Chapter 5

UNIVERSAL ARROWS AND ADJUNCTIONS

In chapter 3, we introduced the notion of functor as a uniform way to describe "acceptable" constructions in Category Theory. The reader may have noticed that in some cases, for a given construction $F: \mathbf{C}\to\mathbf{D}$, there exists a particular object c in $\mathbf{C}$ and an arrow $u: F(c)\to d$, which has a "universal behaviour" with respect to d, in the sense that every other arrow $f: F(c')\to d$ uniquely factorizes through u , i.e., there exists a unique $f': c\to c'$ such that $f = u \circ F(f')$ or the following diagram commutes:

$$\begin{tikzcd}
c' \arrow[d, "f'"'] & F(c') \arrow[r, "f"] \arrow[d, "F(f')"'] & d \\
c & F(c) \arrow[ur, "u"'] &
\end{tikzcd}$$

Consider for example the product functor $\_\times a: \mathbf{C}\to\mathbf{C}$ where $\mathbf{C}$ is a CCC and $a\in Ob_{\mathbf{C}}$. The arrow $eval: b^a\times a\to b$ is universal with respect to b, in the previous sense, since for every arrow $f: c\times a\to b$ there exists a unique arrow $\Lambda(f): c\to b^a$ such that $f = eval \circ \Lambda(f)$, i.e.,

$$\begin{tikzcd}
c \arrow[d, "\Lambda(f)"'] & c\mathrm{x}a \arrow[r, "f"] \arrow[d, "\Lambda(f)\mathrm{x}id"'] & b \\
b^a & b^a\mathrm{x}a \arrow[ur, "eval_{a,b}"'] &
\end{tikzcd}$$

Also the product of two objects may be described in terms of universal arrows. Just take the diagonal functor $\Delta: \mathbf{C}\to\mathbf{C}\mathrm{x}\mathbf{C}$, with $\Delta(c) = (c,c)$ and $\Delta(f) = (f,f)$ and observe that the following diagram commutes:

$$
\begin{tikzcd}
c \arrow[d, "{<f,g>}"'] & \Delta(c) \arrow[r, "{(f,g)}"] \arrow[d, "{\Delta(<f,g>)}"'] & (a,b) \\
a\times b & \Delta(a\times b) \arrow[ur, "{(p_1,p_2)}"'] &
\end{tikzcd}
$$

The aim of this chapter is to study the concept of universal arrow outlined above, and some of its implications. In particular we prove that given a functor $F: \mathbf{C}\to\mathbf{D}$, if for every $d\in\mathbf{D}$ there exists an universal arrow $u_d: F(c_d)\to d$, then the function $g: Ob_{\mathbf{D}}\to Ob_{\mathbf{C}}$ that takes d to c_d may be extended to a functor $G: \mathbf{D}\to\mathbf{C}$. Such a functor G is called (right) **adjoint** to F. Remarkably, if G is a (right) adjoint to F, then F is a (left) adjoint to G, in a dual sense, and each one describes the other up to isomorphism. From another point of view, F and G can be seen as a pair of "mutually representable" functors.

The notion of adjointness is probably the most profound and original contribution of Category Theory to mathematics. The reader must not expect to understand its relevance upon first reading the definition, or the few examples and applications in this chapter: only by a systematic use of adjunctions will she or he become competent on the subject.

5.1 Universal arrows

In addition to the previous remarks, the careful reader has surely noticed that most of the definitions of the constructions defined in chapter 2 (products, coproducts, terminal object, exponents . . .) have the following common pattern:

for all . . . there exist a unique . . . such that . . . .

As a matter of fact, all those definitions can be seen as particular cases of a same notion (or its dual): the **universal arrow**. For historical reasons, universal arrows are defined dually with respect to the examples just mentioned (exponents, products . . .). Couniversal maps, to be defined next, will fit the examples.

5.1.1 Definition. *Let $F: \mathbf{C}\to\mathbf{D}$ be a functor and $d\in Ob_{\mathbf{D}}$. Then $<u,c_d>$ is* ***universal*** *from d to F iff $u\in\mathbf{D}[d,F(c_d)]$, $c_d\in Ob_{\mathbf{C}}$, and $\forall c'\in Ob_{\mathbf{C}}$, $\forall f\in\mathbf{D}[d,F(c')]$, $\exists! f'\in\mathbf{C}[c_d,c']$ $f = F(f')\circ u$.*

As usual, this may be equivalently visualized by

$$\begin{tikzcd}
c' & F(c') & d \arrow[l, "f"'] \arrow[dl, "u"] \\
c_d \arrow[u, "\exists!\, f'"] & F(c_d) \arrow[u, "F(f')"'] &
\end{tikzcd}$$

Example $<(q_1,q_2),(a\#b,a\#b)>$ is universal from (a,b) to $\mathbf{\Delta}$, where $\mathbf{\Delta}: \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ is the diagonal functor.

$$\begin{tikzcd}
c & \Delta(c) & (a,b) \arrow[l, "(f,g)"'] \arrow[dl, "(q_1,q_2)"] \\
a\#b \arrow[u, "f+g"] & \Delta(a\#b) \arrow[u, "\Delta(f+g)"'] &
\end{tikzcd}$$

5.1.2 Definition. *Let $F: \mathbf{C} \to \mathbf{D}$ be a functor and $d \in Ob_{\mathbf{D}}$. $<u,c_d>$ is **(co)universal** from F to d iff $u \in \mathbf{D}[F(c_d),d]$, $c_d \in Ob_{\mathbf{C}}$, and $\forall c' \in Ob_{\mathbf{C}}$, $\forall f \in \mathbf{D}[F(c'),d]$ $\exists!\, f' \in \mathbf{C}[c',c_d]$ $f = u \circ F(f')$.*

The diagrams and examples in the introduction to this chapter refer this "dual" notion. Here are other interesting examples:

5.1.3 Example Let **!C** be the unique functor from the category **C** to the category **1**; if $id_1 = u \in \mathbf{1}[\mathbf{!C}(c),1]$ is universal, then c is terminal in **C**:

$$\begin{tikzcd}
c' \arrow[d, "\exists!\, f'"'] & 1 = \mathbf{!C}(c') \arrow[r, "f = id_1"] \arrow[d, "id_1 = \mathbf{!C}(f')"'] & 1 \\
c & 1 = \mathbf{!C}(c') \arrow[ur, "u = id_1"'] &
\end{tikzcd}$$

5.1.4 Example Let **C** be a category of partial morphisms, $\mathbf{C_t}$ be the associated category of total morphisms, and **Inc**: $\mathbf{C_t} \to \mathbf{C}$ be the embedding functor (see section 2.6). Recall that, by definition, the lifting of $a \in Ob_{\mathbf{C}}$ is an object $a° \in Ob_{\mathbf{C}}$ together with a morphism $ex_a \in \mathbf{C_t}[a°,a]$ such that for every $f \in \mathbf{C}[c,a]$, there exists one and only one $f' \in \mathbf{C_t}[c,a°]$ satisfying the equation $ex_a \circ f' = f$. This says exactly that $ex_a: a° \to a$ is an universal arrow from **Inc** to a :

$$\begin{tikzcd} c \arrow[d, "f'"'] \\ a^* \end{tikzcd} \qquad\qquad \begin{tikzcd} c \arrow[r, "f"] \arrow[d, "f'"'] & a \\ a^* \arrow[ur, "ex_a"'] & \end{tikzcd}$$

in $\mathbf{C_t}$ in $\mathbf{C}$

Exercise Define an initial object as a universal arrow from 1 to **!C**.

5.1.5 Proposition. *Let $F: \mathbf{C}\to\mathbf{D}$ be a functor and $d\in Ob_{\mathbf{D}}$. Assume that $<u,c>$ is universal from d to F. Then*

1. *$<u',c'>$ is universal from d to F $\Rightarrow$ $c \cong c'$;*
2. *$c \cong c'$ via (h,h^{-1}) $\Rightarrow$ $<F(h)\circ u,c'>$ is universal from d to F.*

Proof

1. $\exists!h\in\mathbf{C}[c,c']$ $\exists!h'\in\mathbf{C}[c',c]$ $u'= F(h)\circ u = F(h)\circ F(h')\circ u'= F(h\circ h')\circ u'$. But $u' = F(id)\circ u'$.
By unicity, $h\circ h' = id$. Similarly, $h'\circ h = id$.
2. Exercise. ♦

By the proof of proposition 5.1.5, one even has that the isomorphism is unique. This is a strong property of universal constructions as a very common tool in mathematics (see the examples below). It is hard to think of a more suitable language than the categorical one for expressing properties like this.

We next give two alternative characterizations of the notion of universal arrow: the first one, in theorem 5.1.6, is of an equational nature; the second one, in theorem 5.1.7, makes use only of the notion of a natural isomorphism. In a sense, the definition of universal arrow given in definition 5.1.1 is in the middle way: it is based on one equation and on the existence of an isomorphism (does the reader see any analogy with the various characterizations of products and exponents we have given?).

5.1.6 Theorem *Let $G: \mathbf{C}\to\mathbf{D}$ be a functor, $d\in Ob_{\mathbf{D}}$ and $c_d\in Ob_{\mathbf{C}}$. Then there exists $u\in\mathbf{D}[d,G(c_d)]$ such that $<u,c_d>$ is universal from d to G iff, for every $c\in Ob_{\mathbf{C}}$ there is an operation $\tau_c: \mathbf{D}[d,G(c)]\to\mathbf{C}[c_d,c]$ such that, for every $f\in\mathbf{D}[d,G(c)]$ and every $h\in\mathbf{C}[c_d,c]$,*

1. *$G(\tau_c(f))\circ u = f$*
2. *$\tau_c(G(h)\circ u) = h$*

Proof

($\Leftarrow$) let $<u,c_d>$ be universal from d to G. For every $f\in\mathbf{D}[d,G(c)]$ define $\tau_c(f)$ as the unique arrow f' such that $G(f') \circ u = f$. (1) is then immediate by definition, and (2) follows by unicity.

($\Rightarrow$) let τ_c: $\mathbf{D}[d,G(c)]\to\mathbf{C}[c_d,c]$ be an operation which satisfies (1) and (2) above. We must only prove that it is an isomorphism. Define then, for every $h\in\mathbf{C}[c_d,c]$ $\tau_c^{-1}(h) = G(h) \circ u$; thus, we have:

$$\tau_c^{-1}(\tau_c(f)) = G(\tau_c(f)) \circ u = f \qquad \text{by (1)}$$
$$\tau_c(\tau_c^{-1}(h)) = \tau_c(G(h) \circ u) = h \qquad \text{by (2).} \ \blacklozenge$$

5.1.7 Theorem. *Let $\mathbf{C}$, $\mathbf{D}$ be locally small categories, $G: \mathbf{C}\to\mathbf{D}$, $d\in Ob_{\mathbf{D}}$ and $c_d\in Ob_{\mathbf{C}}$. Then there exists $u\in\mathbf{D}[d,G(c_d)]$ such that $<u,c_d>$ is universal from d to G iff $\mathbf{C}[c_d,\_] \cong \mathbf{D}[d,G\_]$.*

Proof

($\Leftarrow$) For $c'\in Ob_{\mathbf{C}}$ and $f'\in\mathbf{C}[c_d,c']$, set $\tau_{c'}(f') = G(f') \circ u \in \mathbf{D}[d,Fc']$.

Then τ: $\mathbf{C}[c_d,\_]\to\mathbf{D}[d,G\_]$ is a natural transformation, since $\tau_{c'}(g \circ h) = G(g \circ h) \circ u = G(g) \circ \tau_c(h)$. That is, for $g\in\mathbf{C}[c,c']$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C}[c_d,c] \arrow[r, "\tau_c"] \arrow[d, "g\circ_"'] & \mathbf{D}[d,G(c)] \arrow[d, "G(g)\circ_"] \\
\mathbf{C}[c_d,c'] \arrow[r, "\tau_{c'}"] & \mathbf{D}[d,G(c')]
\end{tikzcd}$$

Moreover, $\forall f\in\mathbf{D}[d,Fc']$ $\exists! f'\in\mathbf{C}[c_d,c']$ $f = \tau_{c'}(f')$, by definition. Thus, by proposition 3.2.3, τ is a natural isomorphism.

