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LAMBDA-CALCULUS MODELS AND EXTENSIONALITY

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§ 0. Introduction

This paper grew out of the authors' attempts to understand the ω-rule and extensionality in models of the λ-calculus. One problem we found was that, in contrast to combinators, for λ-equality an explicit definition of model did not exist.

More precisely, there seems to be, firstly an assumption in the literature that the definition is too obvious to need stating, and secondly a disagreement about what the definition should be. For example, when DANA SCOTT constructed his models, e.g. in [18], it is clear from his work that he had in mind a definition more or less equivalent to the one in § 3 below. But he did not state any definition in print.[1]) In contrast, BARENDREGT et al. in [4, Part III] seemed to assume that the concept of λ-model was at the least very similar to that of combinatory model.[2]) And in BARENDREGT's thesis [1], when the topic of models comes up, he carefully restricts himself to combinatory models (chapter III). Finally, in BARENDREGT's [5], which came to the present authors' notice during the preparation of this paper, an explicit definition is at last given. But it is different from both those above.[3]) So, which definition should one use? Now that some variety of models has been constructed, it seems a good time to try to develop a general model-theory; but such a theory will not get very far without a definition of model!

In this paper we shall try to do four things: (i) to clarify the basic concepts involved, (ii) to give an acceptable and natural definition of λ-model, (iii) to explore extensionality and various forms of the ω-rule, and (iv), to set out clearly the relation between λ-models and combinatory models. All these goals are interdependent, and the key concept will turn out to be the relation of "*extensional equivalence*",

$$a \sim b \Leftrightarrow \forall d \; ad = bd.$$

The reader will be assumed to know a little about the three main kinds of model that have appeared in the literature so far. These are: (i) *term models*, which are merely sets of λ-terms modulo a syntactically-defined equivalence, (see for example [1]); (ii) SCOTT's D_∞ and its relatives, which are inverse limits of lattices, (see [16], [17], [20]); (iii) the PLOTKIN-SCOTT *model* $\mathscr{P}\omega$ and its variants, ([12], [18], [14], [15]).

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1) After § 3 below was written, SCOTT showed the authors some unpublished notes of 1976 containing a definition exactly equivalent to the one in § 3 (although formally different from it). See 3.6.

2) This identification is stated explicitly in MILNER [11], but MILNER is only interested in extensional models, and for these it turns out that all the different possible definitions coincide, as will be seen below.

3) In fact the definition (1.14, [5, p. 1099]) can easily be proved equivalent to the definition in § 2 below easily be proved equivalent to the definition in § 2 below of "pseudo-model of $\lambda\beta$", but w.e. λ-algebras (1.15, p. 1099) do not coincide with "models of $\lambda\beta$" of § 3 below (see footnotes 1 and 2 on p. 293).

The present paper will only be concerned with type-free systems. But most of its comments will apply also to models of typed systems, with some obvious changes.

We are very grateful to several people who have read and criticized earlier drafts of this paper and suggested improvements, especially DANA SCOTT, HENK BARENDREGT, GÉRARD BERRY and GORDON PLOTKIN. The present version incorporates some of their suggestions, which will be acknowledged at the places where they are used; responsibility for any errors remaining is entirely our own.

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§ 1. Syntax

Before defining "model of the λ-calculus", we must first define "λ-calculus", or rather, "λ-calculi". The notation will all be from CURRY and FEYS [7] Chapters 3 and 4, or HINDLEY, LERCHER, SELDIN [9], Chapters 1 and 2.

1.1. Definition. *The formal theory* $\lambda\beta$.

λ-terms: variables, (MN), $(\lambda x.M)$.

Formulae: equations $X = Y$ for terms X, Y.

Axioms: three infinite sets of axioms, namely all the instances of the following three axiom-schemes, for particular terms M, N and variables x, y:

(ϱ) $\quad M = M$;

(α) $\quad \lambda x.M = \lambda y.[y/x]\,M \quad$ (y not free in M);

(β) $\quad (\lambda x.M)\,N = [N/x]\,M$.

Deduction-rules: symmetry, transitivity, and

$$(\mu)\ \frac{Y_1 = Y_2}{XY_1 = XY_2} \qquad (\nu)\ \frac{X_1 = X_2}{X_1Y = X_2Y} \qquad (\xi)\ \frac{M = N}{\lambda x.M = \lambda x.N}.$$

1.2. Definition. *The theory* $\lambda\beta\eta$ is the result of adding to $\lambda\beta$ the following infinite set of axioms:

(η) $\quad \lambda x.Mx = M \quad$ (x not free in M).

1.3. Notation. Provability in $\lambda\beta$, $\lambda\beta\eta$ will be denoted by "$\lambda\beta \vdash$", "$\lambda\beta\eta \vdash$". The set of all λ-terms will be called *Terms*, and that of the variables *Vars*. A term without free variables will be called *closed*. The set of free variables of a term M will be called FV(M). A sequence $x_1, \ldots, x_n$ will sometimes be called "$\vec{x}$".

§ 2. Pseudo-models

We shall define "model" by first defining a concept of "λ-structure", and then choosing those structures which are to be models. There is some latitude in the definition of structure, and in this section we shall follow the method which seems the most naive and straightforward.

2.1. Definition. *Pseudo-λ-structures.* For any set D, an *environment* or *valuation of the variables* is any map σ: *Vars* $\to D$. A *valuation of the terms* is any map from *Terms* into D. A *pseudo-(λ-) structure* is a triple $\mathscr{D} = \langle D, \circ, [\![\;]\!]\rangle$ where D is a set with at least two members, $\circ$ is a map from D^2 into D, and $[\![\;]\!]$ is a map which associates to each valuation σ of the variables a valuation $[\![\;]\!]_\sigma$ of all the terms, such that

(i) $[\![x]\!]_\sigma = \sigma(x)$,

(ii) $[\![XY]\!]_\sigma = ([\![X]\!]_\sigma) \circ ([\![Y]\!]_\sigma)$,

(iii) $([\![\lambda x.M]\!]_\sigma) \circ d = [\![M]\!]_{(\sigma_x{}^d)}$ for all x, M and all $d \in D$,

(iv) if $\sigma(x) = \tau(x)$ for all x free in M, then $[\![M]\!]_\sigma = [\![M]\!]_\tau$,

(v) $[\![\lambda x.M]\!]_\sigma = [\![\lambda y.[y/x]\, M]\!]_\sigma$ if $y \notin \mathrm{FV}(M)$.

2.2. Notation. σ_x^d is the map which is the same as σ except that x is given the value d. Sometimes $[\![\;]\!]_\sigma$ will be called "$[\![\;]\!]\sigma$" to simplify printing, and often the "σ" will be omitted altogether.

2.3. Remark. Condition (iii) implies that for each σ, M, x there is a member f of D such that

$$\forall d \in D.\, f \circ d = [\![M]\!]\,(\sigma_x^d).$$

And one such f is chosen to be called $[\![\lambda x.M]\!]_\sigma$. The definition does not insist that f be unique; D might be generous, and offer many such f. If this happens, then $[\![\;]\!]$ is essentially a choice-operator, picking out one f for each σ, x, M; and each possible set of choices determines a different pseudo-structure (as long as it satisfies (iii) and (iv)). Note, by the way, that (iii) does not say that $\mathscr{D}$ satisfies (β) (see 2.9, 2.8).

Condition (iv) is included simply to prevent "crazy" pseudo-structures. If σ and τ differed only on a finite set of variables, and $\mathscr{D}$ was a pseudo-model (see below), then (iv) would follow from the other clauses. But if σ and τ differ on an infinite set, then (iv) seems to be independent. Given (iv), each closed term M defines a fixed member $[\![M]\!]$ of D, independently of σ.

Condition (v) corresponds to (α) of the formal λ-calculus.[1])

2.4. Definition. The *interior domain*, D^0, of a pseudo-structure $\mathscr{D}$ is the set of all $[\![M]\!]$ for all closed terms M; its members are said to be *λ-definable*. (For more about D^0, see § 6.)

2.5. Definition. An equation $X = Y$ is *satisfied in $\mathscr{D}$ by σ* (or *true in $\mathscr{D}$ for σ*) iff $[\![X]\!]_\sigma$ is the same as $[\![Y]\!]_\sigma$. It is *satisfied by $\mathscr{D}$* or *true in $\mathscr{D}$* iff it is true for all σ. (For short, $\mathscr{D} \models X = Y$.)

2.6. Definition. A *pseudo-model* of $\lambda\beta$ or $\lambda\beta\eta$ or of any other formal system is a pseudo-structure which satisfies all the equations provable in the system.

This concept of model is very natural, if one views $\lambda\beta$ merely as a set of equations And it is a close analogue of the concept for combinatory equality. (See § 8.) Some of the properties of combinatory models apply here without change, for example the interior of a pseudo-model also forms a pseudo-model, as one would expect (Lemma 6.2)

[1]) In a previous version of this paper, (v) was omitted, and its inclusion is due to Gérard Berry, who pointed out that without it, the proof of Proposition 3.5 was incomplete.

2.7. Lemma. *Every pseudo-λ-structure has the following properties for substitution of variables for variables*:

(1) *For all* σ, x, y, M *with* y *not free in* M,

$$\sigma(x) = \sigma(y) \Rightarrow [\![M]\!]_\sigma = [\![[y/x]\, M]\!]_\sigma,$$

(2) *For all* σ, τ, M, *if* $\mathrm{FV}(M) = \{x_1, \ldots, x_n\}$ *and* $y_1, \ldots, y_n \notin \mathrm{FV}(M)$, *then*

$$\{\forall i \leqq n \sigma(x_i) = \tau(y_i)\} \Rightarrow [\![M]\!]_\sigma = [\![[\vec{y}/\vec{x}]\, M]\!]_\tau.$$

Proof. For (1): assume $\sigma(x) = \sigma(y) = d$. Then

$$\begin{aligned} [\![M]\!]_\sigma = [\![M]\!]\,(\sigma_x^d) &= [\![\lambda x.M]\!] \circ d && \text{by } \sigma(x) = d \text{ and 2.1 (iii)} \\ &= [\![\lambda y.[y/x]\, M]\!] \circ d && \text{by 2.1 (v)} \\ &= [\![[y/x]\, M]\!]\,(\sigma_y^d) && \text{by 2.1 (iii)} \\ &= [\![[y/x]\, M]\!]_\sigma && \text{since } \sigma(y) = d. \end{aligned}$$

For (2): let $\sigma(x_i) = \tau(y_i) = d_i$ for $i = 1, \ldots, n$. Then

$$\begin{aligned} [\![M]\!]_\sigma &= [\![M]\!]\,(\sigma_{\vec{y}}^{\vec{d}}) && \text{by 2.1 (iv)} \\ &= [\![[\vec{y}/\vec{x}]\, M]\!]\,(\sigma_{\vec{y}}^{\vec{d}}) && \text{by (1) repeated} \\ &= [\![[\vec{y}/\vec{x}]\, M]\!]_\tau && \text{by 2.1 (iv).} \end{aligned}$$

2.8. Lemma. *A pseudo-structure satisfies the axioms* (β) *iff it has the following property for substitution of arbitrary terms for variables*:

(3) $\quad [\![[N/x]\, M]\!]_\sigma = [\![M]\!]\,(\sigma_\sigma^n), \quad$ *where* $\quad n = [\![N]\!]_\sigma$.

