice of counting, to the difficult invention and experience of different possible notations, we got to the abstract concept, as what is common or stable. The historical construction wouldn't probably have been possible without this underlying appreciation of independence of elementary counting that we share with animals. This is its root and cognitive foundation: this is the early, but extremely deep link to the world, which makes counting so "objective" and so effective.

On these grounds, human language could start drawing on a phenomenal veil that we share with other animals and it gave to the underlying elementary invariance an intersubjective content. Husserl, in [Husserl, 1936] stresses also the role of writings: oral communication is still missing the «persisting presence of "ideal objects", which last also in time ... It misses being-in-perpetuity ... This is the crucial role of written linguistic expression, of the expression that stabilizes, that of making possible communication without personal allocution, mediate or immediate, and of becoming, so to say, communication on a virtual mode. By this as well, human communication goes over a further step». Thus, writing adds a further level of objectivity to intersubjective communication: it further clarifies and stabilizes concepts, it increases the passiveness of the individual who accesses by this to human collective memory and it adds by this a further appreciation of the objectivity of our historical constructs.

In short, the invariant concept of number is primarily grounded on phylogenetic memory; it then acquires, by language and writing, a double form of stability. First, by symbols, it is detached form concrete experience; second, the very experience of many possible symbolic notations contributes to its further conceptual stability. It is then definitely given a "formal ontology", in the sense of Husserl. Yet, no platonistic metaphysics nor formal conventionalism is required to understand this human construction and its objective content, which entirely lie on the human praxis, throughout evolution and history.

2.2 - Potential infinity

Language and writing allow to conceive rigorously *indefinite iteration*, as an endless extension of the "go-to-the-next" operation or of a given notation. This is a possible route towards potential infinity. The ever increasing sequence of the natural numbers, a core mathematical structure, is thus the result of shared experiences by language and by writings, along generations, up to the symbolic extension of elementary counting, along potential infinity. This bold further step surely includes metaphysical appreciation of endless space and time, as never closing horizons.

As a result of this cognitive and historical experience, the structure of the potentially infinite collection of numbers is "*there*", for all of us. We even put it back into space, into mental space: we "*see*" the discrete sequence of numbers, the so called "(integer) number line". It is well-known that, in western cultures, this mental line goes from left to right (don't you see it?), the opposite direction for Arabs and Iranians. Here is the influence of writings on the forms of stabilization of a mental construction. Yet, and again, one should not isolate in a vacuum the mathematical construction: the iteration towards infinity could only be dared and have a sense in conjunction to a metaphysical glance on the physical world, which include the sense of expectation, the future, the endless or iterated phenomena, which give us an appreciation of never ending sequences.

I said that we put the number line "back into (mental) space", as, according to many in "Mathematical Cognition" (see Dehaene's book, further references may be found in the journal with this name), the evolutionary counting-process is also analogical, as if represented in space, on a "logarithmic scale": highly precise at the lower end of the scale, but more and more crude when applied to larger quantities. According to numerous neuropsychological experiences, in humans and animals as well, it seems that large quantities, when only slightly differing, are operated upon in an approximate-analogical, but consistent fashion; in humans, they are also represented as appearing distant and ill-defined. These authors suggest then that representation of numbers, counting and comparing quantities are also a spatial, analogical process. Human language confers the precision of linguistically discrete expressions, and allows us to generalise from the precise handling of smaller numbers towards a similarly precise handling of larger quantities, by the arithmetic-linguistic operations. But the analogical counting-comparing process, in a mental space, appears again and again in a thousand different situations: in animals comparing quantities, in human everyday practice, in pathologies, in mathematical apprenticeship and in the "intuitive" work a mathematician does. We consciously and unconsciously "look" at numbers as a "well-ordered" structure, in space (and time, as Brouwer would stress), and we use it in approximate computations or size comparisons. The totally ordered, strictly increasing (well-ordered), sequence of numbers, is the solid *geometric* rock on which it is founded human arithmetic. Logical or formal induction (Peano-Dedekind-Frege) is the very late attempt to found it on purely logical (Frege) or formal language (Hilbert): it turned out to be incomplete. But we will briefly go back to this issue (see [Longo, 1999a] for more).

2.3 - Actual infinity

The few hints given so far on the genesis of numbers concern only the *potentially* infinite sequence of integers (also and correctly called "natural"). The progressive conceptualization of the human notion of *actual* infinity is a further, long and complex story, largely embedded into metaphysical considerations. Weyl hints this, with reference to late Greek mathematics and oriental religions, [Weyl, 1926]; in [Zellini, 1980] one may find a remarkable historical reconstruction of how we gradually got to stabilize the idea of potential infinity, first, and later the difficult concept of actual infinity, a source of major disagreement and conflicts in history (see also [Gardies, 1984]; a linguistic analysis of the metaphores implied, may be found in [Lakof&Nunez, 2000]. Projective geometry, a fall-out of renaissance pictorial perspective, played a crucial role in the conceptual specification of actual infinity: the convergence point of the perspective is "there", into infinity. It is no longer potential, as it is an *actual*, *visible* construction.

It is fair to say that with Cantor, on the grounds of the work of many, from Thomas Aquinas to Pascal, Newton, Leibniz .., we arrived to a robust concept of infinity, as Cantor even inserted it into operational contexts, he dared to work with it. Cantor invented ordinal and cardinal arithmetic and showed, beyond the experience of infinitesimal calculus, that we could have an infinity of infinities and operate on them. There is no better way to give solid ground to a mathematical idea and to turn it into a true conceptual invariant, than showing how it works in different operational contexts, by "manipulating" it as a operationally meaningful symbol. Because, and this should be clear, the elements of Cantor's (and Veblen's) hierarchy, beginning with ω, the actual infinity of the natural numbers, and then $\omega+1$, $\omega+2$ ... $\omega+\omega = \omega\times2$, $\omega\times3$, ... $\omega\times\omega$, ... up to ω^{ω} and ... ω to the ω, ω times (named ε_0), and ε_1, ... ε_2, ... $\varepsilon_{\varepsilon_0}$ and so on so forth, are not just symbols. They are not in the physical world either, as they have no meaning in it, nor they are a mere convention: *they synthesize a principle of construction, generalized iteration and limit; they are the result of an historical praxis, which turned them into a perfectly stable conceptual construction.* This conceptual construction is part of the genesis of our mathematical concept of infinity, as by this I intend both the route and the result of a mental and historical activity. It actually lead to a mathematical structure (see §.4).