($\Rightarrow$) Let $\tau : \mathbf{C}[c_d,\_] \cong \mathbf{D}[d,G\_]$ and set $u = \tau_{c_d}(id)$. Then $u\in\mathbf{D}[d,G(c_d)]$. By the naturality of τ, for all $c,c'\in Ob_{\mathbf{C}}$, $G(g) \circ \tau_c(\_) = \tau_{c'}(g\circ\_)$ and, hence, $\forall f\in\mathbf{D}[d,G(c')]$

$$\begin{aligned} G(\tau_{c'}^{-1}(f)) \circ u &= G(\tau_{c'}^{-1}(f))\circ\tau_{c_d}(id) \\ &= \tau_{c'}(\tau_{c'}^{-1}(f)) \\ &= f . \end{aligned}$$

That is, $\forall f\in\mathbf{D}[d,G(c')]$ $\exists! f'(= \tau_{c'}^{-1}(f))\in\mathbf{C}[c_d,c']$ such that $f = G(f')\circ u$. $\blacklozenge$

Exercise: Give the dual version of theorems 5.1.6 and 5.1.7.

Recall now that a functor $F : \mathbf{C}\to\mathbf{Set}$ is representable if there exists a $c\in\mathbf{C}$ such that $F \cong \mathbf{C}[c,\_]$ naturally.

5.1.8 Corollary *Let $\mathbf{C}$, $\mathbf{D}$ be small categories, $G: \mathbf{C}\to\mathbf{D}$, $d\in Ob_{\mathbf{D}}$ and $c_d\in Ob_{\mathbf{C}}$. Then there exists $u\in\mathbf{D}[d,G(c_d)]$ such that $<u,c_d>$ is universal from d to G iff the functor $\mathbf{D}[d,G\_]: \mathbf{C}\to\mathbf{Set}$ is representable.*

5.2 From Universal Arrows toward Adjunctions

The construction of a universal arrow $u_d: G(c_d)\to d$ from $G: \mathbf{C}\to\mathbf{D}$ to d usually depends on d. If this construction can always be performed, the function $d \vdash c_d$ can be extended to a functor $F: \mathbf{D}\to\mathbf{C}$. We shall see in the next section that such G and F relate in an important way called adjunction; for the moment we concentrate on the construction of the functor F.

In this and in the following section, we assume that we are dealing with locally small categories.

5.2.1 Theorem. *Let $G: \mathbf{C}\to\mathbf{D}$ be a functor such that $\forall d\in Ob_{\mathbf{D}}\ \exists<u_d,c_d>$ universal from d to G Then there exists a functor $F: \mathbf{D}\to\mathbf{C}$ such that*

i. $F(d) = c_d$

ii. $\mathbf{C}[F\_,\_] \cong \mathbf{D}[\_,G\_]$.

(Note that $\mathbf{C}[F\_,\_]$, $\mathbf{D}[\_,G\_] : \mathbf{D}^{op}\times\mathbf{C}\to\mathbf{Set}$). Moreover, the functor F is unique, up to isomorphism.

Proof: By assumption we know that, for all $f\in\mathbf{D}[d,d']$,

$$\begin{tikzcd}
F(d) \arrow[d, "\exists! g"'] & d \arrow[r, "u_d"] \arrow[d, "f"'] & G(F(d)) \arrow[d, "G(g)"] \\
F(d') & d' \arrow[r, "u_{d'}"] & G(F(d'))
\end{tikzcd}$$

Set then $F(f) = g$, that is, $u_{d'}\circ f = G(Ff))\circ u_d$. By the uniqueness property, $G(id) = id$.

Moreover, by twice the definition of F,

$$\begin{tikzcd}
d \arrow[r, "u_d"] \arrow[d, "f"'] \arrow[dd, bend right=60, "h\circ f"'] & G(F(d)) \arrow[d, "G(F(f))"] \arrow[dd, bend left=60, "G(F(h)\circ F(f))"] \\
d' \arrow[r, "u_{d'}"] \arrow[d, "h"'] & G(F(d')) \arrow[d, "G(F(h))"] \\
d'' \arrow[r, "u_{d''}"] & G(F(d''))
\end{tikzcd}$$

And again by unicity of $F(h \circ f)$, one then has $F(f\circ h) = F(f) \circ F(h)$.
We need now to define a natural isomorphism φ: $\mathbf{D}[\_,G\_] \cong \mathbf{C}[F\_,\_]$. Thus we first need to check, for a suitable φ, that for all $g\in\mathbf{D}[d',d]$ and $h\in\mathbf{C}[c,c']$ the following diagram commutes:

$$\begin{tikzcd}
\mathbf{D}[d,G(c)] \arrow[r, "\varphi"] \arrow[d, "G(h)\circ_\circ g"'] & \mathbf{C}[F(c),d] \arrow[d, "h\circ_\circ F(g)"] \\
\mathbf{D}[d',F(c')] \arrow[r, "\varphi"] & \mathbf{C}[F(d'),c']
\end{tikzcd}$$

Equivalently,

1. $\forall f\in\mathbf{D}[d,Fc]\ \ \varphi(G(h)\circ f\circ g) = h\circ\varphi(f)\circ Fg$.

Now write u (u') for u_d ($u_{d'}$, respectively). We know then that $\forall f\in\mathbf{D}[d,G(c)]\ \exists! f'\in\mathbf{C}[F(d),c]$ $f = G(f') \circ u$. Define $\varphi(f) = f'$, that is, $f = G(\varphi(f))\circ u$ (compare with the definition of F). φ is clearly a set-theoretic isomorphism; thus, we have only to prove the naturality (1).
By the definition of the functors G and F, the following diagram commutes:

$$\begin{tikzcd}
d' \arrow[r, "u'"] \arrow[d, "g"'] & G(F(d')) \arrow[d, "G(F(g))"] \arrow[dd, bend left=60, "G(\varphi(f \circ g))"] \\
d \arrow[r, "u"] \arrow[dr, "f"'] & G(F(d)) \arrow[d, "G(\varphi(f))"] \\
& G(c)
\end{tikzcd}$$

That is, $G(\varphi(f \circ g)) \circ u' = G(\varphi(f) \circ F(g)) \circ u'$, since G is a functor. By unicity,

2. $\varphi(f \circ g) = \varphi(f) \circ F(g)$.

Moreover, for all $f \in \mathbf{D}[d,G(c)]$,

$$\begin{tikzcd}
d' \arrow[rr, "u'"] \arrow[dr, "g"'] & & G(F(d')) \arrow[dd, "G(\varphi(f \circ g))"] \arrow[ddr, "G(\varphi(G(h) \circ f \circ g))"] & \\
& d \arrow[dr, "f"'] & & \\
& & G(c) \arrow[r, "G(h)"'] & G(c')
\end{tikzcd}$$

by the definition of G .

Therefore,

$$\begin{aligned} \varphi(G(h) \circ f \circ g) &= h \circ \varphi(f \circ g) && \text{by the diagram and unicity} \\ &= h \circ \varphi(f) \circ F(g) && \text{by (2).} \end{aligned}$$

This proves (1), i.e. the naturality of φ , and by proposition 3.2.3 the proof is completed. ♦

Dually, we have the following:

5.2.2 Theorem. Let $F: \mathbf{D} \to \mathbf{C}$ be a functor such that $\forall c \in Ob_{\mathbf{C}}\ \exists \langle u_c, d_c \rangle$ universal from F to c. Then there exists a (unique) functor $G: \mathbf{C} \to \mathbf{D}$ such that

i. $G(c) = d_c$

ii. $\mathbf{C}[F\_,\_] \cong \mathbf{D}[\_,G\_]$.

Proof The result follows by duality; anyway we explicitly reprove it, but by using a different technique from the one used above. As the reader will see, the difference is essentially notational, but she or he is invited to study both proofs since they are good examples of two common proof styles in Category Theory.

Let $G_{Ob}: Ob_{\mathbf{C}} \to Ob_{\mathbf{D}}$ be the function defined by $G_{Ob}(c) = d_c$, where $u_c: F(d_c) \to c$ is the universal arrow. We have

$$\forall f \in \mathbf{C}[F(d),c] \;\; \exists! g \in \mathbf{D}[d,G_{Ob}(c)] \quad f = u_c \circ F(g)$$

Now define, $\forall g \in \mathbf{D}[d,G_{Ob}(c)]$ $\tau_{d,c}(g) = u_c \circ F(g)$.

For every $d \in Ob_{\mathbf{D}}$ and $c \in Ob_{\mathbf{C}}$, $\tau_{d,c}: \mathbf{D}[d,G_{Ob}(c)] \to \mathbf{C}[F(d),c]$ is clearly a set-theoretic isomorphism. Note that, $\forall h \in \mathbf{D}[d',d]$,

1. $\tau_{d',c}(g \circ h) = u_c \circ F(g \circ h) = u_c \circ F(g) \circ F(h) = \tau_{d,c}(g) \circ F(h)$

By taking $\tau_{d,c}{}^{-1}(f)$ for g in (1), we have $\tau_{d',c}(\tau_{d,c}{}^{-1}(f) \circ h) = f \circ F(h)$, or equivalently,

2. $\tau_{d,c}{}^{-1}(f) \circ h = \tau_{d,c}{}^{-1}(f \circ F(h))$

For simplicity, we now omit the indexes of τ and τ^{-1} .

Let $G_{Mor}: Mor_{\mathbf{C}} \to Mor_{\mathbf{D}}$ be the function defined by

$$\forall k \in \mathbf{C}[c,c'] \quad G_{Mor}(k) = \tau^{-1}(k \circ u_c) \in \mathbf{D}[G_{Ob}(c), G_{Ob}(c')]$$

We want to prove that $G = (G_{Ob}, G_{Mor})$ is a functor:

$$\begin{aligned} G(id_c) &= \tau^{-1}(u_c) \\ &= \tau^{-1}(u_c \circ id_{F(G(c))}) \\ &= \tau^{-1}(u_c \circ F(id_{G(c)})) \\ &= \tau^{-1}(\tau(id_{G(c)})) \\ &= id_{G(c)} \end{aligned}$$

and, for every $f: c' \to c''$, $k: c \to c'$,

$$\begin{aligned} G(f \circ k) &= \tau^{-1}(f \circ k \circ u_c) \\ &= \tau^{-1}(f \circ u_{c'} \circ F(\tau^{-1}(k \circ u_c))) \\ &= \tau^{-1}(f \circ u_{c'}) \circ \tau^{-1}(k \circ u_c) && \text{by (2)} \\ &= G(f) \circ G(k) \end{aligned}$$

(1) proves the naturality of $\tau_{d,c}$ in the component d . We have still to prove the naturality in c, that is, $\forall g \in \mathbf{D}[d,G_{Ob}(c)]$, $\forall k \in \mathbf{C}[c,c']$

3. $\tau_{d,c'}(G(k) \circ g) = k \circ \tau_{d,c}(g)$

We have:

$$\begin{aligned} \tau(G(k) \circ g) &= \tau(\tau^{-1}(k \circ u_c) \circ g) \\ &= \tau(\tau^{-1}(k \circ u_c \circ F(g))) && \text{by (2)} \\ &= k \circ u_c \circ F(g) \\ &= k \circ \tau(g) \end{aligned}$$

By taking $\tau_{d,c}{}^{-1}(f)$ for g in (3) we obtain $\tau_{d,c'}(G(k) \circ \tau_{d,c}{}^{-1}(f)) = k \circ f$, or equivalently:

4. $G(k) \circ \tau_{d,c}{}^{-1}(f) = \tau_{d,c'}{}^{-1}(k \circ f)$.

