Computability and physical predictibility

the *discrete* and the *continuum*

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Laplace's determination

General attitudes:

determination $\Rightarrow$ predictibility

and

determination $\neq$ randomness

[Laplace, Philosophie des Probabilités, 1786]

[J. Monod, Le hasard et la nécessité, 1970]

Key idea and conjecture:

a fine ***analysis of perturbations*** may justify the stability of the Solar system

(Newton; … Laplace: "nuances presque insensibles")

Poincaré's geometry of dynamical systems

- **Three Body Problem** (three physical bodies in their gravitational field)

$$\{ D^2 x_{ij} = \text{non linear}(m_{i/j}, r_{ij}) \qquad i,j \in \{1,2,3\} \quad \text{(9 equations)}$$

Poincaré's analysis (1880-1890):

It has no integral (analytic) solution

Lindstedt-Fourier series diverge: **small divisors**

[Barrow-Green J. "Poincaré and the three body Problem", AMS, 1997]

Thus, **minor variations** may imply (*cause*) **major changes** in the evolution *[Poincaré, II version, published, 1890]*

Determinism ***does not imply*** **Predictibility**

Dramatic change: Crucial role of physical measure and interval (access and closeness)

Geometry of **Dynamical Systems** (Poincaré**):**

Poincaré ***confirms*** by ***broadening*** the role of *classical determinism* (causality):

1.1 measure in Physics: an *interval*

1.2 "local ≠ global" causality: local **perturbations** and **fluctuations** at the core of the *causal relations* (major future events depend on - are caused by - perturbations and fluctuations; this *changes the causal structure* w.r. to Laplace physics)

1.3 **classical randomness** *as* (deterministic) chaos

(random = not iterable in the "same context")

By this, determinism *no more implies* **predictibility**

- Crucial: **no absolute scale of measure** (as under a discrete grid)

Hints to: Dynamical unpredictibility vs. computational undecidability

- **Poincaré's theorem as a (strong) form of undecidability**:

  given the 9 equations of the three bodies problem, decide an assertion F on the future of the system

*(a **finite time** can be given beyond which F is undecidable)*

- **Gödel-Turing undecidability** (halting problem):

  at ***infinity*** (eventually)

Hoyrup M., Kolcak A., Longo G.. Computability and the Morphological Complexity of some dynamics on Continuous Domains. *Invited survey, to appear.*

Summary: Poincaré and deterministic randomness

- **Determinism** (by a set of equations) ***does not imply*** **Predictibility**

Poincaré's Memoire (last chapter, first unpublished, 1889): **Résultats négatifs**

Poincaré's **section**: a geometric paradigm for ***dynamical systems***

[Poincaré, Méthodes Nouvelles, 1892]

The **Geometry** of Dynamical systems, *recent results:*

- **Laskar, Sussman…,** 1990's: chaotic nature of the Solar System:

  Concrete Impredictibility: Pluto: > 1 Mil. Years;

  Earth: > 100 Mil. Years.

*[J. Laskar "**Large scale chaos in the Solar System**", Astron. Astrophys., 287,1994]*

The issue of "access to data" *(measure)*

Physical "access" (measure):

- In Dynamical Systems: *not exact* (and this is crucial)
- In (Relativistic) Physics: *not absolute* (and this is crucial)
- (In Quantum Physics: *indetermination, non-locality, non-separability, exact but as probability… linearity - Shroedinger)*

Discrete (the *natural* topology) **data type:**

- *Sequential*: **exact** and **absolute**
- *Networks* (concurrency, in physical space and time): **exact**, not absolute.

Iteration at the core of Computing:

1. primitive recursion
2. portability of software *(very important!)*
3. Iterate also in networks.

More on : relevance of Measure (in Physics)

- **Classical dynamics** (non-linear; sensitive): initial measure

- **Quantum Physics**: deterministic evolution, but measure as probability value

  (Shroedinger equations - linear! *in Hilbert space* - state spaces - complex variables! *Measure* based on absolute *real* values).

- The invention of **Discrete State Machines** (exact access to data)

***In order to go from a physical process to a (numeric!) computation one has to* perform a measure.**

Double pendulum

Chaos in a metric space

- **Definition** A function $f : X \to X$ is said to be **topologically transitive** if for every pair of open subsets U and V of X, there exists $n \in N$ such that

$$f^n(U) \cap V \neq \emptyset$$

Notes:

A map is topologically transitive if there will always be points that are *eventually mapped* from *any* open set to *any* other open set.

No way to separate the whole set into two disjoint subsets both invariant with respect to this map.

- **Definition** A function $f : X \to X$ has a **sensitive dependence** on initial conditions if there exist $d > 0$ such that, for all $x \in X$ and for all neighborhood V of x, there exist $y \in V$ and $n \in N$ such that

$$|f^n(x) - f^n(y)| > d.$$

Note: A map is sensitive if for any point it is possible to find other points as near as possible so that they will eventually be apart by a distance more than d.

Chaos in a metric space

Definition

A function $f : X \to X$ is **chaotic** if

1. f has *sensitive dependence* on initial conditions.
2. f is *topologically transitive*.
3. the set of periodical points is *dense* in X.

- ***Physical meaning***: unpredictability (1), undecomposability (2) and regularity (3);
- ***Physical consequence***: non-iteratability (under physical mesure).

An example in one dimension (the **logistic function,** a model of ago-antagonistic processes):

$$f_4(x) = 4x(1- x)$$

Discrete-time-trajectories are given by iterating f :

- x_0, $x_1 = f(x_0)$, $x_2 = f(x_1)$, ...

Discrete approximations ?

- **Laplace's analysis of determinism:**

  small perturbation implies *small consequence*

  (except "rare" critical cases)

- In **Newton-Laplace** *(or linear)* systems**:**

  Discretization still possible, as *approximation is preserved*

  – (*exception*: critical states and singularities: badly described in a discrete frame)

Claim: discrete mathematical structures yield an intrinsic laplacian determination

([Turing, 1950] see [Longo: Laplace, Turing and the impossible geometry… , 2007]).

Undecidability = unpredictability ?
= randomness ?

- Existing partial connections between decidability and predictability:

It is possible to **encode** Turing Machine into *some* dynamical systems;
Then, for a given trajectory, "***passing by a point***" is undecidable/unpredictable.

