Chapter 6

CONES AND LIMITS

In chapter 2, we learned how common constructions can be defined in the language of Category Theory by means of equations between arrows of given objects. In chapter 4, we saw that those definitions were based on the existence of an universal arrow to a given functor. The category-theoretic notion of limit is merely a generalization of those particular constructions, as it stresses their common universal character. From another point of view, the limit is a particular and important case of universal arrow, where the involved functor is a "diagonal," or "constant" functor, as we shall see. To help the reader become confident with this new notion, we begin this chapter by looking back at the constructions of chapter 2 and we regard them as particular instances of limits. Then we study some relevant properties concerning existence, creation, and preservation of limits. As for computer science, limits have been brought to the limelight mainly by the semantic investigation of recursive definition of data types: this particular application of the material in this chapter will be discussed in chapter 10.

6.1 Limits and Colimits

The concept of limit embodies the general idea of universal construction, that is, of an entity which has a privileged behavior amongt a class of objects that satisfy a certain property. The only way to define a property in the categorical language is by specifying the existence and equality of certain arrows, that is, essentially by imposing the existence of a particular commutative diagram amongt objects inside the category.

6.1.1 Definition *A **diagram** D in a category **C** is a directed graph whose vertices $i \in I$ are labeled by objects d_i and whose edges $e \in E$ are labeled by morphisms f_e.*

A diagram D in **C** is similar to a subcategory of **C**; however, it does not need to contain identities, nor must it be closed under composition of morphisms.

More formally, a diagram in a category **C** should be defined as a graph homomorphism D from an index graph I to the (graph underlyng the) category **C** . Such a diagram is called "of type I". For the adjunction between graphs and categories, this is exactly the same as a functor from the category **I** freely generated by the graph I (the index category) to **C**. A graph is called small when the index category is small.

6.1.2 Definition. *Let* ***C*** *be a category and* D *a diagram with objects* d_i, $i\in I$. *Then a* ***cone to D*** *is an object* c *and a family of morphisms* $\{f_i\in \mathbf{C}[c,d_i] \mid i\in I\}$ *such that*

$$\forall i,j\in I\ \ \forall e\in E\ \ f_e\in \mathbf{C}[d_i,d_j] \Rightarrow f_e \circ f_i = f_j .$$

A cone may be visualized by

$$\begin{tikzcd}
 & & & c \arrow[dll] \arrow[dl, "f_i"'] \arrow[dr, "f_j"] \arrow[drr] & & & \\
\cdots & d_{i'} \arrow[r] & d_i \arrow[rr, "f"'] & & d_j \arrow[r] & d_{j'} & \cdots
\end{tikzcd}$$

Cocones are defined dually.

Example In a partial order **P**, cones correspond to lower bounds, cocones to upper bounds.

Note now that, given a diagram D, the cones to D form a category, call it $\mathbf{Cones_{C,D}}$. Just take as morphisms from $(c,\{f_i\in \mathbf{C}[c,d_i] \mid i\in I\})$ to $(c',\{h_i\in \mathbf{C}[c',d_i] \mid i\in I\})$ any $g\in \mathbf{C}[c,c']$ such that $\forall i\in I\ \ h_i \circ g = f_i$. That is,

$$\begin{tikzcd}
c \arrow[rr, "g"] \arrow[dr, "f_i"'] & & c' \arrow[dl, "h_i"] \\
 & d_i &
\end{tikzcd}$$

Clearly, $\mathbf{Cones_{C,D}}$ is a category. Dually one defines the category $\mathbf{Cocones_{C,D}}$.

6.1.3 Definition. *Let* ***C*** *be a category and* D *a diagram. Then a* ***limit*** *for* D *is a terminal object in* $\mathbf{Cones_{C,D}}$. ***Colimits*** *are defined dually.*

$(c,\{f_i\in \mathbf{C}[d_i,c] \mid i\in I\})$ is the initial object in $\mathbf{Cocones_{C,D}}$, it may be visualized by the following commutative diagram:

$$\begin{tikzcd}
 & & e & & & & \\
 & & & c \arrow[ul] & & & \\
\cdots & d_{i'} \arrow[urr] & d_i \arrow[l] \arrow[uu, "g_i"] \arrow[ur, "f_i"] & & d_j \arrow[ll, "f"] \arrow[ul, "f_j"'] \arrow[uull, "g_j"'] & d_{j'} \arrow[l] \arrow[ull] & \cdots
\end{tikzcd}$$

Limits are also called universal cones, as any other cone uniquely factorizes via them. Dually, colimits are called universal cocones.

