Space, the foundations of mathematics and the resistible rise of metaphor: the brain is a digital computer[1]

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Motivations and themes

In this short text no empirical or direct comparisons will be made between the "brain" and "digital computers", or it will be done a very little, but it will reflect above all on the fundamental issues of mathematics and knowledge. There are many reasons for doing this. In the first place, computer science and artificial intelligence, in their traditional or "strong" form, come directly from the paradigms of Mathematical Logic, as specified from the end of the nineteenth century onwards, in as an analysis of the foundations of mathematics and of the ways related to it to understand human reason. Furthermore, the philosophy of knowledge is closely related to that of mathematics, since the earliest reflections of the Greeks, and, with great continuity, also computers, mathematical machines were placed at the centre of cognitive analysis in the twentieth century. Finally, and in general, the comparison between scientific paradigms, proposals for knowledge or, even, for the construction of machines, should, if possible, always refer to, and make explicit, underlying epistemological projects: only in this way is a dialogue possible. Starting then from the crisis of the Euclidean intelligibility of space, we will mention the "logico-linguistic turning point ", to formalism and its happy marriage with mechanicism, in the 30's, to get to the point of a proposal, an ongoing scientific project, which returns to enrich the language and logic of the problems of our relationship to space, physical and the living. We should then understand how the abandonment, for over a hundred years and for good reasons, of this component of our forms of knowledge, has contributed to impair the analyses and has allowed the growth of an incomplete, completely resistible vision.

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1. Spaces of rationality and curved rays

Already in the third decade of the nineteenth century, the director of the Gottingen Astronomical Observatory, Carl Friederich Gauss was convinced of the possibility of a geometry different from that of Euclid. To be exact, Gauss thought that an "intrinsic geometry of surfaces " should be able to be developed, regardless of any immersion in Euclidean ambient spaces. A geometry of curved surfaces or rather with "non-zero curvature" (to use a term pointed out later). Perhaps, but only an historian should confirm it, there was forming a transgressive cultural sensitivity that characterises romanticism, especially in the way of "seeing" space: think of the troubled skies of the painters Turner and Constable, the tormented and swirling storms on the horizon, which break the Cartesian (and Euclidean) regularities of Renaissance and neoclassical painting, the perfect geometry of their spaces.

But then, if the universe could be curved, the sum of the internal angles of a triangle could be different from 180 ° ... Gauss climbed three German hills, all in sight of one another, to measure these angles, therefore assuming a curvature of the light rays, as geodesics (minimum distance lines) of a space with non-zero curvature. It has a measurement remarkably close to 180 ° and only a few consequences; but we, today, well know how much these measurements, certainly not made between peaks of hills, but from rays from distant stars, hidden behind a solar eclipse will confirm a radical change in the geometry of physical space, one hundred years later.

Gauss, despite his soon achieved status as a great mathematician, did not dare draw the necessary conclusions from his reflection: the fact that the denial of Euclid's fifth axiom (on a surface, given a straight line and a point outside of it, one can trace one, and only one, parallel line to the line given) can be an interesting mathematically adventure (and not contradictory). Or perhaps, he observed it, but historians say, based on correspondence, he did not dare break the millennial myth of the Euclidean perfection of the world.

Indeed, for two thousand years, the Euclidean organisation of space has been an absolute of thought. Euclidean space coincides with sensible space, which, in turn, is exactly physical space.

Euclidean axiomatics is "a priori", it synthesises "the pure intuition of space itself", anticipating experience, as Kant will summarize. Not only that, but the rational rigor is geometric: the mathematic proof, logic, the reasoning itself is certain when it is conducted "more geometrico". A common thread links Greek thought to that of Descartes up to Kant: the cogito is geometric, or rather a sort of "I move in thought as I move in space" is at the heart of the reflection of those three great moments of human thought; Euclidean organization first, then that of Cartesian "coordinates" in Euclidean Newtonian spaces, dominate rationality. That a little more than the thirty year old director of the Gottingen observatory could oppose all this, if not some unreliable measurements and his mathematical genius?

2. The non-Euclidean break

However, the romantic transgression continues: Lobachevskij and Bolyai developed a geometry entirely based on one of the two possible negations of Euclid's fifth axiom (... yes, they can draw many - therefore an infinity - of lines parallel to the line given). Even they do not play a mere axiomatic game, like the formalist caricature would have us believe, but they make explicit, it seems without knowing it, the Gaussian analysis of curved spaces and its connection to the Euclidean axiomatics. That is, they do not play with the negation of formal axioms either, but they propose a reorganization of space (see [Lobachevskij, 1855]). So much so that they do not ask the problem of the "coherence" (not contradictory) of the proposed axiomatic: who thinks of axioms as sequences of symbols to be manipulated regardless of their meaning (spatial in this case) has, as the only reference of certainty, a coherence test. Instead the mathematicians of that turn of the XIX century are reinventing the relationship between us and physical space: coherence, if it arises, is in the possibility of an interpretation of the axiomatic variant in sensible space.

The operation, as Gauss had guessed, was shocking. In so doing, the pillars of those certainties fell, those pillars that had supported rationality for millennia, that thinking self being in Euclidean space (and time). We can hardly imagine the disturbance in the very restricted world of mathematicians and philosophers aware of the turning point that was proposed.

But something even stronger happens with Riemann. Partly a pupil of Gauss (he wrote his thesis under Gauss's direction, in 1854, in Gottingen), he developed a general theory of curved spaces, based on the notion of "variety" (Riemannian). Through him differential geometry, initiated by Gauss, reached maturity: the geometric analysis of space enriched the tools of differential calculus. It clearly distinguishes between "local" and "global" analysis of space: at a local level, "micro" to say, space can be considered Euclidean (at an "infinitesimal" level, very close to the tangent plane, the Pythagorean theorem: dz2 = dx2 + dy2), but globally it can have a different structure (a different curvature, or variations of curvatures); or vice versa. Riemann analyses a positive curvature in particular spaces, where the sum of the internal angles of a triangle is more than 180 °. He thinks that this is physically interesting especially at the level of the infinitely small and manages to make an incredible conjecture, already in the qualification paper:

"But it seems that the empirical concepts on which the spatial definitions of the physical universe are based, the concept of rigid body and of a light ray, are no longer are in the infinitely small. Thus, it is permissible to think that physical relations in space in the infinitely small and do not correspond to the axiom of [euclidean] geometry. and, in fact, this should be allowed if this would lead to a simpler explanation of the phenomena.

... in a continuous manifold the metric relations must be introduced on different grounds (a linear element does not need to be represented as the square root of a second order differential form). ... Thus, either the real elements ... form a discrete manifold or the foundation of metric relations must be found elsewhere, in cohesive forces that act on it. "

And soon the insults start to arrive. Herr Dühring, a powerful academic philosopher well known to Engels, in a text, dated 1872, awarded as an essay on geometry by the philosophy faculty of Gottingen (!), writes:

«Thus, the late Gottingen mathematics professor, Riemann, who - with his lack of independence except for Gaussian self-mystification - was led astray even by Herbart's philosophistry .... It is not surprising that the somewhat unclearly philosophising physiological professor of physics, H. Helmholtz, commented on this absurdity in a favourable sense. "

The enthusiastic applause to Dühring suggests that perhaps Gauss was right to be cautious in spreading his ideas ... 40 years ago. Riemann, therefore, like his predecessors and great geometers, posed the problem of restructuring physical space, but with even greater scientific and philosophical clarity.

