Chapter 12

EXAMPLES OF INTERNAL MODELS

In this chapter we present three examples of internal models: provable retractions inside a PER model, PER inside **ω-Set**, and PL-Categories inside their Grothendieck completion. The general categorical investigation of internal models developed in the previous chapter allows a deeper and unified analysis of the different aspects of these models, which brings, in our view, to in an original understanding of all of them.

12.1 Provable Retractions

This example continues the analysys of an internal model of retractions developed in chapter 7 (see examples 7.2.2.2, 7.3.9.1.)

12.1.1 Definition *$\lambda\beta(\eta)p$ is $\lambda\beta(\eta)$ plus a fresh constant p such that*

1. $pp = p$
2. $(px)\circ(px) = px$ *where* $s\circ r = \lambda x.s(rx)$

R. $$\frac{a\circ a = a}{pa = a}$$

1+2 imply $p \circ p = p$ and, hence, p is a retraction whose range contains exactly the provable retractions: range(r) = $\{a \mid \lambda\beta\eta p \vdash ra = a\}$. The model constructed over this simple type-free theory is the "distilled" (syntactic) version of the various models in the literature based on categories of retractions: closures, finitary projections, etc. (see the references). It focuses on the general idea of "types as retractions" and, thus, clarifies the basic constructions in these models where "type is a type" (i.e., p itself is a retraction). This extension of type-free λ-calculus is provably non Church-Rosser, w.r.t. the obvious reduction relation, but it is consistent (see references). Consider then the term model of $\lambda\beta\eta p$: it is an applicative structure $\mathsf{T} = (\mathsf{T}, \cdot)$, where $[M]\in\mathsf{T}$ iff $[M] = \{N \mid \lambda\beta\eta p \vdash M = N\}$ and $[M]\cdot[N] = [MN]$. Sometimes, for semplicity, we will make no distinction between a term and is equivalence class.

Of course, T is an extensional λ-model and, thus, by theorem 9.5.10, $\mathsf{T}^{\mathsf{T}} \cong \mathsf{T}$ in $\mathrm{PER}_{\mathsf{T}}$ (remember that by definition $[M]\ \mathsf{T}\ [N]$ iff $\lambda\beta\eta p \vdash M = N$).

It is easy to show that this isomorphism is actually an identity, that is $\mathrm{T}^{\mathrm{T}} = \mathrm{T}$, indeed:

$$\begin{aligned}
[M]\ \mathrm{T}^{\mathrm{T}}\ [N] \quad & \text{iff} \quad \forall[P],\forall[Q] \quad [P]\ \mathrm{T}\ [Q] \ \text{ implies } \ [MP]\ \mathrm{T}\ [MQ] \\
& \text{iff} \quad \forall P,\forall Q \quad \lambda\beta\eta p \vdash P = Q \ \text{ implies } \ \lambda\beta\eta p \vdash MP = NQ \\
& \text{iff} \quad \lambda\beta\eta p \vdash Mx = Nx \\
& \text{iff} \quad \lambda\beta\eta p \vdash M = N \\
& \text{iff} \quad [M]\ \mathrm{T}\ [N]
\end{aligned}$$

As already pointed out, PER_{T} is a Cartesian closed category with all finite limits.

Let $RET_{\mathrm{T}} = (R_0, R_1, dom, cod, comp, id) \in Cat(PER_{\mathrm{T}})$ be the internal category of retractions on T. As we proved in the example 7.3.9.1, RET_{T} is Cartesian closed; we want to prove that RET_{T} yields an (internal) λ_2-model.

We start by giving the explicit definition of R_0 and R_1. Remember that in PER_{T} the equalizer of a pair of morphisms $f,g: P \to Q$ is $(E, h: E \to P)$ where aEb iff (aPb and $f(a)Qg(b)$) and h is the injection from E to P. Note also that $dom(E) = dom(P) \cap \{a \mid f(a)\ Q\ g(a)\}$, and on this domain E coincides with P. Then it is easy to verify that $[M]\ R_0\ [N]$ iff ($[M]\ \mathrm{T}\ [N]$ and $\lambda\beta\eta p \vdash M \circ M = M$.) In other words R_0 coincides with T but is restricted to those terms which are provable retractions.

The domain of R_1 are ($\lambda\beta\eta p$-equivalence classes of) triples $<M, F, G>$, where $< , >$ is the standard encoding of tuples in λ-calculus, F and G are provable retractions, and M is a morphism from F to G, that is, $\lambda\beta\eta p \vdash G \circ M \circ F = M$. Formally:

$dom(R_1) = \{ [<M, F, G>] \mid F \circ F =_{\lambda\beta\eta p} F\ ;\ \ G \circ G =_{\lambda\beta\eta p} G\ ;\ \ G \circ M \circ F =_{\lambda\beta\eta p} M \}$,

$[<M, F, G>]\ R_1\ [<M', F', G'>]$ iff $[<M, F, G>]\ \mathrm{T}\ [<M', F', G'>]$

Note moreover that $[<M, F, G>]\ \mathrm{T}\ [<M', F', G'>]$ iff $M =_{\lambda\beta\eta p} M', F =_{\lambda\beta\eta p} F', G =_{\lambda\beta\eta p} G'$.