2.4 The continuum

In each key mathematical structure, pre-conceptual experiences are at the core of the conceptual construction, which is further specified by mathematical, "structured" invariants. A further crucial example is the "continuum". Since always its specification and use has been at the core of mathematics (see [Longo, 1999] for more discussions and references to part of the enormous literature on the topic).

We all "appreciate" the pre-conceptual continuum of gestures or movement in space and of the ongoing flux of phenomenal time. This is far from being uniquely determined, yet it is a robust experience on which further conceptual constructions are grounded. That is, we are able to *give meaning* to the subsequent concept by reference to the pre-conceptual experiences and actions. Or, the linguistic or geometric concept is meaningful for us, as rooted in our acts of life. These, on their hand, are better understood by the intersubjective specification of the concepts, by language and, then, writing, through history.

For a brief comparison of the continuum of space and that of analysis, consider now some of Euclid's definitions and "axioms". First ... what is a point? «A point is that which has no parts», definition 1. While «a line is a breathless length», definition 2. Observe now that Euclid's lines are not made out of points: «the *extremities* of lines are points», definition 3. By this, Euclid excludes the concept of "open" interval, which would miss exactly one point on each extremity. Thus, lines are parmenidean unities, flux, "compact, unidimensional" length («breathless length»); a segment is a "rigid body" (like all figures of greek geometry), ending, if finite, by points at extremities [Fowler, 1999].

This is why, at their extremities or when they cross, unidimensional lines give rise to a point, as this has no parts, i.e. no "cartesian dimension", in our terms (a point lies at the extremities of lines or is the *result* of two unidimensional lines which meet with no "local" parallelism). No more than this is needed for the continuum in the geometric constructions of Euclid's. In particular, and against the formalist fake reconstruction of history, the first theorem of the first book is perfectly proved, within his *mathematical* concept of *geometric* continuum: given a (finite) straight line, one can construct an equilateral triangle on top of it (proof: just construct the intersection point of the circles centred on the extremities of the line, and draw two straight lines from that point to each point of the extremities - this is an application of axiom 1). As said above, the existence of the intersection point is a *consequence* of Euclid's conception of the geometric continuum and his definition of point and line (see [Longo, 1997; 1999b] for a cognitive analysis of the concept of mathematical line).

The modern use of the continuum in analysis, which uses infinitary limits processes, has been instead consistently found on Cantor-Dedekind "pointwise" construction. This is a possible and beautiful construction of the continuum, grounded on integer numbers (not a necessary one as other approaches are possible, e.g. non-standard constructions or the approach by Veronese, see the paper by R. Peiffer-Reuter in [Salanskis and Sinaceur, 1992]). It is an historical abuse to assume this perspective in order to understand also

Euclid's geometry. Yet, this has been largely done during this century, by the prevailing formalist approach, which forced us to read back history in a biased fashion. Heath in his fundamental analysis of Euclid's books [Heath, 1908] opened the way, but also Leibniz, apparently, had raised the issue of a presumed flaw in Euclid's proof: one has to assume that the intersection point *exists* or *derive it* from a suitable construction (typically, one has to use the metaphysics of Leibniz's monads or, for Heath and followers, Cantor-Dedekind's real numbers!). A surprising way to impose backwards an analytic construction on top of Euclid's geometry of finite figures which does not require it, a geometry of structured lines as flux or parmenidean unities, of compact rigid bodies.

In other words, Euclid had in mind a different concept of continuum, different from the one specified by Cantor-Dedekind pointwise construction and specified it in different structures: a geometry of rigid figures and continuous and straight lines, like light rays, in his understanding (see [Heath, 1908]).

Clearly, also the analysis of time-variation or movement, by Newton and Leibniz, was based on the concept of "flux", yet understood in terms of limits or monads; the latter a remarkable concept, preliminary to the (non-)standard subsequent constructions and structures. Cantor and Dedekind proposed a precise, yet specific, mathematical structuring of this sort of analytic continuum, grounded on and used for limit operations, as a number-theoretic construction: consider the integer numbers, take the quotients (or rational numbers), collect all converging sequences (modulo identity of limits, defined à la Cauchy, say), call them real numbers. Then you have a totally ordered line, with no jumps nor lacunae: movement and its derivatives (i.e. speed and acceleration) are very well described or parametrized over this line. However, it is an «act of violence», as Weyl says, to assume the prefect coincidence of the analytic construction of the continuum with that of phenomenal space and time «... that is, the continuity given to us immediately by intuition (in the flow of time and in motion) has yet to be grasped mathematically» ([Weyl,1918]; see [Longo, 1999] for some reflections inspired by that classic).

It should be clear that there is no need of a transfinite ontology to use or to acknowledge the objectivity of any given specific construction of a mathematical continuum. Exactly as for the integer number line or for the hierarchy of infinities, *the objectivity of Cantor-Dedekind continuum, say, is in the construction itself. Its foundation is in the genesis of the conceptual invariant,* from the early cognitive grounds of the integer numbers to the mathematical construction of the rationals and, then, of the reals. No medieval ontology no "actual existence" of objects is involved in mathematics, but "just" the objectivity of mathematical constructions. The description of this objectivity is a difficult scientific challenge, by the need of a unified analysis of a large variety of often conceptually independent constructions.