[Smale, …]

Technically similar to:

Matiyasevich's undecidability for Diophantine Equations (1970)

(it uses the "algebraic richness")

The approach says nothing on the *intended physical dynamics*.
(**Anthills** and Turing Machines…)

Modelling is the *other way round* (from the system into TMs)

Randomness *as* unpredictability

- **Physical definition (classical):**

A process is ***random*** when iterated in the "*same*" *initial conditions* (interval) it follows a "different trajectory".

(for a comparison to Quantum Mechanics: [Bailly, Longo, 2007])

- **Mathematical definition(s)** (deterministic systems):

Equational: *non-linear* models of physical processes thus:

1 - chaotic (e.g. non-linearity: ***mixing***)

2 - impredictability *(*physical measure)

1. *Chaotic Iterated Systems:* (X, τ (d), f), f: X → X.
2. *Recursion Theoretic*: Martin-Löf (for *infinite* sequences)
3. *Note:* there is no randomness in computing!

More on: **Unpredictability as Randomness** (Hoyrup, Rojas)

2 - Martin-Löf randomness (*infinite* sequences):
(it uses decreasing *effective* open *coverings* on Cantor's set:
Random = not being contained in any *effective covering*
= it stays "eventually outside it")

Consequence:

- No effectively generated ***subsequence***
  (unpredictable = random $\Rightarrow$ not rec. enum. (not decidable))

Recall: Poincaré's unpredictability *implies* undecidability
*But it is **stronger**, since*

there exist *non rec. enum*. sequences which are *not* Martin-Löf random (e.g. $x_1\ e_1\ x_2\ e_2\ x_3 \ldots$ x ML-random, e effective)

An application (with Kolcak, Hoyrup):

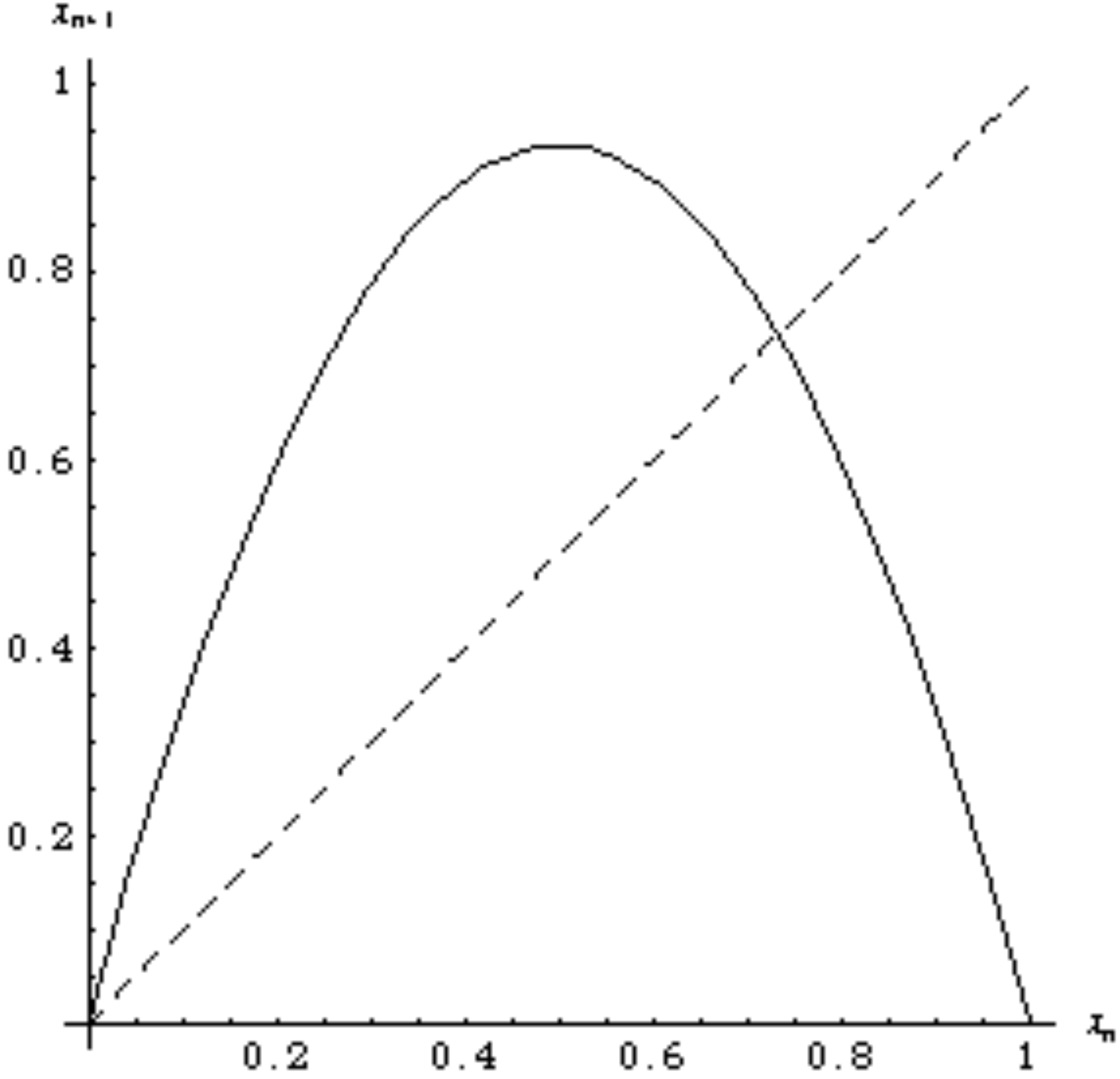
Non-linear unidimensional dynamics
the logistic functions

$$F(x) = kx(1-x)$$

Motivation:

reducibility of many several dimensions problems to one dimension

e.g. Solar system: the distance from the Sun

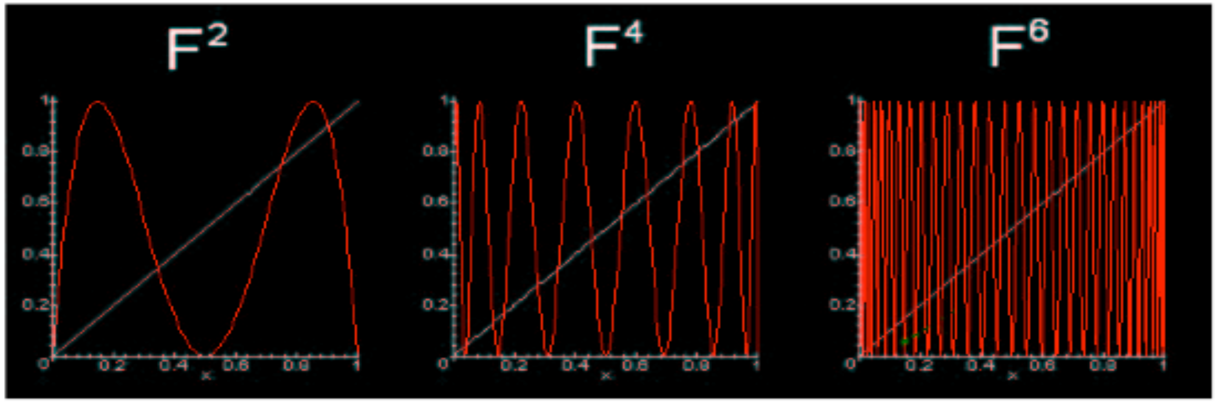


x_{n+1}
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
x_n

$$F(x) = kx(1-x)$$

- Trajectory: x_0 , $x_1 = F(x_0)$, $x_2 = F(x_1)$ …

$$x_{n+1} = F(x_n) = F^n(x_0)$$



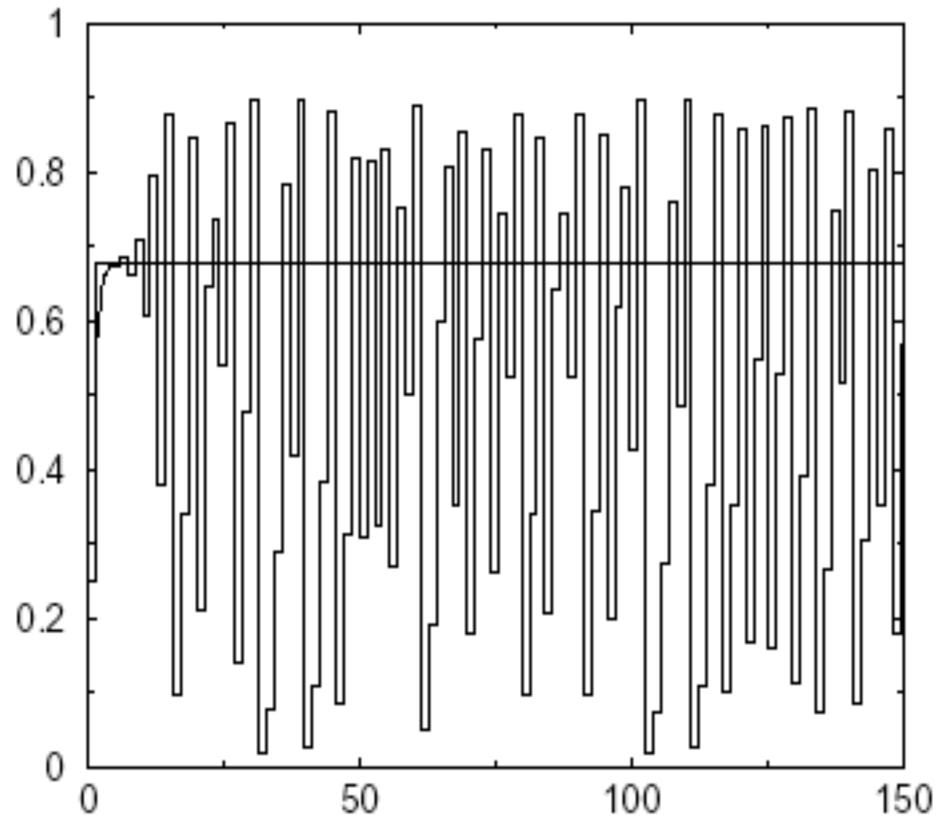
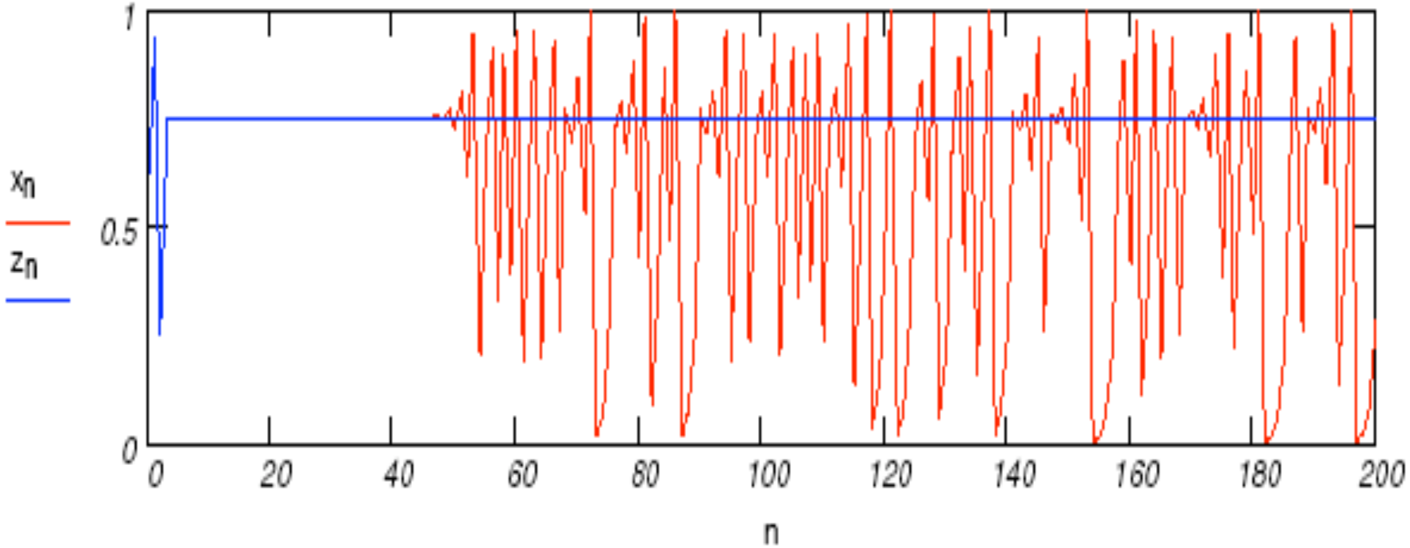