Let $RET_{\mathrm{T}}* = (R_0{}^{R_0}, R_1{}^{R_0}, dom*, cod*, comp*, id*)$ be as in definition 11.4.2. We must prove that the costant functor $K: RET_{\mathrm{T}} \to RET_{\mathrm{T}}*$ has a right adjoint $\forall = (\forall_0: R_0{}^{R_0} \to R_0, \forall_1: R_1{}^{R_0} \to R_1)$. $\forall_0$ is the function realized by $\lambda F\lambda zt.F(pt)(z(pt))$. Thus, we need to show that

1. if F is in the domain of $R_0{}^{R_0}$ then $\forall_0(F)$ is in the domain of R_0 ;

2. $\forall_0$ takes elements that are equivalent in $R_0{}^{R_0}$ to elements that are equivalent in R_0.

As for (1) we have

$$\begin{aligned}
\forall_0 F \circ \forall_0 F &= \lambda x(\ \lambda zt.F(pt)(z(pt))\ )(\ \lambda t.F(pt)(x(pt))\ ) \\
&= \lambda x(\ \lambda t.F(pt)(\ \lambda t.F(pt)(x(pt))\ (pt))\ ) \\
&= \lambda x(\ \lambda t.F(pt)(\ F(pt)(x(pt)))\ ) \\
&= \lambda x\lambda t.\ F(pt)(x(pt)) \qquad \text{because } F(pt) \text{ is a retraction by hypothesis} \\
&= \forall_0 F
\end{aligned}$$

(2) is evident.

An element in $R_1{}^{R_0}$ is a triple $< T, F, G >$, where F and G are in the domain of $R_0{}^{R_0}$, T is in the domain of T^{R_0}, and for every retraction M, one has $TM = GM \circ TM \circ FM$.

$\forall_1$ is the function that takes (an equivalence class in $R_1{}^{R_0}$ of) a triple $<T, F, G>$ in (the equivalence class in R_1 of) the triple $<\lambda t\lambda m\,.T(pm)(\forall_0 Ft),\forall_0(F), \forall_0(G)>$.

Note that $\lambda t\lambda m\,.T(pm)(\forall_0 Ft))$ is a morphism from the retraction $\forall_0 F$ to the retraction $\forall_0 G$; indeed,

$$
\begin{aligned}
\forall_0 G \circ \lambda t\lambda m\,.T(pm)(\forall_0 Ft) \circ \forall_0 F = \\
&= \lambda z.\forall_0 G(\lambda m\,.T(pm)(\forall_0 Fz))\\
&= \lambda z.\ (\lambda x\lambda t.G(pt)(x(pt)))\ (\lambda m\,.T(pm)(\forall_0 Fz))\\
&= \lambda z.\ \lambda t.G(pt)(\ (\lambda m\,.T(pm)(\forall_0 Fz))\ (pt))\\
&= \lambda z.\ \lambda t.G(pt)(\ (T(pt)(\forall_0 Fz)\ )\\
&= \lambda z.\ \lambda t.T(pt)(\forall_0 Fz) \qquad \text{because } G(pt) \circ T(pt) = T(pt) \text{ by hypothesis}
\end{aligned}
$$

It should be clear that $\forall_1$ is indeed a morphism in PER_T its realizer is simply obtained by abstraction on $<T, F, G>$.

Given a variable type G in $R_0{}^{R_0}$ and a type N in R_0, the projection $proj_G(N)$ from $\forall_0 G$ to $G(N)$ is realized by the term $\lambda x.xN$. Indeed, if S has type $\forall_0 G$ that is, $S = (\forall_0 G)S$, then $SN = \forall_0 GSN = G(N)(SN)$, and thus SN has type $G(N)$.

We define now the isomorphism Δ of the adjunction.

Let $<T, \lambda x.M, G>$ be an element in $R_1{}^{R_0}$, where M is a retraction and $\lambda x.M = K_0(M)$.

Define then $\Delta_{M,G}(<T, \lambda x.M, G>) = <\lambda t\lambda m\,.T(pm)t,\ M,\ \forall_0 G>$.

Conversely, given $<S, M, \forall_0(G)>$ in R_1, define $\Delta_{M,G}{}^{-1}(<S, M, \forall_0(G)>) = <\lambda m\lambda t.Stm, \lambda x.M, G>$. Note that $\lambda m.\lambda t.Stm = \lambda m.(\lambda x.xm \circ S) = \lambda m.(\ proj_G(m) \circ S\ )$. Thus, for every retraction N, one has

$$G(N) \circ \lambda t.StN \circ M \ =\ proj_G(N) \circ S \circ M \ =\ G(N) \circ proj_G(N) \circ S \circ M \ =\ \lambda t.StN\,.$$

Moreover,

$$
\begin{aligned}
\Delta_{M,G}{}^{-1}(\Delta_{M,G}(<T, \lambda x.M, G>)) = \\
&= \Delta_{M,G}{}^{-1}(<\lambda t\lambda m\,.T(pm)t\,,\ M, \forall_0(G)>)\\
&= <\lambda m\lambda t.T(pm)t, \lambda x.M, G>\\
&= <\lambda m.T(pm), \lambda x.M, G>
\end{aligned}
$$

and clearly T and $\lambda m.T(pm)$ are equivalent in T^{R_0}.