The a posteriori, formal or logical investigation may help to distil some key proof-principles which unify various aspects of the mathematical deduction. Unfortunately these principles are incomplete even w.r.t. the simplest of these structures, the integer number line, let alone our geometric description of space. This is the result of (recent) incompleteness results, which evidentiate a "gap" between formal proof principles and various mathematical constructions, geometric ones for example (see [Longo, 1999 and 1999a]).

2.5 - Proofs

The notion of "mathematical proof" has surely been evolving in history, yet "its" core structure is very old and based on a strong appreciation of invariance. Consider Pythagora's theorem. Egyptians had a long list of numbers in the ternary relation of the sides of right triangles: $3^2 + 4^2 = 5^2$ etc. ... . Mathematics really begins when the Greek geometer makes a drawing of a right triangle on the sand, of a *specific* triangle, with inevitably some *given* length of the sides (or ratio of lengths). He then gives a proof of the theorem by projecting and comparing the squares on the sides and concludes: "observe that my proof *depends only* on the fact that this angle is right, not on the other properties of the triangle, such as the length I had to drew". The proof is *invariant* w.r.t. (the ratio of) lengths of the sides (which were drawn and, thus, specified in the construction!): this is the key remark that any mathematician has to do at the end of a proof. He has to single out what it depends on, as well as its "independence"; that is, the proof's exact level of generality. An "argument" turns a (universally quantified) statement into a *theorem* when its generality can be explicitly spelled out, or when one can single out w.r.t. to which assumptions it is an invariant. Sometimes a very hard task. Axioms and logical rules may help in this, but they are far from being sufficient. The point is that this key proof-theoretic invariance is more a *practical* invariant than a just a formal invariant; i.e. it is constructed in the conceptual, scientific praxis of mathematics, as part of human reasoning. In other words, this appreciation and specification of invariance is deeply rooted in the forms of invariance that are at the core of our general search for stability and generality in arguments (and in life: memory is the early cognitive ground for the invariance and the normative aspects of mathematics, see [Longo, 1999b]).

The structures of mathematics brings conceptual invariance at its highest human level, by providing, often a posteriori, a re-construction of it as part of the proof, a proof which may well be given, as frequently in geometry, on a specific case (the drawing may be needed and it is always specific), post-factum generalized by an analysis of the proof itself. The use of (universally quantified) variables, in classical formal systems, sets on rigorous basis this analysis, in the algebraic settings, but it is not perfectly adequate in all mathematical frames. Yet, other systems of logic may give different or further levels of information, as it will be hinted in § 4.4 in a comparison of classical and intuitionistic systems.

Methodological Intermezzo: possible stories Vs absolute objectivity

I am aware that, by this approach, I am violating a dogma of the prevailing foundational analysis in mathematics, well established since Frege's foundation of Mathematical Logic: I am referring to "history" and "psychology" (cognition, to be precise, as shared experience, through evolution and history, not just individual experience, as spelled out next). And I am not grounding mathematical knowledge into "universal (logical) laws", nor "absolute objectivity". Indeed, I believe, following the late Husserl, that there is no foundation without epistemological analysis, i.e. without an analysis of the "knowledge process". And that there is no epistemology without a *genetic analysis,* as analysis of a genesis, through history, provided that the meaning of history is meant in a broad sense [Husserl, 1936]. This sense is given both as a (possibly pre-human and) human cognitive experience and as an ongoing intersubjective process, along generations.

In short, mathematical invariants sit on top of or are derived constructions from conceptual ones; these are the result of shared and manifolded experiences, "*acts* of experience" to be precise, as independent and meaningful extensions of our active presence in the world. Intersubjectivity grounds them in this plurality made possible by our communicating community of human beings, through history: we gain the human and historical level of conceptual invariance, by "specifying concepts" toghether, while comparing with the others' experience and, by this, by singling out the stable fragments. Mathematics must then be understood as an extension of a human praxis, the most objective of these communicable extensions, as the one with the highest degree of independence and invariance; it is the tip of the iceberg of human communication, the crystal clear tip, where ambiguities and contextual dependence are excluded as much as possible or as in no other form knowledge.

Of course, we can today work at this bold enterprise and resume the early intuitions on the "cognitive foundation" (in my terms) of mathematics, by Riemann, Helmotz, Mach, Poincare', Enriques and Weyl (see [Boi, 1995] for a broad survey and references), also because we are sitting on the shoulders of giants like Frege and Hilbert, the founders of Mathematical Logic. After the enormous growth of XIX century's mathematics, often with little rigor, and the crisis of Euclidean geometry, it was absolutely needed to evidentiate core logical principles and to provide a precise notion of formal rigor (and even of what it means to give a "good definition", which was far from clear at the time): the advantages of Frege and Hilbert's work have been enourmous for the practice of mathematics, and for their fall outs as well. The main one is the analysis of deduction as computation, which lead to a precise notion of algorithm and computable function and originated Computer Science, the discipline of formal/purely-logical symbol pushers, the computers. Today, though, we can interact with other disciplines, such as biology or cognitive sciences, which have undergone an amazing and recent growth. Thus a phenomenal/cognitive analysis is not anymore a matter of introspection only, as from Riemann to Weyl, but it is becoming a scientific analysis of our forms of knowledge and their interactions. This interdisciplinary investigation does not aim at providing "absolute objectivity" and "unshakeable certainties", the quest of the founding fathers of mathematical logic: it aims at a scientific analysis and, thus, it can only propose "plausible stories" of the world or of our cognitive relation to it, like physics when it tells us plausible stories of the universe or of quanta. An analysis whose developments or very principles may be falsified or improved, discussed and modified, or, here and there, locally, verified, as usual for scientific knowledge, i.e. for our proposal to understand the world, by structuring it. In either case, in mathematics as well as in the other forms of knowledge, the key point, in our view, is that there are no "objets" (mathematical, physical ...) which precede the constituting of objectivity: the foundational analysis, in the various scientific realms, has to investigate this constitutive process.