Larger, because he explicitly raised the problem of the non-arbitrariness of the construction geometric proposal. Under the influence of Herbart, he tried not to throw away the relationship between the knowing subject and the World: He investigated the regularities of space which, according to him, are underlying any geometric proposal. In short, he believed that geometry must remain significant, a privileged place for reasoning and knowledge.

However, he was aware that this place cannot be described by axiomatic choices *"valide a priori",* but is the result of A human proposal, rooted in some of the properties in the interface between us and the world: connectivity, for example, the *"continuità"* (the global topological structure) and the isotropy of space. Also, the geometric variant that states that the further we go from the Euclidean intuition, the generalization to many dimensions or topological or metric manifolds. (Rimannian, we say), refers to these "regularities" of phenomenal space.

3. The logicist response and the turning point "in language". Arithmetic.

Riemann's daring philosophical attempts (and others, including, later, Helmholtz) to save the geometric intuition, however, were too embryonic, too weak in the face of the enormity of the catastrophe of Euclidean certainty. A high-level response, not based on "reactionary" insults, but, in its turn original and profound, it soon prepared, built with completely different tools.

Frege arrived in Gottingen a few years after Riemann's death (who died at the age of 40, in 1866). Aware of the profoundly serious crisis in which the foundations of mathematics (and of knowledge) were, Kantian recognized the "intuitive" value of geometry [Frege, 1873], but precisely for this reason, contrary to Kant, he wanted to exclude his founding role (we see, among the many, [Tappenden, 1995], from which the quotations of Dühring and Frege, 1873 and 1874, come from reported here). In fact, the geometric generalization soon detached itself from intuition e became logical-conceptual:

> "The wildest visions of delirium ... remain so long as they remain intuitable, subject to the axioms of geometry. Conceptual thought alone can after a fashion shake off this joke, when it assumes, say, a space of four dimensions or positive cubature. To study such conceptions is not useless by any means; but it leaves the ground of intuition entirely behind [Frege, 1884; §. 14] "

The intuition can only be Euclidean, and it is lost in the modern generalization, which is only symbolic.
Frege does not speak of Riemann, perhaps he did not dare (he was the true successor of the princeps mathematicorum, Gauss), but gets angry with Herbart, Riemann's philosophical interlocutor, and with Stuart-Mill: once the Euclidean framework is broken, the reference to the intuitive properties of space does not justify, even less found, the delirium visions by which axioms can be interpreted geometrically. It must be recognized that, with great philosophical rigor, Frege certainly makes a clean sweep of "psychologism" and "physicalist empiricism" in fashion at the time: mathematical experience is not subjective nor empirical, but it is rather «objective because it follows a law, expressible in a few words », it is independent of our sensations and representations [Frege, 1884; §. 26]; proof of a theorem "does not depend on the level of phosphorus in the brain" [Frege, 1884; Introd.].
In conclusion, certainty is only in the concept of number, like size (magnitude); in number, again, as a "concept", expression, in language, of the law of thought:
"A concept as comprehensive and as abstract as the concept of magnitude cannot be intuition. There is accordingly a noteworthy difference between geometry and arithmetic. ... The elements of all geometric constructions are intuitions, and geometry refers to intuition as a source of its axioms. Since the object of arithmetic does not have an intuitive character, its fundamental propositions cannot stem from intuition either ... we do not find the concept of magnitude in intuition but create it ourselves. " [Frege, 1874; p. 56-57] ...
"The laws of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable but everything thinkable. " [Frege, 1884; §. 14] «... existence is analogous to number. Stating existence is nothing else, but denying the number 0. " [Frege, 1884; §. 53]
The concept of number is more fundamental and certain than the geometric ones. It even contributes to found geometry, which needs to talk about quantities and relationships.
"Every proposition which states a relation between lengths ... follows from the foundation of analytic geometry and can be derived analytically [Frege, 1873; p. 9-10] ".
«... Euclid, in order to define the identity of two ratios between length, makes use of the

concept of equimultitudes, and equimultitudes bring us back once again to numerical identity. ... the numbers that give the answer to the question "How many?" can answer among other things how many units are contained in a length. [Frege, 1884; §. 19] "

And this is a crucial point. Frege does not seem to want to leave the Euclidean framework: only in it the (numerical) ratios between lengths and angles are invariant and can "establish" geometry.

In fact, only Euclidean geometry contains homothetic (transformations that preserve relationships) between their automorphisms (Euclidean properties do not change by enlarging and shrinking the figures at will, but we will come back to this). In this sense it can be said that the "science of rigid figures", Greek geometry, is based on number, on "how many units are contained in a length ", crucial invariant of non-curved spaces.

Numeric invariants also exist in other geometries, but they depend on specification geometry (from its group of transformations) and are far from having a "founding" role.

In fact, with the Riemannian turning point and the works of Klein (1872) the perspective changes radically.

Geometry is based on the notion of "transformation" of space itself (or "deformation" of space): continuous, differentiable, or other. This inheritance will be collected from the Theory of Categories, a theory of algebraic-geometric structures, which (re-) constructs mathematics around the notions of morphism, functor, natural transformation ... in an approach “essentially geometrical" to mathematics and its foundations.

For Frege, on the other hand, it is the arithmetic laws that are laws of thought, also the foundation of geometry, like the laws of the relationships between lengths (and angles). Or rather, arithmetic is part (central) of logic, due to the "logical nature of arithmetical mode of inference".

The paradigm of this identification is arithmetic induction: logical law and principle of arithmetic proof.

