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THE HEREDITARY PARTIAL EFFECTIVE FUNCTIONALS AND RECURSION THEORY IN HIGHER TYPES[1]

G. LONGO AND E. MOGGI

Abstract. A type-structure of partial effective functionals over the natural numbers, based on a canonical enumeration of the partial recursive functions, is developed. These partial functionals, defined by a direct elementary technique, turn out to be the computable elements of the hereditary continuous partial objects; moreover, there is a commutative system of enumerations of any given type by any type below (relative numberings).

By this and by results in [1] and [2], the Kleene-Kreisel countable functionals and the hereditary effective operations (HEO) are easily characterized.

§1. Hereditary partial effective functionals. Let C be the set of the partial computable (recursive) functions. If one sets $\omega^\perp = \omega \cup \{\perp\}$ and $\perp$ represents undefined computations, the functions in C may be viewed as total maps from ω to $\omega^\perp$.

(*Notation.* For sets A, B, $f: A \to B$ is a total map from A to B.)

Let $\langle\, , \rangle$ be any effective coding of pairs in ω.

Set $C^{(0)} = \omega$ and $C^{(1)} = C$. We now define $C^{(n)}$, for all $n > 1$, as the set of *hereditary partial effective functionals* (HPEF) of integer type n, by induction on n, using a set $C^{(n.5)}$ of maps from $C^{(n-1)}$ to $C^{(n)}$ (functions of "intermediate" type). The maps in $C^{(n)}$ go from $C^{(n-1)}$ to $C^{(n-1)}$. The coding $\langle\, , \rangle$ is extended to higher types in one of the several possible ways (see 3.3).

$\lambda xy \cdot g(x, y)$ is the map $\langle x, y\rangle \mapsto g(x, y)$.

1.1. Definition (HPEF). (i) Let $\phi: C^{(n-1)} \to C^{(n)}$. Then

$$\phi \in C^{(n.5)} \Leftrightarrow \lambda xy \cdot \phi(x)(y) \in C^{(n)}.$$

(ii) Let τ: $C^{(n)} \to C^{(n)}$. Then

$$\tau \in C^{(n+1)} \Leftrightarrow \forall \phi \in C^{(n.5)}\, \tau \circ \phi \in C^{(n.5)}.$$

In [6] the functions in $C^{(n.5)}$ were assumed to be partial. In view of Theorem 3.8 below, this does not make much difference, for each $C^{(n)}, n > 0$, contains an undefined (least) element.

Fix now a canonical (acceptable) Gödel-numbering $\phi_1: \omega \to C$. Let TC be the total computable functions.

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1.2. Lemma. *Let $\phi_a: \omega \to C$ be an acceptable Gödel-numbering. Then one has*
(i) $\phi_a \in C^{(1.5)}$;
(ii) $\phi \in C^{(1.5)} \Leftrightarrow \exists f \in \mathrm{TC}\ \phi = \phi_a \circ f$;
(iii) $\tau \in C^{(2)} \Leftrightarrow \tau \circ \phi_a \in C^{(1.5)}$.

Proof. (i) By the iteration $(s - m - n)$ theorem, for some $f_1 \in \mathrm{TC}$, $\phi_a(n)(m) = \phi_1((f_1(n))(m))$.
(ii) $\Leftarrow$. Trivial.
$\Rightarrow$. By the $s - m - n$ theorem for ϕ_a.
(iii) $\Leftarrow$. By (i) and the definition.
$\Rightarrow$. If $\phi \in C^{(1.5)}$, then $\tau \circ \phi = \tau \circ \phi_a \circ f$, for some $f \in \mathrm{TC}$, by (ii). Clearly $\tau \circ \phi_a \circ f \in C^{(1.5)}$. ■

Effective operations are defined in [7] (see also [4] and [5] at type 2).

1.3. Corollary. $\tau \in C^{(2)} \Leftrightarrow \tau$ *is an effective operation.*

Proof. $\Rightarrow$.

$$\begin{aligned} \tau \circ \phi_1(n)(m) &= g(\langle n, m\rangle) \quad \text{for some } g \in C, \text{ by 1.1,} \\ &= \phi_1(f(n))(m) \quad \text{for some } f \in \mathrm{TC}, \end{aligned}$$

by the $s - m - n$ theorem. Thus $\tau \circ \phi_1 = \phi_1 \circ f$. ■
$\Leftarrow$. Immediate.

By the Myhill-Shepherdson theorem [7], Corollary 1.3 proves that $C^{(2)}$ is the set of computable and continuous operators (the recursive operators) from C to C, when C is given the Scott topology (see [9] and §2 below).

Thus, up to type two, everything is fine and easy. One gets exactly the classical recursive operators.

As for the higher types, the key structural property of the HPEF is the following (see §3 for a proof):

$$\forall n > 0\, \exists \phi_n \in C^{(n.5)}\, \forall \phi \in C^{(n.5)}\, \exists f_n \in C^{(n)}\ \phi = \phi_n \circ f_n. \tag{1}$$

Recall now that for all $f_{n+1} \in C^{(n+1)}$, one has $f_{n+1} \circ \phi_n \in C^{(n.5)}$. Then, by (1), for some $f_n \in C^{(n)}$, $f_{n+1} \circ \phi_n = \phi_n \circ f_n$.

By this, HPEF may be visualized in the integer types by the following diagram (bottom-up):

(2)

$$\begin{tikzcd} \omega \arrow[r, "f"] \arrow[d, "\phi_1"'] & \omega \arrow[d, "\phi_1"] \\ C \arrow[r, "f_2"'] \arrow[d, dashed, no head] & C \arrow[d, dashed, no head] \\ C^{(n-1)} \arrow[r, "f_n"] \arrow[d, "\phi_n"'] & C^{(n-1)} \arrow[d, "\phi_n"] \\ C^{(n)} \arrow[r, "f_{n+1}"'] \arrow[d, dashed, no head] & C^{(n)} \arrow[d, dashed, no head] \\ {} & {} \end{tikzcd}$$

The HPEF were first defined in [6], where (1) was conjectured.

The proof of (1) will be given by proving first another strong property of the HPEF. Namely, for all $n \in \omega$, the functions in $C^{(n.5)}$ and $C^{(n)}$ are "continuous", in the usual sense (see [2], [10] and §3 below, where the proof is given).

Moreover, by results in [1] and [2], modulo a simple equivalence relation (see §4), the HPEF give the Kleene-Kreisel countable functionals and HEO.

By this and by the notion of "relative (Gödel-)numbering", the lemmas in §3 and the above diagram give some information also on classical recursion theoretic hierarchies.