(2) and (4) state the naturality of τ^{-1}.
Note that since G must satisfies (4) , then

$$G(k) = G(k) \circ id = G(k) \circ \tau^{-1}(u_c) = \tau^{-1}(k \circ u_c)$$

which shows that the adopted definition for G was actually forced. This proves the unicity of the functor G. ♦

5.2.3 Example An interesting example of application of theorem 5.2.2 refers to Cartesian closed categories. By the previous section, we know that if **C** is a CCC, then for all a,b in $Ob_{\mathbf{C}}$, ($p: a\times b\to a$, $p_2: a\times b\to b$) is universal from Δ to (a,b), and $eval_{a,b}: b^a\times a\to b$ is universal from $\_\times a$ to b. Then the functions $\_\times\_: Ob_{\mathbf{C}\times\mathbf{C}}\to Ob_{\mathbf{C}}$ and $\_^a : Ob_{\mathbf{C}}\to Ob_{\mathbf{C}}$ which respectively take (a,b) to $a\times b$ and b to b^a, can be extended to two functors $\_\times\_: \mathbf{C}\times\mathbf{C}\to\mathbf{C}$ and $\_^a : \mathbf{C}\to\mathbf{C}$. The explicit definition is the following: for every $f: a\to c$, $g: b\to d$

$$(\_\times\_)(f,g) = f\times g = < f \circ p_1, g \circ p_2 > : a\times b\to c\times d$$

$$(\_^a)(g) = \Lambda(eval_{a,b} \circ g) : b^a\to d^a$$

For every object c in **C**, even the unique arrow $!c: c\to t$ may be seen as universal arrow from the unique functor $!\mathbf{C}: \mathbf{C}\to\mathbf{1}$ to t. In this case, the extension of the function that takes $1\in Ob_{\mathbf{1}}$ to t to a functor $\mathbf{T}: \mathbf{1}\to\mathbf{C}$ is trivial, but it is interesting that the existence of the terminal object t in **C** may be expressed by the natural isomorphism $\mathbf{1}[!\mathbf{C}(c)=1 ,1] \cong \mathbf{C}[c,\mathbf{T}(1)=t]$.

5.2.4 Example Consider **C**, $\mathbf{C_t}$ and **Inc** as in example 5.1.4, and assume that for each object $a\in Ob_{\mathbf{C}}$ there exists the lifting a°. By example 5.1.4 we know that $ex_a: a°\to a$ is an universal arrow from the embedding functor **Inc**: $\mathbf{C_t}\to\mathbf{C_p}$ to a. By theorem 5.2.2 the function $\_°: Ob_{\mathbf{C}}\to Ob_{\mathbf{C}}$ which takes every object a to its lifting a°, may be extended to a functor $\_°: \mathbf{C_p}\to\mathbf{C_t}$. The explicit definition of the functor $\_°$ on a partial arrow $f: b\to c$ is the following

$$(\_°)(f) = f° = \tau(f \circ ex_b)\in\mathbf{C_t}[b°,c°]$$

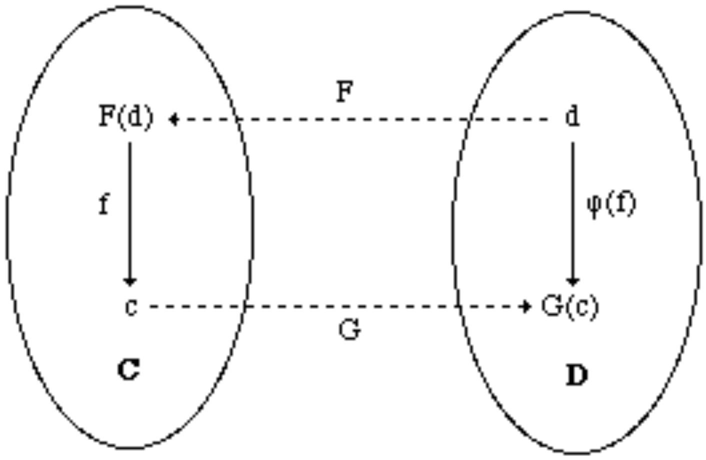
where $\tau(f\circ ex_a)$ is the only arrow such that $ex_c \circ \tau(f\circ ex_b) = f\circ ex_b$.

Note that nearly all the facts about partiality and extendability we proved depend directly on properties of natural transformations and adjunctions. That is, it was not possible to derive the properties of the lifting of b by assuming just a set-theoretic isomorphism between $\mathbf{C_p}[a,b]$ and $\mathbf{C_t}[a,b°]$ for all a, as one may be tempted at first thought. The expressive categorical notion of natural transformation turns out to be essential for these purposes.

5.3 Adjunctions

In this section we derive a general notion from the previous constructions and say that the functors F and G in theorems 5.2.1 and 5.2.2 are "adjoint" to one another. The idea is that an **adjunction** establishes a relation between two categories **C** and **D** through two functors $F: \mathbf{D}\to\mathbf{C}$ and G:

C→D; this relation creates a bijective correspondence φ of arrows in the two categories of the kind described by the following picture:



5.3.1 Definition *Let F: **D**→**C** and G: **C**→**D** be functors. Then an **adjunction** from **D** to **C** is a triple <F,G,φ> such that φ: **C**[F_,_] ≅ **D**[_,G_] is a natural isomorphism. F is called l**eft adjoint** of G, and G is called **right adjoint** of F.*

The naturality of the isomorphism φ deserves to be spelled out. For any f∈**C**[F(d),c], k∈**C**[c,c'] and h∈**D**[d',d], we have

1. $\varphi_{d,c'}(k \circ f) = G(k) \circ \varphi_{d,c}(f)$
2. $\varphi_{d',c}(f \circ F(h)) = \varphi_{d,c}(f) \circ h$

$$\begin{tikzcd}
c \arrow[d, "k"'] & \mathbf{C}[F(d),c] \arrow[r, "\varphi_{d,c}"] \arrow[d, "k \circ _"'] & \mathbf{D}[d,G(c)] \arrow[d, "G(k) \circ _"] \\
c' & \mathbf{C}[F(d),c'] \arrow[r, "\varphi_{d,c'}"] & \mathbf{D}[d,G(c')]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C}[F(d),c] \arrow[r, "\varphi_{d,c}"] \arrow[d, "_ \circ F(h)"'] & \mathbf{D}[d,G(c)] \arrow[d, "_ \circ h"] & d \\
\mathbf{C}[F(d'),c] \arrow[r, "\varphi_{d',c}"] & \mathbf{D}[d',G(c)] & d' \arrow[u, "h"']
\end{tikzcd}$$

It is equivalent to require that φ^{-1} is natural, that is, for any $g\in\mathbf{C}[d,G(c)]$, $k\in\mathbf{C}[c,c']$ and $h\in\mathbf{D}[d',d]$,

3. $\varphi^{-1}{}_{d,c'}(G(k) \circ g) = k \circ \varphi^{-1}{}_{d,c}(g)$
4. $\varphi^{-1}{}_{d',c}(g \circ h) = \varphi^{-1}{}_{d,c}(g) \circ F(h)$.

Examples

1. Let **D**, **C** be partial order categories, and $(Ob_{\mathbf{D}},\leq_D)$, $(Ob_{\mathbf{C}},\leq_C)$ the associated p.o.sets. An adjunction from **D** to **C** is a pair of monotone functions $f: Ob_{\mathbf{D}}\to Ob_{\mathbf{C}}$, $g: Ob_{\mathbf{C}}\to Ob_{\mathbf{D}}$ such that, for every $d\in Ob_{\mathbf{D}}$, $c\in Ob_{\mathbf{C}}$,

$$f(d) \leq_C c \quad\Leftrightarrow\quad d \leq_D g(d) .$$

Consider for example the partial order **Z** of relative numbers, and the partial order **R** of real numbers. Let $I: \mathbf{Z}\to\mathbf{R}$ be the obvious inclusion, and $\lfloor\_\rfloor: \mathbf{R}\to\mathbf{Z}$ be the function that takes a real number r to its lower integer part $\lfloor r\rfloor$. Then I and $\lfloor\_\rfloor$ define an adjunction from **Z** to **R** , since

1. $I(z) \leq_{\mathbf{R}} r \quad\Leftrightarrow\quad z \leq_{\mathbf{Z}} \lfloor r\rfloor$

Conversely let $\lceil\_\rceil: \mathbf{R}\to\mathbf{Z}$ be the function that takes a real number r to its upper integer part $\lceil r\rceil$. Then $\lceil\_\rceil$ and I define an adjunction from **R** to **Z**, since

2. $\lceil r\rceil \leq_{\mathbf{Z}} z \quad\Leftrightarrow\quad r \leq_{\mathbf{R}} I(z)$

Note that $\lfloor\_\rfloor$ and $\lceil\_\rceil$ are respectively the right and left adjoint to the same functor I. Note, moreover, that $\lfloor\_\rfloor$ and $\lceil\_\rceil$ are the unique functions that respectively satisfy conditions (1) and (2) for all r and z.

Another interesting example of adjunctions between partial orders as categories is the following: consider the p.o.set of positive integers **N**. For every natural number n , let $\_\cdot n : \mathbf{N}\to\mathbf{N}$ be the function that takes a natural numbers m to the product $m\cdot n$. The right adjoint to $\_\cdot n$ is the the function $div(\_,n)$: $\mathbf{N}\to\mathbf{N}$ that takes q to (the lower integer part of) q divides n .

Indeed, for every m, q, $m\cdot n \leq q \quad\Leftrightarrow\quad m \leq div(q,n)$

Analogously the "minus" operation is right adjoint to "plus."

2. This further example uses familiar notions and applies the categorical understanding of a fundamental technique in (universal) algebra. Given a category **C** of structures and a category **D** of slightly more general ones, the right adjoint of the forgetful functor from **C** to **D** defines the "free structures" over the objects in the category **D**. This technique is widely explained in several places (see references), so that we just hint at it here.

The category **Graph** was defined in the example 4.1.5. Recall now that a graph **G** is given by:

- a set V of objects (nodes)
- a set T of arrows (edges)
- a function ∂_1: T→O which assigns to each arrow f its range $\partial_1(f)$
- a function ∂_2: T→O which assigns to each arrow f its target $\partial_2(f)$.

Morphisms of graphs **G**, **G**' are pairs <f,g>, where f: T→T' and g: V→V' have the properties in the example 4.1.5. We already mentioned (see the exercise following that example) that each small category **C** may be regarded as a graph **G** = U(**C**), just forgetting identities and composition. Of course, U takes objects to nodes and arrows to edges. Moreover, every functor F: **C**→**D** gives a morphisms H = U(F): U(**C**)→U(**D**) between the associated graphs; the reader should have checked that U: **Cat** →**Graph** is actually a (forgetful) functor. Conversely every graph **G** generates a category **C** = C(**G**) with the same objects of **G**, and, for arrows, the finite strings $(f_1,...,f_n)$ of composable arrows of **G**, i.e., of arrows in the due types (the empty strings are the identities in C(**G**)). Composition in C(**G**) is just string concatenation, that is,

$$(f_1, . . . ,f_n) \circ (g_1, . . . ,g_m) = (f_1, . . . ,f_n,g_1, . . . ,g_m) .$$

Note that $(f_1,...,f_n) = f_1 \circ . . . \circ f_n$. The category C(**G**) is called the **free category** generated by **G**.

This construction may be extended to morphisms of graphs: if H: **G**→**G'** then C(H): C(**G**)→C(**G'**) is the functor that coincides with H on objects, and that is defined on morphisms by:

$$C(H)(f_1,...,f_n) = (H(f_1),... H(f_n)).$$

It is easy to prove that C is a functor from **Grph** to **Cat**. Actually, we have an adjoint situation, since there is an isomorphism Θ : **Cat**[C(**G**), **C**] ≅ **Grph**[**G**, U(**C**)] which is natural in **G** and **C**. The isomorphism Θ takes every functor F: C(**G**)→**C** to the morphism Θ(F): **G**→**C**, which is the "restriction" of F on **G**. For the nature of C(**G**), every functor F: C(**G**)→**C** is uniquely determined by its behavior on the arrows of **G**, indeed if $(f_1,. . . ,f_n)$ is an arrow in C(**G**), by definition of a functor, $F((f_1,. . . ,f_n)) = F(f_1 \circ . . . \circ f_n) = F(f_1) \circ . . . \circ F(f_n)$. This proves that Θ is injective. But Θ is also surjective, since if H: **G**→U(**C**) , we can define a functor F: C(**G**)→**C** by $F((f_1,. . . f_n)) = H(f_1) \circ ... \circ H(f_n)$, and clearly Θ(F) = H. We leave it to the reader to prove the naturality of the isomorphism.

Exercise In section 4.3 we turned each Petri net N into a monoidal category $C^{\otimes}(N)$. Describe $C^{\otimes}(N)$ as a freely generated category.

Exercise Let **C** and **D** be discrete categories (i.e., the only morphisms of the categories are identities). Prove that <G,F,τ>: **C**→**D** is an adjunction if and only if G and F define an isomorphism between **C** and **D**.

In the previous section, we have actually shown how to construct an adjunction when one can uniformly obtain a universal arrow $<u_d,c_d>$ from each object d . Now we show how to obtain universal arrows out of an adjunction, and put together the two results.