Proof. First note that $[\![(\lambda x.M)\, N]\!]_\sigma = ([\![\lambda x.M]\!]_\sigma) \circ n = [\![M]\!]\,(\sigma_x^n)$ by Definition 2.1 (iii). Then, to get $(\beta) \Rightarrow (3)$, evaluate the left-hand side of this by (β). And to get the converse, evaluate the right-hand side by (3).

2.9. Remark. Property (3) is a generalization of (2). It might seem that (3) could be proved by induction on M in all pseudo-structures. And indeed it would be, if M was restricted to terms not containing λ. But for general M, it is not. However, in Remark 8.18 it will be proved that any pseudo-structure can be changed into one satisfying (β) and hence (3), by a systematic alteration of the $[\![\ ]\!]$-map.

2.10. Conclusion. Despite the good points mentioned in 2.6, pseudo-models are too weak. Consider the rule

$(\xi) \quad M = N \vdash \lambda x.M = \lambda x.N.$

It is possible to imagine a pseudo-model of $\lambda\beta$ in which an equation $M = N$ is true for all values of x, but not $\lambda\beta$-provable. And for such an equation, $\lambda x.M = \lambda x.N$ need not be true. How can it fail? Well, for any σ, $[\![\lambda x.M]\!]_\sigma$ is an $f \in D$ such that for all $d \in D$,

$$\begin{aligned} f \circ d &= [\![M]\!]\,(\sigma_x^d) \\ &= [\![N]\!]\,(\sigma_x^d) \quad \text{if } M = N \text{ is true.} \end{aligned}$$

And $[\![\lambda x.N]\!]_\sigma$ is a $g \in D$ such that for all $d \in D$, $g \circ d = [\![N]\!]\,(\sigma_x^d)$. So $f \circ d = g \circ d$ for $d \in D$. But this does not imply $f = g$.

So this "failure" of (ξ) in pseudo-models is possible, even though all the equations that follow from (ξ) in $\lambda\beta$ may be true. This means that not every first-order formal deduction will be valid for each individual valuation in each pseudo-model. For example, suppose one wished to check that a pseudo-model of $\lambda\beta$ was a pseudo-model of $\lambda\beta\eta$. One could not do this simply by checking all instances of (η); one would also have to check all the consequences deducible from (η) by rule (ξ). And in fact an example will be given in § 7 of a pseudo-model which satisfies (η), but does not satisfy the simple consequence

$$\lambda xy.xy = \lambda x.x.$$

Therefore a stronger definition of model is needed. Despite this, pseudo-structures do have some interesting properties: for example they correspond exactly to combinatory models (see § 8).[1])

§ 3. Models

3.1. Definition. A (λ-) *structure* is a pseudo-λ-structure $\langle D, \circ, [\![\ ]\!]\rangle$ which satisfies the rule (ξ) in the following sense:

(ξ) for all σ, x, M, N

$$\{\forall d \in D.\ [\![M]\!]\,(\sigma_x^d) = [\![N]\!]\,(\sigma_x^d)\} \Rightarrow [\![\lambda x.M]\!]_\sigma = [\![\lambda x.N]\!]_\sigma.^{2)}$$

3.2. Definition. A *model* of $\lambda\beta$ (or $\lambda\beta\eta$, etc.) is a λ-structure in which all the equations provable in the formal system are true.

3.3. Remark. A pseudo-structure $\mathscr{D}$ is a model of $\lambda\beta$ in the above sense just when it is a model, in the usual Tarski sense, of the following first-order form of $\lambda\beta$:

formulae: equations between λ-terms, also $\varphi \supset \psi$, $\neg\varphi$, $\forall x\varphi$;

[1]) In fact Barendregt, in [5] Definition 1.14, takes pseudo-models (or rather, a definition which turns out equivalent to them) as his basic definition of λ-calculus model. He also discusses the role of (ξ), and calls models satisfying (ξ) "weakly extensional". But his definition of satisfaction of (ξ) is rather weak: it does not allow a formal deduction to be carried through in the model for one particular valuation of the variables (see Footnote 2 below). Also his definition of model mixes syntactical and semantical structure in what seems to the present authors to be a slightly less natural way than the approach above.

By the way, [5] and the present paper are completely independent, despite similarities in places, including giving the same counterexamples as in § 7 below. The similarities just show that the concepts involved are the natural ones to select.

[2]) It is possible to interpret "satisfies (ξ)" in another sense; namely, for all x, M, N, if $[\![M]\!]_\sigma = [\![N]\!]_\sigma$ for all σ, then $[\![\lambda x.M]\!]_\sigma = [\![\lambda x.N]\!]_\sigma$ for all σ. This is weaker than (ξ) above; in fact, if $v_1, \ldots, v_n$ are the free variables of MN, then the Definition 3.1 sense corresponds to

$$\forall v_1 \ldots v_n((\forall x\, M = N) \supset \lambda x.M = \lambda x.N),$$

and the second sense corresponds to

$$(\forall v_1 \ldots v_n \forall x\, M = N) \supset \forall v_1 \ldots v_n(\lambda x.M = \lambda x.N).$$

The second sense is not the meaning usually intended by deduction rules containing free variables in logic, as such variables are commonly regarded as "parameters"; see for example the discussion in Kleene "Introduction to Metamathematics" § 22, pp. 149–150. Recently however, G. Plotkin and G. Berry have suggested (in correspondence) an interpretation of the weaker sense in terms of the λ-definable functions over D. This will be discussed in the Appendix.

deduction-rules: the usual logical rules for $\supset$, $\neg$, $\forall$, with obvious modifications in the $\forall$-rules to take account of bound variables in terms (see for example BARENDREGT [2], pp. 210–211);

axioms: logical axioms; reflexivity, symmetry and transitivity of equality; (α), (β); also

$$\neg(\lambda xy.x = \lambda xy.y),$$

$$\forall x, y_1, y_2(y_1 = y_2 \,.\supset.\, xy_1 = xy_2 \,\&\, y_1x = y_2x),$$

and finally, for each x, M, N an axiom

$$(\xi) \qquad \forall v_1, \ldots, v_n((\forall x(M = N)) \supset \lambda x.M = \lambda x.N)$$

where $v_1, \ldots, v_n$ are the free variables of MN other than x.[1])

For $\lambda\beta\eta$ only one axiom has to be added (cf. Lemma 4.3), namely

$$\forall y(\lambda x.yx = y).$$

3.4. Lemma. (G. BERRY) *Let $\mathcal{D}$ be a pseudo-λ-structure; then $\mathcal{D}$ satisfies (ξ) iff $\mathcal{D}$ satisfies the following generalization of (ξ):*

$(\xi)'$ *for all σ, τ, x, y, M, N*

$$\{\forall d \in D.[\![M]\!]\,(\sigma_x^d) = [\![N]\!]\,(\tau_y^d)\} \Rightarrow [\![\lambda x.M]\!]_\sigma = [\![\lambda y.N]\!]_\tau.$$

Proof. That $(\xi)'$ implies (ξ) is obvious. For $(\xi) \Rightarrow (\xi)'$, let $\{u_1, \ldots, u_n\} = \mathrm{FV}(M) - \{x\}$ and $\{v_1, \ldots, v_m\} = \mathrm{FV}(N) - \{y\}$, and choose distinct new variables

$$t, z_1, \ldots, z_n, w_1, \ldots, w_m$$

not in $\mathrm{FV}(MNxy)$. Define an environment ϱ by setting

$$\varrho(z_i) = \sigma(u_i) \quad \text{for } i = 1, \ldots, n,$$

$$\varrho(w_i) = \tau(v_i) \quad \text{for } i = 1, \ldots, m.$$

Define $M' = [t/x, \vec{z}/\vec{u}]\,M$, $N' = [t/y, \vec{w}/\vec{v}]\,N$. Then for all $d \in D$ one has

$$\begin{aligned} [\![M']\!]\,(\varrho_t^d) &= [\![M]\!]\,(\sigma_x^d) && \text{by Lemma 2.7 (2)} \\ &= [\![N]\!]\,(\tau_y^d) && \text{by hypothesis of } (\xi)' \\ &= [\![N']\!]\,(\varrho_t^d) && \text{by 2.7 (2).} \end{aligned}$$

So by (ξ) one can deduce

$$[\![\lambda t.M']\!]_\varrho = [\![\lambda t.N']\!]_\varrho.$$

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[1]) By the way, the following ways of expressing (ξ) will not work:

(i) a single axiom $\forall u\,\forall v(\forall x(u = v) \,.\supset.\, \lambda x.u = \lambda x.v)$;

(ii) an axiom $\forall u\,\forall v(\forall x(ux = vx) \,.\supset.\, \lambda x.ux = \lambda x.vx)$;

(iii) an axiom $\forall u\,\forall v(u = v \,.\supset.\, \lambda x.u = \lambda x.v)$;

(iv) an infinity of axioms $\forall v_1 \ldots v_n\,\forall x(M = N \supset \lambda x.M = \lambda x.N)$.

Axioms (i) and (iii) fail because terms containing x free cannot be substituted for u and v; (ii) fails because the equation $\lambda x.x = \lambda x.(\lambda y.y)\,x$ could not be deduced; and (iv) fails because it would imply for example $fI = gI \supset \lambda x.fx = \lambda x.gx$.