In summary, the foundational analysis of mathematics has two complementary aspects. There is a *necessary* investigation of mathematical proof-principles, that tries to set on formally rigorous grounds mathematical theories, in particular by clarifying the rules by which, in each context, one may give "good definitions", by proving relative consistency results etc. ... . This is the relevant job of Mathematical Logic, Proof Theory in particular. But it is *not sufficient*, as Mathematical Logic itself has shown by the many incompleteness and independence results, a remarkable technical achievement (see [Longo, 1999a] for a reflection based on recent "concrete" results). Then, in order to go further, one has to try to investigate the conceptual origin of the mathematical constructions and embed it in our human endeavour towards knowledge (see [Weyl, 1927] as for an early hint towards this distinction between necessary and sufficient conditions in the foundational analysis). These investigations are just "plausible stories" in the typical sense of scientific descriptions mentioned above. This century physics has taught to us how to give up the Newtonian search for "universal laws" and "absolute universes" and develop an analysis of constituting principles of knowledge. The foundational analysis of mathematics has a lot to learn from the methods and epistemology of physics, Quantum Physics, in particular.

3 - The Cognitive Subject: Conceptual constructions Vs Ontologies

But then, are we investigating by this an "independent reality", both in mathematics and in physics? Is this the reason why many observe that, for example, the "sequence of prime numbers has a more stable reality than material reality" ? (A quote from a leading mathematician; but I could also quote the "next door mathematician" in my department, or Gödel: "numbers are at least as real as this table ..."). In this "realistic" perspective, often considered a form of naive platonism, the mathematical objects are "already there", independently of human definitions and conceptual constructions; they are even more stable or permanent than material ones (see [Gödel, 1944; 1947]; for the prevailing interpretation of current "naive platonism" in mathematics, see [Benaceraff, Putnam, 1964; Introduction][2]).

I would agree that electrons or muons, let alone neural synapses, are far from "stable": they are our ongoing proposal to single out objects, in our attempt to organise "reality". A reality which *is* there, as it makes *friction* against our proposals to give *meaning* to it, a meaning which is in the organised interface between us, leaving and historical beings, and that "reality". Thus, electrons and muons as well are conceptual constructions, as theoretical unities, often very temporary ones, grounded on a few symptoms or sparks on a screen of a constructed measure instrument (thus conceived also on the grounds of a theory). In this, they resemble to a mathematical object (and their definition requires a lot of mathematics), except for the possible request to check their definition against a further friction on "reality", by more instruments. Not so differently, we *constructed* tables in (pre-)history at the same time as the concept of table: this individual table will be destroyed soon or late, and the very concept may become obsolete and forgotten, when tea cups will be held by antigravitational forces, in year 2435. As for synapses, not only our description changed over time, but these interconnections between neurones changed during evolution of species and keep evolving in phylogenesis or even ontogenesis.

Thus, there is no doubt, the concept of prime number is "more stable" than these fragments of reality or their related concepts. But, the concept of electron or muon, or even of synapses or of table, *if stabilized* by a mathematical definition or by a praxis which would only rely on the assumption of its perfect conceptual stability, then, this *concept*, may become as stable as the concept of prime number. Indeed, when using it just as formally defined, one would work in a branch of *mathematical* physics or biology (or "mathematical furniture theory", as for tables), such as rational mechanics, say, and not in (theoretical) physics or biology, the loci for (dynamical) conceptual descriptions. The point is that *mathematics is the paradigmatic language for this stabilizing praxis, while organising knowledge*. This is why its "constructed conceptual objects" are very stable, surely more than the (concept of) electron, synapses, table ... which follow the dynamic of (our knowledge of) physics, life or history, and are affected by their direct friction against the world.

And here comes the usual confusion proper to naive realism, in mathematics, a peculiar blend of idealism and empiricism: I call object this table, I appreciate its unity and independence from my "self", as it is invariant w.r.t. to perceiving it by seeing, touching, smelling ... . Enough sensorial invariants, in the common sense experience, usually

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[2] For a coherent and deep, not naive, platonistic understanding of the set-theoretic approach, and a parallel of Topos Theory to Aristotle and Leibniz views, see [Badiou, 1995]. Another well-informed and interesting update of the platonistic view point may be found in [Piazza, 2000].

guaranty that "there really is a table *there*", independently of me, the individual, psychological subject. This common sense remark is then transferred to mathematics: (the concept of) prime number is stable or invariant w.r.t. all reasonable "view points" and mathematical properties and theorems, it is then *there*, before and independently of any conceptual construction. Indeed, it is even more stable than "this shaky table" or uncertain electron, which can actually be broken or split into more fundamental pieces and particles and disappear, even from being considered a useful artefact or an existing unity or object. Of course! Mathematics works only at the conceptual level: it is the very language by which we try to structure the physical world, by singling out, naming and, possibly, formally defining objects; it has an autonomous development, which presents no friction, if not against itself or as interaction of its many structures. This language may be enriched also in reference to physics, by an indirect friction, as when proposing infinitesimal calculus to analize movement or building geometry on top of the notion of curbature of space, as in non-euclidean geometries[3]. But, once given, by definition, its concepts are as stable as Carol's "smile without the cat" or a frozen version of it, when the moving cat has gone: we work only on the smile, integrate, derivate, multiply it by itself ... in a non arbitrary fashion, as its methods of conceptual construction and logical deduction are grounded in our cognitive relation to the world.