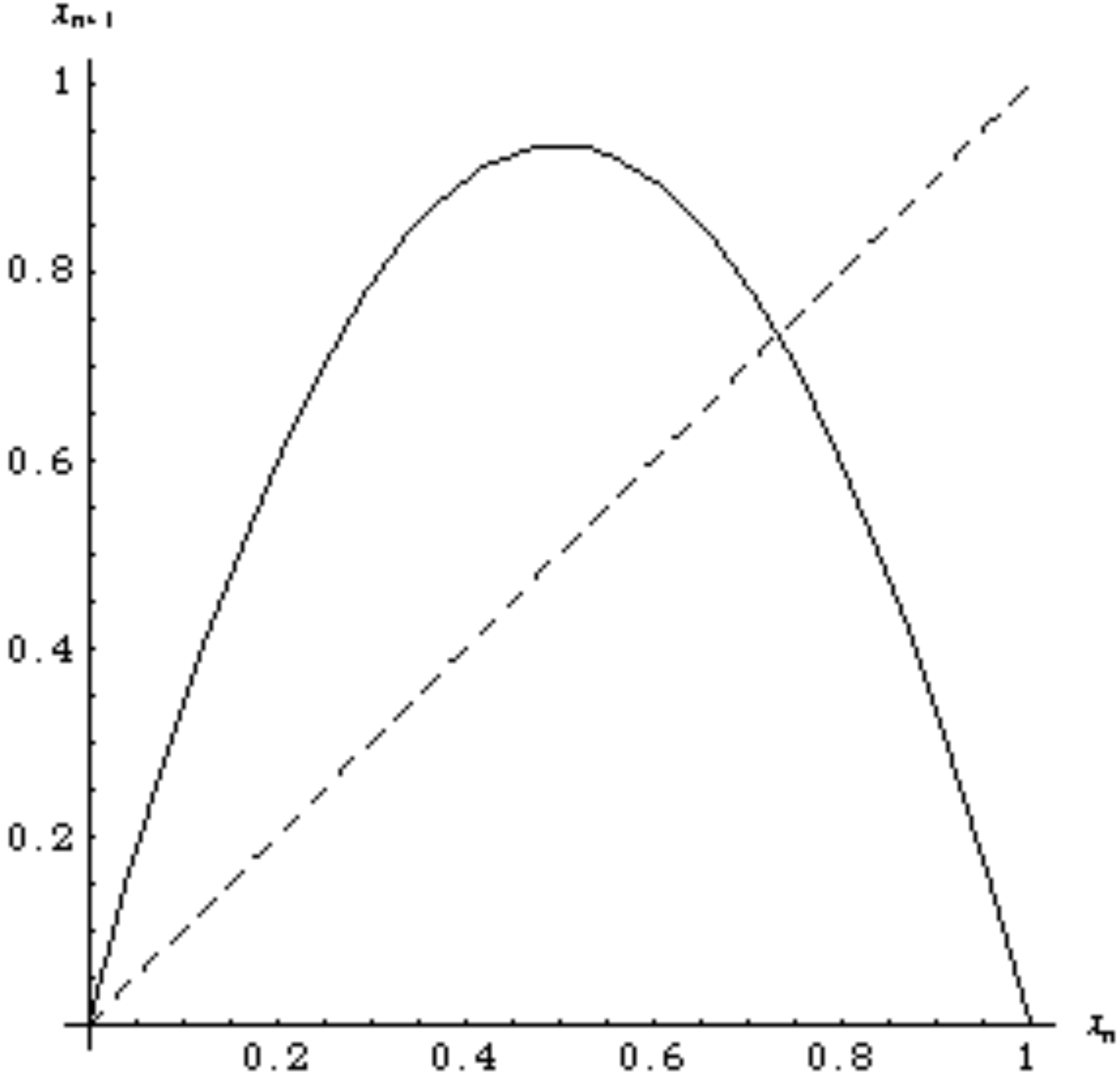
Figure 3: Discretization of the logistic equation (2) with $a = 3.1$, using a time step $h = 3/(a-1) = 1.424$ corresponding to the fully chaotic case $A = 4$, see text, Sec. 2.3.

Logistic function, chaotic case



In red, the *difference* between to input values (x_0) which differ at 10^{-16} (i.e. identical up to the 15th decimal).

The about 50th iteration can be deduced from the *exponent* in the analytic solution $y_n = -\cos(2^n(\pi - k\cos(y_0))$