Examples

1. Let **P** be a partial order. Then limits correspond to greater lower bounds, while colimits correspond to least upper bounds.
2. Let $D = (\{d_i\}_{i\in\omega}, \{f_i\in\mathbf{Set}[d_i, d_{i+1}]\}_{i\in\omega})$ be a diagram in **Set** such that $d_i \subseteq d_{i+1}$, and f_i = incl (the set-theoretic inclusion). Then the colimit of $d_0 \to \ldots\ldots d_i \to d_{i+1} \to \ldots..$ is $\cup\{d_i\}$ (**exercise**: what is the limit of the same diagram?) .

Exercise Prove that the colimits in **C** are the limits in $\mathbf{C}^{op}$ of the dual diagram.

Consider now a diagram as a functor from an index category **I** to **C**. Note first that any object c of the category **C** is the image of a constant functor K_c: $\mathbf{I}\to\mathbf{C}$, and so K_c can be regarded as a degenerate diagram of type I in **C**. Once diagrams are defined as functors, it makes sense to consider natural transformations between diagrams. If D and D' are two diagrams of type I, a natural transformation from D to D' is a family of arrows f_i indexed on objects in **I** such that for each arrow e in **I** (each edge of the graph of type I)

$$\begin{tikzcd} D_i \arrow[r, "f_i"] \arrow[d, "D_e"'] & D'_i \arrow[d, "D'_e"] \\ D_j \arrow[r, "f_j"] & D'_j \end{tikzcd}$$

A cone for a diagram D of type I from an object c is then a natural transformation from the constant diagram K_c to D

$$\begin{tikzcd} K_c(i) = c \arrow[r, "f_i"] \arrow[d, "K_c(e) = id_c"'] & D_i \arrow[d, "D_e"] \\ K_c(j) = c \arrow[r, "f_j"] & D_j \end{tikzcd}$$

Dually, a cocone for a diagram D of type I to an object c is a natural transformation from D to the constant diagram K_c .

6.2 Some Constructions Revisited

Let D be an empty diagram, that is a diagram with no objects and no arrows. By definition, a cone in **C** to D is then just an object c of **C**, with no other structure (and every object of **C** can be seen as a cone). A limit for the empty diagram is then an object t such that for any other object c there is exactly one arrow from c to t, i.e., it is a **terminal** object. Dually, the **initial** object is the colimit of the empty diagram.

A graph is called **discrete** if it has no arrows. For example the set $\{1,2\}$ can be regarded as a discrete graph. A diagram of type $\{1,2\}$ in a category **C** is an ordered pair of objects, (c_1,c_2). A limit for such a diagram is an object d, together with two arrows $f_1: d\to c_1$ and $f_2: d\to c_2$, such that for any other cone $(d',\{g_i\in \mathbf{C}[d',c_i] \mid i\in\{1,2\}\})$ there exists exactly one arrow $h: d'\to d$, with $f_i \circ h = g_i$ for $i\in\{1,2\}$.

$$\begin{tikzcd} d' \arrow[r, "h"] \arrow[d, "g_1"'] \arrow[drr, "g_2"] & d \arrow[dl, "f_1"] \arrow[dr, "f_2"] & \\ c_1 & & c_2 \end{tikzcd}$$

But this is just the definition of **product** d of c_1 and c_2 with $f_1: d\to c_1$ and $f_2: d\to c_2$ as projections.

Dually, the **coproduct** $c_i\#c_j$, if it exists, is just the the colimit of the diagram $\{c_i,c_j\}$.

The product of any indexed collection of objects in a category is defined analogously as the limit of the diagram $D: I\to \mathbf{C}$ where I is the index set considered as a discrete graph. This product is usually denoted by $\Pi_{i\in I}D_i$, although explicit mention of the index set is often omitted.

