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CARTESIAN CLOSED CATEGORIES OF ENUMERATIONS FOR EFFECTIVE TYPE STRUCTURES

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PART I

§.0 INTRODUCTION - (to Part I and II) -. By "data type" one usually intends a set of objects of the same "type" or "kind", suitable for manipulation by a computer program. Of course, computers actually manipulate formal representations of objects. The purpose of the mathematical semantics of programming languages, though, is to characterize data types (and functions on them) in a way which is independent of any specific representation mechanism. Thus the objects one deals with are mostly elements of domains borrowed from Set-Theory, Algebra, Category Theory ..., whose meaning is well understood within each framework and does not depend on the practice of programming. However, by doing so, what is lost is the notion of effective computability, which has an intrinsic operational character. This notion may be recovered by a suitable definition of "computable object" in abstract set-theoretic, algebraic, category-theoretic ... settings.

In particular, a more specific motivation for the study of effectiveness over semantic domains may be suggested by the following analogy.

The categories one needs for interpreting high level programming languages must possess strong completeness and closure properties so

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that the existence of objects, which are formally given by general definitional tools, is "a priori" assured: e.g. we want that cartesian products and morphism spaces still belong to the given category, for these constructs are commonly used in the design of high level languages.

Similarly, completeness and closure properties are the key idea for defining domains and categories in several areas of Mathematics. Banach, Hilbert or Sobolev spaces, say, may be considered as the (metric) completion of the possible solutions of a given set of equations. Once the solution of the problem studied is found in one such a space, it is then time to ask whether it is an "acceptable" solution from the intended viewpoint or whether it has been added by the completion technique. For example, for a given set of partial differential equations, one may (easely) find a solution in the related (Sobolev) space and then check whether it is an acceptable (regular) solution, i.e. whether it is differentiable in the ordinary sense.

Now, acceptable for a computer scientist means computable. It is then worth pursuing a general notion of effectiveness over abstract data types, since computable elements and maps provide the "regular" interpretation of programming constructs over semantic domains. Preliminary investigations on the effectiveness of the semantics of programs may be found in Scott (1976), Giannini & Longo (1983), Kanda (1984).

Unfortunately, the natural numbers, ω, and the partial recursive functions, PR, are not sufficient for this investigation, since, in general, typed and type-free languages cannot be directly interpreted over PR or ω. PR and ω, though, may be used for defining effectiveness over more general data types. The methods are borrowed from higher type Recursion Theory or computability in abstract structures, nowadays strictly interelated topics in view of the work done in the 70's by several authors (see references).

This paper is motivated by the study of completeness and closure properties of natural categories of effectively given data types. Countability, say, is a useful assumption for dealing with effectiveness over abstract data types.

Suppose one is given two countable sets A and B, and two numberings (onto maps) $e_A : \omega \to A$, $e_B : \omega \to B$. There is then a natural definition of computable map between A and B : call $g : A \longrightarrow B$ (<u>effective) morphism</u> iff there exists a recursive function f such that the following diagram commutes:

(1)
$$\begin{tikzcd} \omega \arrow[r, "f"] \arrow[d, "e_A"'] & \omega \arrow[d, "e_B"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

The category of <u>numbered sets</u> (EN) whose objects are pairs such as $\underline{A} = (A, e_A)$ and morphisms defined as in (1), has been studied in Ershov (1973, 1975). An introduction and some applications may be found in Visser (1980) and Bernardi & Sorbi (1983), mainly, or Barendregt & Longo (1982).

The first question one may ask about EN is whether there is natural way to give, effectively, a numbering to the set $EN(\underline{A},\underline{B})$, the set of morphisms from $\underline{A}$ to $\underline{B} = (B, e_B)$. In general, there is no such a "uniform" and "effective" coding of $EN(\underline{A},\underline{B})$, given $\underline{A}$ and $\underline{B}$. As a matter of fact EN is far away from being Cartesian Closed.

Nonetheless EN has several nice properties. We recall a notion and a simple consequence of it, whose relevance should be clear.

(R is the set of the (total) recursive functions; $\underline{\omega}$ is (ω, id)).

0.1 Definition. $\underline{A}$ is a <u>precomplete</u> numbered set if

(2) $\forall f\ PR\ \exists f' \in R\ \forall n(f(n)\downarrow \Rightarrow e_A(f'(n)) = e_A(f(n))$

(i.e. f' extends f w.r.t. (A, e_A)).

$\underline{A}$ is <u>complete</u> if

(2) above holds and $\exists a \in A\ \forall n(f(n)\uparrow \Rightarrow e_A(f'(n)) = a)$

(a is a special (bottom) element).

0.2 <u>Generalized Recursion Theorem</u>. Let (A, e_A) be precomplete. Then

(3) $\forall f \in PR\ \exists n(f(n)\downarrow \Rightarrow e_A(f(n)) = e_A(n))$.

The partial recursive functions suggest an obvious notion of partial morphism for numbered sets.

0.3 <u>Definition</u>. $\underline{A}$ and $\underline{B}$ be numbered sets. Then $f \in EN_p(\underline{A},\underline{B})$ (f is a <u>partial morphism</u>) if $\exists f' \in PR \quad f \circ e_A = e_B \circ f'$.