Aside from the comparison carried out in §4, though, the HPEF approach seems of interest in itself (see, for example, the open problems listed at the end of the paper).

§2. Preliminaries. Most of the notions in this section are borrowed from [1] and [2]. One could also use [10]: as a matter of fact early ideas of Scott inspired Ershov's work (see [1, p. 206]). We follow Ershov's notation in view of the results in [2] we need. The reader may refer to [2] and [3] for an overview and a comparison. We only recall a few definitions.

Given a poset $(X, \leq)$ and $x, y \in X$ we write $x \uparrow y$ for $\exists z \in X\, x \leq z \wedge y \leq z$, i.e. x and y are *compatible*. $\bigsqcup A$ is the least upper bound of A, if it exists.

2.1. Definition. Let $(X, \leq)$ be a poset and $X_0 \subseteq X$. Then $(X, X_0, \leq)$ is an *f-space* iff:

(1) $\forall x \in X\, \exists x_0 \in X_0\, x_0 \leq x$,

(2) $(X_0, \leq)$ is a partial upper semilattice, i.e.

$$\forall x_0, y_0 \in X_0 (x_0 \uparrow y_0 \Rightarrow \exists z_0 \in X_0 z_0 = x_0 \sqcup y_0 = \bigsqcup \{x_0, y_0\}),$$

(3) $\forall x, y \in X (x \not\leq y \Rightarrow \exists x_0 \in X_0\, x_0 \leq x \wedge x_0 \not\leq y)$.

An f_0-space is an f-space with a least element, $\perp$ say; Scott's topology is given by the basis $\{x \in X / x_0 \leq x\}$ for x_0 ranging in X_0. Effective f-spaces, etc. are as in [1], [2] or [3]. In case of completeness they correspond to the effectively given domains in [10].

Let $\boldsymbol{X} = (X, X_0, \nu_0, \leq)$ be an effective f-space, where $\nu_0 \colon \omega \to X_0$ is a numbering of the base X_0. Set $X_c = \{x \in X / \{n/\nu_0(n) \leq x\}$ is r.e.$\}$ (X_c is the set of computable elements in X). $A \subseteq \omega$ is an ideal (with respect to $\boldsymbol{X}$) if $\{\nu_0(n)/n \in A\}$ is an ideal in X_0 (i.e. it is a directed set downward closed). $\boldsymbol{X}$ is *complete over r.e. ideals* (or *effectively complete*) if, for any r.e. ideal $W_i \subseteq \omega$, $\bigsqcup \{\nu_0(n)/n \in W_i\}$ exists in X.

(*Notation.* If there is no ambiguity, by $\boldsymbol{X}$ (effective) $f(f_0)$-space etc. we mean $\boldsymbol{X} = (X, X_0, (\nu_0,) \leq)$, etc.)

Let $\boldsymbol{X}, \boldsymbol{Y}$ be f-spaces. Then $\mathrm{Cont}(X, Y)$ is the set of continuous functions from X to Y.

The category of f_0-spaces, with continuous functions as morphisms, is cartesian closed. The same applies to the various subcategories mentioned above. Thus the notion of finite or computable element is naturally inherited at higher types. For example, $f \in \mathrm{Cont}(X, Y)_0$ iff f is the least upper bound of a finite set of compatible "step" functions (where step $x_0 y_0(z) =$ "if $x_0 \leq z$ then y_0 else $\perp$" for $x_0 \in X_0$, $y_0 \in Y_0$).

Note that if $\boldsymbol{X}$ is an f-space and $\boldsymbol{Y}$ an f_0-space, then $\mathrm{Cont}(\boldsymbol{X}, \boldsymbol{Y})$ is an f_0-space.

2.1. REMARK. Let $\boldsymbol{X} = (X, X_0, \nu_0, \leq)$ be an effective f-space and $\boldsymbol{Y} = (Y, Y_0, \mu_0, \leq)$ an effective f_0-space. Then

$$g \in \mathrm{Cont}(X, Y)_c \quad \text{iff} \quad g \in \mathrm{Cont}(X, Y) \text{ and } \{\langle n, m\rangle / \mu_0(n) \leq g(\nu_0(n))\} \text{ is r.e.}$$

A pair (D, e) is a *numbered set* if $e: \omega \to D$ is surjective (onto).

2.2. DEFINITION. Let $e, e': \omega \to D$ be onto maps. Define

$$e <_{\mathrm{TC}} e' \quad \text{if } \exists h \in \mathrm{TC}\, e = e' \circ h, \tag{1}$$

$$e \equiv_{\mathrm{TC}} e' \quad \text{if } e <_{\mathrm{TC}} e' \text{ and } e' <_{\mathrm{TC}} e. \tag{2}$$

(Recall that $\mathrm{TC} \subseteq C$ is the set of total computable functions.)

2.3. DEFINITION. Let (D, e) be a numbered set. Then $e': \omega \to D$ is an (*acceptable-*) *numbering* of (D, e) if $e' \equiv_{\mathrm{TC}} e$. (We just say "numbering", if there is no ambiguity.)

Given numbered sets (D, e) and (D', e'), $h: D \to D'$ is a *morphism* (of the numbered sets (D, e), (D', e')) if $\exists h' \in \mathrm{TC}\, h \circ e = e' \circ h'$ (*notation*: $h \in \mathrm{Mor}(D, D')$).

Some f-spaces contain interesting numbered sets.

2.4 *Definition.* Let $\boldsymbol{X}$ be an effectively complete f_0-space. Then $\nu: \omega \to X_c$ is a *principal Gödel-numbering* of X_c if there exists $f \in \mathrm{TC}$ such that for all $i \in \omega$ one has:

(1) $W_{f(i)}$ is an ideal,

(2) W_i is an ideal $\Rightarrow W_i = W_{f(i)}$, and

(3) $\nu(i) = \bigsqcup \{\nu_0(n)/n \in W_{f(i)}\}$.

Given $\boldsymbol{X}$ as in 2.4, a principal Gödel-numbering of X_c may be given by an easy recursion theoretic argument. Note that if ν_1, ν_2 are principal Gödel-numberings of X_c, then $\nu_1 \equiv_{\mathrm{TC}} \nu_2$. The numberings of (X_c, ν) are defined as in 2.3, taking ν principal. (Notice that not every numbering needs to be principal.) Since $X_0 \subseteq X_c$, X_c uniquely determines its completion over any ideal, $\bar{X}$ say. Of course $X \subseteq \bar{X}$. Finally, define $\boldsymbol{X}_c = (X_c, \nu, X_0, \nu_0, \leq)$ a *constructive* f_0-space. That is, a constructive f_0-space is the computable part of an effectively complete f_0-space, with the principal Gödel-numbering. Let $\boldsymbol{X}_c$ and $\boldsymbol{Y}_c$ be constructive f_0-spaces; then there is a natural isomorphism $\mathrm{Cont}(X_c, Y_c)_c = \mathrm{Cont}(\bar{X}, \bar{Y})_c$.