Speaking about (iv), it is precisely because (ξ) cannot be expressed as a universally-quantified implication between equations that the class of all models of $\lambda\beta$ is not as simple as for combinatory logic. (See § 8.)

Hence

$$
\begin{aligned}
[\![\lambda x.M]\!]_\sigma &= [\![\lambda t.[t/x]\,M]\!]_\sigma && \text{by 2.1 (v)}\\
&= [\![\lambda t.M']\!]_\varrho && \text{by 2.7 (2)}\\
&= [\![\lambda t.N']\!]_\varrho && \text{by above}\\
&= [\![\lambda y.N]\!]_\tau && \text{by 2.7 (2) and 2.1 (v).}
\end{aligned}
$$

3.5. Proposition. *Every λ-structure is a model of $\lambda\beta$.*

Proof. Let $\mathscr{D}$ be a λ-structure. Then (ξ) holds in $\mathscr{D}$, and hence also $(\xi)'$ by Lemma 3.4. By induction on the structure of M, a straightforward proof gives

(3) $\quad [\![[N/x]\,M]\!]_\sigma = [\![M]\!]\,(\sigma^n_x) \quad$ where $n = [\![N]\!]_\sigma$.

Typical case: $M \equiv \lambda y.P$, $y \not\equiv x$, $y \notin \mathrm{FV}(N)$: for all $d \in D$, by the induction hypothesis one has $[\![[N/x]\,P]\!]\,(\sigma^d_y) = [\![P]\!]\,(\sigma^{n\,d}_{x\,y})$. Hence by Lemma 3.4 $(\xi)'$ applied to σ and σ^n_x,

$$[\![\lambda y.[N/x]\,P]\!]_\sigma = [\![\lambda y.P]\!]\,(\sigma^n_x).$$

Since $[N/x]\,(\lambda y.P) = \lambda y.[N/x]\,P$, one has (3).

After (3) is proved, all the $\lambda\beta$-provable equations follow: (β) comes by Lemma 2.8, (α) is Definition 2.1 (v), and the rules of $\lambda\beta$ all obviously lead from true equations to true equations.

3.6. Discussion. What are structures like? Firstly, for any pseudo-structure $\mathscr{D}$ define *extensional equivalence* $(\sim)$ by $a \sim b \Leftrightarrow \forall d \in D.\, a \circ d = b \circ d$. $\quad (a, b \in D.)$

This relation partitions D into disjoint non-empty equivalence-classes; for each $d \in D$ let $[d]$ be the class containing d.

Each function $\varphi\colon D \to D$ has a set (which might be empty) of *representatives* in D, namely

$$\{f\colon f \in D \text{ and } \forall d \in D.\, f \circ d = \varphi(d)\}.$$

If this set is not empty, it is an extensional-equivalence-class. Conversely, each extensional-equivalence-class determines a representable function, and this correspondence between equivalence-classes and representable functions is one-to-one.

Now, in any pseudo-structure $\mathscr{D}$, each triple M, x, σ determines a function, call it $\psi_{Mx\sigma}$, by the rule

(4) $\quad \forall d \in D.\, \psi_{Mx\sigma}(d) = [\![M]\!]\,(\sigma^d_x)$.

And this $\psi_{Mx\sigma}$ has a representative in D, namely $[\![\lambda x.M]\!]_\sigma$.

If $\mathscr{D}$ is a structure, then by Lemma 3.4, $\mathscr{D}$ satisfies $(\xi)'$, which can easily be seen to imply

(5) $\quad [\![\lambda x.M]\!]_\sigma \sim [\![\lambda y.N]\!]_\tau \Rightarrow [\![\lambda x.M]\!]_\sigma = [\![\lambda y.N]\!]_\tau$.

In other words, if M, x, σ and N, y, τ both determine the same function over D, that is if $\psi_{Mx\sigma} = \psi_{Ny\tau}$, then $[\![\lambda x.M]\!]_\sigma = [\![\lambda y.N]\!]_\tau$. Thus in a λ-structure the value of $[\![\lambda x.M]\!]_\sigma$ does not depend on the formal construction of M at all, nor on x or σ, but only on the function $\psi_{Mx\sigma}$ they generate. In terms of extensional-equivalence-classes, (5) says that in each extensional-equivalence-class $C \subseteq D$ there is one member, call it f_C, such that

(6) $\quad \forall M, x, \sigma\{[\![\lambda x.M]\!]_\sigma \in C \Rightarrow [\![\lambda x.M]\!]_\sigma = f_C\}$.

An equivalent way of looking at λ-structures has been suggested by DANA SCOTT (unpublished notes, 1976, and correspondence, 1978; cf. also VOLKEN [19], Definition 34). Let $\langle D, F, \circ\rangle$ be a two-sorted structure, D being a set, $F \subseteq D$, and $\circ$ being a map: $D^2 \to D$. Define

$$F_0 = D, \qquad F_1 = F, \qquad F_{n+1} = \{f : f \in F_n \text{ and } \forall d \in D. f \circ d \in F_n\}.$$

("$f \in F_n$" is read as "f represents an n-place function".) Then $\langle D, F, \circ\rangle$ is said to *satisfy Scott's conditions* iff the following hold:

(S1) $\quad \forall f, g \in F\{(\forall d \in D. f \circ d = g \circ d) \Rightarrow f = g\}$;

(S2) $\quad \exists k \in F_2 . \forall d, d' \in D.(k \circ d) \circ d' = d$;

(S3) $\quad \exists s \in F_3 . \forall d, d', d'' \in D.((s \circ d) \circ d') \circ d'' = (d \circ d'') \circ (d' \circ d'')$.

Given a λ-structure $\langle D, \circ, [\![\ ]\!]\rangle$, if one defines

$$F = \{[\![\lambda x.M]\!]_\sigma : M \in \mathit{Terms},\ x \in \mathit{Vars},\ \sigma : \mathit{Vars} \to D\},$$

then $\langle D, F, \circ\rangle$ can easily be seen, using (4) or (6), to satisfy Scott's conditions.

Conversely, any $\langle D, F, \circ\rangle$ satisfying Scott's conditions gives rise, in a natural way, to a λ-structure. (Theorems 31 and 35 of [19].) In both directions, (S1) corresponds to $(\xi)'$.

3.7. Notation. When discussing λ-structures, we shall sometimes use an *informal λ-notation*. If $\varphi : D \to D$ is a function defined in the metatheory by some expression

$$\varphi(d) = \ldots d \ldots,$$

and φ is representable in D, then the representatives of φ in D form an extensional-equivalence-class $C(\varphi)$:

$$C(\varphi) = \{f : f \in D \text{ and } \forall d \in D. f \circ d = \varphi(d)\}.$$

In this class by 3.6 above, the structure $\langle D, \circ, [\![\ ]\!]\rangle$ determines a choice of one member $f_{C(\varphi)}$ to be the value of $[\![\lambda x.M]\!]_\sigma$ for all M, x, σ such that $\psi_{Mx\sigma} = \varphi$ (and such M, x, σ always exist if φ is representable; for any $e \in C(\varphi)$, one has $f_{C(\varphi)} = [\![\lambda x.yx]\!]\,(\sigma^e_y)$).

This $f_{C(\varphi)}$ is a "canonical" representative of φ, and will be called $\lambda d. \ldots d \ldots$, where $\ldots d \ldots$ is the expression defining φ. Thus for example one can say $f_{C(\varphi)} = \lambda d.e \circ d$, for e as above. (A similar notation is also used in SCOTT [18].)

This notation has the replacement property; if $A(d)$ and $B(d)$ are two meta-language expressions which define representable functions, then

$$(\forall d \in D. A(d) = B(d)) \Rightarrow \lambda d.A(d) = \lambda d.B(d).$$

3.8. Conclusion. From the above, it seems that Definitions 3.1 and 3.2 give the natural definition of model for λ-calculi, and this coincides with the classical predicate-logic notion when λ-calculi are formulated as in Remark 3.3. Also, all the "models" of $\lambda\beta$ mentioned in the Introduction satisfy Definition 3.2. (For term models, this depends on the fact that single variables are terms; see § 7 for further discussion.) However, the class of all the models given by Definition 3.2 is not as simple as one would wish. (See Corollary 6.6.)

In § 8, models of $\lambda\beta$ will be compared with combinatory models, but first we shall explore extensionality and the ω-rule, because they shed more light on the basic concepts involved.

§ 4. Extensionality

4.1. Definition. A pseudo-structure $\mathscr{D} = \langle D, \circ, [\![\ ]\!]\rangle$ is *extensional* iff, for all $f, g \in D$, when $f \circ d = g \circ d$ for all $d \in D$ then $f = g$.

This is equivalent to saying that $\mathscr{D}$ satisfies the *axiom of extensionality*,

(ext) $\quad \forall x\, \forall y (\forall z (xz = yz) \,.\!\supset\!.\, x = y)$,

or that every extensional-equivalence-class in D is a singleton, or that for each representable function from D into D there is only one possible choice of index.

In the earliest careful study of extensionality, BARENDREGT's [1], the following rule was taken as an expression of the concept in formal systems:

$$(\zeta) \qquad \frac{Mx = Nx}{M = N} \quad (x \text{ not free in } MN).$$

Clearly a pseudo-structure $\mathscr{D}$ is extensional in the sense of Def. 4.1 iff $\mathscr{D}$ satisfies (ζ) in the sense that for all M, N, σ, if $[\![M]\!]_\sigma \circ d = [\![N]\!]_\sigma \circ d$ for all $d \in D$, then $[\![M]\!]_\sigma = [\![N]\!]_\sigma$.

Let $\lambda\beta\zeta$ be the formal system obtained by adding (ζ) to $\lambda\beta$.

4.2. Lemma. *A pseudo-model $\mathscr{D}$ of $\lambda\beta$ satisfies all equations of form*

$(\eta) \qquad \lambda x.Mx = M \quad$ (*x not free in M*)

iff it satisfies the one equation

(7) $\qquad \lambda v.uv = u$.

And a structure is a model of $\lambda\beta\eta$ iff it satisfies the following equation between closed terms:

(8) $\qquad 1 = I \quad (1 \equiv \lambda uv.uv,\ I \equiv \lambda u.u)$.

