Geometry and Cognition
From the foundations of mathematics to theories of knowledge and cognition[1]

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In 1999, in close collaboration with Jean Petitot and Bernard Teissier, the author of this text set up a series of conference-debates titled *Geometry and Cognition* (GeoCo) at Ecole Normale Supérieure in Paris (see the “Startup Conference Cycle”, February - April, 1999, in "http://www.di.ens.fr/users/longo/geocogni.html").

This project focused on the link between geometry and cognition, in a double movement:
1) From Cognition to Geometry: or the cognitive foundations of Mathematics (where in "cognitive" we also want to include evolution conceptual construction and its history);
2) From Geometry to Cognition: or the mathematical analysis of human cognition (vision, in particular).

The consequences were both technical and epistemological:
1) A better understanding of the role of these advances in natural sciences and cognition in the analysis of certain fundamental problems of mathematics traditionally considered philosophical;
2) The development of certain aspects of Differential Geometry and a new role of Geometry in Computer Science.

Our different concerns, referring respectively to cognitive sciences, the geometry of dynamic systems and geometrization in computer science, have found a “common conceptual and working space” thanks to this initiative. The manifesto which accompanied this cycle of conferences and written in collaboration with Jean Petitot and Bernard Teissier, transformed into a workshop project, was then generously financed by the Action Cognitique of the Ministry of Research, for the period 2000-2002.

The volume in footnote 1 is one of three publications which collected some of the contributions to the seminars and symposia organized as part of the workshop (see the conclusions, below). The following text is the general introduction to the volume in footnote 1, partly inspired by the first part of the original manifesto, to the themes of the workshop (for further reflections and bibliographical references, see the article by Bailly-Longo[2]).

1. The origin of a debate.

A great debate was at the origin of the analysis of the foundations of mathematics at the turn of the century. A central moment was the radical opposition between the visions of Riemann and Poincaré,

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1 To appear in the Springer volume *Morphology, Neurogeometry, Semiotics: A Festschrift in honor of Jean Petitot's 80th birthday* (Sarti ed.), 2023. This is an English translation and a revision of the Introduction to the volume "*Géométrie et Cognition*", G. Longo (éditeur), *numéro spécial* de la *Revue de Synthèse*, Editions de la rue d'Ulm, tome 124, 2004. The author wishes to dedicate this translation and revision to Jean Petitot, whose enlightening insights and immense culture provided many fundamental tools and motivations for my work for several years, both by the agreements and the gentle disagreements, in a debate of the highest intellectual depth and honesty I ever witnessed.

2 Bailly F., Longo G. Space, time and cognition. From The Standpoint of Mathematics and Natural Science. Invited paper, *Mind and Causality*, (Peruzzi ed.), Benjamins, Amsterdam, pp. 149-199, 2004 (downloadable from https://www.di.ens.fr/users/longo/download.html).

on the one hand, and those of Frege and Hilbert on the other. By a very short synthesis and in our terms, it is fair to say that Riemann and Poincaré emphasize the role of space and the "constitution of mathematical concepts" in human history, as living beings in the world. Frege proposes logical rules that are universal and independent of human praxes, whose absolute objectivity constitutes the ultimate foundation of mathematics, rules to be expressed in a "format language of thought", rich of meaning. In Hilbert's more formalist view (no reference to "meaning"), as increasingly specified since 1900, these rules and axioms, coded in fine sequences of signs, must satisfy formal coherence alone to "found", if sufficiently expressive, mathematics or its different branches.

During the century, F. Enriques and H. Weyl enriched the reflections of Riemann and Poincaré by adding an appreciation of history, through this analysis of "progressive conceptualizations" in mathematics, which we find in their numerous philosophical writings. Hilbert will develop Frege's Mathematical Logic by also stressing the key role of Arithmetic (or Formal number theory), as a theory of the properly finite. Yet, he went much further than Frege: by seeking "certainty" in the finitary manipulation of coherent formal languages, he will lay the foundations of modern Proof Theory, as a (meta-)mathematical analysis of the formal and linguistic form of mathematics. This was developed especially after the 1930s.