As for the converse, note first that if $S = \forall_0(G) \circ S$, then

$$St(pm) = \forall_0(G)(St)(pm) = (\lambda m.G(pm)(St(pm)))(pm) = G(pm)(St(pm)) = Stm$$

Thus:
$$
\begin{aligned}
\Delta_{M,G}(\Delta_{M,G}{}^{-1}(<S, M, \forall_0(G)>)) = \\
&= \Delta_{M,G}(<\lambda m\lambda t.Stm, \lambda x.M, G>)\\
&= <\lambda t\lambda m\,.St(pm)\,,\ M, \forall_0 G>\\
&= <\lambda t\lambda m\,.Stm\,,\ M, \forall_0 G>\\
&= <S, M, \forall_0 G>
\end{aligned}
$$

Define then

$$\Delta (< M,G, < T, \lambda x.M, G > >) = < M, G, < \lambda t\lambda m .T(pm)t , M, \forall_0 G > >$$

and

$$\Delta^{-1}(< M,G, < S, M, \forall_0 G > >) = <M,G, < \lambda m\lambda t.Stm, \lambda x.M, G > >.$$

12.1.2 Remark It is possible to give an elegant categorical characterisation of the models of $\lambda\beta(\eta)p$. Let A be a reflexive object ($A^A \cong A$) in some category C, and let $RET_A = (R_0, R_1,$ dom, cod, comp, id) be the internal category of retractions on A in PER_A. Let $\xi: R_0 \to A^A$ be as usual the equalizer of the identity and $\lambda f.\ f \circ f$. All we need to turn RET_A in a model for $\lambda\beta(\eta)p$ is that there exist $p: A^A \to R_0$ such that R_0 is a retract of A^A via (ξ,p). Then, in a sense, since R_0 represents all the retractions on A, we can say that the collection of objects of RET_A is itself an object of RET_A: more formally, that RET_A is internal to RET_A w.r.t. PER_A. We have thus the following facts:

1. ($A^A < A$ in a CCC C *and* RET_A internal to RET_A with respect to PER_A)
   imply $A \models \lambda\beta p$.
2. $A \models \lambda\beta p$ *implies*
   ($A^A < A$ in PER_A *and* RET_A internal to RET_A w.r.t. PER_A).

The reader should complete the details for exercise.

12.2 PER inside ω-Set

This example continues our presentation of **PER** an an internal category **M** of **ω-Set** (see examples 7.2.7 and 7.3.9.2, where it is shown that the construction gives also an internal model.)

Let $\mathbf{M}*$ be defined as $\mathbf{c}*$ in 11.4.2. We next define an internal right adjoint to the constant functor $K : \mathbf{M} \to \mathbf{M}*$, that by def. 11.4.4 will complete the construction of a model for λ_2.

Again, we shall take advantage of the wise blend of set-theory and computability on which **ω-Set** is based, in order to avoid the most formal details.

The following lemma motivates the definition of $\forall$ below.

12.2.2 Lemma *If* $<\tau,F,G> \in \boldsymbol{\omega}\textbf{-Set}[\mathbf{M_0},\mathbf{M_1}]$ *then* $\exists n\ \forall A \in \mathbf{M_0}\ \ \tau(A) = \{n\}_{F(A)\to G(A)}$.

Proof. Suppose that $m \Vdash <\tau,F,G>$. Since $0 \Vdash A$ for any $A \in \mathbf{PER}$, then take $n = m\cdot 0$ and observe that $n \Vdash <\tau(A),F(A),G(A)>$. Thus $\tau(A) = \{n\}_{F(A)\to G(A)}$. ♦

As for the definition of $\forall_0$, observe that the intersection of any collection $\{A_i\}_{i\in I}$ of objects in **PER** is still in **PER**, by viewing them as sets of pairs (of numbers). That is, set

$$n\ (\cap_{i\in I} A_i)\ m \quad \text{iff} \quad \forall i\in I\ \ (n\ A_i\ m).$$

12.2.3 Definition

1. $\forall_0 : [\mathbf{M_0} \to \mathbf{M_0}] \to \mathbf{M_0}$, *is given by* $\forall_0(F) = \cap_{A\in \mathbf{M_0}} F(A)$,

2. $\forall_1 : [\mathbf{M_0} \to \mathbf{M_1}] \to \mathbf{M_1}$ *is defined as follows. If* $m \vdash \langle\tau,F,G\rangle \in \boldsymbol{\omega}\textbf{-Set}[\mathbf{M_0},\mathbf{M_1}]$, *set* $\forall_1(\langle\tau,F,G\rangle) = \langle\{m\cdot 0\}_{\forall_0(F)\to\forall_0(G)}, \forall_0(F), \forall_0(G)\rangle$.