4. **Hilbert and certainty in "mechanics".**

Frege revives with great originality Leibniz's idea of a "universal symbolic language of thought", by putting the calculus ratiocinator in a specific logical calculus (of the quantification). For the first time since Leibniz (and Boole), both universal ("for every...") and existential ("there is ...") quantifiers, key features to manage variables (in Maths), have been put under rigorous scrutiny by Frege, paving the way for the birth of Mathematical Logic. However, afterwards, Dedekind and Peano place induction at the core of mathematical reasoning, as a key statements-proving tool in Arithmetic. Being identifiable through logical principles, universalia of human rationality, reasoning can indeed be “signified”, because for Frege, both logical constructions and the concept of number represent an "ontology", something "existing" outside mankind and its psychological, empirical interaction with the world. However, Hilbert’s perspective at the turn of the century is a different one. Similarly to Frege, he gives language a central role (bear in mind that Wittgenstein, "the" philosopher of language analysis, at least in his first phase, explicitly considers himself as being indebted to him), highlighting the importance of its “forms”. Let me explain. Hilbert introduces the formalist analysis of fundamentals (in mathematics, even though frequently referring to "thought”, in general), or the study of the formal structure of language, but also of reasoning as deduction. Moreover, he takes into account the independence of axioms and formal deduction from meaning, both "spatial" and logic. Following his extensive experience in different fields of Mathematics that led him to solve problems of "existence" through the non-constructive proof, Hilbert puts forward this proposal. Basically, he proved that solutions of systems of differential equations exist, answering an important open question: the existence of a finite base point for the space of invariants and covariants of n-ary algebraic structures. In both cases, by "constructively providing" the function or algebraic base point in question, he had already achieved a proof of existence, indicating that it is absurd to assume non-existence. Nonetheless, many would claim that this is theology, not mathematics. As might be expected, Frege almost insults him in a letter, suggesting that he could prove the existence of God… However, this is not the meaning of Hilbert's revolutionary theory. For him, existence, in mathematics, does not refer to "entities" or ideas located somewhere in the empyrean, but to a proof of consistency of said theory. Therefore, he proposes a non-ontological and profoundly original definition of “mathematical existence”: is this a way to prove that a certain mathematical object exists? Well, then we must rigorously employ the same axiomatic system through which the proof of existence was conducted to test in any case and ad absurdum its consistency. So, this is existence in mathematics and nothing else (from here, Frege, not grasping the originality of Hilbert's idea, offers him a short axiomatic theory about God- omnipotent being etc ... – suggesting him to prove its consistency). Despite some misunderstandings in the early stages, the proposal is very popular. As might be expected, it clears away two millenniums of metaphysics (but where are these mathematical objects? triangles, perfect circles and ... functions of a complex variable, all of them housed, in secula seculorum, forever, in the mind of God ...). For the analysis to stand, we need to: - create a formal axiomatic system of all (the main) mathematical theories, starting obviously from Arithmetics, now the focus of attention, until we reach geometry (Hilbert, 1899), the root of so much turmoil; - check that meaning, referred to entities, ideas or intuition (for example space-time), has not in any way used these axioms mathematically or minimally interfered with deduction; - demonstrate the consistency of the theories in question (starting with Arithmetics). Certainty lies in the provable consistency and mechanics of deduction. Nevertheless, its main issues are constituted by references to meaning, commonly associated with metaphysics (mathematical entities), and, even worse, to intuition and to its ambiguities. Therefore, to be generalized, mathematical reasoning must be "as Kroneker says, independent of the existence of God; for Poincaré instead, free from special cases of intuition and anchored to arithmetic induction, but also from each "actual, contentual assumption". Therefore, Hilbert puts forward a “potential mechanisation" of deduction. For example, if a mathematician uses the rule: A A implies B ------------------- (the classical modus ponens) B its application should only depend on the "literal" structure of formulas. If the first A is identical (or syntactically unifiable, as it is nowadays called in automatic proving) to the second A and A and B are separated by an “arrow”, then write (formally deduce) B. This way, deduction is potentially mechanisable, devoid of meaning and finitary. In a nutshell, formulas can well refer to infinite objects like the set of all real and even integer numbers. However, this does not concern deduction, which operates on finite strings and abides by rules of finite strings. Hence, certainty is guaranteed by their finiteness and purely syntactic nature. Here we go, the thinking machine begins to fall into place. In the first decade of the 20th century, Poincaré, and from 1918 on, Hermann Weyl, the most famous of Hilbert's pupils, protest, but with no avail. As Weyl says, when referring to Husserl, mathematics is "contentual “(rich in content) and deduction can be signified, thus rooted in "facts of experience”. For this reason, Hilbert's formalist proposal is a winning one. Among the many on the list, proposed in Paris in 1900, his open problem about the need of a formal, finitist proof of arithmetic consistency is the focus of attention. As I said, his proposal suggests a "final solution" to the problem of foundations, thereby removing metaphysics and any need of intuition .... At the same time, it is robust and precise, and the conjecture has a clear mathematical nature. For this, mathematicians will take care of the problems of foundations, passed on by what Hilbert invented, namely the metamathematics, which involves a mathematical study of mathematical demonstration. This way, a finitary “mathematical proof “of arithmetic consistency can easily be obtained by manipulating only finite strings of symbols (and, of main mathematical theories, once axiomatized). As long as the theory that talks about it (the formal set theory, as it will be called in the future) is finitary or defined by finitely generated axioms, as well as by finite rules, mathematicians will be able to smoothly work with "ideal objects" like complex numbers or infinite entities. As a result, infinity is indeed necessary. Infinitesimal calculus, for example, uses it even in the analysis of the finite, where both velocity and acceleration of the bodies around us require limits or better the actual infinity (derivatives and integrals). The infinitesimals, for Leibniz, metaphysical monads and for Cantor, an infinite with theological significance, are technically necessary and to be justified or formally founded, they do not need neither additional metaphysics nor theology. Therefore, it is sufficient to use finitary or potentially mechanizable calculus to prove the consistency of their rigorous formalization and independence from meaning. I totally disagree with this project not for metaphysical but for scientific reasons (the role of meaning, which I will mention later). However, we must acknowledge the intensity and extreme boldness of Hilbert's first program, without doubt, a great project for mathematics and knowledge. A few years later, instead, he goes beyond, by adding a positivist "drift" to this amazingly original proposal, rich of finitist epistemology and of considerable interest. Without even realizing it, he resumes Laplace's completeness property for differential equation systems as opposed to the physical world. The future is perfectly decidable, save for approximations that "retain, overtime, the same order of magnitude." Unless unavoidable physical measurement errors occur in this context, Laplace considers mathematical (physical) assertions decidable. As both Hilbert and Laplace say, "ignorabimus"(we do not know) does not exist. Obviously, Hilbert applies the program to his formal theories, above all to Arithmetics. Axioms allow you to consider each syntactically well-formed assertion true or false. Oftentimes, we speak in a unified way about "Hilbert's program", combining two moments of thought of the great mathematician, probably because Gödel's incompleteness theorems could make a clean sweep of both in one go. In my opinion, however, the two proposals have a very different philosophical and epistemological "density": the first is revolutionary, the second merely a triviality that every positivist (or pre-post-positivist) scientist attributes to his favourite theory: the fact that it could say everything about the world. Above all, this thought evokes a closed, predetermined world, a Laplacian hypothesis, already beaten by physics and oddly enough, by Poincaré's three-body theorem [Poincaré, 1892]. Conversely, if we move from formal to historical languages, we can observe that having experience with bilingualism is enough to ascertain at least a million times that in each language pair, some expressions are easily and literally translatable, although the expressiveness in the first can sometimes be missed by the second. Therefore, languages are" relatively " and "absolutely" incomplete, with fragments of the world, grasped, learnt or spoken that slip away from a language to another. In other words, every time you choose a plan, a level, in order to represent knowledge, even as rich as our historical languages, this contains signified content (in this case, even emotions, preferences, sensations ...) that is missed, because rich in determinations that stem from a variety of different historical, conceptual and linguist experiences. For instance, mathematics is a small snippet, "sui generis"(distinct), of a man's attempt to describe the world (as H. Weyl says, it forms part of our "historical endeavour towards knowledge") and a tool, a crown jewel, among others, to organize and understand the world. Moreover, it is an open system (see [Cellucci, 1998]) and repeatedly carries, like languages, great significance, in resonance with human practices, which include various forms of presence, not always reducible to a single level of representation. In particular, "crushing" knowledge and putting it on a single representational plane, namely "formal language", constitutes the crucial mistake of linguistic formalism, as opposed to methodological reductionism, partly inevitable in science, thereby excluding the meaning, which epitomizes the point where various practical and conceptual activities come together and integrate different forms of human knowledge (among others, logically signifying reasoning and physical space organization). In fact, although this integration is fulfilled throughout history, it begins and has repercussions during ontogenesis, both in nature and in traces left in our brain, which in turn "integrates" information and a great variety of acts. In the past, conversations among beings fitted with a brain consisted in permanently integrating facts of experience and mental constructs, both in the "private sphere of our minds" and in intersubjectivity. Even so, we could describe and understand it "sufficiently ", only by considering the variety of all the possible modalities of knowledge, but, in any case, we will touch on it later.