In fact, the "linguistic turn" marks the century well beyond Frege's projects: the logical reasoning of Boole and Frege is reified in a mathematics of finite sequences of signs and their effective transformations, the metamathematics of Hilbert, inside which one can pose mathematical problems and precise conjectures:

- the coherence (no formal contradiction is engendered)
- the completeness (all well formed sentences are either proved or disproved)

of any formal system of axioms and rules, thus (meta-)mathematically treated as a pure game of signs.

The conceptual strength, the rigor and the mathematical precision of Hilbert's program will make us forget the informal, although profound, remarks of Riemann, Poincaré and other geometers (including Helmholtz and Mach). Hilbert's Theory of Proofs and more generally Mathematical Logic will pose itself as a new mathematical discipline of great relief and, by making it possible to develop the notion of representation or finitary coding (Gödel) and effective calculation (Herbrand, Gödel, Turing, Church, etc.), it will be at the origin of Computer Science. This is how the absolute objectivity of logical calculations was ultimately objectified in "formally and perfectly logical" machines. These machines have continued, since the 1950s, to change our lives, through their extraordinary efficiency in everything that can be coded by finite sequences of signs and their effective transformations. The consequences of the philosophy specific to this foundational turning point and the machines it generated were enormous in the theory of knowledge and of mind and, consequently, in cognitive sciences.

It is precisely the successes and limits of analyses and applications based on "finitary processing of finite sequences of signs" that push us to go beyond these tools, towards an epistemological and scientific method which also integrates other forms of knowledge and the relationship between man and the world. It is time to take up the ideas outlined by Riemann, Poincaré, Weyl and Enriques, to restart a scientific, in fact mathematical, reflection on the epistemology of mathematics and their cognitive origin. This volume is a first attempt in this direction, developed as part of a series of talks presented at the Geometry and Cognition Workshop.

2. Geometry: from a Science of space to a Science of movement in space.

Greek Geometry was a "science of figures"; with Descartes and, then, Riemann and many others, it became a "science of space". Poincaré went further, emphasizing the role of movement in space: "an immobile being could never have acquired the notion of space since, not being able to correct through its movements the effects of changes in external objects, it cannot "would have had no reason to distinguish them from changes of state" [Poincaré, 1902, p. 78[3]]. More: "locating an object at any point means representing the movement (that is to say the muscular sensations which

3 Poincaré H. 1902. *Science and Hypothesis*, Londres et Newcastle-on-Tyne, The Walter Scott publ. Co. Ltd.

accompany them and which have no geometric character) which must be made to reach it" [Poincaré, 1905, p. 67[4]].

For Poincaré, as for Riemann, there is not an "a priori" geometric theory of the world, given by formal axiomatics. It is rather "the presence of bodies, natural solids" and our own body, our movement and the changes of state which constitute space and which make us apprehend it; "the axioms are only disguised definitions", from which we choose "the most convenient" for us, biological beings living in this world [Poincaré, 1902, p.75-76]. However, for Poincaré, the geometry of the sensible world is not mathematical geometry, because... "muscular sensations... have no geometric character...". It is this divide, which leads to a certain conventionalism, that we must today overcome: we must reconstruct mathematical geometry from that of the sensible world, explain why our explicit choices are "more convenient", using the tools that mathematics, first of all geometry, but also mathematical logic, have given us over the last decades, in combination with the contribution of neuroscience and cognitive sciences.