12.2.4 Proposition. $\forall$ *in definition 12.2.3 is well defined. In particular,* $\forall_1$ *is realized by any number* p *such that, for all* m, $p\cdot m = m\cdot 0$.

Proof. By definition, if $F : \mathbf{M_0}\to\mathbf{M_0}$, then $\forall_0(F)$ is in $\mathbf{M_0}$. $\forall_0$ is realized by (any index of) any total recursive function. By lemma 12.2.2, $\forall_1$ is well defined as its definition does not depend on the choice of the realizer m for $\langle F,\tau,G\rangle$. Therefore we only need to show that $\forall_1$ is realized by p. Namely, that if

1. $m_1,m_2 \vdash \langle\tau,F,G\rangle$
2. $n_1\ \forall_0(F)\ n_2$
3. $A\in\mathbf{M_0}$

then one has $(p\cdot m_1\cdot n_1)\ G(A)\ (p\cdot m_2\cdot n_2)$. Indeed,

4. $p\cdot m_1 = m_1\cdot 0\ (F(A)\to G(A))\ m_2\cdot 0 = p\cdot m_2$, by (1)
5. $n_1\ F(A)\ n_2$, by (2) and (3),

and thus $(p\cdot m_1\cdot n_1)\ G(A)\ (p\cdot m_2\cdot n_2)$ by (4), (5) and the definition of $(F(A)\to G(A))\in\mathbf{M_0}$. ♦

12.2.5 Remark. It should be clear that lemma 12.2.2 is a very simple but crucial lemma. Note first that the morphisms in $\boldsymbol{\omega}\textbf{-Set}[\mathbf{M_0},\mathbf{M_1}]$ are described as triples, $\langle\tau,F,G\rangle$, where $\tau: \mathbf{M_0} \to \cup_{A\in\mathbf{M_0}}Q(F(A)\to G(A))$ is such that $\tau(A)\in Q(F(A)\to G(A))$ and $F, G: \mathbf{M_0} \to \mathbf{M_0}$ give the source and target of $\tau(A)$. This is a sound description, since $\mathbf{M}* = ([\mathbf{M_0}\to\mathbf{M_0}], [\mathbf{M_1}\to\mathbf{M_1}],...)$ may be viewed as the internal category of functors from the discrete category, whose object of objects is $\mathbf{M_0}$, to $\mathbf{M}$. Thus $[\mathbf{M_0}\to\mathbf{M_1}]$ or $\boldsymbol{\omega}\textbf{-Set}[\mathbf{M_0},\mathbf{M_1}]$ are internal natural transformations. By lemma 12.2.2, now, there is a uniform n which realizes $\tau(A)$ for all A. In a sense these internal natural transformations are “almost” constant maps and only depend on the source and target objects.

We need now to prove that there exists an internal natural isomorphism Δ, which gives the adjunction between $\forall$ and K. We can use the simplicity of the intended universe and perform the construction directly. As in the proof of the internal Cartesian closure of **M**, the point is to show that the functional dependencies, usually implicit in the external world (or given by “indexes”: recall $a,b \vdash \Lambda_{a,b}$), can be turned into internal constructions. Once more, this will be straightforward to check within **ω-Set**, as the realizers for the natural isomorphism and its inverse will not depend on their “indexes.”

Let K, with components k_0 and k_1, be the internal constant functor in definition 11.4.3.

12.2.6 Definition. *For* $m \Vdash\!\!- <\tau, k_o(B), G> \in \boldsymbol{\omega}\textbf{-}\boldsymbol{Set}[\boldsymbol{M_o}, \boldsymbol{M_1}]$, *set*

$$\Delta_{B,G}(<\tau, k_o(B), G>) = < \{m\cdot 0\}_{B\to\forall_O(G)}, B, \forall_o(G)> \in \boldsymbol{M_1} .$$

and $$\Delta^{-1}_{B,G}(<\{m\}_{B\to\forall_O(G)}, B, \forall_o(G)>)$$
$$= < \underline{\lambda} X\in\boldsymbol{M_o}.\{m\}_{B\to G(X)}, k_o(B), G> \in \boldsymbol{\omega}\textbf{-}\boldsymbol{Set}[\boldsymbol{M_o}, \boldsymbol{M_1}].$$

12.2.7 Proposition. *$\Delta_{B,G}$ and $\Delta^{-1}_{B,G}$ in 12.2.6 are well defined, for all $B\in\boldsymbol{M_o}$. and G: $\boldsymbol{M_o}\to\boldsymbol{M_o}$ (= $[\boldsymbol{M_o}\to\boldsymbol{M_o}]$). Moreover, they are realized by p and k (respectively), such that $p\cdot m = m\cdot 0$ and $k\cdot m\cdot n = m$.*