Consider the graph I with two vertices and two edges

$$\bullet \rightrightarrows \bullet$$

A diagram of type I in a category **C** is a pair of objects, a and b, and a pair of parallel arrows $f,g\in\mathbf{C}[a,b]$. A cone for this diagram consists of an object d, and two arrows $h\in\mathbf{C}[d,a]$ and $k\in\mathbf{C}[d,b]$ such that $g \circ h = k$ and $f \circ h = k$. A limit is a cone $(d,\{h,k\})$ that is universal, that is, for any other cone $(d',\{h',k'\})$ there exists exactly one arrow $l: d'\to d$ such that $h \circ l = h'$, and $k \circ l = k'$.

$$\begin{tikzcd}
a \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & b \\
d \arrow[u, "h"] \arrow[urr, "k"'] & & d' \arrow[ll, "l"] \arrow[ull, "h'"] \arrow[u, "k'"']
\end{tikzcd}$$

Note now that the existence of two arrows, h and k, such that $g \circ h = k$ and $f \circ h = k$, is equivalent to the existence of an arrow h such that $g \circ h = f \circ h$. Moreover, $h \circ l = h'$ implies $k \circ l = k'$, since $k \circ l = f \circ h \circ l = f \circ h' = k'$, thus the above limit is just the **equalizer** of f and g .

Dually, **coequalizers** are the colimits for the same diagram.

Consider now the following graph:

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

A diagram of this type in a category **C** is a given by three objects, a, c, and b, and two morphisms, $f\in\mathbf{C}[a,c]$ and $g\in\mathbf{C}[b,c]$. A cone to this diagram is an object d, together with three morphisms $h_a\in\mathbf{C}[d,a]$, $h_c\in\mathbf{C}[d,c]$ and $h_b\in\mathbf{C}[d,b]$, such that the following diagram commutes:

$$\begin{tikzcd}
a \arrow[r, "f"] & c & b \arrow[l, "g"'] \\
& d \arrow[ul, "h_a"] \arrow[u, "h_c"] \arrow[ur, "h_b"'] &
\end{tikzcd}$$

A cone $(d, \{h_a, h_b, h_c\})$ is a limit, if for any other cone $(d', \{h'_a, h'_b, h'_c\})$ there exists a unique arrow $k: d'\to d$ such that $h'_i = h_i \circ k$, for $i\in\{a,b,c\}$.

The commutativity of the previous diagram implies that $f \circ h_a = g \circ h_b$; conversely, given two arrows h_a and h_b such that $f \circ h_a = g \circ h_b$, one obtains a cone by defining $h_c = f \circ h_a = g \circ h_b$. Thus, the diagram for the cone $(d, \{h_a, h_b, h_c\})$ is equivalently expressed by giving only the outer commutative "square", i.e., by giving $(d, \{h_a, h_b\})$. In conclusion, a universal cone for a diagram of this type turns out to be just a **pullback**.

As usual, by taking the colimit of the same diagram we obtain the dual notion of **pushout**.

6.3 Existence of limits

In this secton, we study the important question about the existence of limits in a given category. Starting with the familiar category of sets, we generalise a common construction that allows the existence of complex limits to be states, provided that simpler ones exist.

Note first that every diagram D has limit in **Set**. It is obtained as follows.

Let $\{D_i\}_{i\in I}$ be a family of objects in D and consider the object $\Pi_{i\in I}D_i$, i.e., the product indexed by I. The elements of $\Pi_{i\in I}D_i$ are tuples $\{x_0, x_1, x_2, ...\}$ such that $x_i\in D_i$, for all $i\in I$, or equivalently functions $f : I \to \cup_{i\in I}D_i$, such that $f(i)\in D_i$.

$\Pi_{i\in I}D_i$ has projections $p_i : \Pi_{i\in I}D_i \to D_i$ for all $i\in I$, defined by $p_i(\{x_0,x_1,x_2, ...\}) = x_i$. In general these projections do not form a cone on D, that is, if $f_e: D_i\to D_j$ is an edge of D, one may have $p_j \neq f_e\circ p_i$. The idea is to take the subset L of $\Pi_{i\in I}D_i$ of all the tuples that satisfy the condition $p_j = f_e\circ p_i$. That is, $\{x_0, x_1, x_2, ...\}\in L$ if and only if, for all edges $f_e: D_i\to D_j$, one has $x_j = f_e(x_i)$. Let then γ_i be the projection p_i restricted to L. Then $(L,\{ \gamma_i \in \mathbf{C}[L,D_i] \mid i \in I\})$ is the limit (prove it as an exercise).