For the purposes of this paper, partial morphisms will be studied in a general category-theoretic setting, since partial maps come out naturally in computability theory. Note that (3) above is equivalent to $\forall f \; EN_p(\underline{\omega},\underline{A}) \quad \exists n(f(n)\downarrow \Rightarrow f(n) = e_A(n))$.
Completeness may be related to a Least Fixed Point Theorem (see later).

Of course, (3) in Theorem 0.2 is a very desirable property for handling abstract data types, in view of the recursive definitions. But exactly because of this, one may want more; namely the possibility of inheriting completeness and other properties at higher types, i.e. for the set of morphisms on numbered sets, since functions are among the typical data to be mostly defined recursively. This cannot be done in general, in view of the lack of the above mentioned closure properties for EN.

There are two reasonable ways to obtain the Cartesian Closure (CC) of a Category such as EN: one may restrict the attention to a subcategory or enlarge the Category itself. The point is that both ways should be "natural" and should give interesting categories.

In Part I we study a direct, elementary characterization of the "main" types of a well known sub-CCC of EN, Scott's effectively given domains (their computable sub-objects, to be precise). This will be done by a type structure over ω, based on two simple notions: acceptable pairing and relative (Gödel-)numberings (§.1).

§.2 and 3 presents CCC's with partial morphisms and partial objects and relates domains to EN also by using these notions.

Part II will introduce the CCC of Generalized Enumerations, whose definition is inspired by the notion of relative numbering and will relate it, as well as its computability properties, to EN.

§.1 <u>An elementary approach to higher type computability</u>

Let 0 be the type of ω. Then the integer types are defined by $n+1 := n \to n$ and the pure types by $n+1 := n \to 0$. Partial computable functionals in the integer and pure types may be introduced by using only ω and PR, with no mention of the category-theoretic and continuity properties of EN and Scott's domains, at first reading. The Hereditary Partial Effective Functionals were given in Longo (1982) (see Longo & Moggi (1983) for a few results and Longo (1984) for a discussion).

1.1 Definition. Let $L^\circ = \omega$ and fix $L \subseteq \omega \to \omega$. Define then

$L^{n+1.5} = \{\phi : L^n \to L^{n+1} / \lambda xy.\phi(x)(y) \in L^{n+1}\}$

$L^{n+2} = \{f : L^{n+1} \to L^{n+1} / \forall \phi \in L^{n+1.5}\ f \circ \phi \in L^{n+1.5}\}$.

The key idea is that (some) functions in $L^{n.5}$ gödelize L^{n+1} by L^n (see the notion of relative numberings in 3.2.2).

There is another way to look at the HPEF, which makes explicit the role of the pairing function $\langle , \rangle$, implicitely used in the definition of $L^{n+1.5}$.

1.2 Definition. Let U be a set and $F \subseteq U \to U$. Then $\langle , \rangle : U \times U \to U$ is an acceptable pairing w.r.t. F if:

1) $\exists p_1, p_2 \in F\ \forall x, y \in U\ \ p_i(\langle x_1, x_2 \rangle) = x_i$, where p_1 and p_2 are total

2) $\forall f, g \in F\ \ \lambda x.\langle f(x), g(x) \rangle \in F$.

Following the polish tradition in constructive mathematics, an interesting class of (pure) type 2 total functionals on R is defined in Rogers (1967; p. 364). Namely, $f : R \to \omega$ is Banach-Mazur if $\forall g \in R\ \exists h \in R\ f(\lambda y.g(\langle x, y \rangle)) = h(x)$, where $\langle , \rangle : \omega \times \omega \to \omega$ is an effective pairing function (an acceptable pairing w.r.t. R, in our terminology). This can be generalized and extended at higher types as follows.

1.3 Definition. (GBM) Let $BM^\circ = \omega$ and fix $BM^1 \subseteq \omega \to \omega$. Define then $BM^{n+2} \subseteq BM^{n+1} \to BM^{n+1}$ by

$f \in BM^{n+2}$ if $\forall g \in BM^{n+1}\ \exists h \in BM^{n+1}\ \forall x\ BM^n\ \ f(\lambda y.g(\langle x, g \rangle)) = \lambda y.h(\langle x, y \rangle)$,

where $\langle , \rangle : BM^b \times BM^n \to BM^n$ is an acceptable pairing w.r.t. BM^{n+1}.

What remains to verified is that $\langle , \rangle$ actually exists in any type, for a suitable choice of BM^1. This will be done in §.3.

It is now easy to see that, if $L^1 = BM^1$, then $\forall n\ \ L^n = BM^n$. Just notice that

(3) $\quad g \in L^{n+1.5}$ iff $\exists g' \in L^{n+1}\ \forall x, y \in L^n\ \ g(x)(y) = g'(\langle x, y \rangle)$.

Thus, for $f \in L^{n+2}$, $f \circ g(x) = f(\lambda y.g'(<x,y>))$ and, for some $h \in L^{n+1}$, $f \circ g(x) = \lambda y.h(<x,y>)$, by the definition of L^{n+2} and (3) applied to $f \circ g$. The rest is obvious.