2.5. DEFINITION. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be f-spaces. $\boldsymbol{X}$ may be naturally extended to an f_0-space $\boldsymbol{X}^{\perp}$ by adding a least element, if not already in (similarly for $\boldsymbol{Y}^{\perp}$). Moreover, any $g: X \to Y$ may be extended to $g^{\perp}: X^{\perp} \to Y^{\perp}$ by setting $g^{\perp}(\perp_x) = \perp_{y'}$ and $g^{\perp}(x) = g(x)$ else. Otherwise

EXAMPLE. $\boldsymbol{\omega} = (\omega, \omega, \mathrm{id}, =)$ gives $\boldsymbol{\omega}^{\perp} = (\omega^{\perp}, \omega^{\perp}, \mathrm{id}^{\perp}, \leq)$.

A rather broad generalization of the Myhill-Shepherdson theorem (GMS) will be used in several places (see below or [2] for a statement; a proof may be found in [3]).

2.6. THEOREM (GMS). *Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be effective f-spaces such that $\boldsymbol{X}^{\perp}$ and $\boldsymbol{Y}^{\perp}$ are effectively complete. Let also $\nu: \omega \to X_c^{\perp}$ and $\mu: \omega \to Y_c^{\perp}$ be Gödel-numberings of the constructive subspaces. Then*

$$g \in \mathrm{Cont}(\bar{X}, \bar{Y}^{\perp})_c \quad \textit{iff} \quad \exists g' \in \mathrm{TC}\, g^{\perp} \circ \nu = \mu \circ g' \qquad (\textit{i.e. } g^{\perp} \in \mathrm{Mor}(X_c^{\perp}, Y_c^{\perp})).$$

Of course, if $\boldsymbol{X} = \boldsymbol{X}^{\perp}$ and $\boldsymbol{Y} = \boldsymbol{Y}^{\perp}$, 2.6 is exactly GMS as in [2] and [3]. In this case we will often identify $\mathrm{Cont}(X, Y)_c$ and $\mathrm{Mor}(X_c, Y_c)$. (Recall that in our notation a constructive f_0-space is also effectively complete.)

A few more notions are required.

2.7. DEFINITION. Let X, Y be (effective) f-spaces. Then X_0 *can be* (*effectively*) *embedded into* Y_0 (*notation*: $X_0 \hookrightarrow Y_0$ (effectively)), if for some $g: X_0 \to Y_0$ one has:

1. $x_0 \uparrow y_0$ iff $g(x_0) \uparrow g(g_0)$,
2. $g(x_0 \sqcup y_0) = g(x_0) \sqcup g(y_0)$,

(3. $\exists g' \in \mathrm{TC}\, g \circ v_0 = \mu_0 \circ g'$).

Clearly g in 2.7 is continuous.

2.8. LEMMA. *Let* X, Y *be effectively complete* f_0*-spaces. Assume that* $X_0 \hookrightarrow Y_0$ *effectively, via* g. *Then*

$$\exists! \bar{g} \in \mathrm{Cont}(X, Y)_c \textit{ which is one-one and extends } g,$$

$$\exists! h \in \mathrm{Cont}(Y, X)_c \textit{ such that } h \circ \bar{g} = \mathrm{id}_x \textit{ and } \bar{g} \circ h \leq \mathrm{id}_y.$$

PROOF. Set $\bar{g}(x) = \bigsqcup \{g_0(x_0)/x_0 \in X_0 \text{ and } x_0 \leq x\}$ and

$$h(y) = \bigsqcup \{x_0 \in X_0/g(x) \leq y\}.$$

The rest is easy. ■

2.9. LEMMA. *Let* X, Y *be* (*effective*) f_0*-spaces. Assume that* $\omega^\perp \hookrightarrow Y_0$ (*effectively*). *Then* $\omega^\perp \hookrightarrow \mathrm{Cont}(X, Y)_0$ (*effectively*).

PROOF. Let emb: $\omega^\perp \hookrightarrow Y_0$ be the given embedding. Set then $\mathrm{emb}_1(p) = \mathrm{step} \perp (\mathrm{emb}(p))$, for $p \in \omega^\perp$ ■

2.10. DEFINITION. Let X be an effective f_0-space. Assume that $\omega^\perp \hookrightarrow X_0$ effectively, via emb say. Define then $\#: X \to \omega^\perp$ as the inverse map of emb given by Lemma 2.8.

For the purposes of recursion theory in higher types, the $f(f_0)$-spaces one deals with form the type structure of the continuous functions on the natural numbers, in any finite type. Set then

$$E^{(0)} = \omega, \quad E^{(1)} = \mathrm{Cont}(\omega, \omega^\perp) \quad \text{and} \quad E^{(n+2)} = \mathrm{Cont}(E^{(n+1)}, E^{(n+1)}).$$

2.11. LEMMA. $\forall n \geq 1\; \omega^\perp \hookrightarrow E_0^{(n)}$ *effectively*.

PROOF. Clearly $\omega^\perp \hookrightarrow \omega^\perp$. Then use 2.9 inductively. ■

Note that $E_c^{(1)} = C$. By 1.3, $E_c^{(2)} = C^{(2)}$. Theorem 3.8 below proves that, for all $n \in \omega$, $E_c^{(n)} = C^{(n)}$.

§3. Relative numberings and continuity.

3.1. DEFINITION. Let (D, e) and (D', e') be numbered sets. Then $\phi: D \to D'$ is a *relative numbering* (of (D', e') with respect to (D, e)) if $\phi \circ e$ is an (acceptable) numbering of (D', e').

Clearly, if $\phi: D \to D'$ is a relative numbering, then $\phi \in \mathrm{Mor}(D, D')$.

3.2. REMARK. Let ϕ be a relative numbering of (D', e') with respect to (D, e). Let $\phi': D \to D'$ be such that

$$\text{(0)} \qquad \exists f, g \in \mathrm{Mor}(D, D)\; \phi' = \phi \circ f \text{ and } \phi = \phi' \circ g.$$

Then ϕ' also is a relative numbering (cf. 2.3).