x_{n+1}
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
x_n

- K = 2.2; 2.4

  One fix-point

- K = 2.5; 2.7

  One fix-point

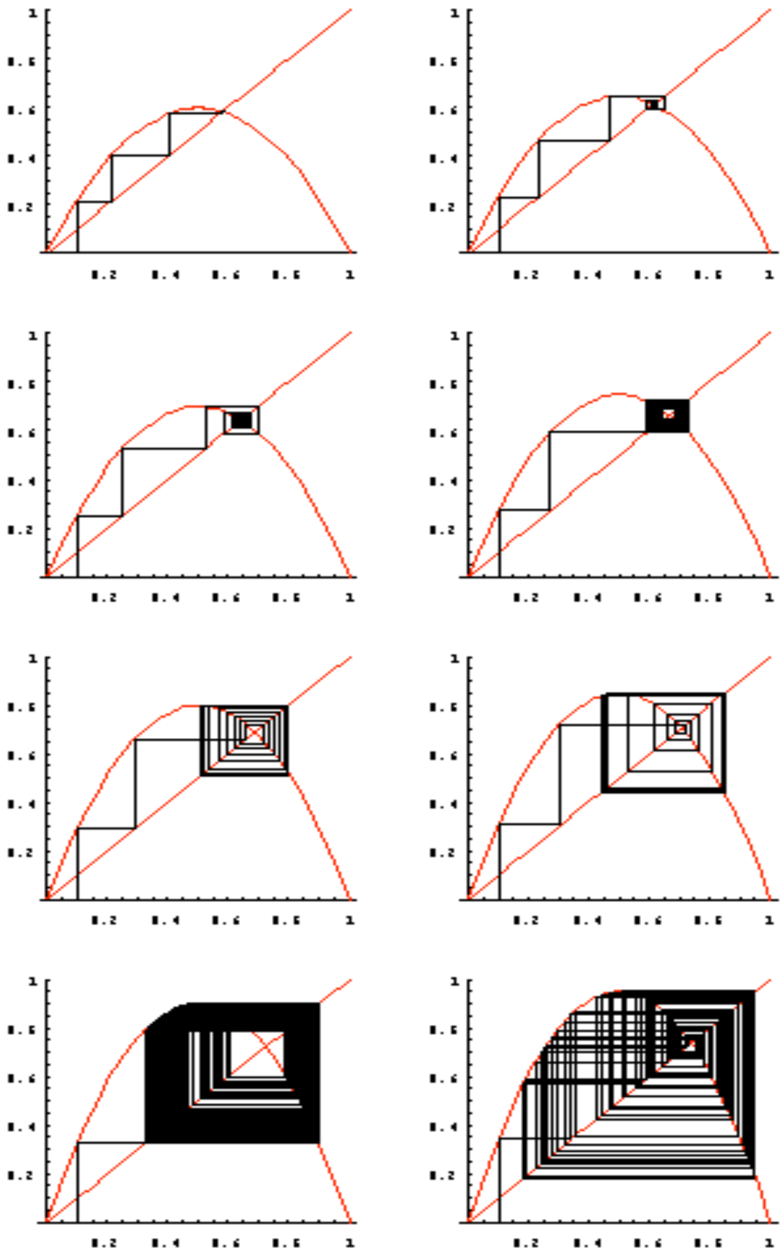
  (slow converge)

- K = 3.2; 3.4

  Stable period 2 and 6 orbits

- K = 3.7; 4

  towards chaos



Non-linear unidimensional dynamics

the logistic functions

$x_{n+1} = f(x_n) = kx_n(1 - x_n) \qquad 2 \leq k \leq 4$ (ago-antagonistic coupling)

- Very low **algebraic** complexity.
- Very high **geometric** (*morphological*) complexity:

Non-linear unidimensional dynamics

the logistic functions

$x_{n+1} = f(x_n) = kx_n(1 - x_n)$ $\quad 2 \leq k \leq 4$ $\quad$ (ago-antagonistic coupling)

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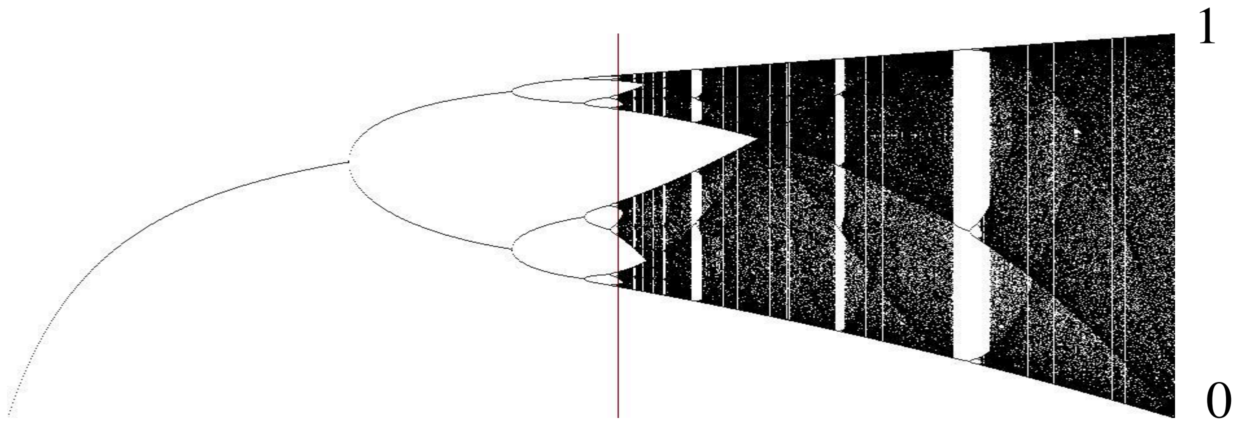
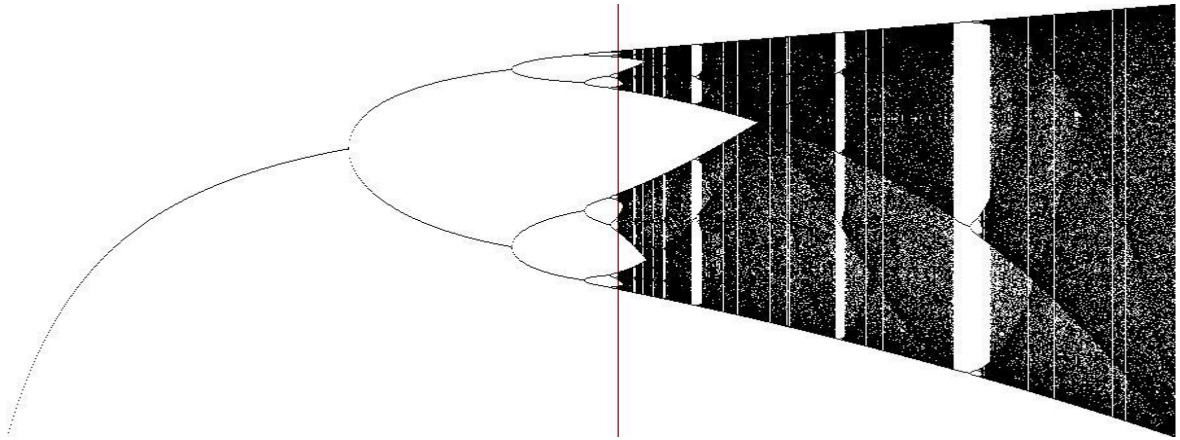