Let's take an example, from the book of a physiologist [Berthoz, 1997][5]: the capture of an approaching ball. According to Berthoz, during the action, there is no centralized reconstruction of the external space, but a multisensory integration of different frames of reference [p.90], each of which makes it possible to "simulate" the space of the perception. That is to say, space does not need to be represented explicitly, in a Cartesian coordinate system or by pixel-by-pixel coding of points in space: the muscular threshold relative to a certain angle of the arm, for example, is itself the frame of reference or the coding of a distance. This is how, when we make a movement to grasp an object, the frame of reference is constituted by the joint space of vision and action and quantified by the muscular thresholds, including those of the eye muscles. Thus, the spatial frame of reference is, first of all, analogically reconstructed on the retina: the enlargement of the two-dimensional image is an "analogue simulation" of the movement of the ball; then, this "simulation" is transferred, through a thousand intermediate passages, to the referential system given by the joints of the arm and quantified by the proprioception of the muscular thresholds. The passage from one analogical representation to another, the integration, by comparison and constitution of invariants (the perception of the stability of certain phenomena), is the first constituent element of this intelligence which allows us to move and act and which, in fact, has its genesis in our movement in the world, in the plurality of our representations of the world and in our action.

In short, this very first form of intelligence, the arm that rises and grasps the approaching ball, resides first of all in the transfer of the analog representation of the ball, its speed and its acceleration onto the retina on another analog representation, those of muscular thresholds, which simulate the direction, speed, acceleration of the ball in their own reference system, that of the joints of the arm. This "geometric" intelligence is far from being independent of coding, on the contrary it is constructed as networks of coding or analog representations; it is acquired through a practice of action in the world, on a body and a cerebral proto-map which make this network possible. The practice of the invariance of objects in the world in relation to the plurality of our frames of reference and our coding allows us to construct or conceive, after the fact, this "invariance" or stability which will be specific to our conscious representations, those of language and space for example, up to the most stable, most invariant conceptual constructions, those of mathematics.

The analysis of the constitution in the praxis of our conceptual invariants requires a remarkable mathematical effort, which starts from dialogue with biologists and physiologists and uses the analysis of the changes of geometric frames of reference. In particular, the space that physiologists talk to us about is always structured by gesture, movement: it is never absolute, but relative to the bodily structure and the space we are acting in. In fact, the analogy reconstructs the essential (the interesting spatial structure) of the action and the simulated phenomenon – the eye jerks and the arm preceding a ball, with the aim of action. The modular frame of reference of thresholds, eye jerks, vestibular system, touch, is chosen and modified according to the structure of the space; it is

4 Poincaré H. 1905. *La Valeur de la Science*. Champs Flammarion, Paris.

5 Berthoz A. (1997). *Le Sens du Mouvement*. Paris: Odile Jacob.

functional to the action of the moment: the eye jerk which simulates the movement of the prey or precedes the pursuit and the action of the arm which simulates the movement of the object to be grasped exactly preserve the structure of the space useful for the action.

However, the analysis of "structured mathematical spaces" is well beyond these early active experiences of action in physical space. Its turning into a conceptual construction, grounded on abstract but meaningful notions (not purely formal), such as the Euclidean notion of “line with no thickness, is a central issue in geometry. It leads to one of its most important generalizations: Category Theory, as a theory of transformations which preserve the structure (that which is of interest). In conclusion, the question that we ask, through this analysis, is that of the relationship between, on the one side, action and mental perception/simulation of physical spaces and our movement in these spaces, in as a living being in the world, and, on the other, the geometric construction - conceptual, mathematical as proposed by humans in history. It is the question of the relationships between mathematical geometry and the geometry of the sensible world. And, quite obviously, the geometry of physical spaces will be an extension of that of the sensible world by focusing on relevant invariants of access and measurement modes (in astrophysics: Riemannian manifolds; in microphysics: non-commutative geometry, see[6]).

3. Epistemology and genesis.

In a 1927 article, Weyl[7] explains that the formal analysis of mathematical theories, in fact of their logical coherence, is only a necessary element for foundational analysis; that this is by no means a sufficient condition, neither from an epistemological point of view, nor for analytical foundational research. It is rather the meanings, built from "our acts of experience" (to use Weyl’s words), i.e. the underlying structures of relations with the sensible world - which we introduce more or less implicitly - which allow the mathematician to "understand" the proof or to formulate conjectures, and which suggest to the logician the sufficient conditions for the constitution of a formal system, the end point, and not the starting point, of a conceptual practice. In short, in our opinion, it is the analysis of the genetic process of constitution of a system which highlights the "necessity" of this system. Husserl will specify:

“The original evidence cannot be interchanged with the evidence of the “axioms”; for the axioms are principially already the results of an original formation of meaning and have this formation itself always already behind them” [Husserl, 1933: p. 192-3].