Proof: By lemma 12.2.2, $\Delta_{B,G}$ does not depend on the choice of the particular realizer m. As for $\Delta^{-1}_{B,G}$, note that, if $m(B\to\forall_O(G))m$, then $\forall n,q$ ($nBq \Rightarrow m\cdot n\ (\cap_{A\in\mathbf{M_o}}G(A))\ n\cdot q$), by definition of $\forall_O$, and hence $m\cdot n\ G(A)\ n\cdot q$, for all $A\in\mathbf{M_o}$. Therefore, $m\ (B\to G(A))\ m$ for all $A\in\mathbf{M_o}$ and $\Delta^{-1}_{B,G}$ too is well defined. By definition, they are uniformly realized by p and k as above. In particular, p and k compute $\Delta_{B,G}$ and $\Delta^{-1}_{B,G}$ for all B and G. ♦

12.3 PL-Categories Inside Their Groethendiek Completion

This example show another way for obtaining an internal model from a PL-category, different from the internalization process of chapter 11.

12.3.1 Definition. *Given any E-indexed category G, define the category* $\boldsymbol{\int G}$*, the* ***Grothendieck completion of G,*** *having as objects the pairs (e, σ) with $e\in Obj_E$ and $\sigma\in Obj_{G(e)}$ and as morphisms pairs (α, f) such that:*

$$(\alpha, f)\in\int G[(e, \sigma), (e', \tau)] \quad \text{iff} \quad \alpha\in E[e,e'] \text{ and } f\in G(e)[\sigma, G(\alpha)(\tau)].$$

The identity of (e, σ) is (id_e, id_σ); the composition of $(\alpha, f)\in\boldsymbol{\int G}[(e,\sigma),(e',\tau)]$ and $(\beta, g)\in\boldsymbol{\int G}[(e',\tau), (d,\rho)]$ is $(\beta, g)\circ(\alpha, f) = (\beta\circ\alpha, G(\alpha)(g)\circ f)$.

Let (E, G, Ω) be an external model (that is an indexed category with the additional requirements on functor G in 12.2.1); then the Grothendieck completion $\boldsymbol{\int G}$ assumes a particularly simple form

Objects: $(e, \sigma)\in\boldsymbol{\int G}$ iff $\sigma\in Obj_{G(e)} = E[e, \Omega]$

(hence we can identify objects of $\int G$ with arrows $\sigma: e\to\Omega$)

Morphisms: $(\alpha,f)\in\boldsymbol{\int G}[\sigma:e\to\Omega, \tau:e'\to\Omega]$ iff

$$\alpha\in E[e,e'] \text{ and } f\in G(e)[\sigma, G(\alpha)(\tau)] = G(e)[\sigma, \tau\circ\alpha].$$

The point is that $\boldsymbol{\int G}$ contains an internal category which, in a sense, internalizes the external model (E, G, Ω).

Warning We are here forcing our terminology, since the category $\int G$ does *not* need to have all finite limits, at least not in general, and hence we could not speak of "internal categories in $\int G$." However, all the needed pullback diagrams exist in $\int G$, as pointed out below.

The internal category $\Gamma = (c_0, c_1, DOM, COD, COMP, ID) \in Cat(\int G)$ is defined as follows:

$c_0 = t_{G(\Omega)} : \Omega \to \Omega$	the terminal object in $G(\Omega)$
$c_1 = [,]_0 : \Omega\times\Omega \to \Omega$	$[,]_0$ is given in lemma 11.2.3
$DOM = (fst, !)$	! is the unique arrow from c_1 to $c_0 \circ fst$ in $G(\Omega\times\Omega)$
$COD = (snd, !)$	
$ID = (<id_\Omega, id_\Omega>, \Lambda_{G(\Omega)}(snd))$	$snd \in G(\Omega)[t_{G(\Omega)} \times id_\Omega, id_\Omega]$, hence $\Lambda(snd) \in G(\Omega)[t_{G(\Omega)}, id_\Omega \to id_\Omega]$, where $id_\Omega \to id_\Omega$

The situation for COMP is more delicate, since $\int G$ does not have all finite limits. However, the pullback of $DOM: c_1 \to c_0$, $COD: c_1 \to c_0$ does exist, and it is given by

$$c_2 = (p_2 \to p_3) \times (p_1 \to p_2),$$

where product and exponents are in the fiber $G(\Omega\times\Omega\times\Omega)$ and $p_i \in E[\Omega\times\Omega\times\Omega, \Omega]$; the pullback projections $\prod_1, \prod_2 : c_2 \to c_1$ are

$$\textstyle\prod_1 = (<p_2, p_3>, id)$$

$$\textstyle\prod_2 = (<p_1, p_2>, id).$$

In order to define COMP, remember that in any CCC, given three objects A, B, C, there exists a morphism $cmp \in Hom[C^B \times B^A, C^A]$ that internalizes the composition, namely

$cmp = \Lambda(eval \circ id\times eval \circ p)$ for $p: (C^B \times B^A) \times A \to C^B \times (B^A \times A)$ an isomorphism.