Note that the converse does not hold. Corollary 3.10 gives, for the HPEF, a set of relative numberings of $C^{(n+1)}$ with respect to $C^{(n)}$, which may be characterized as in (0) (see Remark 3.11).

There is a minor (!) point in the definition of HPEF. Are they well defined?

To check this we only need to define, for all $n \geq 1$, a one-one (onto) coding of pairs, which actually goes from $C^{(n)} \times C^{(n)}$ to $C^{(n)}$. For $C \times C$, say, one cannot use $\langle , \rangle$ on $\omega \times \omega$, for partial maps cause some problems.

3.3 DEFINITION. Given $f, g \in C$, define

$$\langle f, g \rangle(n) = \begin{cases} f(n/2) & \text{if } n \text{ is even,} \\ g((n-1)/2) & \text{else.} \end{cases}$$

For $n > 1$ and $f, g \in C^{(n)}$, define $\langle f, g \rangle(x) = \langle f(x), g(x) \rangle$.

By the following lemma and Theorem 3.8, these codings will do the job.

As a matter of fact, the proof of Theorem 3.8 (Main Theorem) is by combined induction. To make it readable we distilled the inductive step in several lemmas. Here is the first.

3.4. LEMMA. *Let X_c be a constructive f_0-space. Assume that $X_0 \times X_0 \hookrightarrow X_0$ effectively onto. Then*

$$\mathrm{Cont}(X, X)_0 \times \mathrm{Cont}(X, X)_0 \hookrightarrow \mathrm{Cont}(X, X)_0 \quad \textit{effectively onto.}$$

PROOF. Let $\langle , \rangle$ be the given embedding. For $F_I = \bigsqcup_I \text{ step } x_i y_i$ and $F_J = \bigsqcup_J \text{step } u_j v_j$ in $\mathrm{Cont}(X,X)_0$, define

$$\langle F_I, F_J \rangle = \left(\bigsqcup_J \text{step}\, x_i(\langle y_i, \perp \rangle \right) \sqcup \left(\bigsqcup_J \text{ step } u_j(\langle \perp, v_j \rangle) \right)$$

(use 2.7). Then $\langle F_I, F_J \rangle \in \mathrm{Cont}(X,X)_0$ and

$$\langle F_I, F_J \rangle(z) = \left(\bigsqcup_{I_z} \langle y_i, \perp \rangle \right) \sqcup \left(\bigsqcup_{J_z} \langle \perp, v_j \rangle \right), \quad \text{where } I_z = \{ i \in I / x_i \leq z \}$$
$$\text{and } J_z = \{ j \in J / u_j \leq z \},$$
$$= \langle F_I(z), F_J(z) \rangle.$$

Take now an arbitrary $F_H = \bigsqcup_H \text{step } z_h t_h$ in $C(X,X)_0$. Define $F = \bigsqcup_H \text{ step } z_h s_h$, and $G = \bigsqcup_H \text{ step } z_h r_h$, where $\langle s_h, r_h \rangle = t_h$. Note that s_h and r_h exist, for $\langle , \rangle$ is onto. Then $F_H = \langle F, G \rangle$ and $F_H(x) = \langle F(x), G(x) \rangle$. $\langle , \rangle$ is clearly an effective embedding in the sense of 2.7. ∎

Given $\langle , \rangle$ as in Lemma 3.4, by Lemma 2.8, $\langle , \rangle$ can be extended to a computable map, $\langle , \rangle$ say, defined on all $X \times X$ (or $\mathrm{Cont}(X,X) \times \mathrm{Cont}(X,X)$). Moreover, this map is also one-one and onto. The projections p_1, p_2 give the computable inverse (p_1, p_2) of $\langle , \rangle$, by 2.8 again. For any effective and bijective pairing $\langle , \rangle$ for X, Lemma 3.4 essentially gives some information on the higher type pairing $\langle f, g \rangle = \lambda x \langle f(x), g(x) \rangle$, defined as in 3.3.

3.5. LEMMA. *Let $n \geq 1$. Assume that for some constructive f_0-space X_c one has:*

(1) $$C^{(n)} = X_c \quad \textit{and} \quad C^{(n+1)} = \mathrm{Cont}(X, X)_c,$$

(2) $$X_c \times X_c \hookrightarrow X_c \textit{ effectively (via } \langle , \rangle).$$

Then $C^{(n+1.5)} = \mathrm{Cont}(C^{(n)}, C^{(n+1)})_c$.

PROOF. By the cartesian closure of the category we work with,

$$\mathrm{Cont}(X_c \times X_c, X_c)_c \cong \mathrm{Cont}(X_c, \mathrm{Cont}(X_c, X_c)_c)_c.$$

Let $\phi \in C^{(n+1.5)}$. Define $g(\langle x, y\rangle) = \phi(x)(y)$. Then $g \in C^{(n+1)}$, by definition of $C^{(n+1.5)}$. Thus $g \circ \langle , \rangle \in \mathrm{Cont}(X_c \times X_c, X_c)_c$ and

$$\phi = \lambda x(\lambda y \cdot \phi(x)(y)) = \lambda xy \cdot g(\langle x, y\rangle) \in \mathrm{Cont}(C^{(n)}, C^{(n+1)})_c.$$

Similarly for the converse, by using (p_1, p_2). ■

3.6. LEMMA. *Let (X_c, ν) and (Y_c, μ) be constructive f_0-spaces. Assume that $\omega^\perp \hookrightarrow X_0$ effectively* (*via* emb). *Then there exists a relative numbering ϕ of (Y_c, μ) with respect to (X_c, ν).*

PROOF. Let $\#: X \to \omega^\perp$ be the inverse map of emb given in 2.10. Define $\mu^\perp: \omega^\perp \to Y_c$ from μ as in 2.5 and set $\phi = \mu^\perp \cdot \#$.

We need to prove that $\phi \circ \nu \equiv_{\mathrm{TC}} \mu$.

$\mu <_{\mathrm{TC}} \phi \circ \nu$. Let incl: $\omega \to \omega^\perp$ be the inclusion. Clearly $\mathrm{incl} \in \mathrm{Mor}(\omega, \omega^\perp)$. Then $\mathrm{emb} \circ \mathrm{incl} \in \mathrm{Mor}(\omega, X_c)$, i.e.

$$\text{(A)} \qquad \exists h \in \mathrm{CT}\ \mathrm{emb} \circ \mathrm{incl}(\circ\, \mathrm{id}) = \nu \circ h.$$

Thus $\mu = \mu^\perp \circ \mathrm{incl} = \mu^\perp \circ \# \circ \mathrm{emb} \circ \mathrm{incl} = \phi \circ \mathrm{emb} \circ \mathrm{incl} = \phi \circ \nu \circ h$, by (A).