5. Paradoxical Interlude

The vast majority of mathematicians interested in foundational problems and provided with Frege's and Hilbert's new approach, boldly worked at a proof of consistency and completeness in formal Arithmetic. Instead, H. Weyl, the Lone Wolf, as he will later call himself, observes that full mechanizability, even of a theory that considers Arithmetic a "simple structure" (like that of integer numbers), is unsatisfactory. What is more, according to him, some assertions might be eluding the Hilbert's style deduction system, a conjecture of incompleteness (some stunning passages from the book on continuum written in 1918). Although it remains a privilege for a mathematician with deep philosophical intuition, his phenomenological analysis of space-time continuum keeps being a beacon (particularly, for authors past and present).

No matter how illustrious, others will mostly succumb to formalism or to the application of formal tools in foundational analysis (Bernays, Ackermann... then Post, von Neuman, Kleene, Church, Curry, Turing ...), provoking countless negative effects. Yet, I shall refer only to one, historical analysis, before talking about the birth of Computer Science, which I consider a huge positive impact of the debate of that time. As for the history of the "foundational crisis", starting on those years, but also nowadays, the history of the "foundational crisis" is being rewritten, other than being put at the centre through some formal-logical games. Forgetting that, suddenly, some had argued against Euclid, Descartes, Newton and Kant, claiming that the universe is curved, that light can "follow the Geodesic curvature", that the notion of the rigid body, crucial for Greek geometric shapes, is inadequate to the new physical spaces, they also failed to mention that these "paradoxical" proposals change the phenomenal reality on which we represent mathematics, innovate physics, revolutionize geometry and knowledge in its entirety. For decades, some mathematicians have been telling each other funny puns about (ordered) sets in (ordered) sets that they would have never even considered. However, there is worse things, like the stories told at the barbershop on Sunday morning, about barbers shaving everyone and only those who cannot shave by themselves. Mark me, although Cantor's and Frege's set theory, where those paradoxes were generated, is extraordinary, it needed to be reviewed and this can very well happen in the search of expressiveness and new theories as it did with the lambda-calculus introduced by Church in the 1930s and Martin-Löf Type Theory from the 1970s: cannot an expressive theory be inconsistent in the first place, just because it has much to convey? Conversely, when did mathematicians work using the set of all sets or "with no types" (no one had ever dreamed of "sets containing themselves")? Naturally, there are problems to develop a new approach in the language that need to be reviewed and proved, corrected by restricting in different contexts, the axiom of comprehension, which allows you to "separate" or define, sets, or structure of sets of functions.

It is funny to think that a simple "oversight" can become a "foundational crisis "of such a robust and serious discipline as mathematics, in a form that "tells us about the world". However, if this is not the case, then the occurrence should prompt us to reflect on problems inherent to the "self-referential" perspective of language that was devised in those years, where language differs from the world, but at the same time "preaches" about it. Then again, that vision does not stem from logicism, but more from formalism and from that distinction between "syntax and semantics", which is proper to it and ostensibly technical. Free formal assertions imply objective, pre-existing external realities. In addition, predicates, syntactic data or variables, as well as formal grammars (perhaps, innate, sic) collect sets of blue pencils, electrons, positrons or unicorns. Basically, objects that were "already there". As a result, language does not aim to be co-constitutive, but to diverge from meaning. Nevertheless, the philosophical commotion bequeathed to the century, and sequentially, the internal paradoxes to the "linguistic turn" do not have to be considered relevant in terms of "foundational crisis of mathematics", but as the first deep fractures in a predicative-referential notion of language already suggested by Frege.

Moments later, Poincaré, alone, has a very harsh reaction: "la logistique n'est pas stérile: elle a engendré les paradoxes" (logic is not useless, it has caused paradoxes). After the works of Gauss, Riemann, Mach, Helmholtz and Clifford, and at the same time of Lorentz and Einstein, Poincaré was inventing a geometry where the local structure of space is given by field equations (and, therefore, as Riemann predicted, by the presence of bodies ... whose cohesive forces are related to the metric structure of space). In other words, the world he was envisioning combined the foundations of geometry with physical meaning, considered geometry the key to understand physics and fostered Einstein's scientific revolution. Another time, in this hypothetical world, rigid bodies, as well as the structure of matter and light, display astounding mathematical (geometric-cognitive) paradoxes, but also entirely contemporary scientific challenges in relativity, but even more, nowadays, in quantum physics. Nonetheless, some paradoxes are external to the new logical-formal proposals, to puns and games between words, rapidly surpassed by references to mathematical practices.

Take heed, linguistic paradoxes are very meaningful, as in the liar's paradox that states: "this sentence is false." Here is a glimpse of what the early philosophers in ancient Greece thought about the problem of meaning, related with all likelihood to the invention and to the amelioration of phonetic writing, a cognitive turn for mankind. For instance, both an ideogram and a hieroglyph can directly be signifying. As a result, though abstract, their meaning "is there", in the drawing. Still, a mere phoneme, meaningless in nature, can raise dramatic issues for humans, prompting them to ask themselves "what am I saying?" How can these "(archeo-)symbols", phonemes, be signifying? What is this phrase, this set of phonemes, this example of a first set of symbols? Such a paradox is indeed unimaginable, without phonetic writing knowledge. My historical conjecture is utterly arbitrary and may be based on little evidence, but there are no similar paradoxes in ideogrammatic languages: their paradoxes refer to opposing meanings. As might be expected, self-reference represents a potential affinity with the barber paradox, with a bird's eye view on both. However, in the liar paradox, it neglects the correlation to meaning (and to the historical context) and only shows the "form" of this Greek paradox, that can be placed next to the one Greeks, yet again, associate with yet another mankind's reflection on knowledge, namely Zeno's paradoxes.

Even in this latter paradox, the significance of the issue is evident: how does my representation affect segments and points in the space or division, a conceptual operation applied to time and motion? What about physical space-time, that I am explaining through geometry and arithmetic measures?