This set-theoretic construction is better formalized in Category Theory in the following way.

Let $\Pi(D_j / \exists i\in I\ \exists e\in E\ f_e: D_i\to D_j)$ be the product of all codomains of edges in D, with projections $\pi_j: \Pi(D_j / \exists i\in I\ \exists e\in E\ f_e: D_i\to D_j) \to D_j$. By definition of product, there is a unique function

$$\psi_1: \Pi_{i\in I}D_i \to \Pi(D_j / \exists i\in I\ \exists e\in E\ f_e: D_i\to D_j)$$

such that $p_j = \pi_j\circ\psi_1$ for any edge $f_e: D_i\to D_j$ of D. Analogously there is a unique function

$$\psi_2: \Pi_{i\in I}D_i \to \Pi(D_j / \exists i\in I\ \exists e\in E\ f_e: D_i\to D_j)$$

such that $f_e\circ p_i = \pi_j\circ\psi_2$ for any edge $f_e: D_i\to D_j$ of D.

This is visualized in the following diagram:

$$\begin{tikzcd}
 & D_j \\
\Pi_{i\in I}D_i \arrow[ur, "p_j"] \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] \arrow[d, "p_i"] & \Pi(D_j / f_e: D_i\to D_j \text{ edge of } D) \arrow[u, "\pi_e"'] \arrow[d, "\pi_e"] \\
D_i \arrow[r, "f_e"] & D_j
\end{tikzcd}$$

Note now that, in set-theoretic terms, for all the tuples $\{x_0, x_1, x_2, ...\}$ in $\Pi_{i\in I}D_i$ the following properties are equivalent:

1. for all edges $f_e: D_i\to D_j$, $x_j = f_e(x_i)$
2. $\psi_1(\{x_0, x_1, x_2, ...\}) = \psi_2(\{x_0, x_1, x_2, ...\})$

Then, what we are looking for is the maximal subset L of $\Pi_{i\in I}D_i$ whose elements satisfy (2), but we aleady know that this is none other than the equalizer of ψ_1 and ψ_2. By a diagram,

$$\begin{tikzcd}
 & & D_j \\
L \arrow[r, "h"] \arrow[urr, bend left, "\gamma_i"] & \Pi_{i\in I} D_i \arrow[ur, "p_j"] \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] \arrow[d, "p_i"] & \Pi(D_j \,/\, f_e : D_i \to D_j \text{ edge of } D) \arrow[u, "\pi_e"'] \arrow[d, "\pi_e"] \\
L' \arrow[u, "k"] \arrow[ur, "h'"'] & D_i \arrow[r, "f_e"] & D_j
\end{tikzcd}$$

We are now ready to generalize to every category **C** the previous construction of limits in **Set.**

6.3.1 Theorem *Let D be a diagram in **C** with sets I of vertices and E of edges. If every I-indexed family and every E-indexed family of objects has a product, and every pair of morphisms has an equalizer, then D has a limit.*
Proof Exercise (use the previous diagrams). ♦

6.3.2 Corollary *If a category **C** has arbitrary products, and equalizers for every pair of morphisms, then every diagram has a limit.*

6.3.3 Corollary *If a category **C** has all finite products, and coequalizers for every pair of morphisms, then every finite diagram has a limit.*

The relevance of theorem 6.3.1 is that, in general, it is simpler to check the existence of products and equalizers than to prove directly the existence of limits.

Example Corollary 6.3.2 may be used to prove that every diagram has a limit in **CPO**. If $\{C_i\}_{i\in I}$ is a family of c.p.o.'s, let $\Pi_{i\in I}C_i$ be the product indexed by I . $\Pi_{i\in I}C_i$ may be given a c.p.o. structure by the componentwise order, that is, $(c_i)_{i\in I} \le (d_i)_{i\in I}$ iff $\forall\ i\in I\ \ c_i \le d_i$. The projections $p_i: \Pi_{i\in I}(C_i)\to C_i$ are defined by $p_i(\ (c_i)_{i\in I}\ ) = c_i$. It is easy to prove that $\Pi_{i\in I}(C_i)$ is indeed a cpo, that the projections are continuous, and that $\Pi_{i\in I}(C_i)$ satisfies the universal property of the product.