2 $\quad$ 3 $\quad$ 3.5768… $\quad$ 4 $\quad$ = k

Non-linear unidimensional dynamics: *the logistic functions*



2 3 3.5768… 4 = k

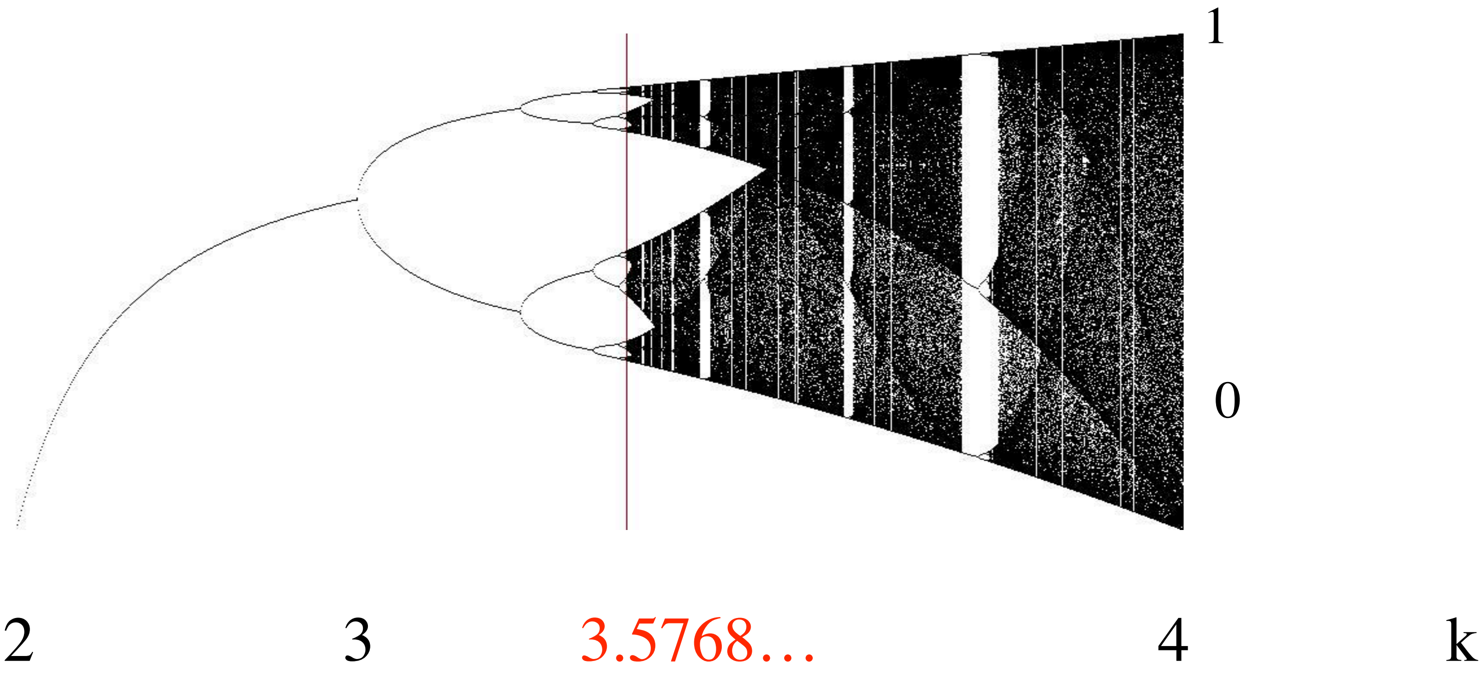
Stable fixed point;

from left: period 2 → period n (stable)…

∞ at $k_\infty = 3.5768\ldots$

Cantor(-Julia) *from right*

Non-linear unidimensional dynamics: *the logistic functions*



Critical transition at $k_\infty = 3.5768\ldots$ *Feigenbaum value*

All unstable periodic orbits of period n, for all n, up to contable infinity

From countable infinity to continuum (Cantor-Julia set)

Transition to Chaos (A. Kolcak)

Theor. 1

For any ε there exit k_1, k_2 with $4 - \varepsilon < k_1, k_2 < 4$

1. f_{k2} is a ***Feigenbaum***
2. For any $k \in [k_1, k_2[$, f_k has a ***stable periodic orbit***.

Theor. 2

$\{k \in [0,4]$ / f_k has a stable periodic orbit$\}$ is ***locally connected*** and ***dense*** in $[0,4]$.

2 - Predictibility in unidimensional dynamics

The logistic function: $\mathbf{f_k(x) = kx(1-x)}$ $k_\infty = 3.5768\ldots \leq k \leq 4$

Measure is an interval **U**:

Question: given a **target** zone **V**, ***whether*** and ***when*** U is taken, by iterating $\mathbf{f_k}$, into V ?

Predictibility Theorem *(A. Kolcak)*
(*Continuous case:* real intervals U, V):
"For any interval U, V in [0,1], whether there exists an m such that $\mathbf{f_k}^m(U) \subseteq V$ may be reduced to a finite number of cases"

Three cases:

1 - $\mathbf{f_k}$ has stable fixed points ($k < k_\infty$):

2 - $\mathbf{f_k}$ is chaotic (e. g. $k = 4$)

3 - $\mathbf{f_k}$ is critical (no stable periodic orbits nor sensitivity to initial conditions)

“For any interval U, V in [0,1], is there an m such that $\mathbf{f_k}^m(U) \subseteq V$?”

1 - $\mathbf{f_k}$ has **stable fixed points** ($k < k_\infty$):

- if k is rational, m can be uniformely effectively computed
- if k is a computable real (and not rational), the non-existence of m is semi-decidable

2 - $\mathbf{f_k}$ is **chaotic**:

- k = 4 : the problem is decidable (easy: wait for m such that $\mathbf{f_k}^m(U) = [0,1]$)
- $k_\infty < k < 4$:
  - if k is rational, m can be uniformely effectively computed
  - if k is a computable real (and not rational), the existence of m is not semi-decidable

3 - $\mathbf{f_k}$ is **critical** (no stable periodic orbits nor sensitivity to initial conditions, e.g. $k = k_\infty$):

- if k is rational, m can be uniformely effectively computed
- if k is a computable real (and not rational), the existence of m is not semi-decidable

Periodic orbits "organize" the dynamics

On computable reals *(M. Hoyrup)* **:**

Theorem 1

The set of periodic points of a chaotic dynamical system is ***not*** *semi-decidable* ***nor*** *co-semi-decidable*.