It is also a simple exercise to give a variant in the pure types of the GBM or the HPEF. Thus these functionals are an easy way to define partial computable functions in higher types, by taking $L^1 = PR$ or $BM^1 = PR$. Partial maps turned out to be essential in computability theory, mainly because they may be effectively numbered and possess universal functions. Moreover, the related type structures yield models of functional languages, namely of typed and type-free λ-calculus, as it will be mentioned below.

Interestingly enough the proof that these hierarchies are well defined (i.e. that $<,>$ exists in any type) goes toghether with the proof of their main properties, which heavely rely on category theoretic and continuity notions for EN and Scott's domains. One cannot avoid, then, some mathematics. Let's first discuss the issue of partiality in a category-theoretic frame.

§.2. Partial morphisms and partial objects

There are at least three different ways to introduce the notion of divergence in categories. By using partial morphisms, partial objects or both. In this section we consider concrete categories (with partial morphisms), i.e. subcategories of Set (Set_p), and see how these ways relate.

2.1 Definition. Set_p is the category whose objects are sets and where $Set_p(x,y) = \{f \mid f:X \to Y \text{ (partial)}\}$, for all objects x,y.

The following notion has been inspired by a talk given in Siena by A. Heller.

2.2 Definition. C is a concrete category with partial morphisms (pC) if:

1) Every hom-set $C(x,y)$ contains an every-where divergent morphism $0_{x,y}$ s.t. for all objects z,v and any $f \in C(z,x)$ and any $g \in C(y,v)$ one has

$$0_{x,y} \circ f = 0_{z,y} \quad \underline{\text{and}} \quad g \circ 0_{x,y} = 0_{x,t}.$$

2) There exists a <u>singleton</u> object t s.t.

$C(t,t) = \{0_{t,t}, id_t\}$ and

$\forall x,y\ \forall f,g \in C(x,y)\ (f = g \iff \forall h \in C(t,x)\ \ f \circ h = g \circ h)$.

Singleton objects clearly coincide to within isomorphism. Thus the category of total morphisms, defined as follows, does not depend on the choice of t.

2.3 <u>Definition</u>. Let C be pC and t a singleton in C. Define then $\underline{C}_T$ with objects in C and morphisms as follows:

$C_T(x,y) = \{f \in C(x,y) / \forall h \in C(t,x)(f \circ h = 0_{t,y} \Rightarrow h = 0_{t,x})\}$.

Clearly EN_p is pC and $(EN_p)_T = EN$.

A pC may be embedded in Set_p in the same way as a concrete category may be emebedded in Set. Namely, the embedding functor $I = C_T(t,-) : C \quad Set_p$ is faithful and $I(C(t,x)) = Set_p(It, Ix)$.

As usual, one may also represent a partial $f : x \rightharpoonup y$ by a total map $\bar{f} : x \to y^{\perp}$; where $y^{\perp}$ is obtained from y by adding a fresh element to y. Recall that, in a category C, $x \triangleleft y$ (x is a <u>retract</u> of y) if there exists a pair (in, out), with in $\in C(x,y)$, out $\in C(y,x)$ s.t. out$\circ$in $= id_x$. By this, we may give a notion of partial object, suitably related to partial morphisms.

2.4 <u>Definition</u>. Let C be a pC. Define then

1) $-^{\perp} : C \to C_T$ is a <u>bottom functor</u> if $C(x,y) \cong C_t(x,y^{\perp})$.

2) x is a <u>partial object</u> if $x \triangleleft x^{\perp}$ in C_T.

(Intuition: ).

2.5 <u>Remark</u>. Let t be a terminal object (a singleton) in C_T. Then $t \triangleleft x^{\perp}$; moreover, if x is a partial object, then $t \triangleleft x$.

Partial morphisms and partial objects may be more fully related within Cartesian Closed Categories. These categories may be defined as in the classical case. One has to take care, though, of the behaviour of functors and natural transformations, which should be preserved on partial morphisms. This may be done using an (implicit) notion of domain (see i,ii,iii in 2.6 below).

2.6 <u>Definition</u>. C is a <u>pCCC</u> if C is pC with the following adjunctions:

1) a terminal singleton object t for C_T;

2) $\langle \Delta -, - \times -, -\hat{\ }- \rangle : C_T \to C_T \times C_T$,

3) for any object a, $\langle - \times a, -^{a}_{p}, \Lambda^{-1} \rangle : C_T \to C$,

where:

i) if $f \in C(x,y)$ and $g \in C(x,z)$, then

$$\forall h \in C_T(t,x) \ (f\hat{\ }g) \circ h = \begin{cases} 0 & \text{if } f \circ h = 0 \text{ or } g \circ h = 0 \\ (f \circ h)\hat{\ }(g \circ h) & \text{otherwise;} \end{cases}$$

ii) if $f \in C(x,y)$ and $g \in C(x',y')$, then

$f \times g = (f \circ p_1)\hat{\ }(g \circ p_2)$;

iii) if $f \in C(x, y^{a}_{p})$, then

$$\forall h \in C(t,x) \ (\Lambda^{-1} f) \circ (h \times id_a) = \begin{cases} 0 & \text{if } f \circ h = 0 \\ \Lambda^{-1}(f \circ h) & \text{otherwise.} \end{cases}$$

Observe that the extensions in the adjunctions in 2.6.2 and 3 are unique. As usual, x^{y}_{p} is an object and represents $C(y,x)$.