In summary, everyday realism refers to tables, electrons and positrons, as "objects", which exist independently of our being humans, who live and act in the world, while trying to organize and understand it; this very realism is then transferred into a naive platonism that confuses the objectivity of mathematics with pre-existing objects. In either case, the relative independence, w.r.t. the *psychological or individual subject*, of tables, particles and ... numbers, is considered as an absolute independence. This is another component of the ontological myth, in either realism: the lack of distinction between the psychological subject, with its individual history and being (I have no doubt that tables, but also electrons and numbers "exist" independently of me and will survive my death), and the *cognitive subject*. The later is the result of a phylogenetic and historical formation of being, it underlies the very life of each psychological subject, it is shaped by the living and historical community. Tables, electrons and numbers are not independent of the cognitive subject, as they are constructed by its own and ongoing constituting in an active presence in the world, simoultaneously to it, in a concurrent-interactive process. The cognitive subject ramifies into the psychological ones, each a peculiar historical instance, whose most stable and invariant part contains the cognitive subject. But the cognitive subject is not independent of the

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[3] Both are dramatic changes in (or broadening of) paradigms. The first by the use of actual infinity to analyse *finite* movement, speed and acceleration. The second, by proposing a geometry of space, independently of the rigidity of bodies and the nature of light rays, in contrast to the Greek geometry of figures and rigid bodies (and straight light rays, to which Euclid explicitly refers as for defining straight lines, see [Heath, 1908]).

individual ones, does not transcend them, as it is the shared, live and historical experience. By living and interacting, the individual egos form themselves, while shaping their own community and as part of it. Thus, the cognitive subject is immanent, as, at the same time, it constitutes and it is shaped by the communicating community of individual subjects[4].

Within this philosophy, mathematics is analyzed here as the result of a cognitive praxis. A key aspect of the present approach is that it tries to relate mathematics to human cognition, both for the sake of mathematics and of cognitive sciences: mathematics is such a "conceptually simple" (even when technically difficult), deep and clear form of knowledge, that it may open the way to the analysis of more complex or involved forms of knowledge. Moreover, its unique generality, which transcends any specific form of knowledge, may help in singling out the "cognitive subject", as the shared part of our humanity, constructed in our phylogenetic and historical interaction with the world, at the various phenomenal levels: mathematics is grounded in core cognitive praxes, those which face the relevant regularities of the world and of life (or, at least, the regularities that matters for us, that we "see"). By the analysis of its foundation, mathematics may thus provide an early, simple and very general conceptual laboratory for human cognition.

4 - Structural invariants

Section 2 hints to those key *conceptual* invariants which are called "integer and real number", "potential and actual infinity", "mathematical proof". Many other mathematical notions and structures are derived from these: from effective computations to ... differential manifolds in geometry.

The idea here is that some "basic" *mathematical structures* are the specification of a conceptual invariant (the integer line, typically) while others are derived from operations on these structures. In a sense, say, there is a crucial difference between the *concept* of actual infinity and the *structure* of Cantor's ordinals or cardinals: the latter helped to specify the concept by operating on it, *within a specific construction*, yet actual infinity stands as a conceptual invariant (or as conceptual synthesis or integration of a variety of "acts of experience"). The concept of actual infinity, say, may be also specified by other mathematical structures: the point at infinity of projective geometry, for example.

Indeed, the order of infinities or projective points in geometry are mathematical structures. The first is "just" the indefinite extension of the successor operation and of

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[4] In the terminology of mathematical logic, I would call this ongoing biological and historical process-formation of the cognitive subject (or its description), as "impredicative": one cannot understand the elements without understanding the whole, which is in turn constituted by its elements (this kind of "circularities" are not vicious, but virtuous and are at the core of the dynamics of physics, life and cognition). The resulting (relatively) most invariant and stable properties, but not absolute, characterise the cognitive subject.

limits; the other is the rigorous geometric handling of a proposal to describe pictorial perspective (L. B. Alberti and Piero della Francesca, the Italian renaissance painters, were the main creators both of the mathematics and of the artistic technique). The concept and the structures interact very fruitfully, as the first gets a specification by the latter, while these derive their "meaning" and external foundation in the historical constituting of the concept of (actual) infinity.

Similarly, the real numbers, as Cantor-Dedekind mathematical specification of the continuum, form also a derived structure (from integer and rational numbers); by this, they also acquire conceptual stability, as the result of a "structural invariance" (w.r.t. certain transformations, the continuous ones, see below). Yet, the phenomenal continua, of space and time, are far from giving uniquely determined mathematical invariants, in contrast to the integer numbers. As a matter of fact, one may provide non isomorphic structures for them, for example the continuum of non-standard analysis. Thus a concept does not need to receive always from mathematics a unique specification (as it may be done in the specific case of the integer numbers[5]). Moreover, the continuum of time is far from being described in a satisfactory way by the (reversible, pointwise) line of real numbers: an ongoing debate since Weyl's 1918 book (see [Longo, 1999]).

The key point, though, is that *the meaning of the intended mathematical construction, the structural invariant, for us human beings, relies in the reference to the underlying conceptual invariant*; or, more precisely, on the "knowledge process" which leads to the mathematical structure. In other words, the understanding, indeed the *foundation*, of the mathematical characterisation(s) is in the underlying pre-conceptual, first, and, then, conceptual experience(s) (numbers, continua ...), as progressively but interactively built on the interface between us and the world (they are "multilayered drawings" on the phenomenal veil, in the terminology already used: the mathematical structure adds the last, most stable and precise "touch"). Intersubjective exchange, of course, contributes essentially to this constituting of meaning: the sharing of the pre-conceptual and conceptual experiences, by language, writing and ... by doing mathematics, stabilizes the concept, allows its very explicitation or allows to single it out from an unfocused praxis.

The invariance and stability of a well established mathematical construction or tool, though, may give it some sort of rigidity, with respect to the underlying concept. As a matter of fact, the greater plasticity of "concepts" may help in proposing novel technical interactions between mathematics and other disciplines.

When Leibniz and Newton tackled the analysis of movement by limits and infinities, they did not use pre-existing mathematical tools and structures: they used the *concept* of

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5 Standard and non standard models of the continuum are non-isomorphic, globally and locally. All models of Arithmetic, instead, have an initial segment which is order-isomorphic (is "identical") to the standard sequence of integer numbers.

actual infinity and, by this, they opened the way to its modern mathematical specification. When interacting with biologists, say, we need to learn from them their methodological tools, not just the technical ones. We should understand the way they handle "concepts", not just learn facts or data and deal with them with already given mathematical tools, mostly inherited from mathematical physics. This may lead to entirely novel mathematical structures or to a change in the discipline as dramatic as the one which followed the birth of infinitesimal calculus.