5.3.2 Theorem. *If < F: **D**→**C** , G: **C**→**D** ,φ > is an adjunction from **D** to **C**, then*
1. < u =φ(id$_{F(d)}$) : d→G(F(d)) , F(d)> is universal from d to G

*u is called **unit** of the adjunction*

2. $<u'=\varphi^{-1}(id_{G(c)}): F(G(c))\to c, G(c)>$ is universal from F to c

*u' is called **counit** of the adjunction.*

Conversely, if $G: \mathbf{C}\to\mathbf{D}$ is a functor and (1) holds (or $F: \mathbf{D}\to\mathbf{C}$ is a functor and (2) holds), then $<F,G,\varphi>$ is an adjunction from $\mathbf{D}$ to $\mathbf{C}$.

Proof. (1) is given by theorem 5.1.6 (⇐) and the definition of G. Note that (2) follows dually. The converse is stated in theorems 5.2.1 and 5.2.2. ♦

Thus, if $< F, G ,\varphi >$ is an adjunction, then the functor F of theorem 5.2.1 is the left adjoint of G and, conversely, G in theorem 5.2.2 is the right adjoint of F. In view of the expressive power of the notion of adjunction, we can now state in one line some of the concepts we introduced in the previous chapters.

5.3.3 Corollary *Let $\mathbf{C}$ be a category. Then*

i. $\mathbf{C}$ has a terminal object iff the unique functor $\mathbf{!C: C \to 1}$ has a right adjoint;

ii. $\mathbf{C}$ has finite products iff the diagonal functor has a right adjoint;

iii. $\mathbf{C}$ is a CCC iff it is cartesian (i.e., $\mathbf{!C: C\to 1}$ and $\mathbf{\Delta : C\to C\times C}$ have right adjoints) and, for each $a\in Ob_{\mathbf{C}}$, the functor $\_\times a : \mathbf{C}\to\mathbf{C}$ has a right adjoint.

Proof. Immediate by theorem 5.2.2 and the considerations in example 5.2.3. ♦

5.3.4 Corollary *Let $\mathbf{C}$ be a category of partial morphisms. The lifting functor $\_^\circ : \mathbf{C}\to\mathbf{C_t}$ is the right adjoint of the embedding functor $\mathbf{Inc}: \mathbf{C_t}\to\mathbf{C}$.*

Proof Immediate by 5.2.2 and the considerations in example 5.2.4. ♦

As the reader probably expects, it is also possible to give a fully equational characterization of adjunctions.

5.3.5 Theorem *An adjunction $<F, G,\tau> : \mathbf{C}\to\mathbf{D}$ is fully determined by the following data:*

- the functor $G: \mathbf{D}\to\mathbf{C}$

- a function $f: Ob_{\mathbf{C}}\to Ob_{\mathbf{D}}$ such that, for every object c of $\mathbf{C}$, $f(c) = F(c)$

- for every object c of $\mathbf{C}$, an arrow $unit_c\in\mathbf{C}[c, G(f(c))]$

- for every object c of $\mathbf{C}$ and d of $\mathbf{D}$, a function $\tau_{c,d}^{-1}: \mathbf{C}[c,G(d)]\to\mathbf{D}[f(c),d]$

such that, for every $h\in\mathbf{C}[c,G(d)]$ and $k\in\mathbf{D}[f(c),d]$,

1. $G(\tau_{c,d}^{-1}(h)) \circ unit_c = h$;
2. $\tau_{c,d}^{-1}(G(k) \circ unit_c) = k$.

Proof The theorem is an immediate consequence of theorems 5.3.2 and 5.1.6. A direct proof is not difficult, and its study is a good exercise for the reader since it summarizes many of the previous results. Here it is:
The function f may be extended to a functor F by setting, for $k \in \mathbf{C}[c,c']$, $F(k) = \tau_{c',d}^{-1}(\mathrm{unit}_{c'} \circ k)$. Note that

$$\begin{aligned} F(id_c) &= \tau_{c,f(c)}^{-1}(\mathrm{unit}_c) \\ &= \tau_{c,f(c)}^{-1}(id_{G(f(c))} \circ \mathrm{unit}_c) \\ &= \tau_{c,f(c)}^{-1}(G(id_{f(c)}) \circ \mathrm{unit}_c) \\ &= id_{f(c)} \,. && \text{by (2)} \end{aligned}$$

and moreover, omitting the indexes for notational convenience,

$$\begin{aligned} F(h \circ k) &= \tau^{-1}(\mathrm{unit} \circ h \circ k\,) \\ &= \tau^{-1}(\, G(\tau^{-1}(\mathrm{unit} \circ h)) \circ \mathrm{unit} \circ k\,) && \text{by (1)} \\ &= \tau^{-1}(\, G(\tau^{-1}(\mathrm{unit} \circ h)) \circ G(\tau^{-1}(\mathrm{unit} \circ k)) \circ \mathrm{unit}\,) && \text{by (1)} \\ &= \tau^{-1}(\, G(\,\tau^{-1}(\mathrm{unit} \circ h) \circ \tau^{-1}(\mathrm{unit} \circ k)\,) \circ \mathrm{unit}\,) \\ &= \tau^{-1}(\mathrm{unit} \circ h) \circ \tau^{-1}(\mathrm{unit} \circ k) && \text{by (2)} \\ &= F(h) \circ F(k)\,. \end{aligned}$$

Let now, for every object c of **C** and d of **D**, $\tau_{c,d}: \mathbf{D}[f(c),d] \to \mathbf{C}[c,G(d)]$ be the function defined by $\tau_{c,d}(k) = G(k) \circ \mathrm{unit}_c$. Equations (1) and (2) express exactly the fact that $\tau_{c,d}$ and $\tau_{c,d}^{-1}$ define an isomorphism. We have still to prove their naturality. Let $k \in \mathbf{D}[d,d']$, $h \in \mathbf{C}[c,G(d)]$, $h' \in \mathbf{C}[c',c]$, and $k' \in \mathbf{D}[f(c),d]$; then

$$\begin{aligned} \text{nat-1.} \quad \tau^{-1}(G(k) \circ h\,) &= \tau^{-1}(G(k) \circ G(\tau^{-1}(h)) \circ \mathrm{unit}_c\,) \\ &= \tau^{-1}(G(k \circ \tau^{-1}(h)) \circ \mathrm{unit}_c\,) \\ &= k \circ \tau^{-1}(h)\,; \\ \text{nat-2.} \quad \tau^{-1}(h \circ k\,) &= \tau^{-1}(G(\tau^{-1}(h)\,) \circ \mathrm{unit} \circ k\,) && \text{by (1)} \\ &= \tau^{-1}(h) \circ \tau^{-1}(\mathrm{unit} \circ k\,) && \text{by (nat-1)} \\ &= \tau^{-1}(h) \circ F(k)\,; \\ \text{nat-3.} \quad \tau(k' \circ F(h')\,) &= \tau\,(\tau^{-1}(\tau(k')) \circ F(k)\,) \\ &= \tau\,(\tau^{-1}(\tau(k') \circ h') && \text{by (nat-2)} \\ &= \tau(k') \circ h'\,; \\ \text{nat-4.} \quad \tau(k \circ k'\,) &= G(\,k \circ k') \circ \mathrm{unit} \\ &= G(\,k\,) \circ G(k') \circ \mathrm{unit} \\ &= G(\,k\,) \circ \tau(k'). \ \blacklozenge \end{aligned}$$

5.3.6 Proposition *Let* $<F,G,\tau>: \mathbf{C} \to \mathbf{D}$ *be an adjunction. Then there exist two natural transformations* $\eta : Id_{\mathbf{C}} \to GF$ *and* $\varepsilon: FG \to Id_{\mathbf{D}}$ *such that, for every c in* **C** *and d in* **D**, $\eta(c)$ *and* $\varepsilon(d)$ *are respectively the unit and counit of the adjunction.*
Proof: exercise. ♦

In other words, one may construct the unit and counit "uniformely" and naturally. Observe also that, if $\eta : Id_C \to GF$ and $\varepsilon: FG \to Id_D$ are the natural transformations in proposition 5.3.6, then the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[rr, bend right, "Id_G"'] & GFG \arrow[r, "G\varepsilon"] & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[rr, bend right, "Id_F"'] & FGF \arrow[r, "\varepsilon F"] & F \end{tikzcd}$$

The previous diagrams fully characterize an adjunction: as a matter of fact many authors prefer to define an adjunction between two categories **C** and **D** as a quadruple $(F, G, \eta, \varepsilon)$ where $F: \mathbf{C} \to \mathbf{D}$ and $G: \mathbf{D} \to \mathbf{C}$ are functors, and $\eta : Id_C \to GF$, $\varepsilon: FG \to Id_D$ are natural transformations such that

$$(G\varepsilon) \circ (\eta G) = id_G ;$$
$$(\varepsilon F) \circ (F\eta) = id_F .$$

We leave it as an exercise for the reader to prove the equivalence of this notion with the one we have adopted. We shall use the definition of adjunction as a quadruple in the next section, since it simplifies the investigation of the relation between adjunctions and monads.

Exercises

1. An adjointness $(F, G, \eta, \varepsilon)$ from **C** to **D** is an **adjoint equivalence** if and only if η and ε are natural isomorphisms. Prove that given two equivalent categories **C** and **D** (see section 3.2) it is always possible to define an adjoint equivalence between them.
2. Given an adjointness $(F, G, \eta, \varepsilon)$ from **C** to **D**, prove the equivalence of the following statements:
    i. $\eta GF = GF\eta$;
    ii. ηG is an isomorphism;
    iii. $\varepsilon FG = FG\varepsilon$;
    iv. εF is an isomorphism.

5.3.7 Example In section 3.4 we defined the CCCs of limit and filter spaces, **L-spaces** and **FIL** respectively, which generalize topological spaces, **Top**. The functorial "embeddings" mentioned in that example are actually adjunctions. Recall that $H : \mathbf{Top} \to \mathbf{FIL}$ is given by

$$H((X,top)) = (X,F) \text{ where } F(x) = \{\Phi \mid \Phi \text{ is a filter and } \forall 0 \in top\ (x \in 0 \Rightarrow 0 \in \Phi)\}.$$

$H(f) = f$ by the definition of continuity. H has a left adjoint $T : \mathbf{FIL} \to \mathbf{Top}$ defined by

$$T((X, F)) = (X,top) \text{ where } 0 \in top \text{ iff } \forall x \in 0\ \forall \Phi \in F(x)\ \ 0 \in \Phi .$$

Also in this case, filter continuity corresponds to topological continuity, i.e., $T(f) = f$. The reader may easily define the natural isomorphism τ.

In general, limits are not unique in filter spaces. A stronger notion of convergence, to be used for computability (see the final remark in section 8.4), may be given as follows. For (X,F) in **FIL** consider (X,top) = T((X, F)) and define

(s-conv.) $\Phi{\downarrow}^{s}x$ iff $\Phi\in F(x)$ and $\forall 0\in\Phi\cap top\ \ x\in 0$.

Then top is T_0 iff s-convergent filters have a unique limit.

Let N = (ω, F) be the natural numbers with the filter structure induced by the discrete topology and $M = N^N$. With some work (see references) one can show that M^M is not topological, i.e. for no (X,top) one has M^M = H((X,top)). The idea is that each topological filter space has a least filter for each F(x), the neighborhood filter at x; deduce from this that the associated adjointness (T,H,η,ε) to (T,H,τ) is not an adjoint equivalence (which one of η and ε is not a natural isomorphism?).

Exercises *(based on the previous example and exercises)*

1. Consider the full subcategory of **FIL** given by the filter spaces (X,F) such that, for each x, there is a least $\Phi\in F(x)$. Give an adjoint equivalence between **Top** and this category.
2. Check that the functors between **L-spaces** and **FIL** defined in the exercise in section 3.4.2 yield an adjunction, which is not an adjoint equivalence.
3. Give directly an adjunction between **Top** and **FIL**, and compare the definition with the adjunction obtained by composition of functors. (*Hint for the direct construction:* given an L-space (X,↓), define (X,top) by $0\in top$ iff $\forall x\in 0, \forall\{x_i\}{\downarrow}x,\ \{x_i\}\subseteq 0$ eventually. Conversely, for (X, top) topological space, define (X,↓) by $\{x_i\}{\downarrow}x$ iff $\forall 0\in top\ (x\in 0 \Rightarrow \{x_i\}\subseteq 0$ eventually).