2.7 <u>Proposition</u>. Let C be a pCCC, x and y objects in C and t a terminal object. Then

(i) $x^{t}_{p} \cong x^{\perp}$, i.e. $-^{t}_{p}$ is a bottom functor,

(ii) x^{y}_{p} is a partial object.

<u>Proof</u>. (i) obvious (ii). We have to prove that $x^{y}_{p} \triangleleft (x^{y}_{p})^{\perp}$ in C_T. Let us identify $x \times t$ with x and $x^{\perp}$ with x^{t}_{p}, by (i). Note then that the following diagrams commute:

$$\begin{tikzcd}
(x^{y}_{p})^{\perp} \times t \arrow[r, "eval"] & x^{y}_{p} \\
x^{y}_{p} \times t \arrow[u, dashed, "(\Lambda id)"] \arrow[ur, "id"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x^{y}_{p} \times y \arrow[rr, "eval"] & & x \\
(x^{y}_{p})^{\perp} \times y \arrow[u, "eval \times id_y"] \arrow[urr, dashed, "\Lambda^{-1} eval"] \arrow[rr, dashed, "(\Lambda(\Lambda^{-1} eval))\, id_y"'] & & x^{y}_{p} \times y \arrow[u, "eval"']
\end{tikzcd}$$

Finally set $in = \Lambda id: x_p^y \longrightarrow (x_p^y)^\perp$ and $out = \Lambda(\Lambda^{-1} eval): (x_p^y)^\perp \longrightarrow x_p^y$ □

2.8 Proposition. Let C be a pCCC, C_T CCC and x a partial object. Then, for any object y, one has:

(i) $x^y \lhd x_p^y$

(ii) $x_p^y \lhd x^y \times t_p^y$

Proof. (i) $x^y \lhd (x^\perp)^y \cong (x_p^t)^y \cong x_p^{t\times y} \cong x_p^y$, by 2.6 (i). (ii) $in_1 = out: x_p^y \longrightarrow x^y$, by (i); $in_2 = out: x_p^y \longrightarrow t_p^y$, by $t \lhd x$ (see 2.5). Moreover $out = \text{-}\hat{\ }\text{-}: x^y \times t_p^y \longrightarrow x_p^y$, by the extended adjunction as defined in 2.6. □

By 2.7 and 2.8, total and partial morphisms, as well as partial objects, are nicely related. In particular, when the target object is partial, partial morphisms do not change the higher type structure in an essential way. In contrast to this, when the target x is not partial, we only know that x^y is a subobject of x_p^y, while nothing can be said about higher types.

We conclude this section by returning to the categories we are interested in for the purposes of computability in abstract data types: domains and numbered sets.

A presentation of the CCC's of domains and effectively given domains, with continuous (and computable) maps as morphisms, may be found in Scott (1982) (see also Giannini & Longo (1983)). A constructive domain is (isomorphic to) the collection of all computable elements in an effectively given domain.

2.9 Generalized Myhill-Shepherdson Theorem (Ershov (1976)). The category of constructive domains is a full sub-CCC of EN.

Proof. (see Giannini & Longo (1983), say). □

We are now in the position to reword a simple result in Ershov (1973/5). A pCC is a partial Cartesian Category in the obvious way.

2.10 Proposition. EN_p is a pCC with a bottom functor.

EN is clearly not a full sub-category of EN_p. However, one may still naturally relate domains to EN_p by the following simple variant of 2.9. Note also that all now empty domains are partial objects.

2.11 Theorem. The category of constructive domains with strict maps is a full sub-pCCC of EN_p.

§.3. Relative numberings and principal morphisms in EN

3.1 Definition. Let $\underline{A},\underline{B}$ be objects in EN and $f,g:A \to B$. Define then

$$f \leq_A g \quad \text{if} \quad \exists h \in EN(\underline{A},\underline{A}) \ f = g \circ h$$

Note that, if $\underline{A} = \underline{B} = \underline{\omega}$, this is a classical notion of recursion-theoretic reducibility. Acceptable Gödel-numberings inspired 3.2.1.

3.2 Definition. Let $\underline{A}$ and $\underline{B}$ be in EN. Define then

1) $f \in EN(\underline{\omega},\underline{A})$ is an acceptable numbering of $\underline{A} = (A,e_A)$ if $e_A \leq_\omega f$.

2) $f \in EN(\underline{A},\underline{B})$ is a relative numbering of $\underline{B}$ w.r.t. $\underline{A}$ if $e_B \leq_\omega f \circ e_A$ (i.e. $f \circ e_A$) is an acceptable numbering of $\underline{B}$).

3) $f \in EN(\underline{A},\underline{B})$ is a principal morphism if $\forall h \in EN(\underline{A},\underline{B}) \ h \leq_A f$.