Given $f,g : A\to B$, their equalizer is $h: A'\to A$, where $A' = \{a\in A \,/\, f(a) = g(a)\}$ with the ordering inherited by A, and h is the injection. A' is a c.p.o. Indeed, let D be a direct subset of

A'; then D is also a direct subset of A, and thus $f(\cup D)= \cup_{a\in D}f(a) = \cup_{a\in D}g(a) = g(\cup D)$. By this, $\cup D\in A'$. The continuity of h and the universal property for equalizers are easy to prove.

In propositions 2.5.5 and 2.5.6 we showed how to define products and equalizers from terminal objects and pullbacks. This suggests an even simpler sufficient (and necessary) condition for the existence of all finite limits.

6.3.4 Corollary *If **C** has a terminal objects and pullbacks for every pair of morphisms, then it has all finite limits.*

Exercise State the dual versions of theorem 6.3.1 and corollaries 6.3.1 to 6.3.4.

6.4 Preservation and Creation of Limits

In this section we study some cases of functors which "preserve" the property of objects to be limits of a diagram.

6.4.1 Definition *Let $G: \mathbf{A}\to\mathbf{X}$ be a functor, and let $(a,\{\tau_i \in \mathbf{A}[a,d_i] \mid i\in I\})$ be an universal cone from a on the diagram D in **A**. We then say that G* ***preserves the limit*** *$(a,\{\tau_i \in A[a,d_i] \mid i\in I\})$ if and only if $(Ga,\{G\tau_i \in \mathbf{X}[Ga,Gd_i] \mid i\in I\})$ is an universal cone from Ga on the diagram G(D) in X.* ***Preservation of colimits*** *is defined dually.*

6.4.2 Theorem *If the functor $G: \mathbf{A}\to\mathbf{X}$ has a left adjoint $F: \mathbf{X}\to\mathbf{A}$, and the diagram $D = (\{d_i\}_{i\in I}, \{f_e\}_{e\in E})$ in **A** has limit $(a, \{\tau_i\in \mathbf{A}[a,d_i] \mid i\in I\})$, then $G(D) = (\{Gd_i\}_{i\in I}, \{Gf_e\}_{e\in E})$ has a limit in **X**, and the limit is $(Ga,\{G\tau_i\in\mathbf{X}[Ga,Gd_i] \mid i\in I\})$.*

Proof By the properties of functors, $(\{Gd_i\}_{i\in I}, \{Gf_e\}_{e\in E})$ is a cone; we only need to prove that it is universal. Let $(x,\{\sigma_i\in\mathbf{X}[x,Gd_i] \mid i\in I\})$ be another cone, and let $\varphi: \mathbf{A}[Fx,d_i] \cong \mathbf{X}[x,Gd_i]$ be the isomorphism of the adjunction. Then $(Fx, \{\varphi^{-1}(\sigma_i)\in\mathbf{A}[Fx,d_i] \mid i\in I\})$ is a cone. Indeed, for all $f_e: d_i\to d_j$ one has

$$\begin{aligned} f_e\circ\varphi^{-1}(\sigma_i) &= \varphi^{-1}(G(f_e)\circ\sigma_i) && \text{by naturality} \\ &= \varphi^{-1}(\sigma_j) && \text{because } (\sigma_i) \text{ is a cone on } G(D). \end{aligned}$$

By the universality of $(a, \{\tau_i\in\mathbf{A}[a,d_i] \mid i\in I\})$ there exists a unique arrow $h: Fx\to a$ such that $\forall i\in I\ \tau_i\circ h = \varphi^{-1}(\sigma_i)$. Take then $\varphi(h): x\to Ga$. Since $G\tau_i\circ\varphi(h) = \varphi(\tau_i\circ h) = \sigma_i$, one has that $\varphi(h)$ is a mediating morphism between the cones $(x, \{\sigma_i\in\mathbf{X}[x,Gd_i] \mid i\in I\})$ and $(Ga, \{G\tau_i\in\mathbf{X}[Ga,Gd_i] \mid i\in I\})$.