$\phi \circ \nu <_{\mathrm{TC}} \mu$. Just note that $\phi \in \mathrm{Mor}(X_c, Y_c)$. ■

3.7. LEMMA. *Let $n \geq 1$. Assume that there exist numbered sets (D, e), (D', e') such that*:

$$\text{(1)} \qquad C^{(n)} = D \quad \textit{and} \quad C^{(n+1)} = D',$$

$$\text{(2)} \qquad C^{(n+1.5)} = \mathrm{Mor}(D, D'),$$

$$\text{(3)} \qquad \exists \phi : D \to D' \textit{ which is a relative numbering.}$$

Then one has

$$C^{(n+2)} = \mathrm{Mor}(C^{(n+1)}, C^{(n+1)}).$$

PROOF. $\supseteq$ follows by 1 and 2, in view of the following diagram:

$$\begin{tikzcd}
\omega \arrow[r, "h"] \arrow[d, "e"'] & \omega \arrow[r, "f'"] \arrow[d, "e'"'] & \omega \arrow[d, "e'"] \\
C^{(n)} \arrow[r, "\psi"'] & C^{(n+1)} \arrow[r, "f"'] & C^{(n+1)}
\end{tikzcd}$$

Conversely, let $f \in C^{(n+2)}$. By definition, since $\phi \in C^{(n+1.5)}$, one has

$$\text{(B1)} \qquad \exists h \in \mathrm{TC}\ f \circ \phi \circ e = e' \circ h$$

by $f \circ \phi \in C^{(n+1.5)}$ and 2.

Moreover, ϕ is a relative numbering of (D', e') with respect to (D, e). Thus

$$\text{(B2)} \qquad \exists h' \in \mathrm{TC}\ e' = \phi \circ e \circ h'.$$

Compute now

$$\begin{aligned} f \circ e' &= f \circ \phi \circ e \circ h', \quad \text{by (B2)}, \\ &= e' \circ h \circ h', \quad \text{by (B1)}. \end{aligned}$$

And we are done. ■

3.8. MAIN THEOREM. *For all* $n \geq 1$ *one has*:

(1) $C^{(n)} = E_c^{(n)}$,

(2) $C^{(n)} \times C^{(n)} \hookrightarrow C^{(n)}$ *effectively onto*,

(3) $C^{(n+1.5)} = \mathrm{Cont}(E_c^{(n)}, E_c^{(n+1)})_c$,

(4) $\exists \phi \in C^{(n+1.5)}$ *relative numbering*.

PROOF. Note first that (1) holds for $n = 1, 2$, by 1.2 and 1.3; (2) holds for $n = 1$, by 3.3; (3) holds for $n = 1$, by 3.5, by (1) for $n = 1, 2$ and (2) for $n = 1$; and (4) holds for $n = 1$, by 3.6, by (1) for $n = 1, 2$ and 2.11.

Let $n > 1$. Assume now that (2), (3), (4) hold for n and that (1) holds for n and $n + 1$. Then one has

(1)′ ($C^{(n+1)} = E_c^{(n+1)}$ and) $C^{(n+2)} = E_c^{(n+2)}$, by 3.7 and the inductive hypothesis on (1), (3), (4),

(2)′ $C^{(n+1)} \times C^{(n+1)} \hookrightarrow C^{(n+1)}$, by 3.4 and the inductive hypothesis on (1) and (2),

(3)′ $C^{(n+2.5)} = \mathrm{Cont}(C^{(n+1)}, C^{(n+2)})$, by 3.5, (1)′ and (2)′, and

(4)′ $\exists \phi \in C^{(n+2.5)}$ relative numbering, by 3.6, (1)′ and 2.11.

This concludes the proof. ■

For $i \geq 0$, let ϕ_{i+1} be a relative numbering of $C^{(i+1)}$ with respect to $C^{(i)}$. Then $\forall n \geq 1\ \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ is an (acceptable) numbering of $C^{(n)}$, as it can be immediately checked.

3.9. PROPOSITION. *Let* X_c, Y_c *be constructive* f_0*-spaces. Assume that*

(1) $\omega^\perp \hookrightarrow X_0$ *effectively* (*via* emb), *and*

(2) $X_c \times X_c \hookrightarrow X_c$ *effectively* (*via* $\langle\, , \rangle$).

Then $\exists \phi\ \in \mathrm{Cont}(X_c, Y_c)_c\ \phi \circ \mathrm{Cont}(X_c, X_c)_c = \mathrm{Cont}(X_c, Y_c)_c$.

PROOF. Let $v\colon \omega \to \mathrm{Cont}(X, Y)_c$ be a numbering. Define $v^\perp$ from v as in 2.5. Let eval: $\mathrm{Cont}(X, Y) \times X \to Y$ be defined by eval $(f, x) = f(x)$; it is easy to show that eval is computable (see [10] or [3; 4.3]). Take finally the inverse maps $\#$ and (p_1, p_2) of emb and $\langle\, , \rangle$ respectively, on X (cf. 2.8–2.10).

Define then

$$\phi = \mathrm{eval} \circ (v^\perp, \mathrm{id}) \circ (\#, \mathrm{id}) \circ (p_1, p_2).$$

That is, $\phi(x) = \mathrm{eval}(v^\perp(\#(p_1(x))), p_2(x))$.

We now show that $\forall \psi \in \mathrm{Cont}(X, Y)_c\ \exists f \in \mathrm{Cont}(X, X)_c\ \psi = \phi \circ f$ (the reverse inclusion is trivial).

For all $i \in \omega$, define $f_i(x) = \langle \mathrm{emb}(i), x \rangle$. Let $\psi = v(j)$. Then

$$\phi \circ f_j(x) = \phi(\langle \mathrm{emb}(j), x \rangle) = \mathrm{eval}(v^\perp(j), x) = v^\perp(j)(x) = v(j)(x) = \psi(x). \quad ■$$

3.10. COROLLARY. $\forall n \geq 1\ \exists \phi_n \in C^{(n.5)} \forall \psi \in C^{(n.5)} \exists f \in C^{(n)} \psi = \phi_n \circ f$ (*see* (1) *in* §1).