3.3 Remark. f $EN(\underline{\omega},\underline{A})$ and f $EN(\underline{A},\underline{B})$ in 3.2.1-2 are equivalent to $f \leq_\omega e_A$ and $f \circ e_A \leq_\omega e_B$. Principal morphisms may be easely generalized to arbitrary categories. In CCC's, principal morphisms characterize models of Combinatory Logic, see Longo & Moggi (1983b).

3.4 Remark. It is easy to prove that, if $f \in EN(\underline{A},\underline{B})$ is a relative numbering, then

$$\underline{A} \text{ (pre-)complete} \Rightarrow \underline{B} \text{ (pre-)complete},$$

see Longo & Moggi (1983).

3.5 Proposition. Let $f \in EN(\underline{A},\underline{B})$ be a relative numbering. Then one has

(i) for $h:B \to C$, $h \in EN(\underline{B},\underline{C})$ iff $h \circ f \in EN(\underline{A},\underline{C})$,

(ii) if $f \leq_A g \in EN(\underline{A},\underline{B})$, then also g is relative.

(Thus, in presence of a relative numbering, any principal morphism is a relative numbering too).

Proof. (i) $\Rightarrow$: obviuos. $\Leftarrow$: $f' \in R$ $h \circ e_B = h \circ f \circ e_A \circ f'$, for f is relative. Thus $\exists h'\ R\ \ h \circ e_B = e_C \circ h'$, by the assumtion.

(ii) $e_B = f \circ e_A \circ f'$, for some $f' \in R$ since f is relative,

$= g \circ l \circ e_A \circ f'$, for some $l \in EN(\underline{A},\underline{A})$ since $f \leq_A g$,

$= g \circ e_A \circ l' \circ f'$, for some $l' \in R$. □

In view of the strict limit on the number of pages imposed by the Publisher, from now on we are forced to skip the proofs. An elementary proof (i.e. with no category theory) of 3.11 may be found in Longo & Moggi (1983). The authors plan an expanded version of the present paper.

Write $\underline{A} \vartriangleleft_p \underline{B}$ (or $\underline{A} \vartriangleleft \underline{B}$) for $\underline{A}$ is a retraction of $\underline{B}$ in EN_p (or in EN).

3.6 Theorem. Let $\underline{\omega} \vartriangleleft_p \underline{A}$ and $\underline{B} \vartriangleleft B^{\perp}$. Then one has

(i) $\exists f \in EN(\underline{A},\underline{B})$ relative numbering,

(ii) if one also has $\underline{A} \times \underline{A} \vartriangleleft_p \underline{A}$ and $\underline{B}^{\underline{A}}$ exists, then $\exists g \in EN(\underline{A},\underline{B})$ principal.

The following Lemma shows how retractions are inherited at higher types.

3.7 Lemma. Assume that, for $\underline{A}$ and $\underline{B}$ in EN, $\underline{B}^{\underline{A}}$ exists. Then

(i) $\underline{\omega} \vartriangleleft_p \underline{B} (\vartriangleleft \underline{B}) \Rightarrow \omega \vartriangleleft_p \underline{B}^{\underline{A}} (\vartriangleleft \underline{B}^{\underline{A}})$,

(ii) $\underline{B} \vartriangleleft \underline{B}^{\perp} \Rightarrow \underline{B}^{\underline{A}} \vartriangleleft (\underline{B}^{\underline{A}})^{\perp}$,

(iii) $\underline{B} \times \underline{B} \vartriangleleft \underline{B} (\cong \underline{B}) \Rightarrow \underline{B}^{\underline{A}} \times \underline{B}^{\underline{A}} \vartriangleleft \underline{B}^{\underline{A}} (\cong \underline{B}^{\underline{A}})$.

The type structure of domains over ω in EN may be defined as follows.

3.8 Definition. Let T be the smallest set of finite types symbols containing 1 (i.e. $1 \in T$; $\sigma, \tau \in T \Rightarrow \sigma \times \tau, \sigma \to \tau \in T$.). Define then $E_c^1 = \omega_p^{\omega} \cong (\omega^{\perp})^{\omega}$, $E_c^{\sigma \times \tau} = E_c^{\sigma} \times E_c^{\tau}$ and $E_c^{\sigma \to \tau} = (E_c^{\tau})^{E_c^{\tau}}$.

Of course, $\{E_c^{\sigma} / \sigma\ T\}$ is the sub-CCC generated by $\omega_p^{\omega} = PR$ in EN. The subscript c recalls that each E_c^{σ} is actually a constructive domain, by 2.8. Thus all the numbered sets in the type structure, are actualy partial objects. By 2.4 and the results following it, the total maps in each $E_c^{\sigma\to\tau}$ may be rightfully considered as partial computable functionals.

3.9 Lemma. $\forall\sigma\in T$ $\underline{\omega} \triangleleft_p E_c^{\sigma}$, $E_c^{\sigma} \triangleleft (E_c^{\sigma})^{\perp}$ and $E_c^{\sigma} \times E_c^{\sigma} \triangleleft E_c^{\sigma}$, for all $n > 0$.

3.10 Theorem. $\forall\sigma,\tau$ T $\exists\phi_{\sigma,\tau}$ $EN(E_c^{\sigma},E_c^{\tau})$ principal morphism and relative numbering.