But, once certain mathematical structures are well specified, how further structural invariants are generated, in general?

Once we have "structures", we may work with the properties which are preserved under various sorts of transformations within or over the intended structure. *Structural invariance* may then be understood as an "internal" notion. It motivates the conceptual construction and, then, it is given *inside* specific and well-defined mathematical structures, within "categories" as pointed out next. And from categories one may obtain derived constructions, as we shall see, and invent new concepts.

In summary, mathematical or structural invariants specify conceptual invariants or are derived constructions from previously specified mathematical invariants. The specification does not need to be unique or "canonical", as already mentioned: the integer numbers are uniquely determined in existing constructions while the various standard and non-standard continua differently specify the concept of continuum or continuous variation. Yet, the mathematical experience helps to determine the pre-existing concept or may propose further ones. In all cases, I claim, the reference to the conceptual underlying invariants steps in the mathematical construction and in *proofs* (see below). By this, the so called "intuition" is permanently enriched by the mathematical work and it is far from stable: any pre-existing concept is affected by the mathematical structure, which specifies it and, by active work and experience, will contribute to a further conceptual construction. This mathematical "intuition or insight" ("... [mathematics] as knowledge or *insight*... whose organ is "seeing" in the widest sense... " [Weyl, 1927]) is *the reference to meaning, as depending on underlying concepts or pre-given structures*; it is "seing" a structure as meaningful and alive, as specification of a meaningful concept. As for continua or the well-ordered sequence of numbers and even more so for all geometric constructions, these force themselves as "visions", since phenomenal space and time are the locus of the pre-conceptual constructions; we *see* them in the reconstructions of our dynamic memory, [Longo, 1999b].

The mathematician's constructed intuition and insight are the reference to a network of meaningful conceptual constructions, often derived from previously obtained concepts: the key ones are in turn grounded on pre-conceptual experiences, largely embedded in phenomenal space and time ([Longo, 1999b] discusses on the role of memory in this process).

4.1 Categories

Category Theory may greatly help to understand this essential aspect of mathematics, the construction of structures. Categories are defined as *structures* (or **objects**) and transformations that *preserve the structures* (or **morphisms**). Thus, each category has its own structural invariants. According to the given category, these may be topological invariants, say, such as the properties that are preserved by continuity, or geometric or algebraic invariants, which are preserved by the intended transformations (the metric or algebraic morphisms). Being a closed line, for example, is preserved by continuous transformations, in the sense of topology, i.e. in the category of topological spaces and continuous morphisms. One may make a more restrictive assumption and call invariants only those properties that are preserved not just by morphisms but by "isomorphisms" or "automorphisms" (transformations that preserve faithfully and fully the structure, and are within the intended object in the case of automorphisms). Distance, for example, is preserved by isometries, which are isomorphic embeddings of metric spaces.

Moreover, categories are inter-related by higher-order maps, called **functors**; functors transform objects into objects and morphisms into morphisms, in a consistent way. Then, functors have their own invariants, which are categorical constructions, i.e. entire collections of structures. Once more, one may just look at full and faithful functors and consider a more restricted class of invariants (or, even, at those functors which are isomorphisms in the category whose objects are categories and morphisms are functors).

This is not the end, as Category Theory was invented to describe (some "**natural**") **transformations** between functors. Then, these transformations, possibly as isomorphisms, preserve invariants at the level of categories, whose objects are functors.

Category Theory is full of examples of invariants constructed at these different orders. Of course, many (most) belong to a previous formation of sense, just specified and unified by the categorical approach. But new ones are derived by further mathematical constructions: duality, adjunction, or, more specifically, Sheaves, as categories of functors, Toposes etc. Toposes, in particular, provide relativized approaches to logic: the invariant properties, w.r.t. the intended transformations in a given topos, form a logical system; this system may be classical, intuitionistic or other, according to its peculiar way of "classifying" objects, [Johnstone, 1977].

One final remark: the role of structural unities in Category Theory reminds the search for unity and interconnections in the various approaches indebted to "Gestalt" (see [Rosenthal and Visetti, 2000]), in contrast to the unstructured approach typical of Set Theory.

4.2 Invariants in the Category of Sets

The category of sets (Set) is a particular category, with all functions as morphisms. The notion of **cardinality** (number of elements) is the invariant w.r.t. to isomorphisms in Set (the bijections). These are "generalized numbers", integer ones but also infinite ones: the latter may be characterized as the cardinalities of sets which are isomorphic to a proper subset (a remarkable mathematical specification of the concept of infinity!).

By this, we transformed into structural invariants much earlier conceptual ones and, at once, we extended the notion beyond finiteness (cardinalities as invariants w.r.t. bijections). However, what has been achieved it is not a *foundation* of the human notion of number, but a relevant *mathematical characterisation* of a concept and, most of all, by defining infinite cardinals by bijections with proper subsets, the way is opened to setting on formal grounds a generalization, towards larger and larger cardinalities. Yet, and exactly for describing infinities, the issue of consistency of the intended formal set-theoretic frames turns out to be crucial: formal theories need to be proved consistent, first. That is, the analysis of consistency becomes a necessary part of these characterization of infinity. While informative and mathematically deep, this analysis is essentially "non-well-founded", as hinted in the introduction and, by this, it is essentially insufficient or incomplete w. r. to its own "foundational" purposes (larger and larger infinities are required to prove relative consistency results). The analysis hinted here of the "cognitive-constitutive path" should complement it.