5.4 Adjunctions and Monads

In this section we study the relation between two seemingly distant concepts as adjunction and monad. As a matter of fact, every adjunction immediately defines a monad, and conversely every monad can be thought of as generated by an adjunction, called a **resolution** for the monad (see 5.4.2 below). Resolutions for a given monad can be build up in a category by introducing a natural notion of morphism between them; it then happens that the Eilenberg-Moore and the Kleisli Categories associated with the monad (see definitions 4.2.3 and 4.2.4) are respectively the terminal and initial object of the category.

The presentation is rather technical; at first reading, the reader may just look at the first theorem, which will be applied in the next section.

5.4.1 Proposition *Let (F, G, η , ε) be an adjunction from **C** to **D**; then (T = GF, η , μ = GεF) is a monad on **C** and (T = FG, δ = GηF, ε) is a comonad.*

Proof Note first that GF, η, and $G\varepsilon F$ have the correct types, i.e., $T = GF$: $\mathbf{C}\to\mathbf{C}$, η: $Id_{\mathbf{C}}\to GF$, and $\mu = G\varepsilon F$: $GFGF\to GF$. We must prove the unity and associative laws for the monad.
As for the unity laws we have

$$\mu \circ T\eta \ = G\varepsilon F \circ GF\eta = G(\varepsilon F \circ F\eta) = G(id_F) = id_{GF}$$
$$\mu \circ \eta T \ = G\varepsilon F \circ \eta GF = \ (G\varepsilon \circ \eta G)F = id_G(F) = id_{GF}$$

For the associative law, note first that

$$\varepsilon \circ \varepsilon FG = \varepsilon \circ FG\varepsilon \ .$$

Indeed, for any $d\in Ob_{\mathbf{D}}$, and letting $f = \varepsilon_d$: $FG(d)\to d$,

$$\begin{aligned}\varepsilon_d \circ \varepsilon_{FG(d)} &= f \circ \varepsilon_{FG(d)} \\ &= \varepsilon_d \circ FG(f) && \text{by naturality of } \varepsilon \\ &= \varepsilon_d \circ FG(\varepsilon_d)\end{aligned}$$

Then one has the following:

$$\begin{aligned}\mu \circ \mu T &= G\varepsilon F \circ G\varepsilon FGF \\ &= G(\varepsilon \circ \varepsilon FG)(F) \\ &= G(\varepsilon \circ FG\varepsilon)(F) \\ &= G\varepsilon F \circ GFG\varepsilon F \\ &= \mu \circ T\mu \ .\end{aligned}$$

The rest is an exercise in duality. ♦

5.4.2 Definition *Let (T, η , μ) be a monad over a category $\mathbf{C}$. A* ***resolution*** *for (T, η , μ) is a category $\mathbf{D}$ and an adjunction $(F, G, \eta , \varepsilon)$ from $\mathbf{C}$ to $\mathbf{D}$ such that $T = GF$ and $\mu = G\varepsilon F$. A* ***morphism*** *between two resolutions $(F, G, \eta , \varepsilon)$: $\mathbf{C}\to\mathbf{D}$ and $(F', G', \eta , \varepsilon')$: $\mathbf{C}\to\mathbf{D}'$ (for the same monad) is a functor $H: \mathbf{D}\to\mathbf{D}'$ such that $F' = H \circ F$, $G = G' \circ H$, and $H\varepsilon = \varepsilon'H$.*

It is easily proved that resolutions with the associated morphisms form a category. Now we are going to prove that the Eilenberg-Moore and Kleisli categories associated with a monad (T, η , μ) both give rise to resolutions. In particular, they are respectively the terminal and initial objects in the category of all resolutions for that monad.

5.4.3 Proposition *Let (T, η , μ) be a monad over a category $\mathbf{C}$, and let $\mathbf{C}^T$ be the Eilenberg-Moore category associated with the monad. Then there exists a resolution for (T, η , μ) which is an adjunction from $\mathbf{C}^T$ to $\mathbf{C}$.*

Proof Let U^T: $\mathbf{C}^T\to\mathbf{C}$ be the forgetful functor that takes every algebra (c,α) to c, and every morphism of algebras h to the same h regarded as a morphism in $\mathbf{C}$. Let F^T: $\mathbf{C}\to\mathbf{C}^T$ be the functor which takes every object c to its free algebra $(T(c),\mu_c)$, and every morhism $f: c\to c'$ to $F^T(f) = T(f)$. Let ε^T: $F^TU^T\to id_{\mathbf{C}^T}$ be the natural transformation defined by $\varepsilon^T(c,\alpha) = \alpha$ (note that α: $T(c)=F^TU^T(c,\alpha)\to c$). We want to prove that $(F^T, U^T, \eta , \varepsilon^T)$ is a resolution for (T, η , μ).

Obviously $U^T \circ F^T = T$. $U^T \varepsilon^T F^T = \mu$, since for any object c one has

$$
\begin{aligned}
(U^T \varepsilon^T F^T)(c) &= U^T(\varepsilon^T{}_{F^T(c)}) \\
&= U^T(\varepsilon^T(T(c), \mu_c)) && \text{by def. of } F^T \\
&= U^T(\mu_c) && \text{by def. of } \varepsilon^T \\
&= \mu_c && \text{by def. of } U^T
\end{aligned}
$$

We still haveto prove that $(F^T, U^T, \eta, \varepsilon^T)$ is an adjunction from $\mathbf{C}^T$ to $\mathbf{C}$, that is,

$$(U^T\varepsilon^T) \circ (\eta U^T) = id_{U^T}$$

$$(\varepsilon^T F^T) \circ (F^T\eta) = id_{F^T}$$

One has, for every T-algebra (c,α),

$$
\begin{aligned}
(U^T\varepsilon^T \circ \eta U^T)(c,\alpha) &= U^T(\varepsilon^T{}_{(c,\alpha)}) \circ \eta_{U^T(c,\alpha)} \\
&= \alpha \circ \eta_c && \text{by def. of } \varepsilon^T \text{ and } U^T \\
&= id_c && \text{by def. of T-algebra}
\end{aligned}
$$

And for every $c \in Ob_{\mathbf{C}}$:

$$
\begin{aligned}
(\varepsilon^T F^T \circ F^T\eta)(c) &= \varepsilon^T{}_{F^T(c)} \circ F^T(\eta_c) \\
&= \mu_c \circ T(\eta_c) && \text{by def. of } \varepsilon^T \text{ and } F^T \\
&= id_{T(c)} && \text{by the unity law of the monad.}\blacklozenge
\end{aligned}
$$

We say that the resolution $(F^T, U^T, \eta, \varepsilon^T)$ from $\mathbf{C}$ to the Eilenberg-Moore category $\mathbf{C}^T$, and given by proposition 5.4.3, is **associated with $\mathbf{C}^T$**.

5.4.4 Proposition *Let (T, η, μ) be a monad over a category $\mathbf{C}$ .Then the resolution $(F^T, U^T, \eta, \varepsilon^T)$: $\mathbf{C} \to \mathbf{C}^T$, associated with the Eilenberg-Moore Category $\mathbf{C}^T$, is a terminal object in the category of all the resolution for the monad (T, η, μ).*

Proof Let $(F, G, \eta, \varepsilon)$: $\mathbf{C} \to \mathbf{D}$ be another resolution for (T, η, μ). We must prove that there exists a unique arrow from $(F, G, \eta, \varepsilon)$ to $(F^T, U^T, \eta, \varepsilon^T)$. Remember (cf. definition 5.4.2) that such an arrow is a functor $H: \mathbf{D} \to \mathbf{C}^T$, such that $F^T = H \circ F$, $G = U^T \circ H$, and $H\varepsilon = \varepsilon^T H$.

Define, for any object d, and any morphism f of $\mathbf{D}$,

$$H(d) = (G(d), G(\varepsilon(d)))$$

$$H(f) = G(f).$$

Then one has, for any $c \in Ob_{\mathbf{C}}$, any $h \in Mor_{\mathbf{C}}$,

$$H(F(c)) = (G(F(c)), G(\varepsilon(F(c)))) = (T(c), \mu_c) = F^T(c)$$

$$H(F(h)) = G(F(h)) = T(h) = F^T(h)$$

that proves the equality $H \circ F = F^T$.

Moreover, for any $d \in Ob_{\mathbf{D}}$, and any $f \in Mor_{\mathbf{D}}$,

$$U^T(H(d)) = U^T(G(d), G(\varepsilon(d))) = G(d)$$

$U^T(H(f)) = U^T(G(f)) = G(f)$, as U^T is the identity on morphisms.

That proves the equality $G = U^T \circ H$.

Finally, for any $d \in Ob_{\mathbf{D}}$,

$$\varepsilon^T{}_{H(d)} = \varepsilon^T{}_{(G(d),\, G(\varepsilon(d)))} = G(\varepsilon(d)) = H(\varepsilon(d))$$

that proves the equality $H\varepsilon = \varepsilon^T H$.

We have still to prove that H is the unique morphism from $(F, G, \eta, \varepsilon)$ to $(F^T, U^T, \eta, \varepsilon^T)$. Let H' be another morphism; then, for any $f \in Mor_{\mathbf{D}}$,

$$H'(f) = U^T(H'(f) = G(f) = U^T(H(f) = H(f)$$

and, for any $d \in Mor_{\mathbf{D}}$,

$$\begin{aligned} H'(d) &= (\, U^T(H'(d), \varepsilon^T{}_{H'(d)}) && \text{by def. of } U^T \text{ and } \varepsilon^T \\ &= (\, G(d), H'(\varepsilon^T(d))\,) && \text{as } G = U^T \circ H' \\ &= (\, U^T(H(d), H(\varepsilon^T(d))\,) && \text{as } H'(f) = H(f) \\ &= (\, U^T(H'(d), \varepsilon^T{}_{H(d)}) \\ &= H(d). \end{aligned}$$

This completes the proof. ♦

The unique functor to $(F^T, U^T, \eta, \varepsilon^T)$ in the category of all resolutions for a given monad (T, η, μ) is called **comparison functor** and it is usually denoted by K^T.

The category of resolutions of a monad has also an initial object, which is based on the Kleisli category associated with the monad.

5.4.5 Proposition *Let (T, η, μ) be a monad over a category $\mathbf{C}$, and let $\mathbf{C}_T$ be the Kleisli category associated with the monad. Then there exists a resolution for (T, η, μ) that is an adjunction from $\mathbf{C}_T$ to $\mathbf{C}$.*

Proof: Let $U_T: \mathbf{C}_T \to \mathbf{C}$ be the functor defined by the following:

for any object c of $\mathbf{C}_T$ (i.e., of $\mathbf{C}$), and any morphism $h \in \mathbf{C}_T[c,c']$ (and thus $h \in \mathbf{C}[c,T(c')]$)

$$U_T(c) = T(c);$$
$$U_T(h) = \mu_{c'} \circ T(h).$$

Let $F_T: \mathbf{C} \to \mathbf{C}_T$ be functor defined by the following:

for any object c of $\mathbf{C}$, and any morphism $f \in \mathbf{C}[c,c']$

$$F_T(c) = c;$$
$$F_T(f) = \eta_{c'} \circ f \;\; (= T(f) \circ \eta_c\,).$$

Let $\varepsilon_T: F_T U_T \to id$ be the natural transformation defined by the following:

for any object c of $\mathbf{C}_T$

$$\varepsilon_T(c) = id_{T(c)} \quad (\text{in } \mathbf{C}).$$

We want to prove that $(F_T, U_T, \eta, \varepsilon_T): \mathbf{C} \to \mathbf{C}_T$ is a resolution for (T, η, μ).

Obviously, $U_T \circ F_T = T$.