Define then

$COMP = (<p_1, p_3>, cmp)$ for $cmp \in G(\Omega\times\Omega\times\Omega)[\, c_2, p_1 \to p_3]$

(recall that, in $G(\Omega\times\Omega\times\Omega)$, $p_1 \to p_3 = [,]_0 \circ <p_1, p_3>$).

The verification that these data define an internal category is straightforward.

We now prove that if (E, G, Ω) is an external model, than $\int G$ is Cartesian closed.

The terminal object is given by $t_{G(t)}$, the terminal object of the local category $G(t)$ (for t terminal in E). As for products, they can be defined by using the Cartesian structure of the local categories: $(f{:}e \to \Omega) \times_{\int G} (g{:}d \to \Omega) := x_0 \circ f \times_E g$, where $x_0: \Omega\times\Omega \to \Omega$ is the arrow obtained in lemma 11.2.3; hence $f \times_{\int G} g = (f \circ fst) \times_{G(e\times d)} (g \circ snd)$. Projections are obtained as follows. We need FST: $c_0 \times c_0 \to c_1$, where $FST = (\alpha, f)$ for some $\alpha: \Omega\times\Omega \to \Omega\times\Omega$ and $f \in G(\Omega\times\Omega)[c_0 \times c_0, c_1 \circ \alpha]$. As for α, which intuitively takes a pair of objects (σ, τ) to $(\sigma\times\tau, \sigma)$, we can take $\alpha = <x_0, fst>: \Omega\times\Omega \to \Omega\times\Omega$.

In order to define $f \in G(\Omega\times\Omega)[\, t_{G(\Omega\times\Omega)}, x_0 \to fst]$, we can use again the Cartesian closed structure of $G(\Omega\times\Omega)$ and the fact that, by lemma 11.2.3, one has

$$x_0 = x_0 \circ id = x_0 \circ <fst, snd> = fst \times_{G(\Omega\times\Omega)} snd.$$

Define then $f = \Lambda(fst)$ where $fst \in G(\Omega\times\Omega)[fst \times snd, fst]$. SND is defined analogously.

It remains to give the pairing isomorphism $\langle\!\langle,\rangle\!\rangle$; note first of all that the required pullbacks (see definition 7.3.6, and also appendix A at the end of chapter 7) are given by the following data:

$X = [,]_0 \times_{\int G} [,]_0 \circ <\mathrm{id}\times\mathrm{fst}, \mathrm{id}\times\mathrm{snd}> : \Omega\times(\Omega\times\Omega)\to\Omega$
$\prod_X : X\to c_1\times c_1$ (that is $X \to [,]_0 \times_{\int G} [,]_0$)
$\prod_X = (<\mathrm{id}\times\mathrm{fst}, \mathrm{id}\times\mathrm{snd}>, \mathrm{id})$
$\rho : X \to c_0\times c_0\times c_0$ (that is $X \to t_{G(\Omega\times\Omega\times\Omega)}$)
$\rho = (\mathrm{id}, !)$

$Y = [,]_0 \circ \mathrm{id}\times x_0 : \Omega\times(\Omega\times\Omega)\to\Omega$
$\prod_Y : Y\to c_1$ (that is $Y \to [,]_0$)
$\prod_Y = (\mathrm{id}\times x_0, \mathrm{id})$
$\rho' : Y \to t_{G(\Omega\times\Omega\times\Omega)}$
$\rho' = (\mathrm{id}, !)$

Observe now that, as we are dealing with objects of $G(\Omega\times(\Omega\times\Omega))$, by interpreting all products and exponentials locally in $G(\Omega\times(\Omega\times\Omega))$), one has

$X = (\mathrm{fst} \to \mathrm{fst} \circ \mathrm{snd})\times(\mathrm{fst} \to \mathrm{snd} \circ \mathrm{snd})$
$Y = \mathrm{fst} \to ((\mathrm{fst} \circ \mathrm{snd})\times(\mathrm{snd} \circ \mathrm{snd}))$

and hence $X \cong Y$, since $G(\Omega\times(\Omega\times\Omega))$ is Cartesian closed.

The isomorphism $\langle\!\langle,\rangle\!\rangle: X\to Y$ (in $\int G$) is then given by $\langle\!\langle,\rangle\!\rangle = (\mathrm{id},f)$, where f is the isomorphism between X and Y in $G(\Omega\times(\Omega\times\Omega))$. The required equations are now easily verified.

Let us now come to the exponents $\Rightarrow_0 = ([,]_0 , !) : c_0\times c_0\to c_0$, where $[,]_0$ is given again by lemma 11.2.3.

EVAL : $c_0\times c_0\to c_1$ is given by a pair: EVAL = (α,f), for $\alpha:\Omega\times\Omega\to\Omega\times\Omega$ and $f\in G(\Omega\times\Omega)[c_0\times c_0, c_1\circ \alpha]$. As for α, whose intuitive content is to send a pair of objects (σ, τ) into $(\tau^\sigma\times\sigma, \tau)$, we can set $\alpha = <x_0 \circ < [,]_0, \mathrm{fst}>, \mathrm{snd}>$.