The axiomatic “conventions”, therefore, come after the “formation of original meaning”. On the contrary, as mentioned above, the founding fathers of mathematical logic, at the beginning of the century, took as their foundation axioms and rules, as "logical atoms" of a language of thought, and were able to offer this remarkable discipline thanks to the ...“almighty dogma of the principled break between epistemological elucidation and historical explanation as well as psychological explanation in the order of the sciences of the mind, of the break between the epistemological origin and the genetic origin; this dogma, to the extent that we do not limit in an unacceptable way, as is usual, the concepts of "history", "historical explanation" and "genesis", this dogma is reversed from top to bottom.” [Husserl, 1933: p.201][8].

It is this reversal that we want to work towards, because it is possible today. It is possible, also thanks to breakthroughs in mathematical logic: the numerous results of incompleteness of recent decades confirm to us the essential role of geometry (at least order) as well as of mathematical infinity in general, even in the proof of finitary theorems, in number theory for example (see [Longo, 2002, revised 2011[9]]).

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6 Longo G. (2001) Space and Time in the Foundations of Mathematics, or some challenges in the interactions with other sciences. Invited lecture, **First AMS/SMF meeting**, Lyon, July, 2001 (downloadable from https://www.di.ens.fr/users/longo/download.html).

7 Weyl H. (1967) Comments on Hilbert's second lecture on the foundations of Mathematics. [1927] in van Heijenoort J., *From Frege to Goedel*, Harvard U. Press**.**

8 Husserl E. (1970) ‘The Origin of Geometry’ in *The Crisis of the European Sciences and Transcendental Phenomenology* (German: 1933). Evanston: Nortwest. Univ. Press.

9 Longo G. “Reflections on Concrete Incompleteness,” in **Philosophia Mathematica**, 19(3): 255-280, 2011

4. The constitution of mathematical invariants: their cognitive interest.

Returning to Weyl's "distinction", when we consider the question of identifying sufficient conditions for the emergence of a theory, those which require this theory and make it possible, the issue, this time, is not internal to mathematics. It requires an examination of the historical insertion of mathematics in its operational context, in the world, an examination of the interface between our mind and the world, where mathematical conceptualizations begin. In fact, the objectivity of mathematical conceptualizations has its sources in their constitutive processes.

Real numbers, for example, do not have objectivity by themselves, they acquire both existence and objectivity through the Cantor-Dedekind construction which constitutes the standard real numbers from the integers induced by their historical, very ancient invention. Numbers, even integer ones, are not “already there”: singling out the invariant of small counting and the twofold, discrete progression of time, as Brouwer would put it, requires a complex conceptualization, stabilized by writing, as Husserl stresses in the “Origin of Geometry”. Cantor-Dedekind approach uses uncountably may limits of these integer numbers to get the intuitive continuum of a trajectory, a border with no thickness, the great invention of Euclid’s geometry. In addition, there are also other, non-standard constructions of the real numbers which follow other paths and lead to other results. In this sense the reals are not platonic entities, already-there-always-and-"in-themselves"-such-as-they-are.

On the one hand, the constitutive processes of mathematical conceptualizations are therefore cognitive in nature and they are then subject to historicity. A satisfactory analysis of these processes leads to a re-examination of the theory of proofs according to which the bases of mathematical conceptualizations are found in linguistic formalisms and in the general principles of syntactic proof. In particular, the role of topologies and geometry in logic - and, for us, with its cognitive origins - must be reaffirmed, both on a technical level and on a philosophical level as well as its construction from our relationship with the world and of living in the world (from Riemann, Helmoltz, Poincaré, Weyl, as we said). This role is also already present in the topological structures and Topos for Intuitionist Logic, with their applications to Computer Science initially proposed by D.S. Scott, and, above all, thanks to the very original ideas of J.-Y. Girard in Proof Theory – in Linear Logic, in particular, where geometry does not only concern the underlying structures, but it also enters "through the window", into the structure of proofs itself, through the "networks of proofs."