Proof. Let $\mathscr{D}$ satisfy $\lambda v.uv = u$ ($u \not\equiv v$); we deduce (η). By (α), $\mathscr{D}$ satisfies $\lambda x.ux = u$ for all $x \not\equiv u$. Now let M not contain x free; then $\mathscr{D}$ satisfies $\lambda x.Mx = M$, because

$$\begin{aligned}
[\![\lambda x.Mx]\!]_\sigma &= [\![[M/u]\,(\lambda x.ux)]\!]_\sigma \\
&= [\![\lambda x.ux]\!]\,(\sigma_u^m) \quad \text{by } (\beta) \text{ and Lemma 2.8} \quad (m = [\![M]\!]_\sigma) \\
&= [\![u]\!]\,(\sigma_u^m) \quad \text{by } \lambda x.ux = u \\
&= m = [\![M]\!]_\sigma.
\end{aligned}$$

The converse is trivial. Finally, if $\mathscr{D}$ is a structure, then (8) is equivalent to (7) by (ξ); and every structure is a model of $\lambda\beta$, so (8) is equivalent to $\lambda\beta\eta$ by above.

4.3. Proposition. (i) *For pseudo-models (p.m.) of $\lambda\beta$ the following implications hold*:

extensional p.m. of $\lambda\beta$ $\Leftrightarrow$ p.m. of $\lambda\beta$ satisfying rule (ζ)

$\Downarrow$

p.m. of $\lambda\beta\zeta$ $\Leftrightarrow$ p.m. of $\lambda\beta\eta$ $\Leftrightarrow$ p.m. of $\lambda\beta$ satisfying $1 = I$

$\Downarrow$

p.m. of $\lambda\beta$ satisfying (η).

(ii) *For structures all the above six properties are equivalent (and the "pseudo's" can be omitted).*

Proof. (i) First, it is well known that the equations provable in $\lambda\beta\eta$ are the same as those provable in $\lambda\beta\zeta$. The rest of (i) follows from Lemma 4.2 or else is obvious.

(ii) For structures, the only extra step is to prove that satisfying (η) implies extensionality. Let a structure satisfy (η) and hence (7); so $\lambda d.e \circ d = e$ for all $e \in D$. But $\lambda d.e \circ d$ is a member f_e of the extensional-equivalence-class $[e]$, and by definition of structure, f_e is the same for all members of this class. But $f_e = e$; hence $[e]$ must be a singleton, so $\mathscr{D}$ is extensional.

4.4. Warning. For pseudo-models in general, both the one-way implications in 4.3 are irreversible. In particular for $\lambda\beta\zeta$ and rule (ζ), the snag is the same as for (ξ); a pseudo-structure can satisfy all the equations provable in $\lambda\beta\zeta$ without satisfying rule (ζ) itself. (One could have $\mathscr{D} \models Mx = Nx$ for an equation not provable in $\lambda\beta\zeta$, and this need not imply $\mathscr{D} \models M = N$; a concrete example will be given in § 7).

4.5. Comparison with (ξ). Note that extensionality is much stronger than rule (ξ); (ξ) says that $[\![\ ]\!]$ chooses one member out of each extensional-equivalence-class of a structure, but (ext) says further that each such class only contains one member. This gives:

4.6. Proposition. *Every extensional pseudo-structure is a structure.*

By this proposition, for extensional models one need not worry much about the definition of "model"; none of the problems discussed here will arise.

4.7. Remark. Dana Scott has preferred to view (ξ) as a principle of extensionality rather than (ζ) ([16], p. 164). Because, referring to Scott's conditions in § 3.6, (ξ) is equivalent to (S 1) which says that if the members of F are viewed as functions, then these "functions" behave extensionally:

$$\forall f, g \in F(f \sim g \Rightarrow f = g).$$

In contrast, (ext) says that all the members of D behave extensionally:

$$\forall f, g \in D(f \sim g \Rightarrow f = g).$$

We prefer to give the name "extensionality" to (ext) rather than (ξ), following Barendregt [1] and the well-established usage in typed λ-calculus, for example Church [6]. Barendregt calls (ξ) "weak extensionality".

§ 5. Models of the ω-rule

A formal rule related very closely to the concept of extensionality is *Barendregt's ω-rule*, which has an infinity of premises and says:

$$(\omega) \qquad \frac{MP = NP \text{ for all closed } P}{M = N}.$$

It only expresses extensionality when every member of D is the value of a closed term. But it is an interesting rule in its own right (see Barendregt [3]), and some of its properties help to illuminate points made earlier about structures, so it is worth looking at for a page or two here.

First, notice that according to the definition in [3] the M and N in (ω) may be any terms, even variables. But in the informal introduction of [3], the ω-rule was said to express extensionality of the interior of a structure, and this suggests that the author was thinking of M and N as being closed terms only. Thus the following form of (ω) is worth looking at as well:

$$(\omega^\circ) \qquad \frac{MP = NP \text{ for all closed } P}{M = N} \quad (M, N \text{ closed}).$$

A pseudo-structure satisfies (ω°) iff its interior is extensional.

Finally, a very weak form of (ω°), already mentioned in [2, p. 218: rule (tr)], will play a key technical role in § 6:

$$(\omega^-) \qquad \frac{MP = NP \text{ for all closed } P}{Mx = Nx} \quad (M, N \text{ closed}).$$

5.1. Definition. $\lambda\beta\omega$, $\lambda\beta\omega^\circ$, $\lambda\beta\omega^-$ are the formal systems obtained by adding rules (ω), (ω°), (ω^-) respectively to $\lambda\beta$.

It is well known (ζ) is provable in $\lambda\beta\omega$, so every model of $\lambda\beta\omega$ is extensional. But the converse is false, by PLOTKIN's counterexample [13]; this example shows that the term model of $\lambda\beta\zeta$ cannot be a model of $\lambda\beta\omega$ or $\lambda\beta\omega^\circ$.

5.2. Lemma. *The provable equations of $\lambda\beta\omega^\circ$ are exactly the same as those of $\lambda\beta\omega$.*

Proof. It is enough to derive (ω) in $\lambda\beta\omega^\circ$. To do this, let M, N be terms with variables $v_1, \ldots, v_n$ $(n > 0)$ free in MN. (Some v_i might be free in only one of M, N.) Define $M' \equiv \lambda v_1 \ldots v_n.M$, $N' \equiv \lambda v_1 \ldots v_n.N$. Suppose that $MP = NP$ is provable in $\lambda\beta\omega^\circ$ for all closed P. Then $\lambda\beta\omega^\circ \vdash M'v_1 \ldots v_n P = MP = NP = N'v_1 \ldots v_n P$. Now for all X, Y, v, Q, an easy induction on proofs shows that

$$(9) \qquad \lambda\beta\omega^\circ \vdash X = Y \Rightarrow \lambda\beta\omega^\circ \vdash [Q/v]\, X = [Q/v]\, Y \quad (Q \text{ closed}).$$

Hence, for all closed $Q_1, \ldots, Q_n, P$,

$$\lambda\beta\omega^\circ \vdash M'Q_1 \ldots Q_n P = N'Q_1 \ldots Q_n P.$$

Therefore, by rule (ω°) $n + 1$ times, $\lambda\beta\omega^\circ \vdash M' = N'$, and so finally,

$$\lambda\beta\omega^\circ \vdash M = M'v_1 \ldots v_n = N'v_1 \ldots v_n = N.$$

5.3. Definition. Let $\mathscr{D} = \langle D, \circ, [\![\;]\!]\rangle$ be a pseudo-structure:

(i) *$\mathscr{D}$ satisfies* (ω) iff for all σ, M, N,

$$\{\forall P \text{ closed } [\![MP]\!]_\sigma = [\![NP]\!]_\sigma\} \Rightarrow [\![M]\!]_\sigma = [\![N]\!]_\sigma;$$

(ii) *$\mathscr{D}$ satisfies* (ω°) iff for all σ and all closed M, N,

$$\{\forall P \text{ closed } [\![MP]\!]_\sigma = [\![NP]\!]_\sigma\} \Rightarrow [\![M]\!]_\sigma = [\![N]\!]_\sigma;$$

(iii) *$\mathscr{D}$ satisfies* (ω^-) iff for all σ and all closed M and N,

$$\{\forall P \text{ closed } [\![MP]\!]_\sigma = [\![NP]\!]_\sigma\} \Rightarrow [\![M]\!]_\sigma \sim [\![N]\!]_\sigma.^{1})$$

[1]) The definition of satisfaction of a rule given here is the same as that used for (ξ) in 3.1 and for (ζ) in the remarks before 4.2. However, just as for (ξ), a weaker way of defining satisfaction is possible. Such a definition does not seem to us so natural as the one above, but it is worth investigating and will be discussed in the Appendix.

5.4. Definition. An *ω-structure* (*ω-pseudo-structure*, *ω-model*) is a structure (pseudo-structure, etc.) $\mathscr{D}$ such that for all $m, n \in D$,

(10) $\quad (\forall d \in D^\circ . m \circ d = n \circ d) \Rightarrow m = n.$

(Cf. Barendregt's *ω-models*.) We shall call a $\mathscr{D}$ an *ω°-structure* (etc.) iff (10) holds for all m, n in D°, not necessarily throughout D.

5.5. Lemma. *Let $\mathscr{D}$ be a pseudo-structure; then*

(i) *$\mathscr{D}$ satisfies (ω) $\Leftrightarrow$ $\mathscr{D}$ is an ω-pseudo-structure*
$\Leftrightarrow$ $\mathscr{D}$ is an ω-structure.

(ii) *$\mathscr{D}$ satisfies (ω°) $\Leftrightarrow$ $\mathscr{D}$ is an ω°-pseudo-structure*
$\Leftrightarrow$ D° is extensional.

Proof. (i) Let $\mathscr{D}$ satisfy (ω). To prove (10), let $m \circ d = n \circ d$ for all $d \in D^\circ$. In (ω), take $M \equiv u$, $N \equiv v$, and σ such that $\sigma(u) = m$, $\sigma(v) = n$; then $[\![uP]\!]_\sigma = [\![vP]\!]_\sigma$ for all closed P, and hence by (ω), $[\![u]\!]_\sigma = [\![v]\!]_\sigma$. That is $m = n$, so $\mathscr{D}$ is an ω-pseudo-structure. The converse implication is obvious. For the second equivalence: every pseudo-structure satisfying (10) is extensional, so by Proposition 4.6 it is a structure.

(ii) Like (i).