Moreover, $\varphi(h)$ is unique, for, if $\varphi(h')$ is another mediating morphism, then h' is a mediating morphism between $(Fx,\{\varphi^{-1}(\sigma_i)\in \mathbf{A}[Fx,d_i] \mid i\in I\})$ and $(a, \{\tau_i\in \mathbf{A}[a,d_i] \mid i\in I\})$. By universality, $h' = h$ (see the diagram below). ♦

The proof of theorem 6.4.2 may be visualized by the following commutative diagrams:

in **A**

$$\begin{tikzcd}
D_i \arrow[r, "f_e"] & D_j \\
a \arrow[u, "\tau_i"] \arrow[ur, "\tau_j"] & Fx \arrow[l, "h"] \arrow[u, "\varphi^{-1}(\sigma_j)"'] \arrow[ul, "\varphi^{-1}(\sigma_i)"]
\end{tikzcd}$$

in **X**

$$\begin{tikzcd}
GD_i \arrow[r, "Gf_e"] & GD_j \\
Ga \arrow[u, "G\tau_i"] \arrow[ur, "G\tau_j"] & x \arrow[l, "\varphi(h)"] \arrow[u, "\sigma_j"'] \arrow[ul, "\sigma_i"]
\end{tikzcd}$$

Exercise Give the dual statement of theorem 6.4.2 .

An example of application of (the dual of) theorem 6.4.2 is the following.

6.4.3 Theorem *In every Cartesian closed category **C**, products distribute over colimits.*
Proof Just note that by definition of CCC the functor $-\times a$: $\mathbf{C}\to\mathbf{C}$ has a right adjoint for each $a\in Ob_{\mathbf{C}}$, and apply the dual of theorem 6.4.2. ♦

6.4.4 Corollary *Let **C** be a CCC. Suppose, moreover, that it contains an initial object 0, and coproducts for each pair of objects. Then, for all $X,Y,Z\in Ob_{\mathbf{C}}$, one has*

i. $0 \times Z \cong Z$
ii. $(X + Y) \times Z \cong (X \times Z) + (Y \times Z)$

Exercises (Huwig-Poigné) A category **C** has **fixpoints** if for every morphism $f: X\times X'\to X'$ there exists a morphism $Y(f): X\to X'$ such that $f \circ \langle id_X,Y(f)\rangle = Y(f)$. Prove then the following facts:

1. **CPO** has fixpoints.
2. If **C** is a CCC and it has an initial object 0 and fixpoints, then it is inconsistent , i.e. all objects are isomorphic. (*Hint*: let t the terminal object, and consider the projection p_2: $t\times 0\to 0$. Then $Y(p_2)$: $t\to 0$. Deduce from this an isomorhism between 0 and t ...) .
3. (difficult) If **C** is a CCC and it has fixpoints and binary coproducts, then **C** is inconsistent. *Hint*: consider the object $2=t+t$ and interpret the injection **tt:** $t\to 2$ and **ff**: $t\to 2$ as denoting "truth" and

"falsehood." Then all finitary truth tables can be expressed by morphisms in $2\times2\times....\times2 \to 2$. The existence of a fixpoint for "**not**" induces the following identities:

$$\mathbf{tt} = \mathrm{Y}(\mathbf{not})\ \mathbf{or}\ \mathbf{not}(\mathrm{Y}(\mathbf{not})) = \mathrm{Y}(\mathbf{not})\ \mathbf{or}\ \mathrm{Y}(\mathbf{not}) = \mathrm{Y}(\mathbf{not})$$

$$\mathbf{ff} = \mathrm{Y}(\mathbf{not})\ \mathbf{and}\ \mathbf{not}(\mathrm{Y}(\mathbf{not})) = \mathrm{Y}(\mathbf{not})\ \mathbf{and}\ \mathrm{Y}(\mathbf{not}) = \mathrm{Y}(\mathbf{not})$$

Hence the injections **tt** , **ff** : $t\to2$ are identified. As, for all objects X in **C**, $X+X = (t\times X) + (t\times X) = (t+t)\times X$, one may deduce the equality of the coproduct injections $u,v: X\to X\times X$ for all X . By this it is easy to obtain the inconsistency.

Fixed points will be widely discussed in chapter 8. The reader may already understand, though, that from the point of view of denotational semantics, this is a negative result: coproducts (i.e. disjoint sums) are incompatible with fixed point operators. As is well known, both constructions are rather relevant in semantic domains.