In both of these classic examples, the linguistic problem is completely analogous to that of the geometric representation of sensitive space. During the second half of the nineteenth century, while geometry was disrupting

the world of mathematics and physics, some logicians created word games about small formalization mistakes in mathematics, even unfamiliar to the discipline itself. Nevertheless, for decades, logicians and formalists have been confirming us that that was the foundational crisis, failing to remember, erasing, as a psychoanalyst would say, the problem and the tension, of space and of its geometrical intelligibility. I am aware that my overly polemic attitude in my observations on set theory and on its internal crisis, which inspired much mathematical beauty, may sound paradoxical. Perhaps, after having finished the unilateral logical-linguistic analysis of knowledge, we will have found some balance to better locate both Burali-Forti's (on ordinal numbers) and Russell's paradox (on self-containing sets), which I mentioned before. Indeed, these intuitions about formalizations, under way then in mathematics, were remarkably interesting. As noted above, at the time, the most prominent logicians and formalists were also great mathematicians. Nonetheless, the conundrum is not mathematical, but rather more philosophical and historical and in this case, the latter, which hinders, at the turn of the two centuries, an analysis on the foundations and excludes the "crisis" between local and global that, from Riemann to Einstein, will revolutionize the geometric intelligibility of space. In addition, Helmholtz and Poincaré's studies reflections on the living and its action in space constitute a unique approach on the concepts of vision and motion.
For this reason, presently, you may demand to be intentionally provocative (and paradoxical), even towards theorists (for example, Russell's Type Theories) that, thanks to the Computability Theory (see below), have for long allowed me to make a living (see the author's web page). Moreover, the latter mathematical theories are very appealing and full of practical applications. Yet, if we succeeded to overcome the philosophical monomanias deriving from logicism and formalism, we could, as I said, more consistently highlight all the aspects of the "foundational crisis", including logical-linguistic paradoxes, important enough to represent an early rupture in the aforementioned self-referential view of language. At the beginning of the 20th century, Zermelo's solution continues to dominate the Formal Set Theory, which restricts the ability of freely writing predicates to define sets of mathematical "objects". Furthermore, a well-formed formal predicate is not enough to isolate or define a set, it needs an "already given" construct, within a semantic universe, even solely created from an empty set. As a result, the latter needs to include meaning in constructive linguistics, although the formalists keep making attempts to constrain it to a minimal "ontological hypothesis"(as it will later be called!), that is the "existence" of empty sets.

6. Thinking machines

The formal theories, under way at the turn of the century, were so well-devised that they allowed, in the 1930s, to rigorously explain "artificial thought". In reality, it is well-known that Curry, in 1929, with his Combinatory Logic, studied and worked on Hilbert's program either to confirm it or negate it (among other things, on the proof of consistency, of decidability and of completeness in Arithmetic), followed by Herbrand and Gödel, and all of them proposed languages for effective, "mechanical" deduction. Furthermore, both the computation of functions (the first three) and the systems of equations (Kleene) contribute to make Hilbert's intuition on formalist deduction rigorous. As a result, the new discipline, the mathematics of computation or Computability Theory, intended as mathematical formalization of mathematical *reasoning, was invented and this had to be used in different systems, each provided with their own field of application*.

In 1931, Gödel introduced the notion of computable function, with which he defined metatheory in Arithmetic. In addition, thanks to his extraordinary technical and conceptual expertise, he managed to code deduction, a metatheoretical concept in formal theory, that is the axiomatization of Arithmetic conjectured by Peano and Dedekind, to which Hilbert referred in his program. Subsequently, the liar paradox is rewritten in mathematical language as "this sentence is *not provable"* (notice the difference) and in Hilbert's hypothesis that "truth" and "formal provability" concur (axiom of completeness). In other words, with the supposition that Arithmetic is consistent, he effortlessly demonstrates that "this sentence is not provable", thereby encoding it as an arithmetical assertion is unprovable in Mathematics, together with its negation. Hence, this gives an assertion, the latter, that is undecided, thus Arithmetic is incomplete.
Furthermore, in a second theory, he shows that "this sentence is not provable", but detectable in Arithmetic itself, equivalent to (the sentence that encodes) arithmetical consistency, which, if that was the case, could

not be demonstrable through finalist formal methods, codable in Arithmetic. According to the two aspects that I previously “philosophically” discerned, it almost looked like the conclusion of Hilbert’s program. However, while commenting his theorem, Gödel promptly noticed that something was missing. In fact, the computable functions he contextually defined need not always denote effective deduction in all its forms.

In 1936, the problem was eventually overcome, when Turing and Kleene validated the equivalence of several computing systems, including the one proposed by Gödel to prove his theory. This resulted in the encoding of any metatheoretical deduction form in Arithmetic and therefore, incompleteness is “absolute”. Or better it is an invariant of all formal (Hilbertian) systems containing Arithmetic.

The above-mentioned result marked the end of Hilbert’s Laplacian Dream. Nevertheless, his philosophy inspired the beginning of a new scientific adventure, Computer Science. inspired by that philosophy. Indeed, those years are the start and, perhaps, those scientifically happier than the marriage between formalism and mechanism that changed the century.

So, while intending to rigorously define deductive calculus, Turing invents the distinction between hardware and software and introduces a precise notion of program (a set of formal rules or instructions), which considerably differs from the "machine" that has to do it. The programming of machines that can be materialised in any way began: electrical, mechanical, hydraulic, electronic ... neuronal; if they allow you to implement a Turing-complete programming language, that is, with the level of expressiveness of the various just proved equivalent systems, then they can "formally reason", simulating human deductive reasoning. Rationality, for almost all the mathematicians involved with the problem, is exactly in the "if ... then ... otherwise" logical-formal so well described by nascent programming. The schizophrenic operation reaches its completion: the limits that Gödel, with the incompleteness theories, posed that formal deduction is the same for a machine and for man. Human rationality is mechanizable or it is not. The intellect moves outside man: rational certainty is in the machine