Theorem 2

The set of points of a chaotic dynamical system whose orbits never fall into a given closed interval is ***not*** *semi-decidable*.

Theorem 3

k_∞ is a computable real.

The shadowing lemma

Computational problem: the **round-off**

– **Mathematically:** "*shadowing lemma*" for hyperbolic dynamical systems (D, f)

For any x_0 and δ there is an ε such that, for any ε-approximated (or rounded-off $\leq \varepsilon$) trajectory, there is one in the continuum which goes δ -close to it, at each step.

(M. Hoyrup: on the Logistic function)

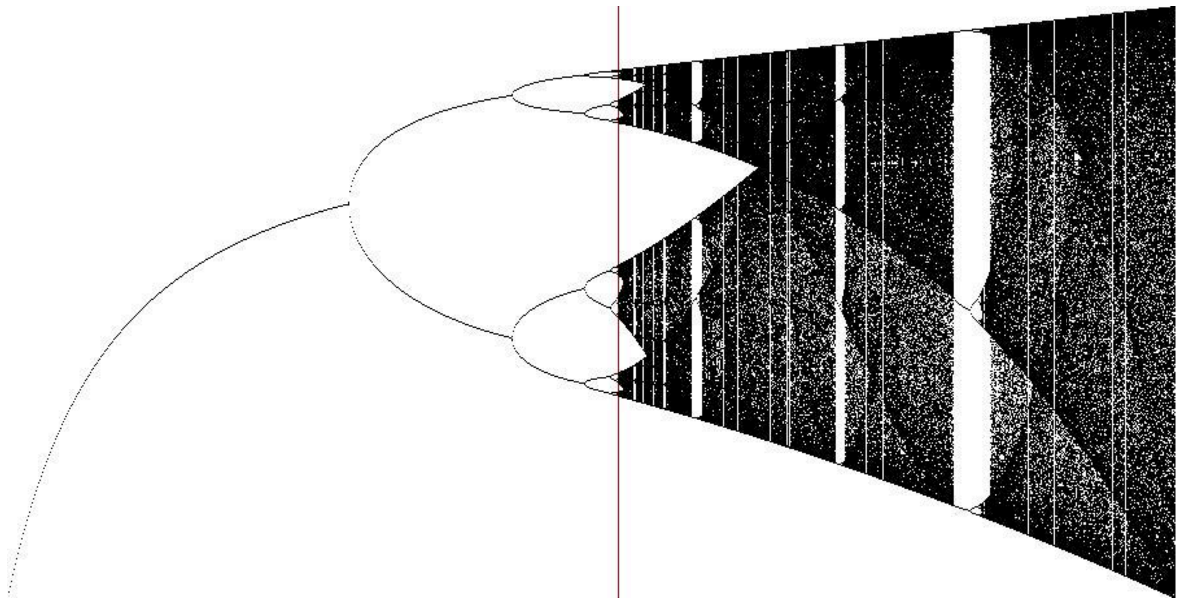
– **Informally:**

Given a "sufficiently regular" non-linear iterated function system, *any digital sequences* can be actually approximated by a continuous one **(but, in general *not the converse!*)**

Or … there are so many continuous trajectories, that, given a discrete one, you can find a continuous sequence which goes close to it, see the reflection of the *double pendulum* in [Longo, 2002; 2003].

Ongoing work: extend Shadowing; relevant counterexamples

A critical point in Physics:
a singularity, one point-parameter



0

2 3 3.5768… 4 k

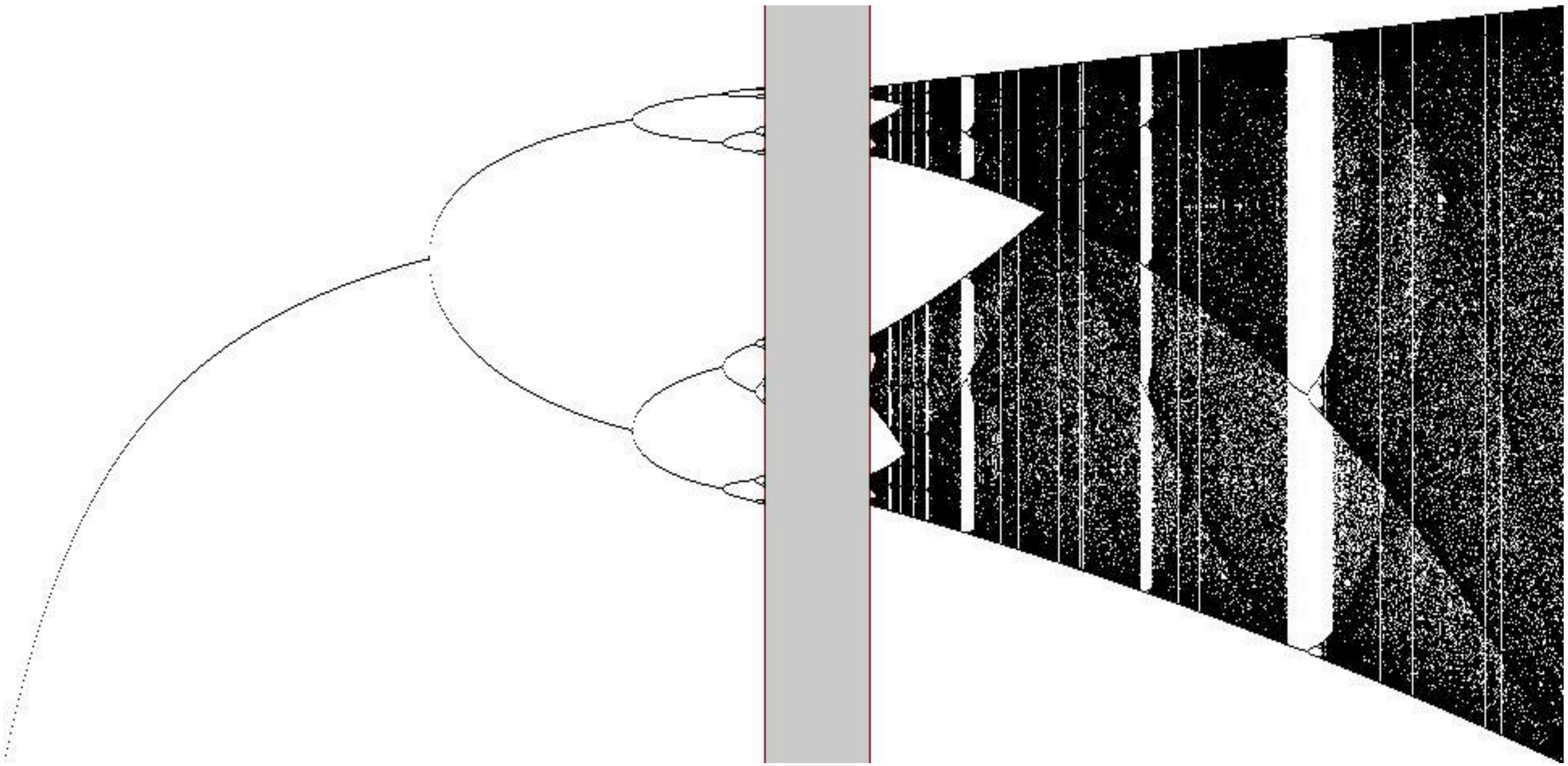
Logistic function: Critical transition at k_∞ = 3.5768…