5.6. Proposition. *For structures the following implications hold:*

Satisfies (ω) $\Leftrightarrow$ ω-structure

$\Downarrow$

satisfies (ω°) $\Leftrightarrow$ ω°-structure $\Leftrightarrow$ interior is extensional

(11) $\Downarrow$

model of $\lambda\beta\omega^\circ$ $\Leftrightarrow$ model of $\lambda\beta\omega$

(12) $\Downarrow$

extensional model of $\lambda\beta$.

For pseudo-structures all the implications hold except possibly (11) *and* (12).

Proof. By the previous lemmas.

5.7. Corollary. *Every structure with an extensional interior is extensional.*

5.8. Remark. The one-way implications in 5.6 are unlikely to be reversible in general.

To reverse the first implication, one could try to carry out the proof of Lemma 5.2 in an arbitrary structure, for a particular valuation σ, but the attempt would fail at (9). The most one could prove would be that (ω) is true in the weak sense discussed in the Appendix; namely that if $[\![MP]\!]_\sigma = [\![NP]\!]_\sigma$ for all σ and all closed P, then $[\![M]\!]_\sigma = [\![N]\!]_\sigma$ for all σ.

Regarding the second implication; a model of $\lambda\beta\omega^\circ$ need not satisfy (ω°), because if equations $MP = NP$ were true in the model but not provable in $\lambda\beta\omega^\circ$, then $M = N$ would not be demanded in a model of $\lambda\beta\omega^\circ$ (the same as for (ζ): § 4 and § 7).

The third implication is irreversible, by Plotkin's counterexample. However, using (ω^-) we can easily get the following partial converse.

5.9. Corollary. *Let $\mathscr{D}$ be a structure; then $\mathscr{D}$ satisfies (ω°) iff $\mathscr{D}$ is extensional and satisfies (ω^-).*

5.10. Note on (ω^-). From Definition 5.3, it can be seen that a pseudo-structure $\mathscr{D}$ satisfies (ω^-) iff for all $m, n \in D^\circ$, when $m \circ d = n \circ d$ for all $d \in D^\circ$ then $m \circ d = n \circ d$ for all $d \in D$.

Suppose a relation $\sim^\circ$ (*extensional equivalence relative to D°*) is defined in D° by

$$m \sim^\circ n \Leftrightarrow \forall d \in D^\circ .\, m \circ d = n \circ d \quad (m, n \in D^\circ).$$

And for $m \in D^\circ$ let $[m]^\circ$ be the $\sim^\circ$-equivalence-class containing m: $[m]^\circ = \{n \in D^\circ : n \sim^\circ m\}$. Then for all $m, n \in D^\circ$

$$m \sim n \Rightarrow m \sim^\circ n,$$

so $[m]^\circ \supseteq [m] \cap D^\circ$. Then (ω^-) can be characterized as follows: $\mathscr{D}$ satisfies (ω^-) iff

$$\forall m, n \in D^\circ .\, m \sim^\circ n \Leftrightarrow m \sim n,$$

or in other words, $[m]^\circ = [m] \cap D^\circ$.

Moreover, the failure of (ω^-) in a pseudo-structure $\mathscr{D}$ may be seen as a "largeness" property of $D - D^\circ$: the members of $D - D^\circ$ play a non-trivial role in distinguishing between functions. Therefore we shall call $\mathscr{D}$ *large* (for want of a better word) iff $\mathscr{D}$ does not satisfy (ω^-).

5.11. Definition. We end this section with a couple of special kinds of model that have appeared in the literature. A *hard* pseudo-structure $\mathscr{D}$ is one for which $D^\circ = D$, i.e. whose members are all λ-definable. A *sensible* pseudo-structure $\mathscr{D}$ is one for which $[\![M]\!]_\sigma = [\![N]\!]_\sigma$ for all unsolvable terms M, N (and all σ).

In sensible pseudo-structures, terms are distinguished only by their activities as functions; all the "passive" terms are identified.

In hard pseudo-structures, (ω^-) is always satisfied; also extensionality and satisfaction of (ω) and (ω°) coincide.

5.12. Proposition. *Any sensible structure satisfying (ω^-) also satisfies (ω°).*

Proof. Let $\mathscr{D}$ be the sensible structure. Then $\mathscr{D} \models \lambda x.Mx = M$ for all closed M. This is obvious if M is unsolvable, because then $\lambda x.Mx$ is also unsolvable. And for solvable closed M one has by Wadsworth's result,

$$\lambda\beta \vdash M = \lambda x_1 \ldots x_n .\, x_i M_1 \ldots M_j$$

for some $x_1, \ldots, x_n, M_1, \ldots, M_j$. Then

$$\lambda\beta \vdash \lambda x.Mx = \lambda x x_2 \ldots x_n .\, x_i M_1 \ldots M_j = M.$$

Next, let $m \in D^\circ$; then for some closed M, $m = [\![M]\!] = [\![\lambda x.Mx]\!] = \lambda d.[\![M]\!] \circ d = \lambda d.m \circ d$, in the informal λ-notation. Thus if, for some $m, n \in D^\circ$, $m \circ d = n \circ d$ for all $d \in D^\circ$, then by (ω^-) one would have $m \circ d = n \circ d$ for all $d \in D$, and so since $\mathscr{D}$ is a structure, $m = \lambda d.m \circ d = \lambda d.n \circ d = n$.

5.13. Corollary. *Every sensible structure satisfying (ω^-) is extensional; in particular, every sensible hard structure is extensional.*

§ 6. Interiors of models

Let $\mathscr{D} = \langle D, \circ, [\![\ ]\!]\rangle$ be a pseudo-structure, and D° be the set of all its λ-definable members as usual. On this set, it is possible to define a pseudo-structure $\mathscr{D}^\circ$, as follows.

6.1. Definition. The *interior*, $\mathscr{D}^\circ$, of a pseudo-structure $\mathscr{D}$ is $\langle D^\circ, \circ^\circ, [\![\ ]\!]^\circ\rangle$ where $\circ^\circ$ is the restriction of $\circ$ to D°, and $[\![\ ]\!]^\circ$ is the restriction of $[\![\ ]\!]$ to maps σ into D°; that is, for all σ: *Vars* $\to D^\circ$,

$$[\![X]\!]^\circ_\sigma = [\![X]\!]_\sigma .$$

(In future we omit the superscript $^\circ$ from $\circ^\circ$ and $[\![\ ]\!]^\circ$ whenever no confusion is likely).

6.2. Lemma. *For all pseudo-structures $\mathscr{D}$, $\mathscr{D}^\circ$ is indeed a pseudo-structure, and satisfies all equations satisfied by $\mathscr{D}$. Hence if $\mathscr{D}$ is a pseudo-model of $\lambda\beta$, etc., then so is $\mathscr{D}^\circ$.*

Proof. First, D° must be proved to be closed under $\circ^\circ$. Let $f, g \in D^\circ$. Then $f = [\![F]\!]_\sigma$ and $g = [\![G]\!]_\tau$ for some closed F, G and some σ, τ. Since F and G are closed, by Definition 2.1 (iv) we can assume $\sigma = \tau$. Hence $f \circ g = [\![FG]\!]_\sigma \in D^\circ$.

Next we must show that $[\![M]\!]_\sigma \in D^\circ$ for all σ: *Vars* $\to D^\circ$. If $\mathscr{D}$ satisfies (β), this comes from Lemma 2.8. But if not, we argue as follows. Let $M' \equiv \lambda v_1 \ldots v_n . M$ where $v_1, \ldots, v_n$ are the free variables of M, and let $g_i = \sigma(v_i)$. Then $[\![M']\!]_\sigma, g_1, \ldots, g_n$ are all in D°, and

$$\begin{aligned} [\![M]\!]_\sigma &= [\![M]\!]\,(\sigma^{g_1 \ldots g_n}_{v_1 \ldots v_n}) \\ &= (([\![M']\!]_\sigma) \circ g_1) \circ \ldots) \circ g_n \quad \text{by 2.1 (iii)} \\ &\in D^\circ. \end{aligned}$$

Finally, $\mathscr{D}^\circ$ satisfies Definition 2.1 (i)–(v) because $\mathscr{D}$ does and because $\circ^\circ$ and $[\![\ ]\!]^\circ$ are restrictions of $\circ$ and $[\![\ ]\!]$. And for the same reason, $\mathscr{D}^\circ$ satisfies all equations that $\mathscr{D}$ does.

6.3. Lemma. *For all pseudo-structures $\mathscr{D}$, $\mathscr{D}^\circ$ is hard; i.e. $\mathscr{D}^{\circ\circ} = \mathscr{D}^\circ$.*

Proof. For closed M, $[\![M]\!]^\circ_\sigma = [\![M]\!]_\sigma$.

6.4. Remark. Lemma 6.2 shows that the class of pseudo-models of $\lambda\beta$ is closed under the operation of forming the interior, just as one would expect for an equational theory.

But what about the class of models, whose definition includes (ξ)? Unfortunately, if $\mathscr{D}$ is a model of $\lambda\beta$, then $\mathscr{D}^\circ$ need not satisfy (ξ), as will now be shown.

6.5. Proposition. *Let $\mathscr{D}$ be a pseudo-structure:*

(i) *if $\mathscr{D}$ is large (i.e. it does not satisfy (ω^-)), then $\mathscr{D}^\circ$ is not a structure;*

(ii) *if $\mathscr{D}$ is a structure, then $\mathscr{D}$ satisfies (ω^-) iff $\mathscr{D}^\circ$ is a structure.*

Proof. (i) We shall show that if $\mathscr{D}^\circ$ is a structure, then $\mathscr{D}$ satisfies (ω^-), or in other words, $m \sim^\circ n \Rightarrow m \sim n$ for all $m, n \in D^\circ$. Let $m, n \in D^\circ$ with $m = [\![M]\!]$ and $n = [\![N]\!]$ for closed M and N, and let $m \sim^\circ n$. Then for any σ: *Vars* $\to D^\circ$,

$$\begin{aligned} [\![\lambda x.Mx]\!]_\sigma &= [\![\lambda x.Mx]\!]^\circ_\sigma \\ &= \lambda^\circ d.m \circ d \quad \text{using informal } \lambda\text{-notation } \lambda^\circ \text{ in } D^\circ\ (d \in D^\circ) \\ &= \lambda^\circ d.n \circ d \quad \text{since } m \sim^\circ n \\ &= [\![\lambda x.Nx]\!]_\sigma \quad \text{similarly.} \end{aligned}$$

Hence for all d in D, not just in D°,

$$\begin{aligned} m \circ d = [\![Mx]\!]\,(\sigma_x^d) &= ([\![\lambda x.Mx]\!]_\sigma) \circ d && \text{by Definition 2.1 (iii)} \\ &= ([\![\lambda x.Nx]\!]_\sigma) \circ d && \text{by above} \\ &= n \circ d && \text{similarly.} \end{aligned}$$

(ii) By (i), it is enough to show that if $\mathscr{D}$ satisfies (ω^-) then $\mathscr{D}^\circ$ satisfies (ξ). Let $[\![M]\!]\,(\sigma_x^d) = [\![N]\!]\,(\sigma_x^d)$ for all $d \in D^\circ$; then by 2.1 (iii) and (ω^-), $[\![M]\!]\,(\sigma_x^d) = [\![N]\!]\,(\sigma_x^d)$ for all $d \in D$. Hence, since $\mathscr{D}$ is a structure, $[\![\lambda x.M]\!]_\sigma = [\![\lambda x.N]\!]_\sigma$.