Proof. By 3.6 (ii), 3.9 and 2.8, $EN(E_c^{\sigma},E_c^{\tau})$ contains a principal morphism. Moreover, by 3.6 (i) and 3.9, it also contains a relative numbering. By 3.5 (ii) we are done. □

3.11 Remark. By this and by results in Longo & Moggi (1983b), each E_c yields a type-free Combinatory Algebra; actually a model of λ-calculus.

In view of all the information we have on numbered sets, we are now in the position to give the main theorem in Longo & Moggi (1983) as a simple corollary. This proves that the BM^n's and the HPEF (see §.1) give the integer types in the type structure over ω. The pure type variant of 3.12 is easely given.

3.12 Corollary. Let $L^1 = PR$. Then, for all $n \geq 0$,

n) $L^n = E_c^n$

n+1.5) $L^{n+1.5} = EN(E_c^n,E_c^{n+1})$.

Proof. (By induction) 0), 1) by definition

1.5) by a simple argument (show that $L^{1.5}$ contains all acceptable gödel-numberings of $\underline{PR}$).

n+2) $EN(E_c^{n+1},E_c^{n+1}) \subseteq L^{n+2}$, by definition and n+1.5.

Conversely, let $\phi_{n+1}\in EN(E_c^n,E_c^{n+1})$ as in 3.10. Then

$$\tau\in L^{n+2} \Rightarrow \tau\circ\phi_{n+1}\ EN(E_c^n,E_c^{n+1}) \quad \text{by } n+1.5),$$
$$\Rightarrow \tau\in EN(E_c^{n+1},E_c^{n+1}) \quad \text{by } 3.5\ (i).$$

n+2.5) $E_c^{n+1} \times E_c^{n+1} \triangleleft E_c^{n+1}$, via $(<,>,(p_0,p_1))$, by 3.9. The pairing is clearly acceptable w.r.t. E_c^{n+2}, in the sense of definition 1.2; hence

$$\phi \in L^{n+2.5} \Rightarrow g = \lambda x.\phi(p_0(x))(p_1(x)) \in E_c^{n+2}$$
$$\Rightarrow \phi = \lambda x.(\lambda y.g(<x,y>)) \in EN(E_c^{n+1},E_c^{n+2})$$

Conversely,

$$\phi \in EN(E_c^{n+1},E_c^{n+2}) \Rightarrow f = \lambda xy.\phi(x)(y)\ EN(E_c^{n+1} \times E_c^{n+1},E_c^{n+1})$$
$$\Rightarrow \lambda x.\phi(p_0(x))(p_1(x)) = f\circ(p_0\hat{\ }p_1)\ EN(E_c^{n+1},E_c^{n+1}) = L^{n+1}. \qquad \Box$$

3.13 <u>Remark</u>. The key issue in this part has been the study of partial morphisms and objects in EN and the related sub-CCC's. Note that, in precomplete numbered sets, partial morphisms may be always extended to total ones. As for complete numbered sets one can say more, in view of 2.4: with some work, it may be actually shown that complete numbered sets and partial objects concide in EN.

(For references, see end of part II).

CARTESIAN CLOSED CATEGORIES OF ENUMERATIONS AND PARTIAL MORPHISM FOR EFFECTIVE TYPE STRUCTURES

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PART II

§.1. Introduction. The type structure $\{L^n\}_{n\in\omega}$ studied in §.3 of part I (i.e. with $L^1 = PR$) actually gives the partial computable functionals (see Ershov (1975)) in the integer types. The key fact was the possibility of enumerating each type $n+1$ by type n, via a principal relative numbering. This generalizes the fact that PR, i.e. L^1, can be effectively numbered by ω, i.e. L^0.

If one takes $L^1 = R$ (the total recursive maps) this is no longer possible, i.e. there is no effective numbering of R by ω, therefore $\{L^n\}_{n\in\omega}$ (with $L^1 = R$) is not representable in EN.

As pointed out in §.1 of part I, the definition of HPEF is rather general, and still works if we take as L^1 a set L of partial maps from ω to ω (instead of PR or R).

We give a characterization of $\{L^n\}_{n\in\omega}$, for L^1 enumeration-acceptable (see 1.1 below), in terms of a concrete CCC (and pCCC) based on the notion of numbering:

1.1 Definition. Let $L \subseteq \omega \to \omega$, then L is enumeration-acceptable iff:

1) $L \circ L \subseteq L$, $id \in L$;

1') $0 \in L$ (0 is the everywhere divergent function);

2) $\forall n\in\omega\ \lambda x.n \in L$;

3) there is an acceptable pairing of ω w.r.t. L (see 1.2 of part I);

4) equality in ω is decidable w.r.t. L, i.e.

$\forall f,g\in L\ \exists h\in L\ h(\langle x,y,z\rangle) = \text{if } (x = y) \text{ then } fz \text{ else } gz$ $\square$

1.2 Definition. The category of numbered sets on L (EN_p^L) is defined by:

i) $\underline{A} = (A, e_A) \in EN_p^L$ iff $e_A : \omega \to A$ is onto;

ii) $f \in EN_p^L(\underline{A}, \underline{B})$ iff $f : A \to B$ and $\exists g \in L \; f \circ e_A = e_B \circ g$

Let L be enumeration-acceptable and 1)-4) be the assumptions on L in (1.1), then one easely has:

1) implies that EN_p^L is a category;

1') implies that EN_p^L has null morphisms;

2) implies that EN_p^L has a singleton object;

2) and 3) imply that EN_p^L is cartesian.