Moreover, $U_T\, \varepsilon_T\, F_T = \mu$, since for any object c one has

$$(U_T\, \varepsilon_T\, F_T)(c) = U_T(\varepsilon_T\,(c)\,) \qquad \text{by def. of } F_T$$

$$\begin{aligned} &= U_T(id_{T(c)}) && \text{by def. of } \varepsilon_T \\ &= \mu_c. && \text{by def. of } U_T \end{aligned}$$

We have still to prove that $(F_T, U_T, \eta, \varepsilon_T)$ is an adjunction from $\mathbf{C}$ to $\mathbf{C}_T$, that is,

$$(U_T\varepsilon_T) \circ (\eta U_T) = id: U_T \to U_T$$
$$(\varepsilon_T F_T) \circ (F_T\eta) = id: F_T \to F_T$$

One has, for every object c of $\mathbf{C}_T$:

$$\begin{aligned} (U_T\varepsilon_T \circ \eta U_T)(c) &= U_T(id_{T(c)}) \circ \eta_{T(c)} && \text{by def. of } \varepsilon_T \text{ and } U_T \\ &= \mu_c \circ T(id_{T(c)}) \circ \eta_{T(c)} && \text{by def. } U_T \text{ on morphisms} \\ &= \mu_c \circ \eta_{T(c)} && \text{as T is a functor} \\ &= id_c. && \text{by the unity law of the monad} \end{aligned}$$

And, for every $c \in Ob_{\mathbf{C}}$,

$$\begin{aligned} (\varepsilon_T F_T \circ F_T\eta)(c) &= \varepsilon_T(c) \circ (\eta_{T(c)} \circ \eta_c) && \text{by def. of } F_T \\ &= id_{T(c)} \circ (\eta_{T(c)} \circ \eta_c) && \text{by def. of } \varepsilon_T \\ &= \mu_C \circ T(id_{T(c)}) \circ \eta_{T(c)} \circ \eta_c && \text{by def. of composition} \circ \text{ in } \mathbf{C}_T \\ &= \mu_C \circ \eta_{T(c)} \circ \eta_c && \\ &= \eta_c && \text{by the unity law of the monad} \\ &= id_c && \text{by def. of the identity in } \mathbf{C}_T. \ \blacklozenge \end{aligned}$$

5.4.6 Proposition *Let (T, η, μ) be a monad over a category $\mathbf{C}$. The resolution $(F_T, U_T, \eta, \varepsilon_T)$: $\mathbf{C} \to \mathbf{C}_T$ associated with the Kleisli Category $\mathbf{C}_T$ is an initial object in the category of resolutions for the monad (T, η, μ).*

Proof Let $(F, G, \eta, \varepsilon)$: $\mathbf{C} \to \mathbf{D}$ be another resolution for (T, η, μ). We must prove that there exists a unique arrow from $(F_T, U_T, \eta, \varepsilon_T)$ to $(F, G, \eta, \varepsilon)$, that is a unique functor $K: \mathbf{C}_T \to \mathbf{D}$, such that $F = K \circ F_T$, $U_T = G \circ K$, and $K\varepsilon_T = \varepsilon K$.

Define, for any object c of $\mathbf{C}_T$, and any morphism $f \in \mathbf{C}_T[c,c']$,

$$K(c) = F(c);$$
$$K(f) = \varepsilon_{F(c')} \circ F(f).$$

where c and f are regarded as object and morphism of $\mathbf{C}$.

Then one has, for any $c \in Ob_{\mathbf{C}}$, any $h \in \mathbf{C}[c,c']$,

$$K(F_T(c)) = K(c) = F(c)$$

$$\begin{aligned} K(F_T(h)) &= K(\eta_{c'} \circ h) && \text{by def. of } F_T \\ &= \varepsilon_{F(c')} \circ F(\eta_{c'} \circ h) && \text{by def. of K} \\ &= \varepsilon_{F(c')} \circ F(\eta_{c'}) \circ F(h) && \text{as F is a functor} \\ &= F(h) && \text{as } (F, G, \eta, \varepsilon) \text{ is an adjunction} \end{aligned}$$

This proves the equality $K \circ F_T = F$.

Moreover, for any object c of $\mathbf{C}_T$, and any morphism $f \in \mathbf{C}_T[c,c']$,

$$G(K(c)) = G(F(c)) = T(c) = U_T(c)$$

$G(K(f))$	$= G(\varepsilon_{F(c')} \circ F(f))$	by def. of K
	$= G(\varepsilon_{F(c')}) \circ G(F(f))$	as G is a functor
	$= \mu_{c'} \circ T(f)$	as $(F, G, \eta, \varepsilon)$ is a resolution
	$= U_T(f)$	by def. of U_T

that proves the equality $U_T = G \circ K$.

Finally, for any $d \in Ob_{\mathbf{D}}$,

$K(\varepsilon_T(c))$	$= K(id_{T(c)})$	by def. of ε_T
	$= \varepsilon_{F(c')} \circ F(id_{T(c)})$	by def. of K
	$= \varepsilon_{F(c')}$	as F is a functor
	$= \varepsilon_{K(c')}$	by def. of K

that proves the equality $K\varepsilon_T = \varepsilon K$.

We have still to prove that K is the unique morphism from $(F_T, U_T, \eta, \varepsilon_T)$ to $(F, G, \eta, \varepsilon)$.

Let $K': \mathbf{C}_T \to \mathbf{D}$ be another morphism; then, for every object c of $\mathbf{C}_T$,

$K'(c)$	$= K'(F_T(c))$	by def. of F_T
	$= F(c)$	as $K' \circ F_T = F$
	$= K(F_T(c))$	as $K \circ F_T = F$
	$= K(c)$	by def. of F_T

and, for any $f \in \mathbf{C}_T[c,c']$,

$K'(f)$	$= K'(id_{c'} \circ f)$	
	$= K'(\mu_{c'} \circ \eta_{T(c')} \circ f)$	by the unitary law of the monad
	$= K'(\mu_{c'} \circ T(id_{T(c')}) \circ \eta_{T(c')} \circ f)$	
	$= K'(id_{T(c')} \circ (\eta_{T(c')} \circ f))$	by def. of composition $\circ$ in $\mathbf{C}_T$
	$= K'(id_{T(c')}) \circ K'(\eta_{T(c')} \circ f))$	as K' is a functor
	$= K'(\varepsilon_T(c')) \circ K'(F_T(f)))$	by def. of ε_T and F_T
	$= \varepsilon_{K'(c')} \circ F(f)$	as $K'\varepsilon_T=\varepsilon K'$ and $F=(K' \circ F_T)$
	$= \varepsilon_{F(c')} \circ F(f)$	as $K'(c) = K(c) = F(c)$
	$= K(f).$	by def. of K

This completes the proof. ♦

For consistency with the terminology adopted for the comparison functor, we shall denote by K_T the unique arrow from the initial object $(F_T, U_T, \eta, \varepsilon_T)$ in the category of all resolutions for a monad (T, η, μ).

Consider now the comparison functor from the initial to the terminal object. Of course, it must be $K^T = K_T$; let us check this explicitly.

For any object c in $\mathbf{C}_T$,

$K^T(c)$	$= (U_T(c), U_T(\varepsilon_T(c)))$	by def. of K^T
	$= (T(c), \mu_c \circ T(id_{T(c)}))$	by def. of U_T and ε_T

$$
\begin{array}{rll}
 & = (T(c), \mu_c) & \\
 & = F^T(c) & \text{by def. of } F^T \\
 & = K_T(c). & \text{by def. of } K_T
\end{array}
$$

And for any morphism $f \in \mathbf{C}_T[c,c']$,

$$
\begin{array}{rll}
K^T(f) & = U_T(f) & \text{by def. of } K^T \\
 & = \mu_{c'} \circ T(f) & \text{by def. of } U_T \\
 & = \varepsilon^T{}_{(T(c'),\ \mu(c'))} \circ T(f) & \text{by def. of } \varepsilon^T \\
 & = \varepsilon^T{}_{F^T(c')} \circ F^T(f) & \text{by def. of } F^T \\
 & = K_T(f)\ . & \text{by def. of } K_T
\end{array}
$$

Exercises

1. Prove that the comparison functor $K_T = K^T : \mathbf{C}_T \to \mathbf{C}^T$ is full and faithful.
2. Prove that the Kleisli Category is isomorphic to the full subcategory of $\mathbf{C}^T$ consisting of all free algebras.

5.5 More on Linear Logic

In this section, we complete an introductory presentation of linear logic and its categorical meaning, initiated in section 4.4. As already mentioned, the leading idea of this system refers to the "linear" use of "resources" or logical assumptions. From assuming A, one derives less than from assuming A, A, i.e., twice A . The logic-oriented reader, mostly used to classical or intuitionistic reasoning, may find this a little strange. Probably, though, this habit hides a nonconstructive view which may result in a limitation of our understanding of effective processes. Indeed, the alternative approach proposed by linear logic, which enriches and complements the traditional ones, seems to suggest formalizations and understanding of processes, such as parallel ones, which have so far escaped to a description by usual tools (see the examples on monoids and Petri nets in section 4.3 and the references, for recent developments of this idea).

As the reader may recall, the changes in the structural rules motivate a duplication of the connectives (see section 4.4). However, there is a way to recover the usual possibility, in classical as well as in intuitionistic logic, of an iterated use of assumptions. The idea is to introduce a connective "!" (read "of course"), which allows to assume as (finitely) many times one wishes a given assumption. This connective has a categorical meaning, which may be given in the terms of adjunctions and monads, following the previous section. The interesting categorical significance of this relation to classical and intuitionistic logic, as well as the categorical understanding we described in section 4.4, via structures such as the categories **Stab** and **Lin**, is probably what makes the difference between linear logic and previous formal experiments with the structural rules in other areas of logic.

The rules below are meant to extend the system in section 4.4. Observe that the structural rules of weakening and contractions apply with respect to the connective !. This is exactly what it is introduced for: it is meant to allow copies of assumptions and observe that they lead to the same consequences. This is expressed also by the rules (!,r) and (!,l) . Following Girard's work, we explicitly introduce the dual "?" (read "why not"), of the connective ! . Its operational behaviour is described by the rules, which mimic those for ! on the other side of the entailment (cf. the duality of $(\text{-})^{\perp}$ in 4.4.3), and by the equivalence of !A and $(?(A^{\perp}))^{\perp}$, proved below.
The exponential fragment of Linear Logic is as follows:

5.5.1 exponential unary connectives: ! (of course), ? (why not)

5.5.2 exponential rules

$$(\text{weak-l})\ \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad (\text{weak-r})\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}$$

$$(\text{contr-l})\ \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad (\text{contr-r})\ \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}$$

$$(!,l)\ \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad (!,r)\ \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}$$

$$(?,l)\ \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta} \qquad (?,r)\ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta}$$

The duality between ! and ? is easily obtained by the following deductions:

$$\begin{array}{ll}
 & A \vdash A \\
(!,l) & \overline{} \\
 & !A \vdash A \\
(\perp,l) & \overline{} \\
 & !A, A^{\perp} \vdash \\
(?,l) & \overline{} \\
 & !A, ?A^{\perp} \vdash \\
(\perp,r) & \overline{} \\
 & !A \vdash (?A^{\perp})^{\perp}
\end{array}
\qquad
\begin{array}{ll}
 & A \vdash A \\
(\perp,r) & \overline{} \\
 & \vdash A^{\perp}, A \\
(?,r) & \overline{} \\
 & \vdash ?A^{\perp}, A \\
(!,r) & \overline{} \\
 & \vdash !A, ?A^{\perp} \\
(\perp,l) & \overline{} \\
 & (?A^{\perp})^{\perp} \vdash !A
\end{array}$$

Exercise Prove that $!(A \cap B) \vdash\; !A \otimes !B$ and $!A \otimes !B \vdash\; !(A \cap B)$.

5.5.3 Remark The connective "!" is exactly what is needed to recover the intuitionistic calculus: it is possible to prove that there is an embedding of intuitionistic (and classical) logic into linear logic. The embedding maps every intuitionistic formula A to a linear formula $\underline{A}$ in the following way:

$\underline{A} = A$ if A is an atomic formula

$\underline{A \wedge B} = \underline{A} \cap \underline{B}$

$\underline{A \vee B} = !\underline{A} \oplus !\underline{B}$

$\underline{A \Rightarrow B} = !\,\underline{A} \multimap \underline{B}$

(! is supposed to bind tighter than $\multimap$ and $\cup$).

The absurdum F of intuitionistic is translated into 0, the identity for $\oplus$. Thus $\underline{\sim A} = !\underline{A} \multimap 0$. In other words, the iterated use of the premises, in a linear implication, gives exactly the intuitionistic implication. Then, if, $\Gamma \vdash_i A$, in intuitionistic Logic, then $!\,\Gamma \vdash \underline{A}$ in Linear Logic.

Our aim now is to give categorical meaning to the connective !, of course. By duality, in linear categories (see definition 4.4.4), we also obtain an interpretation for ?, why not.

We have already remarked that a resource such as !A can be duplicated and erased, and in a sense these properties characterize the meaning of the connective ! . Thus, at the sematic level, we expect to have two morphisms $\delta: !A \to !A \otimes !A$, and $\varepsilon: !A \to 1$ where 1 is the identity of the monoidal category. (Commutative) comonoids in 4.3.4 seem the right structure for this, as they are characterized by a sort of diagonal map, such as δ, and a map ε which dualizes the map η in definitions 4.2.1 and 4.3.3.