6.6. Corollary. *The interiors of $\mathscr{P}\omega$ and of the term model of $\lambda\beta\eta$ are not structures, even though they are pseudo-models of $\lambda\beta$.*

Proof. $\mathscr{P}\omega$ and the term model of $\lambda\beta\eta$ are large; see § 7 for proof.

6.7. Remark. Is something wrong perhaps with our definition of "interior"? Could a change in the definition of $[\![\ ]\!]^\circ$ make the interior of a structure always a structure? No, it could not. Whatever way $[\![\ ]\!]^\circ$ is defined, provided it has the property

$$[\![M]\!]^\circ = [\![M]\!] \quad \textit{for all closed } M,$$

the proof of 6.5 (i) will go through, as can easily be checked.

6.8. Sensible structures. Recall that sensibleness meant that all unsolvable terms were identified. Now for non-closed terms M, it can be shown that M is unsolvable iff $\lambda x_1 \ldots x_n.\ M$ is unsolvable, where the sequence $x_1, \ldots, x_n$ *includes* all free variables in M; also, for pseudo-models of $\lambda\beta$, if $\mathscr{D} \models \lambda x_1 \ldots x_n.\ M = \lambda x_1 \ldots x_n.\ N$, then $\mathscr{D} \models M = N$. Hence for pseudo-models of $\lambda\beta$, $\mathscr{D}$ is sensible iff $\mathscr{D}^\circ$ is sensible.

6.9. Lemma. *Let $\mathscr{D}$ be a sensible pseudo-structure*:

(i) *if $\mathscr{D}$ is a structure, then $\mathscr{D}^\circ$ satisfies (η)*;

(ii) *If $\mathscr{D}^\circ$ is a structure, then $\mathscr{D}^\circ$ satisfies (ω).*

Proof. (i) Given any σ: *Vars* $\to D^\circ$ and any variable y, there exists a closed term M such that $[\![y]\!]_\sigma = [\![M]\!]_\sigma$. By the proof of 5.12, $[\![\lambda x.Mx]\!]_\sigma = [\![M]\!]_\sigma$. Hence

$$\begin{aligned} [\![\lambda x.yx]\!]_\sigma^\circ &= [\![\lambda x.yx]\!]_\sigma \\ &= \lambda d.([\![y]\!]_\sigma) \circ d \quad \text{noting that } \mathscr{D} \text{ is a structure} \\ &= \lambda d.([\![M]\!]_\sigma) \circ d \\ &= [\![\lambda x.Mx]\!]_\sigma \\ &= [\![M]\!]_\sigma \\ &= \sigma(y). \end{aligned}$$

So $\mathscr{D}^\circ$ satisfies (η) by Lemma 4.2.

(ii) $\mathscr{D}^\circ$ is a hard sensible structure; hence by 5.13 it is extensional, i.e. it satisfies (ω).

6.10. Remark. In Case (i) of the lemma one cannot say $\mathscr{D}^\circ$ is extensional; in fact, if $\mathscr{D}$ is large, then $\mathscr{D}^\circ$ will certainly not be extensional. So if a large structure $\mathscr{D}$ can be found which is a model of $\lambda\beta\zeta$ (or in which all unsolvables are identical), then $\mathscr{D}^\circ$ will be an example of a pseudo-model of $\lambda\beta\zeta$ which is not extensional. (See 4.4 and next §).

§ 7. Two Counterexamples

Let $\mathscr{P}\omega = \langle P\omega, \circ, [\![\ ]\!]\rangle$ be the model of $\lambda\beta$ defined in SCOTT [18]. This $\mathscr{P}\omega$ is a structure (SCOTT [18], pp. 530), and is sensible (HYLAND [10], p. 370). But it is not extensional; using Scott's notation, if d is a finite non-empty element of $P\omega$, then d and graph(fun(d)) are extensionally equivalent elements of $P\omega$, but are not equal. (SCOTT [18], p. 526.) Hence the following

7.1. Fact. *By Corollary* 5.13, *$\mathscr{P}\omega$ is large.*

7.2. Fact. *$\mathscr{P}\omega^\circ$, the interior of $\mathscr{P}\omega$, is not a structure (by* (6.5 (i)), *and hence is not extensional.*

7.3. Fact. *By Lemma* 6.9 (i), *$\mathscr{P}\omega^\circ \models (\eta)$; in particular $\mathscr{P}\omega^\circ \models \lambda x.yx = y$.*

From the last fact, $P\omega^\circ$ is contained in the set FUN of SCOTT [18], p. 531. But it is not extensional. In fact, $\mathscr{P}\omega^\circ$ does not even satisfy all the equations of $\lambda\beta\eta$ (i.e. it is only a pseudo-model of $\lambda\beta$, not of $\lambda\beta\eta$), despite satisfying those of $\lambda\beta$ together with (η). (Cf. Remark 2.9.) What fail are some of the (ξ)-consequences of (η), for example:

7.4. Fact. *Not $\mathscr{P}\omega^\circ \models 1 = I$.*

(Proof. If $1 = I$ was true in $\mathscr{P}\omega^\circ$, then it would be true in $\mathscr{P}\omega$, since 1 and I are closed, and so $\mathscr{P}\omega$ would be extensional, by 4.3 (ii)).

7.5. Remark. Let $m = [\![1]\!] = \lambda de.d \circ e$, and $i = [\![I]\!] = \lambda d.d$. Then m, i provide a concrete example to show that $\mathscr{P}\omega^\circ$ is large. In fact, for all $d \in P\omega^\circ$ and σ: $Vars \to P\omega^\circ$,

$$\begin{aligned} m \circ d &= [\![\lambda y.xy]\!]\,(\sigma_x^d) && \text{by 2.1 (iii)} \\ &= [\![x]\!]\,(\sigma_x^d) && \text{by } (\eta) \text{ in } \mathscr{P}\omega^\circ \\ &= d. \end{aligned}$$

And $i \circ d = d$ for all $d \in P\omega$; so $m \sim^\circ i$. But $m \not\sim i$; in fact as we noticed $m \circ d \neq i \circ d$ for all finite $d \in P\omega$.

Putting it another way, $[m]^\circ = [i]^\circ$ but $[m] \cap [i] = \emptyset$: or again, if φ and ψ are the functions defined by

$$\varphi(d) = m \circ d, \qquad \psi(d) = d \quad (d \in P\omega),$$

then φ and ψ have different indices in $\mathscr{P}\omega$; but in $\mathscr{P}\omega^\circ$, if $\mathscr{P}\omega^\circ$ were a structure, they would both be forced to have the same index, which is impossible since the indices are chosen by $[\![\ ]\!]$ which is the same in $\mathscr{P}\omega^\circ$ as in $\mathscr{P}\omega$.

From $\mathscr{P}\omega^\circ$, we now turn to another example.

7.6. Definition. $\mathscr{M}(\lambda\beta\eta)$ is the term model of $\lambda\beta\eta$ in the sense of BARENDREGT [1], namely the set of all equivalence-classes of λ-terms with respect to the relation $\lambda\beta\eta \vdash X = Y$, with $[\![\ ]\!]$ defined by letting $[\![M]\!]_\sigma$ be the class containing $[T_1/v_1, \ldots, T_n/v_n]\, M$, where $v_1, \ldots, v_n$ are the free variables of M and T_i is any member of $\sigma(v_i)$.

7.7. Lemma. *$\mathscr{M}(\lambda\beta\eta)$ is a structure.*

Proof. The only non-routine property to check is (ξ). Let $[\![U]\!]\,(\sigma_x^d) = [\![V]\!]\,(\sigma_x^d)$ be true in $\mathscr{M}(\lambda\beta\eta)$ for all d, and let $v_1, \ldots, v_n$ be the variables free in UV other than x, and $T_1, \ldots, T_n$ be members of $\sigma(v_1), \ldots, \sigma(v_n)$ respectively. Then for all terms P,

$$\lambda\beta\eta \vdash [P/x, T_1/v_1, \ldots, T_n/v_n]\, U = [P/x, T_1/v_1, \ldots, T_n/v_n]\, V.$$

Taking in particular $P \equiv y$, a new variable, and applying (ξ) in $\lambda\beta\eta$ gives

$$\lambda\beta\eta \vdash \lambda y.[y/x, T_1/v_1, \ldots] U = \lambda y.[y/x, T_1/v_1, \ldots] V,$$

which is equivalent to

$$\lambda\beta\eta \vdash [T_1/v_1, \ldots] (\lambda y.[y/x] U) = [T_1/v_1, \ldots] (\lambda y.[y/x] V),$$

which by (α) implies that $[\![\lambda x.U]\!]_\sigma = [\![\lambda x.V]\!]_\sigma$.

(By the way, putting $P \equiv y$ is the key to this proof; if P was restricted to closed terms, the proof would be trying to show $\mathscr{M}(\lambda\beta\eta)^\circ$ is a structure, which it is not (see below).)

7.8. Remark. The interior of $\mathscr{M}(\lambda\beta\eta)$ is another example of a pseudo-model which is not a model. Because by Plotkin's counterexample [13] $\mathscr{M}(\lambda\beta\eta)$ is large, and hence by 6.5 (i), $\mathscr{M}(\lambda\beta\eta)^\circ$ is not a structure, but only a pseudo-model of $\lambda\beta\eta$.