Remark. Note that the notion of enumeration-acceptable class of function is also a sound recursion-theoretic generalization of basic properties of PR. As a matter of fact, if $(\omega, \cdot)$ is a Uniformely Reflexive Structure, then $(\omega \to \omega) = \{f : \omega \to \omega / \exists a \in \omega \; \forall b \in \omega \; f(b) = a \cdot b\}$ is enumeration acceptable EN_p^L is not a pCCC, hence the type structure generated from ω does not need to exist in it.

However every category C may be embedded in the category of presheaves on C, $Set^{C^{op}}$ (which is a CCC), by a full and faithful functor, which preserves products and representations of morphisms (see Scott (1980), McLane (1971)). We will define a full sub-CCC (GEN^L) of $Set^{(EN^L)^{op}}$ with the following property:

the embedding functor of EN^L in $Set^{(EN^L)^{op}}$ (I_{EN}) factorizes through that of GEN^L in $Set^{(EN^L)^{op}}$ (I_{GEN})

i.e.: $\exists I$

$$\begin{tikzcd}
EN^L \arrow[rr, dashed, "I"] \arrow[drr, "I_{EN}"'] & & GEN^L \arrow[d, "I_{GEN}"] \\
& & Set^{(EN^L)^{op}}
\end{tikzcd}$$

1.3 Definition. Let L be enumeration-acceptable, define then the category of generalized numbered sets on L by:

i) $\underline{X} = (X, E_x) \in GEN_p^L$ iff

1) $E_x \subseteq \omega \longrightarrow X$,

2) $\bigcup\{\mathrm{img}\ e \mid e \in E_x\} = X$,

3) $\forall e_0, e_1 \in E_x\ \exists e \in E_x\ \ \exists f_0, f_1 \in L$ s.t.

$$\begin{tikzcd} & X & \\ & & \\ \omega \arrow[uur, "e_0"] \arrow[r, dashed, "f_0"'] & \omega' \arrow[uu, dashed, "e"] & \omega \arrow[uul, "e_1"'] \arrow[l, dashed, "f_1"] \end{tikzcd}$$

(i.e. E_x is a directed sets w.r.t. L-reducibility);

ii) $f \in GEN^L_p(\underline{X}, \underline{Y})$ iff $f: X \longrightarrow Y$ and

$\forall e \in E_x\ \exists g \in L\ \exists e' \in E_Y\ \ f \circ e = e' \circ g$, i.e.

$$\begin{tikzcd} \omega \arrow[r, dashed, "g"] \arrow[d, "e"'] & \omega \arrow[d, dashed, "e'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(Intuition: one cannot gödelize all of R, but one can effectively enumerate it piecewise).

<u>Notation</u>. EN^L and GEN^L are the categories of total morphisms.

<u>Lemma</u>. GEN^L has coproducts.

<u>hint</u>: $\underline{X} \sqcup \underline{Y} = (X \sqcup Y, \{e \sqcup e' \mid e\ E_X \wedge e'\ E_Y\})$,
where $(e \sqcup e')(\langle x,y,z\rangle) \equiv$ if $(x = y)$ then ez else $e'z$, i.e. it is the sup of e, e' w.r.t. L-reducibility. □

1.4 <u>Theorem</u>. i) GEN^L is a CCC and

ii) GEN^L_p is a pCCC.

<u>hint</u>: $\underline{X} \times \underline{Y} = (X \times Y, \{e\hat{\ }e' \mid e\ E_X \wedge e'\ E_Y\})$;

$$\begin{tikzcd} X & X \times Y \arrow[l, "\Pi_X"'] \arrow[r, "\Pi_Y"] & Y \\ & & \\ & \omega \arrow[uul, "e"] \arrow[uu, dashed, "e\hat{\ }e'"] \arrow[uur, "e'"'] & \end{tikzcd}$$

use 3) and 4) (in Definition 1.1) for proving that $E_{X\times Y}$ is directed; let $\underline{\omega} = (\omega, \{id\})$ GEN^L, then

i) $\underline{Y}^{\underline{X}} = (GEN^L(\underline{X},\underline{Y}), \Lambda(GEN^L(\underline{\omega} \times \underline{X},\underline{Y})))$;

ii) $\underline{Y}^{\underline{X}}_p = (GEN^L_p(\underline{X},\underline{Y}), \Lambda(GEN^L(\underline{\omega} \times \underline{X},\underline{Y})))$

where Λ is the curry operator on maps, use 3) and 4) (in 1.1) for proving that $\underline{\omega} \sqcup \underline{\omega} \lhd \underline{\omega}$ and $(\underline{\omega} \times \underline{X}) \sqcup (\underline{\omega} \times \underline{X}) \simeq (\underline{\omega} \sqcup \underline{\omega}) \times \underline{X}$, then it follows easely that E_{Y^X} is directed.

<u>Remark</u>. In general, if C_p is pCCC, it does not follow that C is CCC (the problem are objects s.t. $Y \not\lhd Y^{\perp}$).