We need now $f\in G(\Omega\times\Omega)[t_{G(\Omega\times\Omega)}, [,]_0 \circ <x_0 \circ < [,]_0, \mathrm{fst}>, \mathrm{snd}>]$; observe that

$[,]_0 \circ <x_0 \circ < [,]_0, \mathrm{fst}>, \mathrm{snd}> =$
$= (x_0 \circ < [,]_0, \mathrm{fst}>) \to \mathrm{snd}$ in $G(\Omega\times\Omega)$
$= ([,]_0 \times \mathrm{fst}) \to \mathrm{snd}$
$= ((\mathrm{fst}\to\mathrm{snd}) \times \mathrm{fst}) \to \mathrm{snd}.$

Take then $\mathrm{eval}_{\mathrm{fst},\mathrm{snd}}\in G(\Omega\times\Omega)[(\mathrm{fst}\to\mathrm{snd}) \times \mathrm{fst}, \mathrm{snd}]$ and set

$$f = \Lambda(\mathrm{eval}_{\mathrm{fst},\mathrm{snd}})\in G(\Omega\times\Omega)[t_{G(\Omega\times\Omega)}, ((\mathrm{fst}\to\mathrm{snd}) \times \mathrm{fst}) \to \mathrm{snd}].$$

Before giving the isomorphism Λ, we need to express the pullback diagrams of theorem 7.3.7 (see also appendix A). They can be instantiated as

$$\begin{tikzcd} X' \arrow[r, "\Pi_{X'}"] \arrow[d, "\sigma"'] & {[,]_0} \arrow[d, "(id_{\Omega\times\Omega},\,!)"'] \\ t_{G(\Omega\times\Omega\times\Omega)} \arrow[r, "(x_0\times id_\Omega,\,!)"] & t_{G(\Omega\times\Omega)} \end{tikzcd} \qquad \begin{tikzcd} Y' \arrow[r, "\Pi_{Y'}"] \arrow[d, "\sigma'"'] & {[,]_0} \arrow[d, "(id_{\Omega\times\Omega},\,!)"'] \\ t_{G(\Omega\times\Omega\times\Omega)} \arrow[r, "(id_\Omega\times[,]_0,\,!)"] & t_{G(\Omega\times\Omega)} \end{tikzcd}$$

where:

$X' = [,]_0 \circ (x_0\times id_\Omega)$ $\qquad$ $Y' = [,]_0 \circ (id_\Omega \times [,]_0)$

$\prod_{X'} = (x_0\times id_\Omega, id)$ $\qquad$ $\prod_{Y'} = (id_\Omega\times[,]_0, id)$

$\sigma = (id_{\Omega\times\Omega\times\Omega}, !)$ $\qquad$ $\sigma' = (id_{\Omega\times\Omega\times\Omega}, !)$

As before, observe that

$X' = fst\times(fst \circ snd)\to(snd\circ snd)$

$Y' = fst \to ((fst \circ snd)\to(snd\circ snd))$.

Therefore $X' \cong Y'$, by the Cartesian closure of $G(\Omega\times(\Omega\times\Omega))$.

The internal Cartesian closed category $\Gamma\in Cat(\int\mathbf{G})$ is actually an internal model when (E, G, Ω) is an external one. In order to define the required adjunction, we need first to show that we can indeed construct the internal category $\Gamma*$; that is, the exponents $c_0{}^{c_0}$ and $c_1{}^{c_0}$ exist in $\int\mathbf{G}$ (as for the limits, not all the exponents exist in $\int\mathbf{G}$).

12.3.2 Lemma

i. For any object e of E, $c_0{}^{t_{G(e)}}$ exists in $\int\mathbf{G}$ and it is given by $t_{G(\Omega^e)}$.

ii. For any object e of E, $c_1{}^{t_{G(e)}}$ exists in $\int\mathbf{G}$ and it is given by $\forall([,]_0 \circ eval)$.

Proof Set $\sigma = t_{G(e)}$.

i. Note first that in $\int\mathbf{G}$, for any e' and τ, every arrow $(\beta,g):\tau\to t_{G(e')}$ has $g = !$. We must then show that, for $c_0{}^\sigma = t_{G(\Omega^e)}$, the following diagram commutes, for any object τ and arrow $(\alpha,!)$:

$$\begin{tikzcd} \tau\times\sigma \arrow[r, "(\alpha,\,!)"] \arrow[d, "(\Lambda(\alpha),\,!)\times id"'] & c_0 \\ c_0{}^\sigma\times\sigma \arrow[ur, "(eval,\,!)"'] & \end{tikzcd}$$

which is immediate.