As already stated, the dismay, engendered by the collapse of Euclidean certitudes (and Frege is very clear about the causal link between his proposal and the crisis of geometry), caused a break in the relation to space, which substantially contributed, along with other evident elements of crisis, to conjecture a theory, whose purpose is to impair “human” knowledge and to place it in the machine. Nevertheless, in the 1930s, Husserl regarded the crisis *of knowledge* (and of human relations: especially in Nazi Germany), as a "loss of meaning", built by all individuals in the course of history and in various contexts of life. In the "Origin of Geometry", the relation to space is put at the centre of the epistemological analysis, as a study of “conceptual genesis”: « geometry... is generated in our space of humanity from a human activity » [Husserl, 1936]. On the negative side, the impairment of one of the three pillars of mathematics, among others logic, formal calculus and the principles of geometric construction (symmetry, connectivity ...) has aided to eradicate the constitutive aspects of meaning; in particular, the one that is mostly anchored in human practices (as the aforementioned moving through thought like through space).
As foreseen in the 1930s, Artificial Intelligence (Strong A.I. as we say today, the by-product of linguistic formalism in mathematics, would be launched with a paper written by Turing in 1950. Before, we pointed out that this was achieved because knowledge changed its nature and stopped being a dialogue between humans and the world, encompassing, in mathematics, geometry, logic and calculus (all of which coexist for proof theory), but also an attempt, while developing and using these tools, to understand both space and language; and finally formal sign calculus, independent of meaning, in particular spatial and logical. As might be expected, in the long run, this results in a precise, finitary, and mechanical manipulation, albeit not yet “signifying”.
We are aware of the enormous positive consequences following this schizophrenic operation. Far from replacing humans in cognitive tasks, machines help them, far beyond what had been expected, by enriching both action and thought with their extraordinary speed, unfaltering iteration skills through huge communication networks and databases, but also with their flawless "logic", unrelated to contexts and meanings. What is more, some interactive systems also manage to participate in the mathematical proof. For instance, when you hand them a well-formalized lemma, provided with a well-ordered induction (a previously well-chosen induction load), and its demonstration requires hideous computations; the technical instrument, the computer, is brilliant. Undeniably, these experiences make us realise that mathematical abstraction is different from formalization, a huge fallacy in Hilbertism (rather than Hilbert's) and that "intelligence" is not “independent of encoding", to which we will get back later. In sums, mathematics is not only formal, but also abstract. It manipulates signifying symbolic objects, which are not purely formal, that is they are not merely intended as “strings of symbols” and “objects of manipulation” Mathematics is normative, yet its norm, is not the “formal” rule, absolutely certain and devoid of meaning.

Consequently, mathematics focuses and is *based* on conceptual invariants, although every single conceptual construction cannot be considered "invariant", regarding “everything" (to “implementation”). Therefore, only the mathematician has the last word on the identification of the proper level of invariance. For instance, it depends on what is done during the proof, on the hypotheses formulated; it is independent of these and its invariance lies in here and not there. *In like manner*, human intelligence *operates th*rough encoding or representation, to which it may be unrelated. Anyhow, I will endeavour to confirm (during this ongoing project), that both the substrate and the technique of knowledge representation, as well as their own spatial organization, contain vital information.

7. Portability and Metempsychosis

We have already mentioned that the distinction between "program" and "machine"(purely mathematical, because in 1936, these were merely rigorous clarifications of the meaning of mechanizability of deduction), made by Turing, paved the way for the birth of Computer Science as a scientific discipline. In the same years, through methods that define them, namely programming language and machine (finitist) computations, it was possible to demonstrate the independence of the class of functions from the specific choice of formalism (programs and machines). Not only, but Turing using the notion of Gödelization, that is Gödel's representation of metatheory in the theory, He constructed a "universal machine". As an archetype of the modern compiler, the machine takes a number as input and encodes a sequence of numbers as a program, which is later evaluated on that particular argument. The independence of programming, of computer languages and of expressiveness from specific implementation and from the physical machine is really a key concept in Computer Science, And it is also for the theories of "functionalist" knowledge that are derived from it: what is interesting is the "function", that of the brain, to think, to deduce .... Once well described formally, this function can be transferred to any other machine, even different from the biological one we have in our skull.

In Computer Science, "software portability" is the practical aspect of functional independence, which is unavoidable, if one wants to approach this discipline in a sufficiently general (and scientific) way. What is more, programs, languages, compilers, databases ... must be defined, regardless of the machines with which they are implemented (the software used by Microsoft often makes an exception to this, mainly due to trade monopoly). Also, if your personal computer "is dying" of old age or from aches, and pains, you can transfer everything, including your data and programs, as well as its compiler… and operating system to another computer. In fact, "metempsychosis" is an essential part of the technical and scientific application of Computer Science. While impressively easy from a technological point of view, this is a poor notion, if shifted with the metaphor "the brain is a digital computer", to the scrutiny of human cognition for those who are not Hindu.

Nevertheless, this is a vital challenge for all those who want to examine human cognition in a materialist and monist perspective, without being mechanistic, but paying particular attention to avoid the neo-dualistic approach, put forward by functionalism: how can our mind, at the same time, be a place of remarkably general and stable conceptual constructions, if it is inhabited by hardware and software, a single biological matter entity, comfortable only in the braincase of an alive person, existing throughout history?

The answer, as stated by the succession of logicism, formalism and even functionalism, is schematically summarized here: identify universal laws of thought, Boolean algebra, induction or Fregean quantifiers; write them in finite sequences of symbols, ultimately codable as 0 and 1, and propose formal manipulation rules, independent of ambiguities linked to meaning, of references to space or to other places of incertitude; redraft all this in formal language, namely programming, and you will obtain the rational human being replicated in the machine.

Unfortunately, though, even for arithmetic, automatic theorem proving, other than being a point of mechanization, stops when faced with the fairly non-trivial issue of selecting an induction load, and concrete incompleteness results acknowledge meaning, including that of infinity, but also order and tree structures, like geometric notions, essential for mathematical proof (see [Longo, 1999a]).Not to speak about those robots of classical AI, who walked in the garden of the Carnegie Mellon University, solving differential equations with extraordinary speed, with 0 and 1 following each other in nanoseconds. However, every so often they stopped along the path… in the shadow of a tree. The new AI, a connectionist approach, which works in a completely different way, by stabilizing invariants of images or sound. This is obtained by interpolating and filtering them through layers of networks of formal neurons. The approach performs immensely better that the previous logicist parody of human action – typically it would make a difference between a tree and the shadow of a tree, a remarkable performance for a machine. The change is dramatic, in particular since the observation that the old two-dimensional nets of points invented by Rosenblatt in the

‘50s, could be placed in layers, thus in three dimensions (Deep Learning). Of course, the paradigm does not change, software is split from hardware, the machine is an input-output one … Our brain instead is an always active dynamics of connections and of electric and chemical flows, constrained by the interface with the environment mediated by a material body - a totally different paradigm. If this constrained is lowered, one goes crazy by the uncontrolled super activity of the brain, not by a passive lack of input. This crucial difference, joined to the fact that we do not act on the grounds of “recognized pictures” of the environment, liked self-driving cars, but by “preceding all that moves”, anticipating by eye jerks, like when we hunt or drive, a totally different way of being in the world. Some new AI people, in view of the limits of their approach as well as trying to extend it by enriching it with some reference to the previous, old AI, logicist-formalist approach, at the origin of computing. If going back to mathematical logic may add some further features to machine intelligence, we need to welcome this extension. Yet, how to resume, for the analysis of human knowledge, the themes eliminated from the conceptual path described here, while retaining the richness of an experience of great mathematical depth? In fact, Mathematical Logic was one of the most profound mathematical sectors and originals of the century, and among the least sterile, being the mother of a discipline of great importance, computer science: the foundational analysis it proposes should be enriched, not forgotten. The paradoxes of knowledge, posed by geometry and its relationship to physical space, can be a first starting point.