3.11. REMARK. Given $n \geq 1$, let ϕ_n be as in 3.10. Clearly ϕ_n is a relative numbering of $C^{(n)}$ with respect to $C^{(n-1)}$. Not every relative numbering, though, has the strong property of ϕ_n. Call a *principal relative numbering any* $\psi_n \in C^{(n.5)}$ such that $\forall \psi \in C^{(n.5)} \exists f \in C^{(n)} \psi = \psi_n \circ f$. Then $\phi: C^{(n-1)} \to C^{(n)}$ is a principal relative numbering iff $\exists f, g \in C^{(n)}\ \phi = \phi_n \circ f$ and $\phi_n = \phi \circ g$ (cf. Remark 3.2).

3.12. REMARK. The HPEF, as defined, are maps in the "integer" types, i.e. $n + 1: n \to n$. By the same technique as in Definition 1.1, it is easy to define functionals in the "pure" types, i.e. $n + 1: n \to 0$.

Set $PC^{(0)} = \omega, PC^{(1)} = C$. Define then, for $\phi: PC^{(n-1)} \to PC^{(n)}$,

$$\phi \in PC^{(n.5)} \Leftrightarrow \lambda xy \cdot \phi(x)(y) \in PC^{(n)}.$$

For $\tau: PC^{(n)} \to \omega^{\perp}$,

$$\tau \in PC^{(n+1)} \Leftrightarrow \forall \phi \in PC^{(n.5)}\ \tau \circ \phi \in PC^{(n)}.$$

The corresponding version of Theorem 3.8 holds in the pure types, by the same argument.

Intermezzo (following a discussion with Henk Barendregt). In the category of numbered sets with morphisms defined as in §2 (see after 2.3), Ershov has given two notions, which yield on arbitrary numbered sets a generalized version of the recursion theorem (see [11]). Namely, (D, e) is *pre-complete* if

$$(1) \qquad \forall f \in C\ \exists f' \in \mathrm{TC}\ \forall n(f(n)\downarrow \Rightarrow e(f'(n)) = e(f(n)))$$

(i.e. f' extends f over (D, e)).

(D, e) is *complete* if (1) above holds and

$$\exists a \in D\ \forall n(f(n)\uparrow \Rightarrow e(f'(n)) = a)$$

(a is a special (bottom) element of D).

PROPOSITION. *Let $\phi: D \to D'$ be a relative numbering of (D', e') with respect to (D, e). Then, if (D, e) is (pre-)complete, so is (D', e').*

PROOF. Let $h, g \in \mathrm{TC}$ be such that $\phi \circ e = e' \circ h$ and $e' = \phi \circ e \circ g$.

Given $f \in C$, $e' \circ f = \phi \circ e \circ g \circ f$. Since $g \circ f \in C$, let $h' \in \mathrm{TC}$ extend $g \circ f$ over (D, e). Then $f(n)\downarrow$ implies

$$e' \circ f(n) = \phi \circ e \circ g \circ f(n) = \phi \circ e \circ h'(n) = e' \circ h \circ h'(n)$$

and $h \circ h'$ extends f over (D', e').

Let now $a \in D$ be a special element of D and consider $\phi(a) \in D'$. If $f(n)\uparrow$, then $g \circ f(n)\uparrow$ and $e(h'(n)) = a$. Thus $e' \circ h \circ h'(n) = \phi \circ e \circ h'(n) = \phi(a)$. ∎

By this, the first recursion theorem is immediately inherited from C at higher types.

§4. HPEF and the (total) continuous and computable functionals. In the previous section we proved the continuity of the HPEF. Ershov (see [2]) related its continuous functionals to the Kleene-Kreisel countable functionals and the computable ones to the HEO.

A key point to be taken into account in such a comparison is that both Ershov's $CE^{(n+1)}$'s (see below) and the $C^{(n+1)}$'s are sets of "partial" functions, in the weak but satisfactory sense that each $C^{(n)}$, say, contains a least (undefined) element, for $n > 0$. The always divergent function $\varnothing$ for $C^{(1)} = C$, $\lambda x \cdot \varnothing$ for $C^{(2)} \ldots$

On the other hand the countable functionals and the HEO, in any type, are total maps.

Moreover $\mathbf{C}E = \{CE^{(n)}/n \in \omega\}$ does not coincide with our $\mathbf{E} = \{E^{(n)}/n \in \omega\}$. In this section we compare at once $\mathbf{C}E$ and the HPEF, by using the model $\mathbf{E}$, as well as their subsets of total functionals.

4.1. DEFINITION. $CE^{(0)} = \omega^{\perp}$.

(i) $CE^{(n+1)} = \mathrm{Cont}(CE^{(n)}, CE^{(n)})$.

(ii) $E^{(1)} = \mathrm{Cont}(\omega, \omega^{\perp})$, $E^{(n+2)} = \mathrm{Cont}(E^{(n+1)}, E^{(n+1)})$.

4.2. DEFINITION. $\sim_0 \equiv \simeq_0 \equiv \sim_0^c \equiv \simeq_0^c = \mathrm{id}_\omega$ (the identity on ω).

(i) $G^{(n)} = \mathrm{dom} \simeq_n$,

$$f \simeq_{n+1} f' \Leftrightarrow f, f' \in CE^{(n+1)}\ \forall x \in G^{(n)}\ f(x) \simeq_n f'(x).$$

(ii) $M^{(n)} = \mathrm{dom} \sim_n$,

$$f \sim_{n+1} f' \Leftrightarrow f, f' \in E^{(n+1)}\ \forall x \in M^{(n)}\ f(x) \sim_n f'(x).$$

(iii) $G_c^{(n)} = \mathrm{dom} \simeq_n^c$,

$$f \sim_{n+1} f' \Leftrightarrow f, f' \in E^{(n+1)} \forall x \in M^{(n)}\ f(x) \sim_n f'(x).$$

(iv) Similarly for $M_c^{(n)}$ and $\sim_n^c$, using $E_c^{(n)}$.

Each $G^{(n+1)}$ and $M^{(n+1)}$ is hereditarily a set of total maps on $G^{(n)}$ and $M^{(n)}$. The functionals in $G_c^{(n+1)}$ and $M_c^{(n+1)}$ are computable and hereditarily total (on $G_c^{(n)}$ and $M_c^{(n)}$).

In the following statements the type index n is intended as universally quantified (over ω or $\omega\backslash\{0\}$, if needed). Each time $G^{(n)}$ and $CE^{(n)}$ (or $M^{(n)}$ and $E^{(n)}$) appear one can consistently substitute $G_c^{(n)}$ and $CE_c^{(n)}$ (or $M_c^{(n)}$ and $E_c^{(n)} = C^{(n)}$, respectively).