As already mentioned, one may also "experience" infinities in different conceptual contexts, as limits and iterations of limits (ordinals) or as geometric (projective) limits. These yield further and relevant *mathematical characterizations* of the *concept* of actual infinity, each presented in a different structural context or *category*. Functors relate them and unify the mathematical descriptions.

4.3 More invariants

Some structural invariants may also help to define new abstract concepts and by this they sit on the bordering line with conceptual invariants, as in the case of categorical constructions. Yet another familiar example is given by the Recursive Functions, as the collection of number-theoretic functions that are computed by any of the systems for computations, so far formalized in Mathematical Logic (Herbrand-Gödel recursion, Turing Machines, Church's lambda-calculus, Kleene's equations ...). They are defined as the *extensional notion of function* that is as the invariant (class of functions) w.r.t. the various formalisms for computing.

Similarly, the three different geometries (Euclid, Bolyai-Lobacewskij, Riemann) may be formally described as the invariant properties w.r.t. different groups of transformations. Either construction may be carried on in suitable categories.

4.4 Proofs as invariants

One of the examples above of conceptual invariant concerned *proofs* as invariants (the proof of Pythagora's theorem, §. 2.5). Again here, Proof-Theory, indeed Hilbert's metamathematics, looked upon proofs as objects of study, and, by this, it transformed proofs into mathematical "entities" (I do not dare to say structures, because, as I will say, this is exactly what classical formalism is still missing). Hilbert's proposal was a remarkable step, yet limited by the attempt to analyse proofs only as formal-linguistic (algebraic) entities, as sequences of symbols, independent of meaning and handled "mechanically". By this, for example, the classical (Tarski's) description and interpretation of the universal quantifier, "for all x...", as "for all instances ..." is inadequate to grasp the nature of the generality of "for all right triangle ...", say, in the proof of a theorem of geometry as in §. 2.5. That treatment of universal quantification is eminently linguistic/algebraic: in algebra (often in analysis) one proves a result for an arbitrary element of the intended structure, by the explicit formal/algebraic manipulation of a variable, as a "generic" instance. The universally quantified variable stands for an arbitrary element and there is no need to use any of its specific properties during the proof and, then, prove that the proof is invariant w.r.t. them (like one has to do, instead, in Pithagora's proof, with the specific length of the sides, or their ratio, which must be drawn).

Of course, similarly as in that theorem, also in algebra or in analysis, one has to check that no other properties of the intended variable are used, i.e. that only its "type" is used in the proof, but this is easy then, as no explicit use of its individual properties has, in general, been made. When there is no (possible) reduction to an algebraic treatment, the geometric interpretation of "For all ..." is qualitatively different from the linguistic/algebraic one, which guided Frege's logical approach, Hilbert's formalisms and their tarskian semantics. With reference to our example (Pythagora's theorem), one cannot formally manipulate a variable, subsequently interpreted (instantiated) by any, generic, right triangle: there is no such a proof.

A better insight is given by the intuitionistic and the categorical (Lawvere-Grothendiek) description/interpretations of quantification. These approaches depart from Tarski's and are much more insightful, yet still different from the one hinted here for geometry

In short, in Intuitionistic Theories, both implications and universal quantifications are understood as *functions* or, more precisely, as *effective transformations*, i.e. computations with a well defined structure of function, see [Troelstra, 1973; vanDalen&Troelstra, 1988]. In a perfect correspondence to intuitionistic systems (by the extended "Curry-Howard isomorphism": "Types - as - Propositions - as - Objects of Categories") Category Theory interprets proofs as morphisms: the nature of proofs as invariants is then understood by the morphisms which structure the intended categories. By this, universal quantification becomes a (fibred) product, where fibration is a deep way to understand "variations", or as indexed product (in the second order case). This interpretation departs from Tarski's and it is much more insightful, yet still different from the one hinted here for geometry (see the Lawvere-Grothendiek interpretation of quantification [Johnstone, 1977; Lambek&Scott, 1989] and, for the second order case, [Hyland and Pitts, 1987; Asperti&Longo,1991; Longo&Moggi, 1991].)

4.4.1 Proofs as Prototypes

Following then the intuitionistic approach , let's try to introduce some basic ideas for a type-theoretic analysis of "universal" proofs, which stresses the role of "invariance". Given a mathematical theory which allows universal quantification, i.e. sentences such as $\forall x.P(x)$, how does one prove these kind of propositions? If the range of quantification is infinite, even uncountable, there is no way to explore and check each individual case. For the analysis of proofs, the understanding of $\forall x.P(x)$ as "for all x ..." (the tarskian semantics) has little interest, or is even misleading (for the second order, impredicative case, in particular, see [Carnap, 1931], [Longo, 2000]). Indeed, the practice of mathematics is usually the following: check what is the "intended range of quantification", i. e. over which set, domain or structure the universally quantified x is meant to be interpreted (the set of real numbers, for example, or any other countable or uncountable structure). Then prove P(a), where a is an arbitrary or **generic** element of that domain ("take a to be a real", typically), that is, give a proof of P(a) where the only property of a, used in the proof, is that a belongs to the intended domain. In Type Theory we would say: only the *type* of a is used in the proof. By this, the proof of P(a) is a **prototype** or paradigm or pattern for the proof of P(b), for any other b in that domain ([Longo, 2000] presents a detailed type-theoretic approach, for second order theories). Thus, one has a proof of $\forall x.P(x)$, i.e. a proof that P(x) holds everywhere in the intended domain of interpretation of x. In Category Theory, a prototype proof should be understood as a fiber in a fibred product (or a morphism, in an indexed product, as for the second order case): an issue of further investigation.