ii. Let τ: e'$\to\Omega$ be any object of $\int\mathbf{G}$; we are looking for the unique $\Lambda(\alpha, f)$ such that the following diagram commutes:

$$\begin{tikzcd}
\tau\times\sigma \arrow[r, "{(\alpha, !)}"] \arrow[d, dashed, "{\Lambda(\alpha,f)}"'] & {[,]_0} \\
c_1^{\sigma}\times\sigma \arrow[ur, "eval"'] &
\end{tikzcd}$$

Since α: e'×e$\to\Omega\times\Omega$, if we set $\Lambda(\alpha,f) = (\underline{\alpha},\underline{f})$, a natural choice for $\underline{\alpha}$ is $\underline{\alpha} = \Lambda(\alpha)$:e'$\to(\Omega\times\Omega)^e$.
Moreover, for $(\alpha, f):\tau\times\sigma\to[,]_0$, we have the following:

$f\in G(e'\times e)[\tau\times\sigma, [,]_0\circ\alpha\ ] \cong$

$\cong G(e'\times e)[\tau\circ \text{fst}, [,]_0\circ \alpha]$	by $\sigma = t_{G(e)}$ and the definition of product in $\int\mathbf{G}$
$\cong G(e')[\tau\circ \text{fst}, \forall([,]_0\circ \alpha)]$	via the isomorphism Δ of the adjuction giving the external model
$\cong G(e')[\tau\circ \text{fst}, \forall([,]_0\circ \text{eval}) \circ \Lambda(\alpha)]$	by naturality of $\forall$.

We can then define

$c_1^{\sigma} = \forall([,]_0 \circ \text{eval}) : (\Omega\times\Omega)^e\to\Omega$

and take $\underline{f} = \Delta(f)$; the previous diagram then commutes for eval = (eval, $\text{proj}_{c0}\sigma$).

We can eventually give the data for the adjunction

$\forall_0 : c_0^{c0} \to c_0$

$\forall_0 = (\forall_0 : \Omega^{\Omega} \to \Omega, !)$.

As for PROJ = (α, f): $c_0^{c0}\to c_1^{c0}$, we need $\alpha : \Omega^{\Omega}\to(\Omega\times\Omega)^{\Omega}$ and $f\in G(\Omega^{\Omega})[c_0^{c0}, c_1^{c0}\circ \alpha]$.
For eval : $\Omega^{\Omega}\times\Omega\to\Omega$ and fst: $\Omega^{\Omega}\times\Omega\to\Omega^{\Omega}$, set $\alpha = \Lambda(<\forall_0 \circ \text{fst}, \text{eval}>)$. As for f, note first that

$c_0^{c0} = t_{G(\Omega^{\Omega})}$

and

$c_1^{c0}\circ \alpha = \forall([,]_0 \circ \text{eval}) \circ \Lambda(<\forall_0 \circ \text{fst}, \text{eval}>)$	
$= \forall([,]_0 \circ <\forall_0 \circ \text{fst}, \text{eval}>)$	by naturality of $\forall$
$= \forall((\forall_0 \circ \text{fst}) \to \text{eval})$	by the usual isomorphims.

We are then looking for $f\in G(\Omega^{\Omega})[t_{G(\Omega^{\Omega})}, \forall((\forall_0 \circ \text{fst}) \to \text{eval})\ ]$; as we did in the proof of the previous lemma, let us look for an arrow $h\in G(\Omega^{\Omega}\times\Omega)[t_{G(\Omega^{\Omega})}\circ \text{fst}, (\forall_0 \circ \text{fst}) \to \text{eval}]$. Then the f we need will then be $\Delta(h)$. Observe now that $\text{proj}_{\text{eval}}\in G(\Omega^{\Omega}\times\Omega)[\forall_0 \circ \text{fst}, \text{eval}]$; this immediately gives the required h in the following diagram:

$$
\begin{tikzcd}
t_{G(\Omega^{\Omega}\times\Omega)} \times (\forall_0 \circ \mathrm{fst}) \cong \forall_0 \circ \mathrm{fst} \arrow[r, "\mathrm{proj}_{\mathrm{eval}}"] \arrow[d, "h \times id"'] & \mathrm{eval} \\
(\forall_0 \circ \mathrm{fst} \to \mathrm{eval}) \times (\forall_0 \circ \mathrm{fst}) \arrow[ur, "\mathrm{eval}"'] &
\end{tikzcd}
$$

In conclusion, the Grothendieck completion provides yet another example of the general categorical construction of model in chapter 11. Moreover, it gives further insight into the external versus internal approach by giving a setting where the external models may be viewed as internal constructions.

References The examples in this chapter may be found, in part, in the many papers mentioned at the end of the previous chapter. Since Moggi's hint for the small completeness of the **PER** construction, in particular, these models have been thoroughly explored by many authors. The three sections above develop the ideas in Amadio and Longo (1986) and in Longo and Moggi (1988), and follow Asperti and Martini (1989), respectively. By this, the first two sections extend to higher order the understanding of simple types as partial equivalence relations and retractions in Scott (1976). The consistency of $\lambda\beta\eta p$ is shown in Berardi (1988).