4.3. LEMMA. (i) $\forall x \in G^{(n)}\ \forall y \in CE^{(n)}\ (x \le y \Rightarrow x \simeq_n y)$.

(ii) $\forall x, y \in G^{(n)}\ (x \simeq_n y \Rightarrow x \sqcap y \simeq_n x)$.

The same holds for $M^{(n)}$ (*with respect to* $E^{(n)}$ *in* (i)).

PROOF. An easy induction. ■

4.4. LEMMA. *Let* $g \in G^{(n+1)}$ ($g \in M^{(n+1)}$). *Then*

$$x \simeq_n y \Rightarrow g(x) \simeq_n g(y).$$

Similarly for $\sim_n$.

PROOF. By the monotonicity of g, $g(x \sqcap y) \le g(x)$ and $g(x \sqcap y) \le g(y)$. As a matter of fact $g(x \sqcap y) \in G^{(n)}$, by 4.3(ii). Then, by 4.3(i), $g(x) \simeq_n g(x \sqcap y) \simeq_n g(y)$. ■

Lemma 4.3 gives the "maximality" of the functionals in $G^{(n)}$ or $M^{(n)}$ (they are total functions). Lemma 4.4 says that they are "extensional" with respect to $\simeq_n$ or $\sim_n$.

4.5. LEMMA. $E_0^{(n)} \hookrightarrow CE_0^{(n)}$ *effectively*.

PROOF. Set $I_0 = \mathrm{id}_\omega$ and define $I_1 \colon E_0^{(1)} \to CE_0^{(1)}$ by $I_1^{(g)} = g^{\perp}$ (see 2.5).

Assume now that, for $n > 0$, $E_0^{(n)} \hookrightarrow CE_0^{(n)}$ effectively, via I_n. Define

$$I_{n+1}\left(\bigsqcup_J \mathrm{step}\ x_j y_j\right) = \bigsqcup_J \mathrm{step}\ I_n(x_j) I_n(y_j).$$

It is then easy to check the requirements in Definition 2.7. ■

4.6. DEFINITION. Let $i_n \in \mathrm{Cont}(E^{(n)}, CE^{(n)})_c$ (and $p_n \in \mathrm{Cont}(CE^{(n)}, E^{(n)})_c$) be the computable extension of I_n (and its "inverse" map) as given by Lemma 2.8. Set also $e_n = i_n \circ p_n$.

Recall that $p_n \circ i_n = \mathrm{id}_{E^{(n)}}$ and $e_n \le \mathrm{id}_{CE^{(n)}}$.

4.7. LEMMA. *For f, g, x and y in the due types, one has:*

(i) $i_{n+1}(f)(i_n(x)) = i_n(f(x))$,

(ii) $e_{n+1}(g)(e_n(y)) = e_n(g(e_n(y)))$.

PROOF. (i) $i_{n+1}(f) = \bigsqcup\{\text{step } I_n(y)I_n(z)/y, z \in E_0^{(n)} \textit{ and } \text{step } yz \le f\}$. Then

$$i_{n+1}(f)\left(\bigsqcup_{y \le x} I_n(y)\right) = \bigsqcup\{I_n(z)/y \le x \textit{ and } y, z \in E_0^{(n)} \textit{ and } \text{step } yz \le f\}$$
$$= i_n(f(x)).$$

(ii) By definition, $e_n(u) = \bigsqcup\{I_n(x)/I_n(x) \le u\}$. An easy computation gives the result. ■

4.8. LEMMA. $\forall x \in G^{(n)}\ x \simeq_n e_n(x)$.

PROOF. This is trivial for $n = 1$.

Assume the result for $n > 1$ and let $f \in G^{(n+1)}$. Then, by induction and Lemma 4.4,

$$\begin{aligned}\forall x \in G^{(n)}\ f(x) &\simeq_n f(e_n(x))\\ &\simeq_n e_n(f(e_n(x))) \quad \text{by the inductive hypothesis}\\ &= e_{n+1}(f)(e_n(x)) \quad \text{by 4.7(ii)}\\ &\le e_{n+1}(f)(x) \quad \text{for } e_n(x) \le x \text{ and } e_{n+1}(f) \text{ is monotone.}\end{aligned}$$

Then, by 4.3(i) and $f(x) \in G^{(n)}$, $f(x) \simeq_n e_{n+1}(f)(x)$; that is, $f \simeq_{n+1} e_{n+1}(f)$. ■

4.9. LEMMA. (i) $x \in M^{(n)} \Leftrightarrow i_n(x) \in G^{(n)}$.

(ii) $y \in G^{(n)} \Leftrightarrow p_n(y) \in M^{(n)}$.

PROOF. The results are trivial for $n = 1$. We first prove the implication from left to right by combined induction.

(i) $\Rightarrow$. Let $f \in M^{(n+1)}$ and $y \in G^{(n)}$. We need to prove that $i_{n+1}(f)(y) \in G^{(n)}$. Note first that $i_n(f(p_n(y)) \in G^{(n)}$, by the inductive hypothesis on (i) and (ii). Then

$$\begin{aligned}i_n(f(p_n(y))) &= i_{n+1}(f)(e_n(y)) \quad \text{by 4.7(i)}\\ &\le i_{n+1}(f)(y) \quad \text{by monotonicity.}\end{aligned}$$

By 4.3 (i), we are done.

(ii) $\Rightarrow$. Let $g \in G^{(n+1)}$ and $x \in M^{(n)}$. Note now that

$$p_n(e_{n+1}(g)(i_n(x))) = p_n(g(i_n(x))) \in M^{(n)}$$

for $e_{n+1}(g) \simeq_{n+1} g$, by 4.8 and by the inductive hypothesis. Then

$$\begin{aligned}p_n(e_{n+1}(g)(i_n(x))) &= p_n \circ i_n(p_{n+1}(g)(x)) \quad \text{by 4.7(i)}\\ &= p_{n+1}(g)(x),\end{aligned}$$

and we are done.

As for the reverse implications, one has

$$i_{n+1}(f) \in G^{(n+1)} \Rightarrow f = p_{n+1}(i_{n+1}(f)) \in M^{(n+1)} \quad \text{by (ii)} \Rightarrow,$$

and

$$p_{n+1}(g) \in M^{(n+1)} \Rightarrow g \simeq_{n+1} e_{n+1}(g) \in G^{(n+1)} \quad \text{by 4.8 and (i)} \Rightarrow. \qquad ■$$

4.10. LEMMA. (i) $f \sim_n f' \Leftrightarrow i_n(f) \simeq_n i_n(f')$.

(ii) $g \simeq_n g' \Leftrightarrow p_n(g) \sim_n p_n(g')$.