8. The geometric structuration of information

So, let us look at two first examples, which in my opinion are paradigmatic, in which show that a geometric organization of “information” is unavoidable. In 1877, Cantor laid the foundations of the set theory, by demonstrating that you can “encode" the Cartesian plane, and therefore, any finite-dimensional space, with a straight line. Even though, nowadays, for us, the technique looks easy, its theory, a new "paradox" in the geometry of space, shocked the author: in the long run, does it destroy the Cartesian notion of dimension? I can determine a point on the plane or on the space with a single coordinate, likely written as a sequence of 0 and 1... Notwithstanding, Dedekind sent him a reassuring letter, claiming that its bijective correspondence is everywhere discontinuous: we will later define that dimension a topological invariant (that is, only preserved by isomorphisms of topological spaces). The intelligibility of space is lost by Cantor's “coding". In other words, the straight line and the plane have no *mathematical sense* without their topological or metric structure or even that of vector spaces, where the "isomorphism" is absolutely false (the continuous curve or Peano curve, containing the square [0,1] is not bijective). Yet, this result continues to linger “in the back of the mind" of many, a sort of archetype of today’s so common attitude... as “in the meantime" everything" can be encoded through sequences of 0 and 1". As a result, the latter does not only appear in functionalist "simulations "of human vision, but also in a 1993 paper by Jonhson-Laird: we could start encoding each pixel of the image with a camera...
In another example, this is exemplified by the relation between different types of geometry, Euclidean and non-Euclidean, which can be “algebraically” unifiable. In that event, each of them becomes the set of invariant properties, as compared to other transformation groups. Differently from others, at a glance, Euclidean geometry is the only one, whose group of automorphisms contains homotheties (i.e. solely Euclidean properties do not vary in case of arbitrary reducing and enlargement of geometric structures). In Physics, the lack of this property is acceptable and to this day, it does not claim or fail to unify microphysics (quantum physics) with astrophysics (namely relativity, although huge progress is being made, geometrically), by using, when and where appropriate, even non-Euclidean geometries: we do not demand or know how to pass through homotheties, from extremely small to extremely large. Much before the end of the 19th century, while working on a unifying algebraic framework, Beltrami and Klein demonstrated how you can immerse (give a relative interpretation of) one kind of geometry in another. At the same time, with two-dimensions, we can obtain an intuitively effective representation (such as the Riemann sphere). However, with multiple dimensions, of which three really matter, it is possible to lose the "physical sense" of the different geometries. To put it differently, this way, the equicoherence demonstrated is one of the weakest forms of equivalence, as it entails the loss of "information that matters", namely the geometrical structuration of a physical space at the origin of Riemann’s choice. Therefore, if you immerse, or "encode", a Riemannian space into a Euclidean one, its translation will easily deprive it of ... "physics" (Relativity). As has been noted, the physical problem, already raised by Gauss and Riemann, was addressed during the development of the theory of curved surfaces, which are not, “per se”, immersed in varieties of Euclidean space.

Nowadays, a similar phenomenon, still under scrutiny, can be found in Computer Science. In addition, the problems of concurrent, distributed and asynchronous processes (basically, computer “networks ") are also space/time *problems: for example, the distribution* of machines that *concur with* the same process, in space, incurs over *time* in synchronization problems. At present, there have been interesting attempts to translate and encode some concurrent systems into essentially sequential theories (among others, Robin Milner's Calculus for Concurrent Systems, CCS, in lambda-calculus), that is to “translate” the CCS into one of the fundamental computability theories of the 1930s. Nevertheless, when and if possible, it is necessary to make a "passage through the quotient", to impose equivalence relations that seem to elude what matters: i.e. the Computer Science of concurrent calculus. Hence, there is an open problem of crucial interest, as, even in this case, the possible encodings, which set aside, among other things, the role of space, do not seem to be transparent at all. Once again, the "intelligence "of the system can be found in its space/time structure, but if the element of "sense”, present in spatial structuring ,was lost in one of its translations, then it would not be possible to retain the pertinent invariants.

At this point, we cannot fail to mention an approach to representation and automatic knowledge processing, as well as an alternative to digital computing, nowadays in rapid development. In the 1940s, in parallel with the infancy of digital computers, McCullogh, Pitts and, independently, Hebb, put forward the notion of "formal neural networks". Although inspired, in different ways, by the structure of the "neural nets" located in the brain, they mathematically drew machines, whose events were encoded by those geometric dynamics, deriving from networks of binary threshold elements (i.e. neurons "discharge" signal, when their “stimulus" overcomes a specific threshold). In fact, with a theoretical, but certainly brilliant idea, Hebb thought that the information is stored and processed in the brain, thanks to changes in interneuronal connections. Particularly, the strengthening and weakening of synaptic connections could constitute the physical place, where mental processing occurs: in this case, information storage could also depend on the stabilization (strengthening) of a neural circuit.

The approach is radically different from the one adopted by "Turing”, namely, geometry of the connections. This difference was further accentuated, primarily because of the way "connectionist" cognitive theories (that is how they are defined) developed. Instead of encoding strings of 0 and 1 as "universal laws of thought" like Boole, Frege, Hilbert ..., (through a "top-down" approach to knowledge), the mathematics of neural networks attempts to describe the formation of "bottom-up” intelligence: the automaton interacts with the world (movement, minimal signals ...) in a very basic way and rebuilds “from the bottom-up" fragments of representations that will be dynamically processed. Both statistical physics and mathematics of dynamical systems (see [Hertz et al., 1991]) have substantially contributed to the evolution of connectionism, certainly with a much closer approach to the one observed here (role of time and space in knowledge representation). Nonetheless, two problems arise: even though the theory of neural networks will someday give us exceptional machines, the "intentionality “of representation *in* the living, mentioned at the end of this paper, can, among other things, elude it. Moreover, at present, no actual "neural machines" exist: for the implementation of mathematical networks, our colleagues are obliged to shift their beautiful geometry of dynamic systems to binary codes, a very difficult programming exercise, a real bottleneck and a complete change in their approach.

Besides, the reason why the many approaches to analog computing, from Wiener’s ideas to formal neural networks, have been overwhelmed by digital computers is a story yet to be told. Mainly owing to compelling technological reasons, digital processing and broadcasting are by far the most prevalent. Concerning broadcasting, there is no doubt: nowadays both music and voice are transmitted on digital channels, with such efficiency and reliability, so far unachieved by the analog system (in fact, our voice, our brain, produced analogically, sound so unreliable, so slow ... yet so emotive and intelligent...). Indeed, on the subject of processing, the above-mentioned hegemony of vision on the foundations of mathematics (and knowledge), along with technological superiority, has fulfilled an important role. And is now providing new machines, far away from Turing Logical
Computing Machines (even though, in the end, we are forced to implement ourmathematics of continuous network deformations into them). Therefore, it is necessary to review it, not only for a better understanding of the living and the thinking, but also to build better-quality machines.