On the other hand, mathematics, while being, in its genesis, intimately linked to other general forms of knowledge, presents a very strong specificity when it is already accomplished. No other form of representation of the world is so essentially and explicitly based on the requirement for independence of structures with respect to representations, with respect to the notations and codings used. This requirement is the very heart of mathematical conceptualization, of its particular type of generality. Furthermore, mathematics, at any level, is in the hunt for maximum "simplicity", even or especially when it is profound. "Elegance", the search for what is "essential", "minimal", "stable", in short the search for sorts of abstract "geodesics" of thought - as a method linked to a principle that one is tempted to qualify as "aesthetic" – this search is a constituent part of it. However, in spite of the specificity of mathematics, the role of invariance in its sense can illuminate efforts to analyze any processes of human cognition. It can also guide the identification of the constraints which implicitly governed the structuring of forms of knowledge representation more complex than abstract mathematical systems, as embedded in the complexity of everyday live and diversity, and whose principles are therefore more difficult to isolate directly, more hidden than in pure mathematics.

For all these reasons, mathematics has already very often been at the center of philosophical reflections in epistemology, and it must be kept there. In particular, it seems to us that the role of geometry is the most central, because "geometry is more compelling", as D.S. Scott likes to say (but J.Y. Girard would also agree). It is a fact that, in mathematics, geometry is the preferred place of "meaning": to analyze this fact, we must develop our thesis which links mathematical geometry to sensitive space.

5. Infinity and Geometry: from continuity to dynamical systems.

Enough has been said about our interest in aspects of the non-linguistic, non-formal foundations of mathematics and the role of geometry. As for the concept of mathematical infinity, this is one of the most interesting examples of "progressive conceptualization" throughout history. A "metaphysical" concept which has become a "mathematical practice", which has materialized in operational contexts, from infinitesimal analysis to Cantor's theory of ordinals. But infinity has acquired a very solid meaning also thanks to its geometric meaning, and first of all in projective geometry: the points at infinity in the painting of Piero della Francesca, one of the inventors of this geometry, is one of the earliest, strongest and most rigorous human "conceptualizations" of the concept of *actual* infinity, which has a theological origin[10].

It is therefore necessary to develop an analysis of the role of infinity in the foundations of mathematics in two essential aspects. On the one hand, many results over the last twenty years show that infinity (actual, of course) plays a massive role in the proof of theorems whose statements are finitary (arithmetic) such as the Paris-Harrington theorem or Friedman's result proving the formal unprovability of the finite version of Kruskal's theorem ([Longo, 2002, revised 2011, quoted]). However, it seems to us that a "fine", epistemological and mathematical analysis of this phenomenon is lacking. Geometric judgements and continua seem to step in, but what notion of calculation can we transfer to these latest "continuous" systems? How can the mathematical analysis, using continuous tools, of certain discrete systems (formal and computer's languages) inform us about their meaning and their expressiveness? What links exist between the continuum of structures used for the semantics of languages and formal systems and, through their new role in dynamical systems, the continuum of morphodynamic analyses, in particular Thomian, of language?

6. Computing

The project of reification of rationality in "formal rules", which developed over the course of the century, is at the basis of the invention of computers in the 1930s and 1940s: the finite sequences of symbols in as coding of axioms and deduction rules, make it possible to manipulate enormous quantities of data. Intelligence has transformed into the finitary manipulation of finite sequences of 0 and 1.