$\mathscr{M}(\lambda\beta\eta)^\circ$ has slightly different properties from $\mathscr{P}\omega^\circ$; like $\mathscr{P}\omega^\circ$ it is not extensional (because if it was, it would be a structure), but unlike $\mathscr{P}\omega^\circ$ it satisfies all the provable equations of $\lambda\beta\zeta$, including $1 = I$. So $\mathscr{M}(\lambda\beta\eta)^\circ$ and $\mathscr{P}\omega^\circ$ together show that the one-way implications in Proposition 4.3 (i) are irreversible.

7.9. Remark. In contrast to the above, with the splendid model D_∞ everything goes perfectly smoothly; it is a structure, satisfies (ω^-) and (ω), and hence it possesses an extensional interior which must therefore also be a structure.

§ 8. Combinatory models

How do λ-models and their complications connect up with models of combinators? Models of combinatory logic look very similar to models of the λ-calculus, and yet they are much simpler to define; how can this be?

Well, in fact it cannot be; and we shall show why in this section.

8.1. Definition. *The theory CLw of weak combinatory equality* is a first-order theory with equality in the usual predicate-logic sense (cf. Remark 3.3):

combinatory terms: S, K variables, (XY);

formulae: $X = Y$, $\varphi \supset \psi$, $\neg\varphi$, $\forall x\varphi$;

logical axioms and rules: as usual;

non-logical axioms: $\forall x\, \forall y\, \forall z(Sxyz = xz(yz))$, $\forall x\, \forall y(Kxy = x)$, $\neg(S = K)$.

(The relation $CLw \vdash X = Y$ can be shown to be the same as weak equality as it is commonly defined; cf. Barendregt [2] Theorem 2.12.)

8.2. Definition. *The theory CLβ of combinatory β-equality* is obtained by adding to CLw the extra axioms $[K]$, $[S]$, $[SK]$, $[SI]$ of [7, p. 203].

8.3. Definition. *The theory CL$\beta\eta$ of combinatory $\beta\eta$-equality* is obtained by adding to $CL\beta$ the extra axiom $[I_1]$ of [7, p. 203].

8.4. Definition. A *model* of CLw (or $CL\beta$ or $CL\beta\eta$) is a quadruple $\mathscr{C} = \langle C, \circ, s, k\rangle$ where C is a set, $\circ$ maps C^2 into C, and $s, k \in C$, such that $\mathscr{C}$ satisfies CLw (or $CL\beta$ or $CL\beta\eta$) in the usual Tarski sense of first-order logic. For each combinatory term X and each valuation σ of the variables, $\langle\langle X\rangle\rangle_\sigma$ is the interpretation of X defined by σ in the obvious way.

8.5. **Comment.** The β and $\beta\eta$ axioms contain no free variables, and their purpose is to replace rules (ξ) and (ζ). For example, consider $CL\beta\eta$; in [7] § 6C, it was proved that

(13) *if* $CL\beta\eta \vdash Ux = Vx$ *(x not in UV), then* $CL\beta\eta \vdash U = V$.

Thus rule (ζ) is *admissible* in $CL\beta\eta$; that is, if (ζ) were added to $CL\beta\eta$ no new provable equations would result. But on the other hand, (ζ) is not *directly derivable* in $CL\beta\eta$; that is, as 8.13 will prove, one does not have in $CL\beta\eta$

$$Ux = Vx \vdash U = V$$

for x not in UV; equivalently, not $CL\beta\eta \vdash ((\forall x. Ux = Vx) \supset U = V)$. Similarly with ($\xi$): for none of Curry's definitions of $[x]$ does one have $X = Y \vdash ([x].X = [x].Y)$, although rule ($\xi$) is admissible. Consequently, a model $\mathscr{C}$ of $CL\beta\eta$ need not satisfy (ξ) or (ζ) (in the sense of 3.1 or 4.1). This means that deductions using (ξ) or (ζ) cannot be carried out in arbitrary models, so combinatory models certainly do not do the same job as λ-models when it comes to deductions. They correspond in fact to pseudo-λ-models as will be shown.

For $CL\beta$ the position is the same as for $\beta\eta$; the axioms give admissibility, but not direct derivability. Of course for $CL\beta$ rule (ζ) is not admissible in its full strength; all one gets is rule (ζ') of [7] p. 201, namely

(14) *if* $CL\beta \vdash Ux = Vx$ *and x is not in UV and U, V are O_1-obs* (i.e. *they weakly reduce to terms of form SXY, SX, S, KX or K), then* $CL\beta \vdash U = V$.

8.6. **The λ- and H-transforms.** Recall that for combinatory terms the analogue of λx is $[x]$, which can be defined recursively by any of several algorithms ([7] § 6A), and which has the property

(15) $CLw \vdash ([x]\,U)\,V = [V/x]\,U$.

Each algorithm for $[x]$ determines a map $M \mapsto M_H$ from λ-terms to combinatory terms; and there is a natural map $X \mapsto X_\lambda$ in the opposite direction ([7], p. 212). In what follows, $[x]$ will be assumed to be defined by algorithm (abf), namely

(a) $[x].U \equiv KU$ *if x not in U*,
(b) $[x].x \equiv I$ $(\equiv SKK)$,
(f) $[x].UV \equiv S([x]\,U)\,([x]\,V)$,

(where "$\equiv$" denotes identity of terms). This choice of algorithm is due to Barendregt [1], who has pointed out that it is particulary convenient for comparing β-equalities. (In contrast, for comparing reductions other algorithms have more advantages; see [8], for example.) This algorithm has the property

(16) $([N/x]\,M)_H \equiv [N_H/x]\,M_H$,

and hence by (15),

(17) $CLw \vdash ((\lambda x M)\,N)_H = ([N/x]\,M)_H$.

From these properties one gets by straightforward inductions:

(18) $\lambda\beta \vdash M = N \Rightarrow CL\beta \vdash M_H = N_H$,
(19) $CL\beta \vdash X = Y \Rightarrow \lambda\beta \vdash X_\lambda = Y_\lambda$,
(20) $CL\beta \vdash X_{\lambda H} = X$,
(21) $\lambda\beta \vdash M_{H\lambda} = M$.

8.7. Definition. For each model $\mathscr{C}$ of CLw, $\mathscr{C}_\lambda$ is the pseudo-λ-structure whose domain is C and whose valuation-map is defined by $[\![M]\!]_\sigma = \langle\langle M_H\rangle\rangle_\sigma$.

8.8. Definition. For each pseudo-λ-structure $\mathscr{D}$, $\mathscr{D}_H$ is the model of CLw whose domain is D, $s = [\![\lambda xyz.xz(yz)]\!]$, $k = [\![\lambda xy.x]\!]$ and whose valuation-map is defined by $\langle\langle X\rangle\rangle_\sigma = [\![X_\lambda]\!]_\sigma$.

It is easy to check that $\mathscr{C}_\lambda$ and $\mathscr{D}_H$ are indeed pseudo-λ-structures and models of CLw (respectively). Incidentally, to prove that $\mathscr{D}_H$ satisfies the axioms for S and K, one does not need (ξ), but only 2.1 (iii).

Now let *Models* ($CL\beta$) be the class of all models of $CL\beta$, and *P-models* ($\lambda\beta$) be the class of all pseudo-models of $\lambda\beta$, and similarly for the other formal systems.

8.9. Proposition (BARENDREGT). *The mappings $\mathscr{C} \mapsto \mathscr{C}_\lambda$ and $\mathscr{D} \mapsto \mathscr{D}_H$ are mutual inverses, and give a one-to-one correspondence between Models ($CL\beta$) and P-models ($\lambda\beta$).*

Proof. If $\mathscr{C}$ is a model of $CL\beta$, then $\mathscr{C}_\lambda$ is a pseudo-model of $\lambda\beta$, by (18) and the definition of $\mathscr{C}_\lambda$. Conversely, if $\mathscr{D} \in$ *P-models* ($\lambda\beta$) then $\mathscr{D}_H \in$ *Models* ($CL\beta$) by (19). Also, by (20) and (21) it follows that

(22) $\qquad \mathscr{C}_{\lambda H} = \mathscr{C}, \qquad \mathscr{D}_{H\lambda} = \mathscr{D}.$

Hence the mappings form a one-to-one correspondence.

8.10. Remark. Note that the above mappings, which are "natural" in an intuitive sense, relate models of $CL\beta$ with pseudo-models of $\lambda\beta$, not models. And since there are pseudo-models of $\lambda\beta$ which are not models, there are models of $CL\beta$ which do not correspond to any model of $\lambda\beta$. (Take $\mathscr{D}_H$, for any $\mathscr{D}$ which is a pseudo-model but not a model of $\lambda\beta$.)

8.11. Proposition. The *mappings $\mathscr{C} \mapsto \mathscr{C}_\lambda$ and $\mathscr{D} \mapsto \mathscr{D}_H$ give a one-to-one correspondence between Models ($CL\beta\eta$) and P-models ($\lambda\beta\eta$).*

Proof. Since *Models* ($CL\beta\eta$) is a subclass of *Models* ($CL\beta$), and similarly for λ, by 8.9 it is enough to prove that $\mathscr{C} \in$ *Models* ($CL\beta\eta$) implies $\mathscr{C}_\lambda \in$ *P-models* ($\lambda\beta\eta$) and vice-versa. But these results follow from the analogues of (18) and (19) for $\beta\eta$, which are easy to prove.

8.12. Remark. The next proposition will emphasize yet again the contrast between combinators and λ; although in λ-models (η) implies extensionality, in combinatory models $CL\beta\eta$ is strictly weaker than extensionality. This is because, as was mentioned in 8.5, the $\beta\eta$-axioms only give admissibility of rule (ζ) not derivability, and it corresponds to the fact that for λ, a pseudo-structure may satisfy (η) without being extensional.

8.13. Proposition. *The term model of $CL\beta\eta$ is extensional, but there is a model of $CL\beta\eta$ which is not extensional.*

Proof. The term model of $CL\beta\eta$ is extensional because if $CL\beta\eta \vdash UZ = VZ$ for all Z, then in particular for a variable x not in UV, $CL\beta\eta \vdash Ux = Vx$, and so $CL\beta\eta \vdash U = V$ by (13).