9. From the intelligibility of physical space to the space of living beings

Previously, we have discussed the problem of physical space, at the core of the long mentioned “crisis”, which mostly stems from Relativity Theory and its mathematical aspects. Nevertheless, Quantum Physics seems to pose a major challenge. Recently, a number of proposals, concerning noncommutative structures, have provided new insights on the potential geometrical organizations of microphysics ([Connes, 1990]). Indeed, the analysis conducted in connection with strings and superstrings is even more mind-blowing and difficult (still inexplicable for those who write): in that event, the attempt to merge Relativity and Quanta resembles a reunification of all those geometries under review (in these areas, we talk about non-localities of phenomena or ubiquity of particles, solved, in the broad sense, by considering, instead of points, strings' segments basic elements of space: in a nutshell, massive intellectual challenges imposed by the geometric intelligibility of nowadays' physical experiences).

Then again, the problems posed by the "geometry of the living" are of a similar, yet partially or radically different nature. It is worth specifying that these issues have only recently been clarified, especially when compared to... Zeno's paradoxes or to the geometry of Riemannian spaces. For this reason, and for space limits (I mean, page limits), see the bibliography.

Succinctly, in the first place, Biology made it possible to establish significant differences and connections between "external and internal" spaces and the living being. For example, visual spaces represent a continuous game between the two: permanent reconstructions of visual clues, explorations of a "feeling by gaze", characterized by extreme complexity (see [Ninio, 1989], [Maffei, Fiorentini, 1995], [Berthoz, 1997] and numerous others, already mentioned). Therefore, whilst continuously recreated through our active presence, image and space are far from being passively "absorbed".

Envisioned as "visual gestalt", mathematical analysis of vision is making extraordinary progress (see [Morel, Solimini, 1995]). Hence, without reconstructing an underlying unit, without interpreting and projecting an inner space, there can be no visual image. Philosophically speaking, the difference between sensation and mental construction is becoming increasingly problematic. Other than being rich in phylogenetic, human and ontogenetic history, the interplay between the two, is at the core of our activity as living beings. Thanks to the modern studies of motion and vision, we are moving from a geometry of figures (Euclid), towards one of space (from Descartes to Riemann), of motion and action (Poincaré and, see the more recent [Petit, 1997] and the quoted project "Géométrie et Cognition" on the web).

Nonetheless, the analysis of the geometric perspective on genetics continues to be enriched by additional subtler elements and currently, the linear "encoding" of the ontogenetic development is non-existent. This is why the space-time organization of information is crucial. Symmetries and succession of growth overtime, in different spatial directions, contribute to the determination of an individual ([Prochiantz, 1997]). For instance, both ecosystem geometry and interaction with the ambient space store vital information. According to Prochiantz, the "genetic program", essential for ontogenesis, is contained in the interaction between humans and the ecosystem, which takes place in space and time.

Dwelling for a while on the nervous system, the hundred billion neurons in our brain are connected to each other with about ten thousand synapses each and their relative geometries undergo permanent reorganization. Among other things, in fact, synapses continuously change position and contact points. On top of that, information is encoded by the finest structures of intersynaptic communication and it is carried by exchanged proteins, also depending on their three-dimensional biochemical structure. Nevertheless, with the same components, interaction between enzymes, a key process in living beings, is accomplished. As well as being conceptually wrong, it is exceedingly difficult to think of encoding only two types of brain functions structuring into strings of 0 and 1, even neglecting geometry of interaction. Erroneously, but promptly, Cantor hypothesized that the real line could "encode" the Cartesian plane: basically, an information contained in the geometric structuring, which is similar to the topology in the Cartesian dimension of the plane. In both cases, "what matters" is left out of the linear encoding.

In fact, geometric structuring of information, in the brain, is a key element of its processing, both at local (geometrical dynamics such as protein structure and synaptic plasticity) and global level, as in the case of "neuronal assemblies" (see [Edelman, 1992]).

While briefly discussing the problem of vision and action, we can quote the example of an object in motion, about to be picked, whose projected image on the retina is two-dimensional. Velocity and acceleration are represented in the eye as extension and variation of said extension. In addition, the image is transmitted through a relay, from the lateral geniculate body or nucleus to the primary visual cortex (V1), involving binary thresholding and digital encodings and the aforementioned geometric subtle structures that form synaptic connections.

In the first place, the image leaves a "trace" of its edges in the primary cortex. What is more, recent, astounding experiences show the "analog" activation of V1 neurons, precisely along contour lines, later deformed by the fovea (the "focus" centre, in the retina), tasked with focusing image details. Afterwards, the three-dimensional nature of the image is reconstructed. Thanks to the signals emitted by saccades, by eye or body movements, cortical thickness is involved in the analysis of depth and motion: primary cortex seems to be organized into "fibrations", in terms of geometry/ categories (see [Tondut, Petitot, 1998]). This results in a mathematical way of three-dimensionally/analogically structuring information that permits to recreate the three-dimensionality of space (thereby allowing/ suggesting us to consider it three-dimensional).

Let us move on and grab the object. For this, it is necessary to set up a new coordinate system, an angular one, which takes into account the configuration of the arm, measured in muscle thresholds (see [Berthoz, 1997]). If needed, this could constitute another analogue reproduction or even a dual representation of the motion of the object. Consequently, the brain integrates the plurality of coordinate systems and, in synchronizing the action, changes from one to the other. As a result, it is not possible to centrally reproduce the world in a system of Cartesian axis with pixel-decomposed images. On the other hand, it can be built/rebuilt with the purpose of incorporating a *plurality of dynamically- managed coordinate systems.* In other words, the brain does not behave like a digital database, it works with the geometric structuralization of information, coupled with its mathematical, geometric analysis, which must become indispensable for the study of human intelligence.

Against functionalism, we could claim that (almost) all the intelligence put in the single gesture of grabbing an object is shown through codings and representations. Therefore, it is anything but independent from coding. In this case, then, intelligence *may be* in the phase, when the simulation of the event, starting from the retina and ending in the arm, takes place through several intermediate, analog / digital / biochemical codings... , or perhaps "only" in the (geometric) coding of the physical phenomenon "*in the living being*", because that's the whole point.

The person who wants to grab the object has a large amount of intentionality and this makes a difference even in the choice of the analogy. Hence, this is how the structure of representation operates. Unlike the extremely "faithful " pixel by pixel digital representation, typical of electronic computers, the analog one can be considered an intentional, pre-conscious choice, taking place in the living beings, on a cell and in the body system. In fact, the analogy "decides" to represent/ encode what matters (the contours, the relative speed or depth variation ...).

After mentioning intentionality and its thousands of levels, including the hugely controversial and problematic preconscious and conscious, I would like to close this very hasty section. I hope that the reader feels motivated to go beyond all those shortcuts that claim to encode human intelligence in strings of 0 and 1, and tend to forget, in the first place, the essential incompleteness of these representations, also caused by the limited aims of the foundation of mathematics. The space time analysis of intentional contexts becomes unavoidable, if you want to mention more than the complete and specific conceptualizations, such as mathematics and all the other forms of intelligence embodied by a human being and immersed in the network of intersubjectivity, but also in history. This remarkably important interdisciplinary challenge can happen, only if modern mathematics and the reflection on its foundations influence and support the disciplines that study humans as living beings, living in history.