This project, as for human (and animal!) cognition, although so effective for constructing machines, came up against very specific limits. In fact, behind the results of incompleteness, which logicism and formalism themselves have been able to give us, there is the human praxis of mathematical proving, which largely uses direct references to space (mental and physical): what is "missing" from formal or mechanical deduction is the correct geometric order in which we arrange the numbers in our mental spaces – well ordering. There are the explicit judgments on the spatial continuum, its connection, the symmetries, the analogies between shapes and forms, movement. These conceptual practices have often been "hidden" under a vague reference to "intuition". It is time to develop a scientific analysis, in "cognitivist" terms, of these implicit aspects of reasoning, in particular spatial, as rational as those governed by "finite logico-linguistic rules". It is computing itself that requires it and this for at least two different reasons:

10 Arasse, D. (1999). *L'Annonciation Italienne. Une Histoire de Perspective,* Hazan, Paris ; Longo S. (2022) *Daniel Arasse et les plaisirs de la peinture*, Editions de la Sorbonne, Paris; Longo G., Longo S. (2020) Infinity of God and Space of Men in Painting, Conditions of Possibility for the Scientific Revolution. In *Mathematics in the Visual Arts* (R. Scheps and M.-C. Maurel ed.), ISTE-WILEY Ltd, London (downloadable from https://www.di.ens.fr/users/longo/download.html).

- the remarkable growth of computing systems in which space and time play a central role (distribution and asynchrony of concurrent systems) and which are poorly represented by certain variants of essentially sequential languages and methods,
- the immense difficulties faced by Artificial Intelligence, in particular Robotics, in the simulation of the simplest human or animal tasks, in particular when the relationship to space and movement are central.

7. Conclusions

The articles in the special issue of the **Revue de Synthèse** (see title's footnote), where an early French version of this introduction first appeared, take up the problem of space and its mathematical intelligibility in different forms: its history, its relationship to time and causality in physics, its role in natural languages. The analysis of action in space as well as of the internal space of a single cell, in biology, is followed by a phenomenological reflection on the constituent dynamics of the consciousness of the body in space. After an original approach to algebra, a sort of geometrization of algebra, the volume analyzes the possible historical origin of reading the world in terms of potentially mechanizable linguistic formalisms.

Some of the themes outlined in this introduction have been also developed by technical work, published as special issues of two very well-known international journals – they are the proceedings of two major conferences organized within the "GeoCo" research project. For some aspects of geometry in concurrent systems computing, one of the members of the workshop edited the volume:

> *Geometry and Concurrency*, E. Goubault (Editor), Special issue of **Mathematical Structures in Computer Science** (G. Longo, Editor-in-chief), Cambridge U. Press, vol. 10, n. 4, August 2000.

The proceedings of a conference on the foundations of mathematics, organized as part of the workshop, appeared in:

> *New Programs and open problems in the Foundations of Mathematics*, G. Longo, P. Scott (Editors), Special issue of **The Bulletin of Symbolic Logic**, ASL, vol. 9, n. 2, June 2003.

Jean Petitot further developed his seminal work in Neurogeometry, which strongly influenced the GeoCo project, by a fundamental book:

> J.Petitot "*Elements of Neurogeometry: Functional architectures of vision*", Springer, 2017.

Petitot's collaboration with Alessandro Sarti led to new developments, by the latter, in the geometric analysis of brain dynamics, relating brain activity to its "bodily meaning":

> A. Sarti, D. Barbieri, Le cerveau comme virtuel incarné, **Intellectica**, 69 pp. 347-367, 2018

up to the new insights as analysis of "heterogeneous" dynamics:

> A. Sarti, G. Citti, D. Piotrowski, "Differential heterogenesis and the emergence of semiotic function", **Semiotica**, Vol. 230: 1-34, 2019.

During 2002, the debate and the results of this workshop contributed to the establishment of a more specific research project, in the form of a new team within the Department of Computer Science of ENS, "Morphological complexity and information" (http://www.di.ens.fr/~longo/CIM/projet.html), which formally ended in 2011 – yet the team survived as a very active seminar till 2022 ("CIM"). It is now followed by a more biologically oriented seminar series, Philosophy of Biology and Ecology (PhiBE): https://republique-des-savoirs.fr/events/event/7231/

The debt of these activities to the seminal enterprise with Jean Petitot and Bernard Teissier is immense.