Another important case of limit-preserving functor is the hom-functor.

6.4.5 Theorem *Let* **C** *be a small category. For any object* $c\in Ob_{\mathbf{C}}$*, the hom-functor hom[c,_]:* $\mathbf{C}\to\mathbf{Set}$ *preserves limits.*

Proof: Consider the diagram $D = (\{d_i\}_{i\in I}, \{f_e\}_{e\in E})$ in **C**, and let $(a, \{\tau_i\in\mathbf{A}[a,d_i] \mid i\in I\})$ be a limit. We must prove that the diagram $S = (\{hom[c,d_i]\}_{i\in I}, \{f_e \circ \_\}_{e\in E})$ has a limit in **Set**.

Take $L = (hom[c,a],\{\tau_i \circ \_ : hom[c,a]\to hom[c,d_i] \mid i\in I\})$ as a limit.

Since hom[c,_] is a functor, L is a cone for S. We have only to prove that it is universal. Suppose then that $L' = (X, \{\gamma_i : X\to hom[c,d_i] \mid i\in I\})$ is another cone for S. This means that for any $f_e: d_i\to d_j$, and any $x\in X$, $f_e \circ \gamma_i(x) = \gamma_j(x)$. For any $x\in X$, $(c, \{\gamma_i(x): c\to d_i \mid i\in I\})$ is then a cone for D, and by universality of $(a, \{\tau_i\in\mathbf{A}[a,d_i] \mid i\in I\})$, there exists a unique morphism h_x: $c\to a$ in **C** such that $\gamma_i(x) = \tau_i \circ h_x$ for all i. Define then $h: X\to hom[c,a]$ by $h(x) = h_x$. We have $(\tau_i \circ \_) \circ h = \gamma_i$, since for every $x\in X$, $\tau_i \circ h(x) = \tau_i \circ h_x = \gamma_i(x)$. Unicity follows by unicity of h_x for any x. ♦

Exercise Use theorem 6.4.5 and prove theorem 6.4.2 in case the categories considered are small.

4.6 Definition *A functor* $F: \mathbf{A}\to\mathbf{X}$ ***creates*** *limits for a given diagram D if, whenever* $(x, \{\sigma_i\in\mathbf{X}[x,F(d_i)] \mid i\in I\})$ *is a limit for F(D) in* **X***, then there exists a unique cone* $(a, \{\tau_i \in\mathbf{A}[a,d_i] \mid i\in I\})$ *over D in* **A***, such that* $F(a) = x$ *and* $F(\tau_i) = \sigma_i$ *for every* $i\in I$*, and* $(a,\{\tau_i \in\mathbf{A}[a,d_i] \mid i\in I\})$ *is a limit.*

Example The forgetful functor U from **Grp** to **Set** creates all limits. For instance, the fact that it creates products is another way of stating that, given two groups G and G', there is a unique group structure on $U(G)\times U(G')$, which gives their product in Grp.

6.5 ω-limits

An important case of diagrams in a category **C** is that of infinite chains of objects. These diagrams, and the associated limits, are particularly relevant for the denotational semantics of programming languages, since they provide the base for the solution of recursive domain equations with the so-called *least fixed point technique* (see chapter 10)

6.5.1 Definition

*i) An **ω-diagram** in a category **C** is a diagram with the following structure:*

$$D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} D_2 \cdots\cdots \longrightarrow D_n \xrightarrow{f_n} D_{n+1} \cdots$$

*(dually, one defines **ω^{OP}-diagrams** by just reversing the arrows).*
*ii. A category **C** is **ω-complete** (**ω-cocomplete**) iff it has limits (colimits) for all ω-diagrams.*
*iii. A functor F: **C**→**C** is **ω-continuous** iff it preserves all colimits of ω-diagrams.*