PROOF. This is clearly true for $n = 1$. Then, as for (i) $\Rightarrow$,

$$\begin{aligned} f \sim_{n+1} f' &\Rightarrow \forall y \in G^{(n)}\, f(p_n(y)) \sim_n f'(p_n(y)) \quad \text{by 4.9 (ii)} \\ &\Rightarrow \forall y \in G^{(n)}\, i_n(f(p_n(y))) \simeq_n i_n(f'(p_n(y))) \quad \text{by induction} \\ &\Rightarrow \forall y \in G^{(n)} i_{n+1}(f)(e_n(y)) \simeq_n i_{n+1}(f')(e_n(y)) \quad \text{by 4.7(i)} \\ &\Rightarrow i_{n+1}(f) \simeq_{n+1} i_{n+1}(f') \quad \text{by 4.8.} \end{aligned}$$

As for (ii) $\Rightarrow$,

$$\begin{aligned} g \simeq_{n+1} g' &\Rightarrow \forall x \in M^{(n)} g(i_n(x)) \simeq_n g'(i_n(x)) \quad \text{by 4.9(i)} \\ &\Rightarrow \forall x \in M^{(n)} p_n(g(i_n(x))) \sim_n p_n(g'(i_n(x))) \quad \text{by induction.} \end{aligned}$$

An easy computation actually gives

$$p_n(g(i_n(x))) = p_{n+1}(g)(x) \quad \text{and} \quad p_n(g'(i_n(x))) = p_{n+1}(g')(x).$$

And we are done.

The reverse implications easily follow: (i) from (ii), and (ii) from (i) and 4.8. ■

As pointed out after 4.2, $G_c^{(n)}$ and $M_c^{(n)}$ can be consistently substituted for $G^{(n)}$ and $M^{(n)}$ in the above statements, since computable functions take computable elements to computable elements (e.g. $\forall x \in E_c^{(n)}\ i_n(x) \in CE_c^{(n)}$).

Recall that a continuous projection of complete posets A and B (*notation*: $A \lhd B$) is a pair $f \in \mathrm{Cont}(A, B)$ and $g \in \mathrm{Cont}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g \leq \mathrm{id}_B$ (e.g. 2.8, 4.6). Then the above lemmas give the following result.

4.11. THEOREM. (i) *$E^{(n)} \lhd CE^{(n)}$ and $E_c^{(n)} \lhd CE_c^{(n)}$, via i_n and p_n.*

(ii) *$M^{(n)} \lhd G^{(n)}$ and $M_c^{(n)} \lhd G_c^{(n)}$, via $i_n \restriction M_{(c)}^{(n)}$ and $p_n \restriction G_{(c)}^{(n)}$.*

(iii) *$M^{(n)}/\sim_n \cong G^{(n)}/\simeq_n$ and $M_c^{(n)}/\sim_n^c \cong G_c^{(n)}/\simeq_n^c$, via the induced maps.*

4.12. CONCLUDING REMARKS. (i) $\mathbf{G} = \{G^{(n)}/\simeq_n / n \in \omega\}$ is exactly Ershov's model of Kleene-Kreisel countable functionals; $\mathbf{0} = \{G_c^{(n)}/\simeq_n^c / n \in \omega\}$ is isomorphic to the HEO (see [2]).

Note that one may be willing to deal directly with the pure types. In this case use Remark 3.12: the analogues of the results in this section hold by the same arguments.

(ii) Note that $G^{(n)} \subseteq CE^{(n)}$ and $G_c^{(n)} \subseteq CE_c^{(n)}$ ($M^{(n)} \subseteq E^{(n)}$ and $M_c^{(n)} \subseteq E_c^{(n)} \subseteq E^{(n)}$), but $G_c^{(n)}$ and $G^{(n)}$ ($M_c^{(n)}$ and $M^{(n)}$) need not be related by "$\subseteq$". Actually one even has that $G^{(n)} \cap CE_c^{(n)} = G_c^{(n)}$ holds *only* for $n = 1$, since $G^{(2)} \cap CE_c^{(2)} \subsetneq G_c^{(2)}$ and at higher types... we are lost (similarly for $M^{(n)}$).

(iii) $E^{(n)}$ is just the completion by ideals of $C^{(n)}$, for $E_c^{(n)} = C^{(n)}$. Thus (i) and 4.11(iii) fully relate the HPEF to the countable functionals and the HEO.

Note that a recursively countable functional need not be computable as an element of the corresponding type in $\boldsymbol{G}$, with the induced f-space structure. Consider for example $F \in \mathrm{Ct}(2)$ (of pure type 2; see [8]), given by

$$F(f) = \begin{cases} 1 & \text{if } \{f(0)\}(f(1)) \text{ halts within } f(2) + 1 \text{ steps,} \\ 0 & \text{otherwise.} \end{cases}$$

Then F has a recursive associate, but its neighbourhoods' filter in $G^{(1\to 0)}$ does not have an r.e. presentation.

However the equivalence class of F, defined similarly as in 4.2(i), contains a computable element.

Open problems. Definition 1.1 (the HPEF) can be easily extended to other hierarchies just picking up a different base set C of partial functions, instead of the partial computable functions. Moggi has recently shown that if one takes C as the set of total computable functions, then $C^{(2)}$, given as in 1.1, coincides with the Banach-Mazur functionals. The higher types are probably unrelated to known hierarchies.

More work can be done by taking C as the primitive recursive functions: it is clear, say, that $\tau \circ \phi \in C^{(1.5)}$ of 1.1(ii) is a strong condition on τ (note that no $\phi \in C^{(1.5)}$, in this case, may be onto). The $C^{(n)}$'s, then, may yield a natural (and easy to define) notion for "primitive" recursive functionals. The relation to known type structures, if any, will surely be hard to work out.

Similar questions may be raised by taking C as the set of polynomial computable functions, and so on.

In all cases, Definition 1.1, essentially because of the generalization of the notion of relative (Gödel-)numbering it is based on (the intermediate types $C^{(n.5)}$), seems a strong and natural way for inheriting at higher types properties of classes of number theoretic functions.

The main fall-out the authors had, so far, from this approach to effective *typed* structures is the relation they bear to *type-free* models of computability. As a matter of fact, in cartesian closed categories principal (relative) numberings (morphisms), plus two simple conditions, characterize models of combinatory logic. By this and by 3.10, $\forall n > 0$ $C^{(n)}$ yields one such model [in preparation].

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PISA, ITALY