*from **countable** infinity to **continuum*** (Cantor-Julia multifractal)

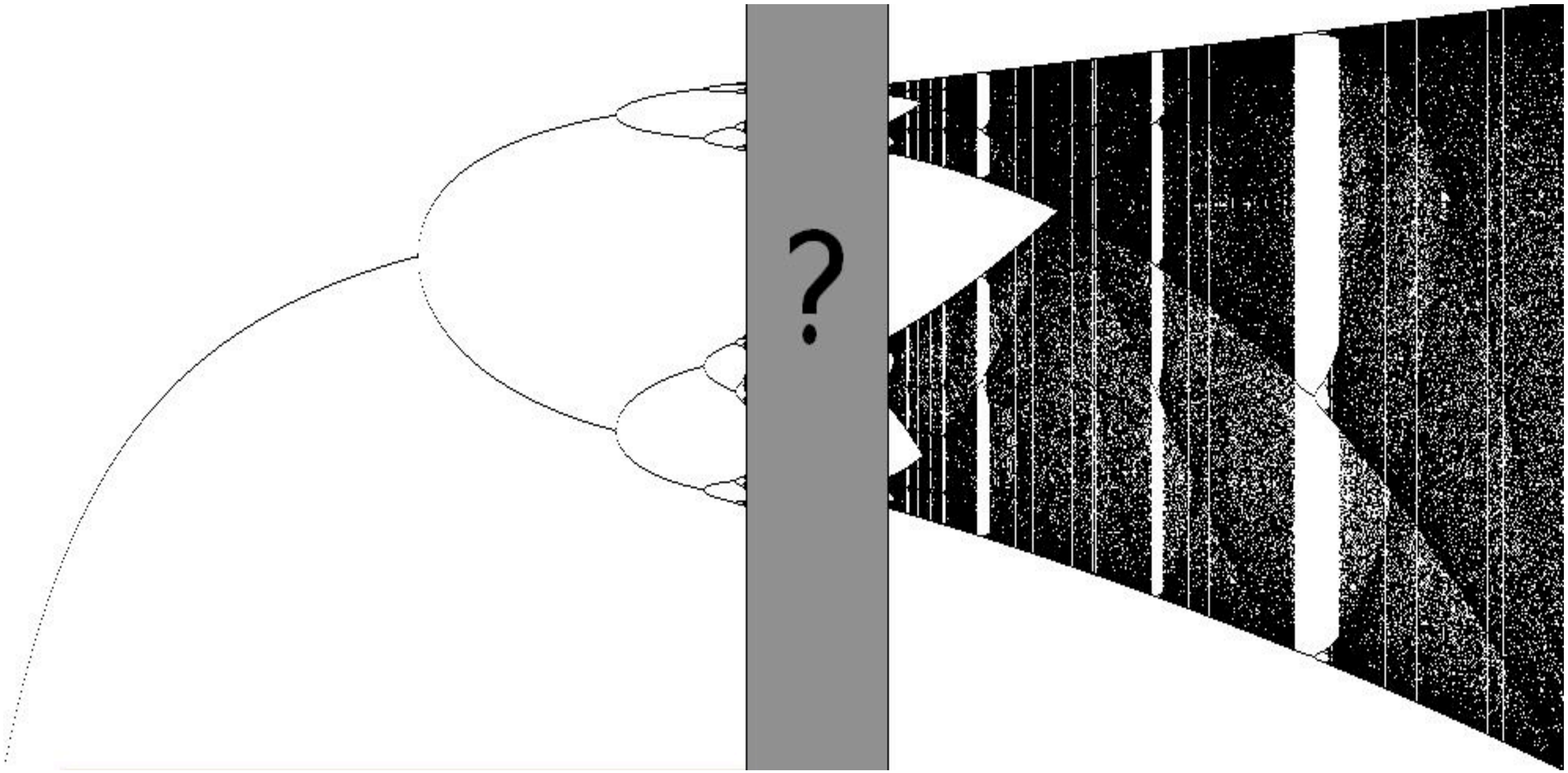
Extending criticality

An "infinitary" transition: ***countable infinite*** --> ***continuum***

(no CH)



Extending criticality



Physical Singularity of Life Phenomena

- Physical variation vs. **Variability (individuation)**

- Physics*: Specific* trajectoires (geodetics) and *generic* objects
  Biology: ***generic trajectories*** (compatible) and ***specific objects***

- Physics: *determistic impredictibility* or *indetermination* within a given phase space
  Biology: Intrinsic indetermination due to change of the phase space - phylogenesis (ontogenesis?)

- **Biology: Extended critical situations** (from *physical criticality*)

- Physics***:*** **energy** as operator***,*** **time** as parameter;
  Biology*: **time** as operator, **energy** as parameter*

Bailly F., Longo G. **Mathématiques et sciences de la nature. La singularité physique du vivant,** *Hermann, Paris, 2006.*

Some references

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