Consider now the special case when the (intended) domain of interpretation is the set of natural numbers or any countable well-ordering. How does this technique relate to induction? The implicit "regularity" of a prototype proof - all proofs are given by the same reasons relatively to a given formal frame - may be not present in induction. Once proved P(0), which may be "ad hoc", one proves, in an inductive argument, the implication $P(n) \Rightarrow P(n+1)$, *for all n*. This is the proof that, *soon* (in general, at the first level of the application of the rule) *or late* (in case of nested induction), *must be* prototype in a generic n. Note also that this proof is based only on the assumption of P(n) not on its proof (as for derivable vs. admissible rules in logic) So, formally, P(n) and P(n+1) may be true for different raisons or their individual proofs may follow different patterns and yet, the proof of $P(n) \Rightarrow P(n+1)$ may be prototype (in case of nested induction, this will show up after a finite number of nesting).

In summary, intuitionistic systems of Types and Category Theory allow a finer analysis of (universal) quantification in proofs, by giving a mathematical structure to the conceptual underlying invariant. More work should be done in order to understand universal quantification in proofs of geometry, as we lack a proof theory of this discipline.

4.4.2 The geometry of Proofs

Going back to Geometry, a *dual* link may be established. G. Gentzen first, in the '30, but more broadly J.Y. Girard, in these days, gave to proofs also a geometric structure. Some rules, for Girard, are there just because of symmetries or dualities; deductions are carried on the ground of nets, as spatial connections of formulae by lines, designed along the derivations; nodes of these lines are inspected, as three dimensional properties, in order to go to the next "deductive" step. This may be unrelated to the logical meaning of formulae; in a sense, Girard's Geometry of Interactions let geometry come back "through the window" into proofs. Proofs are investigated in a structured space, a novel mathematical construction, given on top of deductions, well beyond the formalist perspective and a turning point for Proof Theory. Once more, mathematics (geometry) is giving some structure to the independent "game of symbols" by which the formalists planned to found mathematics, independently of concepts and disregarding space (and time).

Conclusion: a two ways foundational relation.

It should be clear that the interaction between conceptual and structural invariants is a two ways interaction. First, we may turn into mathematical or structural invariants some conceptual ones. Moreover, purely formal treatments may be proposed for the mathematical structures: symbols are then detached from meaning and manipulated according to mechanical rules. A further step, not a starting point. Both these "clarifications" are a crucial part of the mathematical activity, in particular as they may provide the technical tools for considering (structural) specifications and (formal) generalizations of the very notion, in various mathematical settings. The first, at least, is a necessary step, in order to set on rigorous grounds some general, often ambiguous, practical and conceptual experiences (e.g. the continuum, the infinite); but also full (axiomatic) formalization may contribute to this.

Yet, this activity is insufficient to "found", in the epistemological sense we mean here, the originating or underlying concept, which sets the "meaning" of the mathematical construction. As already said here, *this meaning or sense is in the cognitive and historical path that lead us to the abstract notion, as a mathematical structure*.

Thus, on one hand, a mathematical structure, with its properties, possibly in its complete formalization, may be considered to "found" a preconceptual or conceptual experience, only in the limited sense that it specifies it (and possibly gives an account of its formal derivability from least axiomatic frames). On the other hand, the foundation of the mathematical construction (and of its formalization, whenever made or possible) lies on its "meaning", as reference to a pre-existing or ongoing formation of sense. This underlying meaning may be possibly "pulled back", up to the level of axioms, in case of an axiomatic treatment, but even then it is grounded in our practices of life and conceptual constructions which give sense to the axiomatic choice.

Clearly, structural or formal invariants may in turn suggest new concepts, by operations or by conceptual contaminations: further infinite cardinalities, in Sets, or the many "universal notions" or the derived constructions obtained by functors, duality or adjunctions in Category Theory. Moreover, the practice of mathematics may change the very "intuition", and, thus, the nature and extent of the underlying concepts: the search for invariance and conceptual stability of (pre-)conceptual experiences is at the core of the mathematical work.

Category Theory is very effective in setting the "right level" of invariance which is being used: (iso-)morphisms, functors, natural transformations ... w.r.t. the intended categories. This is the relevant foundational role it plays, which accommodates the novel constructions of Mathematics, as a growing, open-ended human activity. New categories may be proposed and the unity of mathematical knowledge is recomposed by relating them by functors and transformations, at the right (intended) level of invariance. This part of the foundational analysis brings in the sense of "relativized", yet not "relativistic", constructions so typical of modern science, physics in particular. As in modern physics, Category Theory does not search for "absolute and ultimate" constructions: proposals are made of new categories (similarly as for theories, in physics), in order to make intelligible other parts of mathematics (of the physical universe), or to single out brand new constructions (objects of investigation); unity is brought back by looking for links and mutual explanations, as functors and natural transformations. This practice is enriching and it departs from the *Newtonian absolute* of Set Theory, with its fixed universe of reference, a bunch of axioms, sort of "universal laws of thought", into which all possible mathematical constructions (present and future ones!) already exist or can be embedded or derived.

The interaction with other disciplines may require a further digging into the "knowledge process" which leads to the mathematical invariants. We may need to go back to conceptual experiences and specify them into novel mathematical constructions. This was done in the fruitful relation between mathematics and physics; it may require a further change in our mathematical tools when interacting with biology. Or, perhaps, we must be ready to modify even the perfect stability of the mathematical structures and allow the "ambiguities" and individualization (or context dependence) which are so strong in biological phenomena: dynamics may need to get not only into the structures and theorems of the mathematics of "dynamical systems", typically, but into the underlying concepts as well, following the distinction proposed here between "concepts" and "structures". Probably then, mathematics would become a new discipline or at least a broader understanding of it may be required, grounded on new conceptual experiences. A change comparable to the one which lead us from the Greek geometry of figures and the algebra of Arabs to infinitesimal analysis and to the geometry of spaces as riemannian manifolds. In these cases, the phenomenal description radically changed, beginning with the very concepts of space and time involved; the intelligibility of movement, first, then of space and time themselves was enriched by (and lead to) entirely novel mathematical tools. But this is how the growth of (mathematical) knowledge goes: surely not by derivations from fixed sets of axioms, but as a mutually enriching interplay between us and the world, on continually changing and restructured phenomenal plans.

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