We start then with a monoidal category **C**. By definition, **C** must satisfy certain natural isomorphisms, given in 4.3.1, which we rebaptize in this section, for convenience, with more suggestive names

1. assoc: $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$
2. ins-l: $X \cong 1 \otimes X$
3. exch: $X \otimes Y \cong Y \otimes X$

Let also:

4. ins-r = exch $\circ$ ins-l : $X \cong X \otimes 1$

As mentioned in section 4.4, the connective of linear implication $\multimap$ is interpreted by the right adjoint to the tensor product $\otimes$, when **C** is a monoidal closed category, as defined in section 4.3.

Recall that the category $\mathbf{CoMon_C}$ of commutative comonoids over a monoidal category **C**, has as objects, for c in **C**, $(c, \delta: c \to c \otimes c, \varepsilon: c \to 1)$, and morphisms $f: (c, \delta, \varepsilon) \to (c', \delta', \varepsilon')$, for each arrow $f: c \to c'$ in **C** , such that

$\delta' \circ f = (f \otimes f) \circ \delta : c \to c' \otimes c'$,

$\varepsilon' \circ f = \varepsilon: c \to 1$.

Given a commutative comonoid $(c, \delta: c \to c\otimes c, \varepsilon: c \to 1)$ observe that the following equations hold:

$$(\varepsilon\otimes id_c) \circ \delta = \text{ins-l} : c \to 1\otimes c$$
$$\text{exch} \circ \delta = \delta : c \to c\otimes c$$
$$\text{assoc} \circ (id_c\otimes\delta) \circ \delta = (\delta\otimes id_c) \circ \delta : c \to (c\otimes c)\otimes c \,.$$

Exercise As pointed out after definition 4.3.6, if **C** is Cartesian, in the sense that $\otimes$ is actually a cartesian product $\cap$ and the isomorphisms are the canonical ones, then all the maps and isomorphisms above can be constructed for each object in **C** (in particular, recall that δ and ε are canonically given; namely, $\delta = \langle id,id\rangle : c \to c\cap c$ is the diagonal and $\varepsilon: c \to t$ is the unique map to the terminal object). Prove that, if **C** is Cartesian, then **C** is actually isomorphic to $\mathbf{CoMon_C}$. Does the converse hold?

5.5.4 Definition *A* ***! - model*** *is a linear category* ***C*** *and a comonad* $(!,D,E)$ *such that there exist natural isomorphisms*

$$I : !(A\cap B) \cong !(A)\otimes!(B)$$
$$J : !t \cong 1$$

where t *and* 1 *are the identities for* $\cap$ *and* $\otimes$*, the Cartesian and tensor products in* ***C****.*

Indeed, by definition, in a linear category one has both a monoidal and a Cartesian structure. The relation established by the natural isomorphisms gives the monoids we need.

5.5.5 Lemma *Let* $\langle \mathbf{C},(!,D,E)\rangle$ *be an !-model. Then, for each object* c *in* ***C****, there exist maps* $\delta':!c \to !c\otimes!c$ *and* $\varepsilon': !c \to 1$*, such that* $(!c, \delta', \varepsilon')$ *is a comonoid.*

Proof. Just set $\delta' = I\circ!\delta : !c \to !(c\cap c) \to !c\otimes!c$

$\varepsilon' = I\circ!\varepsilon : !c \to !t \to 1$

where $\delta = \langle id,id\rangle : c \to c\cap c$ and $\varepsilon: c \to t$ are the monoidal maps in the remark above, w.r.t. the Cartesian product $\cap$. The rest is easy.♦

Thus, the comonad $(!: \mathbf{C} \to \mathbf{C}, D: ! \to !\circ!, E: ! \to Id_\mathbf{C})$ associated with a !-model gives all the ingredients for the interpretation of the connective !, of course. In view of the above lemma, we can define the functor $\mathbf{!}: \mathbf{C} \to \mathbf{CoMon_C}$ by

$$\mathbf{!}(c) = (!c, \delta':!c \to !c\otimes!c, \varepsilon': !c \to 1) \,.$$

This gives the required monoids, while the natural transformations D and E uniformly yield maps $D_c : !c \to !\circ!c$ and $E_c : !c \to c$, which are needed to interpret the rules in (!,r) and (!,l).

We already mentioned in section 4.4 that the idea of the interpretation relies on viewing entailments as morphisms. In short, observe that, with an informal blend of syntax and semantics,

the rules	*are interpreted by*	*the fact that*
(weakenings)		each morphism $f: 1 \to \Delta$ gives a morphism $f \circ \varepsilon_a: !a \to \Delta$
(contractions)		each morphism $f: !a \otimes !a \to \Delta$ gives a morphism $f \circ \delta_a: !a \to \Delta$
(!,l)		each morphism $f: a \to \Delta$ gives a morphism $f \circ E_a: !a \to \Delta$
(!,r)		each morphism $f: !c \to a$ gives a morphism $!f \circ D_c: !c \to !a$.

As for the rules which contain ?, their meaning is easily derivable by duality. The idea is to define a functor $? : \mathbf{C} \to \mathbf{C}$ by

$$? = * \circ ! \circ *$$

that is $?A = (!A^*)^*$. Then the following theorem gives the categorical meaning of the modality ?, why not.

5.5.6 Theorem *Let $<\boldsymbol{C},(!,D,E)>$ be an !-model. Then there exist a monad $(?, D': ? \circ ? \to ?, E': Id \to ?)$ and natural isomorphisms*

$$I': ?(A \oplus B) \cong ?(A) \cup ?(B)$$

$$J': ?0 \cong \perp ,$$

where 0 and $\perp$ are the identities for $\oplus$ and $\cup$, the duals of the Cartesian and tensor products in $\boldsymbol{C}$.

Proof. Set $? = * \circ ! \circ * : \mathbf{C} \to \mathbf{C}$ and, for each object A, $D'_A = (D_{A^*})^*$ and $E'_A = (E_{A^*})^*$. As $D: ! \to ! \circ !$, one has

$D_{A^*}: !A^* \to ! \circ !A^*$	
$(D_{A^*})^*: (! \circ !A^*)^* \to (!A^*)^*$	by def.of $(-)^*$
$(D_{A^*})^*: (! \circ * \circ * \circ !A^*)^* \to (!A^*)^*$	by $Id \cong (-)^{**}$
$D'_A: ? \circ ?A \to ?A$	

Each of these steps is an isomorphism, uniform in A, and gives a natural transformation $D': ? \circ ? \to ?$. Similarly, from $E: ! \to Id_{\mathbf{C}}$ one has $E_{A^*}: !A^* \to A^*$ and, thus, $E'_A = (E_{A^*})^*: A \to ?A$. The properties required for a monad follow by duality.
As for the natural isomorphisms, compute

$?(A \oplus B) = * \circ ! \circ *(A \oplus B)$	
$= * \circ !(A^* \cap B^*)$	by theorem 4.4.6

$\cong (!(A^*)\otimes!(B^*))^*$ by def. of !-model

$= ?A\cup ?B$ by theorem 4.4.6.

Finally, $?0 = * \circ ! \circ *0 \cong * \circ ! \; t \cong 1^* \cong \perp$, by definition and theorem 4.4.6. ♦

Exercise Endow a structure of monoid over each object in a monoidal category whose tensor product is actually a Cartesian coproduct. Then give the details of the interpretation of the rules for ?.

Next we find, within any categorical model of linear logic, an interpretation for the intuitionistic connectives $\cap$ and $\Rightarrow$, by using the comonad construction in the !-model. Namely, given an !-model **C**, one may interpret intuitionistic "and" and "implication" by Cartesian product and exponential in a suitable category derived from **C**. As the purpose of the iterator ! was to take us back to intuitionistic logic, we use its categorical meaning to construct this new category.

As a matter of fact, in the remark 5.5.3, we hinted how to derive intuitionistic connectives from linear ones, once the connective ! is available. The following result gives the categorical counterpart of that construction.

Observe that in general, given a comonad (T, δ, ε) over **C**, the co-Kleisli category **K** is the category whose objects are those of **C**, and the set **K**[A,B] of morphisms from A to B in **K** is **C**[T(A),B]. The identity in **K**[A,A] is ε_A: T(A)→A. The composition of f∈**K**[A,B] and g∈**K**[B,C] in **K** is

$$g \circledcirc f = g \circ T(f) \circ \delta_A : T(A) \to T^2(A) \to T(B) \to C$$

(see definition 4.2.4 where Kleisli categories over monads were defined).

5.5.7 Theorem *If* **C** *be an !-model. Then the co-Kleisli category* **K** *associated with the comonad (!,D,E) is Cartesian closed.*

Proof (*hint*) The exponent of two objects B and C is (!B—oC). We then have the following chain of isomorphisms:

K[A∩B, C]	≅ **C**[!(A∩B), C]	by definition of **K**
	≅ **C**[!(A)⊗!(B), C]	as !(A∩B) ≅ !(A)⊗!(B)
	≅ **C**[!(A), !B—oC]	as **C** is monoidal closed
	≅ **K**[A, !B—oC]	by definition of **K**. ♦

Example In section 2.4.2 we defined the category **Stab** of coherent domains and stable functions. In that section (see exercise 4) the subcategory **Lin**, with linear maps, was also introduced and, later (see section 4.4), it was given as an example of linear category. We also defined a function ! on coherent domains as follows: if X is a coherent domain, then !X is the coherent domain defined by:

i. |!X| = {a / a∈X, a finite};

ii. a↑b [mod !X] iff a∪b∈X.

We need now to extend it to a functor $! : \mathbf{Stab} \to \mathbf{Lin}$. Recall that a linear map $g: Z \to Z'$ is uniquely determined by its behavior on the points of the coherent domain Z, i.e., on the elements of $|Z|$. Moreover, any stable map may be equivalently described in terms of its trace. Set then, for each stable map $f : X \to Y$,

$$Tr(!f) = \{(\{a\}, b) \mid b \in Y,\ b \text{ finite}, a \in X,\ a \text{ finite and least such } b \subseteq f(a) \}.$$

Next, we define an adjunction between $!$ and the obvious inclusion functor **Inc** from **Lin** into **Stab**. This is given by a natural isomorphism

(iso) $\qquad \varphi : \mathbf{Lin}[!A,B] \cong \mathbf{Stab}[A,B]$

where the inclusion functor is omitted.

Once more we use traces, that is, for each $g \in \mathbf{Lin}[!A,B]$ set

$$Tr(\varphi(g)) = \{(a, y) \mid (\{a\}, y) \in Tr(g) \} .$$

The reader may prove for exercise the naturality of φ. In particular, the unit and counit of the adjunction are given, as usual, by

$\eta_A = \varphi(id_{!A}) : A \to !A$ where $Tr(\eta_A) = \{(a, a) \mid a \in A \text{ finite} \}$

$\varepsilon_A = \varphi^{-1}(id_A) : !A \to A$ where $Tr(\varepsilon_A) = \{(\{x\}, x) \mid x \in |A| \}$.

Exercise Check, by actual computations in the structure, that $!f = \varphi^{-1}(\eta_A \circ f)$ and $f = \varphi(\varepsilon_A \circ !f)$.

Following theorem 5.4.1, $(!, \mathrm{Inc}, \eta, \varepsilon)$ yields a comonad

$$(! = !\circ \mathrm{Inc}: \mathbf{Lin} \to \mathbf{Lin},\ D = !\eta\mathbf{Inc}: ! \to !\circ!,\ E = \varepsilon: ! \to Id_{\mathbf{C}})$$

as required to turn **Lin** into an !-model. Moreover, it is a matter of a simple observation on the "hardware" of coherent domains to show that the isomorphisms needed to complete the definition hold in **Lin**, namely, that $!(A \cap B) \cong !(A) \otimes !(B)$ and $!t \cong 1$ are uniformly valid in this model (see the example in section 4.4).

Interestingly enough, by (iso) above, **Stab** is the co-Kleisli category associated with the comonad $(!, D, E)$ on **Lin**.

We conclude this section by identifying a class of categories which yield an interesting interpretation of the modality $!$. The idea is to interpret $!A$ as the commutative comonoid freely cogenerated by A, not just as a comonoid in the intended linear category.