For the second part, take the interior of the term model; this is not extensional by PLOTKIN's counterexample [13] (see also § 7).

8.14. Remark. By the preceding results, for the β and $\beta\eta$ systems there is an exact correspondence between combinatory and pseudo-λ-models. Now it only remains to deal with weak equality; but unfortunately here the correspondence is not so neat.

It can be made a bit tidier by changing the $[x]$-algorithm in the definition of H (for example to algorithm (abcf)), but then the correspondence for β is lost. So we shall keep to algorithm (abf).

The first task is to define a formal system of λ-equality which corresponds to the formal theory of combinatory weak equality. This has in fact already been done, by W. Howard in an (unpublished?) work in proof-theory, and his definition is equivalent to the following ([8], p. 172).

8.15. Definition. *Howard's weak λ-equality* (λw) is the theory defined by axioms (α), (β), (ϱ) and the rules of symmetry, transitivity and

$$(23) \qquad \frac{M = N}{[M/x]\,P = [N/x]\,P}.$$

8.16. Proposition. (i) *For pseudo-structures, $\mathscr{D}$ satisfies (α) and (β) iff $\mathscr{D}$ is a pseudo-model of λw.* (ii) *If $\mathscr{C}$ is a model of CLw, then $\mathscr{C}_\lambda$ is a pseudo-model of λw; conversely if $\mathscr{D}$ is a pseudo-model of λw, then $\mathscr{D}_H$ is a model of CLw.*

Proof. (i) Let $\mathscr{D}$ satisfy (α) and (β). It is enough to show that $\mathscr{D}$ satisfies rule (23). By lemma 2.8, (3) in 2.8 holds; so if $[\![M]\!]_\sigma = [\![N]\!]_\sigma$ for some σ, then $[\![[M/x]\,P]\!]_\sigma = [\![P]\!]\,(\sigma_x^m) = [\![[N/x]\,P]\!]_\sigma$, where $m = [\![M]\!]_\sigma = [\![N]\!]_\sigma$. The viceversa part is obvious (see Def. 2.6). (ii) By [8], pp. 171–172,

$$CLw \vdash X = Y \Rightarrow \lambda w \vdash X_\lambda = Y_\lambda.$$

Conversely, a straightforward induction (cf. (19)) shows that

$$\lambda w \vdash M = N \Rightarrow CLw \vdash M_H = N_H.$$

Then (ii) follows from these by definitions 8.7 and 8.8.

8.17. Remark. Proposition 8.16 (i) shows that satisfying (β) does not imply being a pseudo-model of $\lambda\beta$: in fact, if y occurs free in M or N, not $\lambda w \vdash \lambda y.((\lambda x.M)\,N) = \lambda y.([N/x]\,M)$, since (23) is stronger than (μ) and (ν) but not than (ξ). Moreover, the proof of 8.16 (i) incidentally points out that pseudo-models of λw are "models" of λw in the usual Tarski sense (as in 3.3): the only non-routine rule of inference, (23), is always satisfied in the stronger sense.

8.18. Remark. Incidentally, an arbitrary pseudo-λ-*structure* $\mathscr{D}$ can always be made into a pseudo-model of λw by changing the valuation-map $[\![\;]\!]$; namely by taking $\mathscr{D}_{H\lambda}$ instead of $\mathscr{D}$ (i.e. taking $[\![\;]\!]'$: $[\![M]\!]'_\sigma = [\![M_{H\lambda}]\!]_\sigma$). ($\mathscr{D}_H$ is a model of CLw by the sentence just after 8.8, and then $\mathscr{D}_{H\lambda}$ is a pseudo-model of λw by 8.16 (ii).) In general $\mathscr{D}_{H\lambda} \neq \mathscr{D}$, of course.

8.19. Conclusion. Combinatory models seem to correspond most naturally, not to models of λ, but to pseudo-models. For β and $\beta\eta$ the "natural" correspondence is one-to-one, but for weak equalities it is not.

This correspondence has influenced us a bit in our choice of which class, pseudo-models or models, to adopt as the "real" models of λ-calculi. Both classes seem interesting but the study of pseudo-models can probably be most efficiently carried out by considering them as models of combinatory logics, leaving the study of models to be done through λ-calculi.

One last point: for extensional structures, all the different concepts of model coincide; more precisely, the three kinds of combinatory model (models of CLw, of $CL\beta$, and of $CL\beta\eta$) coincide, and for λ all six kinds coincide (pseudo-models of λw, models of λw, etc.), and the H-transform is a one-to-one correspondence between extensional combinatory and extensional λ-models.

Appendix. The weaker sense of satisfaction

In footnote 2 on p. 293 an alternative, weaker, way of defining "D satisfies (ξ)" was described. This appendix will look at this weak sense of satisfaction for the three rules (ξ), (ζ) and (ω). For (ζ), we shall show that in structures the weaker and stronger senses coincide. For (ω), the weak sense will turn out equivalent to (ω°). For (ξ), we shall give an explanation of the weak sense in terms of the λ-definable functions of the model, due to GORDON PLOTKIN (in correspondence).

A1. Definition. A pseudo-structure $\mathscr{D}$ *weakly satisfies* (ζ) iff for all M, N and all $x \notin FV(MN)$, $\{\forall\sigma.[\![Mx]\!]_\sigma = [\![Nx]\!]_\sigma\} \Rightarrow \{\forall\sigma.[\![M]\!]_\sigma = [\![N]\!]_\sigma\}$. Equivalently,

$$\{\forall\sigma.[\![M]\!]_\sigma \sim [\![N]\!]_\sigma\} \Rightarrow \{\forall\sigma.[\![M]\!]_\sigma = [\![N]\!]_\sigma\}. \tag{24}$$

A2. Definition. A pseudo-structure $\mathscr{D}$ *weakly satisfies* (ω) iff for all M, N.

$$\{\forall\sigma\ \forall P \text{ closed } [\![MP]\!]_\sigma = [\![NP]\!]_\sigma\} \Rightarrow \{\forall\sigma.[\![M]\!]_\sigma = [\![N]\!]_\sigma\}.$$

A3. Definition. A pseudo-structure $\mathscr{D}$ *weakly satisfies* (ξ) iff for all M, N,

$$\{\forall\sigma.[\![M]\!]_\sigma = [\![N]\!]_\sigma\} \Rightarrow \{\forall\sigma.[\![\lambda x.M]\!]_\sigma = [\![\lambda x.N]\!]_\sigma\}.$$

A4. Remark. The corresponding strong senses of satisfaction were defined, respectively, just after Definition 4.1, in Definition 5.3, and in Definition 3.1. For (ζ) and (ω) the strong and weak senses coincide when MN contains no variables. (Hence, in particular, for (ω°) the two senses would always coincide.) For (ξ), the two senses coincide when MN contains only x free.

A5. Proposition. *For structures or hard pseudo-structures, weak satisfaction of* (ζ) *is equivalent to stronger satisfaction, i.e. to extensionality.*

Proof. Obviously the stronger sense implies the weaker. For the converse; if $\mathscr{D}$ is hard, then if $m \sim n$ there exist closed terms M, N such that $[\![M]\!]_\sigma = m$ and $[\![N]\!]_\sigma = n$ for all σ, so we can apply (24) to get $m = n$. On the other hand if $\mathscr{D}$ is a structure, then $\mathscr{D}$ is extensional iff $\mathscr{D}$ is a model of $\lambda\beta\zeta$, by Proposition 4.3. But if $\mathscr{D}$ weakly satisfies (ζ), induction on $\lambda\beta\zeta$-proofs shows that $\mathscr{D}$ is a model of $\lambda\beta\zeta$.

A6. Note. For non-hard non-structure pseudo-structures there seems to be no reason why weak (ζ) should coincide with ordinary (ζ).

A7. Proposition. *Let $\mathscr{D}$ be any pseudo-model of $\lambda\beta$; then $\mathscr{D}$ weakly satisfies (ω) iff $\mathscr{D}$ satisfies (ω°).*

Proof. For "$\Rightarrow$", use Definition 43. For "$\Leftarrow$", translate the proof of Lemma 5.2 into $\mathscr{D}$, as suggested in Remark 5.8.

A8. Discussion of weak (ξ). In the authors' opinion, current logical practice leads most naturally to the stronger sense (cf. also [18], p. 523); but GORDON PLOTKIN and GÉRARD BERRY have pointed out in correspondence that there exist constructions interesting from the computer-science point of view, which give pseudo-structures satisfying (ξ) only in the weak sense. So the weak sense seems worth studying, too. (BARENDREGT in [5] used the weak sense, but says that he would now prefer the stronger.)

Weak (ξ) is best interpreted not in terms of the objects in D, but the λ-definable functions over D (G. PLOTKIN, correspondence). Let $\langle D, \circ, [\![\ ]\!]\rangle$ be a pseudo-λ-structure and let F_n ($n = 0, 1, \ldots$) be the set of all λ-definable functions from D^n into D. That is, $F_0 = D^\circ \subseteq D$, and for $n \geqq 1$,

$$\varphi \in F_n \Leftrightarrow \varphi: D^n \to D \quad \text{and} \quad \exists M.\forall d_1, \ldots, d_n \in D.\varphi(\vec{d}) = [[M]]\,(\sigma^{\vec{d}}_{\vec{x}}),$$

where $x_1, \ldots, x_n$ are the first n variables in some standard list of variables and they include all the free variables of M. Then the $[\![\ ]\!]$-map determines for each $\varphi \in F_{n+1}$ a set $\Lambda_n(\varphi) \subseteq F_n$ as follows: if $\forall d_1, \ldots, d_n, e \in D.\ \varphi(\vec{d}, e) = [\![M]\!]\,(\sigma^{d_1 \ldots d_n, e}_{x_1 \ldots x_n, x_{n+1}})$, then define ψ_M by $\psi_M(\vec{d}) = [\![\lambda x_{n+1}.M]\!]\,(\sigma^{\vec{d}}_{\vec{x}})$ and put this ψ_M into $\Lambda_n(\varphi)$.

Weak (ξ) says that if M and N both λ-define the same function φ, then $\psi_M = \psi_N$. In other words, $\mathscr{D}$ satisfies weak (ξ) if and only if for each n and each $\varphi \in F_{n+1}$; $\Lambda_n(\varphi)$ has exactly one member.

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