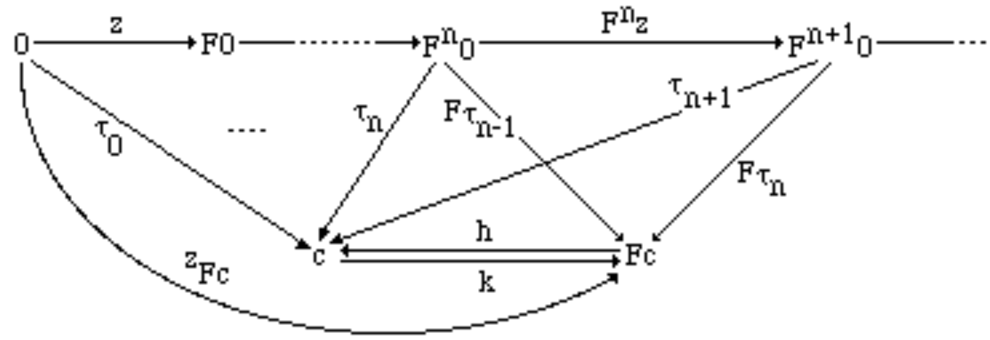
If **C** is a partial order then,

i. an ω-diagram in **C** is an ω-chain
ii. **C** is ω-cocomplete if and only if **C** is a cpo
iii. a functor F: **C**→**C** is ω-continuous iff the associated function on object of **C** is continuous.

6.5.2 Theorem *Let **C** be a category with initial object 0. Let F: **C**→**C** be an ω-continuos (covariant) functor and $z\in\mathbf{C}[0,F(0)]$ be the unique arrows defined by the initiality of 0. Assume also that $(c, \{\tau_i\in\mathbf{C}[F^i(0),c]_{i\in\omega}\})$ is a colimit for the ω-diagram $(\{F^i(0)\}_{i\in\omega}, F^i(z)\}_{i\in\omega})$, where $F^0(0) = 0$ and $F^0(z) = z$. Then $c \cong Fc$.*

Proof By the hypothesis, one has that $(Fc, \{F\tau_i\in\mathbf{C}[F^{i+1}(0),Fc]_{i\in\omega}\})$ is a limit for $(\{F^{i+1}(0)\}_{i\in\omega}, \{F^{i+1}(z)\}_{i\in\omega})$ and $(c, \{\tau_{i+1}\in\mathbf{C}[F^{i+1}(0),c]_{i\in\omega}\})$ is a cone for the same diagram. Thus, by universality, there exists a unique arrow $h: Fc\to c$ such that $\forall i\in\omega\ \ h \circ F\tau_i = \tau_{i+1}$. Now add to $(Fc, \{F\tau_i\in\mathbf{C}[F^{i+1}(0),Fc]_{i\in\omega}\})$ the unique arrow $z_{Fc}\in\mathbf{C}[0,Fc]$. This gives a cone for $(\{F^i(0)\}_{i\in\omega}, \{F^i(z)\}_{i\in\omega})$ and, by the universality of $(c,\{\tau_i \in \mathbf{C}[F^i(0),c]_{i\in\omega}\})$, there exists a unique arrow $k: c\to Fc$ such that $\forall i\in\omega\ \ k \circ \tau_{i+1} = F\tau_i$ (of course $k \circ \tau_0 = z_{Fc}$). But, then, $\forall i\in\omega$ $h \circ k \circ \tau_{i+1} = h \circ F\tau_i =\tau_{i+1}$ (and $h \circ k \circ \tau_0 = \tau_0$), thus $h\circ k$ is a mediating morphism between $(c, \{\tau_i\in\mathbf{C}[F^i(0),c]_{i\in\omega}\})$ and itself. Thus, by unicity, $h\circ k = id$.
In the same way, one proves that $k \circ h = id$. ♦

This is all summarized by the following diagram:



Theorem 6.5.2 tells us how to give meaning to recursive definitions of data types under certain circumstances. Very informally, assume that types are interpreted as objects of a category. Then in a recursive definition X = [...X...] of a data type of data X, the transformation [... _ ...] may be understood as an endofunctor F(_) for which we are seeking a fixed point. Indeed, if F satisfies the properties in theorem 6.5.2, then the theorem "solves" the equation (or recursive definition) X = [...X...]. In a sense, this construction gives meaning to X = [...X...], over a suitable categorical structure, in the same way that the equation $x = x^2+7$ is "given meaning" over the complex numbers by finding a solution for it.

However, the assumptions on F are too strong and leave out several significant cases (e.g., hom-functors or exponents). Chapter 10 is entirely devoted to a nontrivial extension of this technique in order to handle a more relevant class of recursive definitions of data types.

References Main textbooks.