5.5.8 Definition *Let $\mathbf{C}$ be a linear category and $\mathbf{U}: \mathbf{CoMon_C} \to \mathbf{C}$ be the forgetful functor which takes (c, δ, ε) to c. Then $\mathbf{C}$ is a **free !-model** if there exists a right adjoint to $\mathbf{U}$, that is a functor $! : \mathbf{C} \to \mathbf{CoMon_C}$ and a natural isomorphism $\Omega: \mathbf{C}[c,a] \cong \mathbf{CoMon_C}[(c,\delta,\varepsilon), !(a)]$.*

We need to show that free !-models are indeed !-models. This follows from the simple, but powerful, adjointness property stated in 5.5.8. As already recalled, by proposition 5.4.1, each adjunction yields a comonad. We explicitly reconstruct the units and counits as they bear some information.

5.5.9 Lemma *Let* **C** *be a free !-model and* <**U**,**!**,**Ω**> *be the given adjunction. Then, for* $! = \mathbf{U}\circ\mathbf{!}$, *there exist natural transformations* $D: ! \to !\circ!$ *and* $E: ! \to Id_{\mathbf{C}}$ *such that*

$$(!: \mathbf{C}\to\mathbf{C},\ D: ! \to !\circ!,\ E: ! \to Id_{\mathbf{C}})$$

is the comonad associated with **C**, *in the sense of proposition 5.4.1.*

Proof By the definition of morphism in $\mathbf{CoMon}_{\mathbf{C}}$, for every $h\in\mathbf{C}[c,a]$, the morphism $\Omega(h)\in \mathbf{CoMon}_{\mathbf{C}}[(c,\delta,\varepsilon),\mathbf{!}(a)]$ satisfies the following equations:

hom-1. $\delta_a \circ \Omega(h) = (\Omega(h)\otimes\Omega(h)) \circ \delta : c\to !a\otimes !a$;

hom-2. $\varepsilon_a \circ \Omega(h) = \varepsilon : c\to 1$.

Moreover, the naturality of Ω is expressed by the following equations:
for every $h\in\mathbf{C}[c,a]$, $f\in\mathbf{C}[a,b]$, $g\in\mathbf{CoMon}_{\mathbf{C}}[(c',\delta',\varepsilon'), (c,\delta,\varepsilon)]$:

nat-1. $\Omega(f \circ h) = \mathbf{!}(f) \circ \Omega(h)$;

nat-2. $\Omega(h \circ \mathbf{U}(g)) = \Omega(h) \circ g$.

The counits of the adjunction $(\Omega, \mathbf{U}, \mathbf{!})$: $\mathbf{CoMon}_{\mathbf{C}}\to\mathbf{C}$, are arrows $E_c=\Omega^{-1}(id_{\mathbf{!}(c)})$: $!c\to c$. By equation (nat-1) above, for $h = E_c$, we obtain $\mathbf{!}(f) = \Omega(f \circ E_c)$, and by equation (nat-2), $E_c \circ \mathbf{U}(\Omega(h)) = h$. The family of arrows $\{E_c\}_{c\in\mathbf{C}}$ defines a natural transformation $E: (\mathbf{U} \circ \mathbf{!})\to\mathbf{I}$. Dually the units of the adjunction define a natural transformation $H : \mathbf{I}\to(\mathbf{!} \circ \mathbf{U})$, where: $H_{(c,\delta,\varepsilon)} = \Omega(id_c)$: $(c,\delta,\varepsilon)\to !c$. The adjunction between $\mathbf{CoMon}_{\mathbf{C}}$ and **C** is thus equivalently expressed by the parameters $(\mathbf{U}, \mathbf{!}, H: \mathbf{I}\to(\mathbf{!} \circ \mathbf{U}), E: (\mathbf{U} \circ \mathbf{!})\to \mathbf{I})$.

Remember now that a comonad over a category **C** is a comonoid in the category of endofunctors from **C** to **C** (with composition as product, see 4.2.2).

By proposition 5.4.1, every adjunction $(F, G, \eta: Id_{\mathbf{C}}\to G\circ F, \varepsilon: F\circ G\to Id_{\mathbf{C'}})$ from **C** to **C'** determines a comonad $(T = F\circ G, \delta = F\eta G: T\to T\circ T, \varepsilon : T\to Id_{\mathbf{C'}})$ over **C'**.

In particular the adjunction $(\mathbf{U}, \mathbf{!}, H: \mathbf{I}\to(\mathbf{!} \circ \mathbf{U}), E: (\mathbf{U} \circ \mathbf{!})\to \mathbf{I})$: $\mathbf{CoMon}_{\mathbf{C}}\to\mathbf{C}$, defines a comonad $(! = \mathbf{U} \circ \mathbf{!} , D = \mathbf{U}H\mathbf{!} : ! \to ! \circ ! , E : ! \to Id_{\mathbf{C}})$ over the !-model **C**. ♦

Finally we derive the natural isomorphisms in definition 5.5.4.

5.5.10 Theorem *Let* **C** *be a free !-model and* $(!: \mathbf{C}\to\mathbf{C}, D: ! \to !\circ!, E: ! \to Id_{\mathbf{C}})$ *be the comonad associated with it by the lemma. Then there exist natural isomorphisms*

$I: !(A\cap B) \cong !(A)\otimes!(B)$

$J: !t \cong 1$

where t *and* 1 *are the identities for* $\cap$ *and* $\otimes$, *the Cartesian and tensor products in* **C**.

Proof Consider the comonoid $(!(A)\otimes!(B),\delta,\varepsilon)$ where

$$\delta = mix \circ (\delta_A\otimes\delta_B) : !(A)\otimes!(B) \to (!(A)\otimes!(B))\otimes(!(A)\otimes!(B))$$

$$\varepsilon = \text{ins-r}^{-1} \circ (\varepsilon_A \otimes \varepsilon_B) : !(A)\otimes!(B) \to 1$$

and

$$\text{mix} : (!(A)\otimes!(A))\otimes(!(B)\otimes!(B)) \to (!(A)\otimes!(B))\otimes(!(A)\otimes!(B))$$

is the obvious isomorphism.

Then, by hypothesis, we have an isomorphism

$$\Omega: \mathbf{C}[!(A)\otimes!(B), A\cap B] \cong \mathbf{CoMon_C}[\ (!(A)\otimes!(B),\delta,\varepsilon),\ \mathbf{!}(A\cap B)\ ]$$

The isomorphism $I_{A,B}$ from $!(A\cap B)$ to $!(A)\otimes!(B)$, which we write I for short, is given by

$$I = (!\text{fst} \otimes !\text{snd}) \circ \delta_{A\cap B} : !(A\cap B) \to !(A)\otimes!(B)$$

Note that I is a morphism of comonoids, that is, as it is easily verified,

$$\delta \circ I = (I\otimes I) \circ \delta_{A\cap B}$$

$$\varepsilon \circ I = \varepsilon_{A\cap B}$$

The inverse image of I is defined in the following way.

Let

$$k_1 = E_A \circ \text{ins-r}^{-1} \circ (\text{id}_{!A} \otimes \varepsilon_B): !(A)\otimes!(B)\to A$$

$$k_2 = E_B \circ \text{ins-l}^{-1} \circ (\varepsilon_A \otimes \text{id}_B): !(A)\otimes!(B)\to B$$

and

$$k = \langle k_1, k_2\rangle : !(A)\otimes!(B)\to A\cap B$$

Then the inverse image of I is $\mathbf{U}(\Omega(k)) = \Omega(k) : !(A)\otimes!(B)\to!(A\cap B)$, indeed:

$\Omega(k) \circ I =$

$= \Omega(k \circ I)$ by (nat-2)

$= \Omega(\ \langle k_1, k_2\rangle \circ I\ )$ by def. of k

$= \Omega(\ \langle k_1 \circ I, k_2 \circ I\ \rangle)$

$= \Omega(\ \langle E_A\circ\text{ins-r}^{-1}\circ(\text{id}_{!A}\otimes\varepsilon_B)\circ(!\text{fst}\otimes!\text{snd})\circ\delta_{A\cap B}, E_B\circ\text{ins-l}^{-1}\circ(\varepsilon_A\otimes\text{id}_B)\circ(!\text{fst}\otimes!\text{snd}) \circ \delta_{A\cap B}\ \rangle)$

by def. of k_1, k_2 and I

$= \Omega(\ \langle E_A\circ\text{ins-r}^{-1}\circ(\text{id}_{!A}\otimes\varepsilon_A)\circ(!\text{fst}\otimes!\text{fst})\circ\delta_{A\cap B}, E_B\circ\text{ins-l}^{-1}\circ(\varepsilon_B\otimes\text{id}_B)\circ(!\text{snd}\otimes!\text{snd}) \circ \delta_{A\cap B}\ \rangle)$

as $\varepsilon_B\circ!\text{snd} = \varepsilon_{A\cap B} = \varepsilon_A\circ!\text{fst}$

$= \Omega(\ \langle\ E_A \circ \text{ins-r}^{-1} \circ (\text{id}_{!A}\otimes\varepsilon_A) \circ \delta_A \circ !\text{fst}, E_B \circ \text{ins-l}^{-1} \circ (\varepsilon_B\otimes\text{id}_B) \circ \delta_B \circ !\text{snd}\ \rangle)$

as !fst and !snd are comonoid morphisms

$= \Omega(\ \langle\ E_A \circ !\text{fst}, E_B \circ\ !\text{snd}\ \rangle)$ by properties of comonoids

$= \Omega(\ \langle\ \text{fst} \circ E_{A\cap B}, \text{snd} \circ E_{A\cap B}\ \rangle)$ by naturality of E

$= \Omega(\ E_{A\cap B})$

$= \text{id}_{!(A\cap B)}\ .$ by def. of E

$I \circ \Omega(k) =$

$= (!\text{fst} \otimes!\text{snd}) \circ \delta_{A\cap B} \circ \Omega(k)$

$= (!\text{fst} \otimes!\text{snd}) \circ (\ \Omega(k)\otimes\Omega(k)\ ) \circ \delta$ by (hom-1)

$= (!\text{fst} \circ \Omega(k)\ )\otimes(!\text{snd} \circ \Omega(k)\ ) \circ \delta$

$= (\ \Omega(\ fst \circ k)\ )\otimes(\ \Omega(\ fst \circ k)\ )\circ \delta$ by (nat-1)

$= (\ \Omega(k_1)\ )\otimes(\ \Omega(k_2)\ )\circ \delta$ by def. of k

$= (\ \Omega(\ E_A \circ ins\text{-}r^{-1} \circ (id_{!A} \otimes \varepsilon_B)\ )\otimes(\ \Omega(E_B \circ ins\text{-}l^{-1} \circ (\varepsilon_A \otimes id_B)\ )\circ \delta$

by def. of k_1 and k_2

$= (\ ins\text{-}r^{-1} \circ (id_{!A} \otimes \varepsilon_B)\ )\otimes(\ ins\text{-}l^{-1} \circ (\varepsilon_A \otimes id_B)\ )\circ \delta$

by (nat-2) and def. of E

$= (\ ins\text{-}r^{-1} \circ (id_{!A} \otimes \varepsilon_B)\ )\otimes(\ ins\text{-}l^{-1} \circ (\varepsilon_A \otimes id_B)\ )\circ mix \circ (\delta_A\otimes\delta_B)$

by def. of δ

$= (\ ins\text{-}r^{-1} \circ (id_{!A} \otimes \varepsilon_A)\ )\otimes(\ ins\text{-}l^{-1} \circ (\varepsilon_B \otimes id_B)\ )\circ (\delta_A\otimes\delta_B)$

by application of mix

$= (\ ins\text{-}r^{-1} \circ (id_{!A} \otimes \varepsilon_A)\circ \delta_A\ )\otimes(\ ins\text{-}l^{-1} \circ (\varepsilon_B \otimes id_B)\ \circ \delta_B)$

by properties of comonoids

$= id_{!(A)}\otimes id_{!(B)}$

$= id_{!(A)\otimes !(B)}$.

The construction is clearly uniform in A and B.

As for the natural isomorphism J, note that $!A \cong !(A\cap t) \cong !A\otimes!t$, $!A \cong !(t\cap A) \cong !t\otimes!A$ and that the right and left identity, in a monoidal category, are unique. ♦

References Universal arrows and adjunctions are fundamental notions in Category Theory. Their treatment, in various forms, and references to their origin may be found in all textbooks we mentioned in the previous chapters. References for Linear Logic have been given in chapter 4.