1.5 <u>Definition</u>. The embedding functor, I, of EN^L_p into GEN^L_p is defined by:

i) $I(A,e_A) = (A,\{e_A\})$ on objects and

ii) I is the identity on maps □

The properties of I are summarized by theorem 1.7 below.

1.6 <u>Lemma</u>. Let $f:X \longrightarrow Y$, then

$f \in GEN^L_p(\underline{X},\underline{Y}) \Longleftrightarrow f \circ GEN^L(\underline{\omega},\underline{X}) \subseteq GEN^L_p(\underline{\omega},\underline{Y})$,

$f \in EN^L_p(\underline{X},\underline{Y}) \Longleftrightarrow f \circ EN^L(\underline{\omega},\underline{X}) \subseteq EN^L_p(\underline{\omega},\underline{Y})$ □

1.7 <u>Theorem</u>. i) I is full and faithful,

ii) I preserves products and representations of total and partial morphisms. □

The main reason, for using generalized numbered sets instead of presheaves, is that the former are more similar to numbered sets than the latter, thus we can easely extend meaningful concepts from EN^L to GEN^L (such as the notion of partial morphism and relative numbering), whereas this seems impossible for presheaves.

§.2 <u>HPEF and generalized numbered sets</u>.

This section is devoted to the characterization of the generalized

HPEF $\{L^n\}_{n\in\omega}$, in the integer (or pure) types (see §.1 of part I), with the corresponding type structure in GEN^L_p. For L is an arbitrary enumeration-acceptable function set, the full generality of GEN^L_p is required.

The main step is to find the right counterpart to the notion of relative numbering given in EN (see 3.2 part I).

2.1 Definition. $\underline{X}$ factorizes $\underline{Y}$ iff

$$GEN^L(\underline{\omega},\underline{Y}) = GEN^L(\underline{X},\underline{Y}) \circ GEN^L(\underline{\omega},\underline{X}) \qquad \square$$

or equivalently $\forall e' \in E_Y \;\; \exists e \in E_X \;\; \exists f\; GEN^L(\underline{X},\underline{Y})$ s.t.

$$\begin{tikzcd} \omega \arrow[d, dashed, "e"'] \arrow[dr, "e'"] & \\ X \arrow[r, dashed, "f"'] & Y \end{tikzcd}$$

Remark. Let $\underline{A},\underline{B} \in EN^L$, then
$\exists f \in EN^L(\underline{A},\underline{B})$ relative numbering $\Longleftrightarrow$ $I\underline{A}$ factorizes $I\underline{B}$.

2.2 Proposition. If $\underline{X}$ factorizes $\underline{Y}$ and $f:Y \quad Z$, then
$f \in GEN^L(Y,\underline{Z}) \Longleftrightarrow f \circ GEN^L(\underline{X},\underline{Y}) \subseteq GEN^L(\underline{X},\underline{Z})$. (see 3.5 part I)

2.3 Theorem. Let $\underline{\omega} \triangleleft_p \underline{X}$ and $\underline{Y} \triangleleft \underline{Y}^{\perp}$, the $\underline{X}$ factorizes $\underline{Y}$.
(see 3.6 part I) $\square$

The integer type structure in GEN^L_p is defined as in (3.8 part I):

2.4 Definition. $E^0_L = \underline{\omega}, E^1_L = \underline{\omega^{\omega}_{p}}, E^{n+1}_L = E_L^{nE^n_L}$ $\square$

We have that:

2.5 Lemma. $\underline{\omega} \triangleleft_p E^n_L, E^{n+1}_L \triangleleft (E^{n+1}_L), E^n_L \times E^n_L \triangleleft E^n_L$.

hint: use $\underline{\omega} \sqcup \underline{\omega} \triangleleft \underline{\omega}$ for proving that $E^L_1 \times E^L_1 \triangleleft E^L_1$, where

$$\begin{tikzcd} \underline{\omega} \arrow[r, "in_0"] \arrow[dr, "f"'] & \underline{\omega} \arrow[d, dashed, "<f,g>"] & \underline{\omega} \arrow[l, "in_1"'] \arrow[dl, dashed, "g"] \\ & \underline{\omega} & \end{tikzcd}$$

then it follows by induction that $\forall n > 0 \;\; E^L_n \times E^L_n \triangleleft E^L_n$ $\square$

From (2.5) and (2.3) it follows

2.6 Theorem. E_L^n factorizes E_L^{n+1} for $n \geq 0$ □

2.7 Theorem. Let L be enumeration-acceptable, then $\{L^n\}_{n \in \omega}$ is defined and for all $n \geq 0$:
n) $L^n = E_L^n$, n+1.5) $L^{n+1.5} = GEN^L(E_L^n, E_L^{n+1})$.

hint: (see also (3.12) in part I)
0), 1) by definition,
n+1.5) follows from n), n+1) and $E_L^n \times E_L^n \lhd E_L^n$,
n+2) follows from n+1.5, (2.6) and (2.2) □

The existence of principal morphisms in $L^{n+1.5}$ does not follow from (2.6) (compare to 3.10 in part I), in fact it requires stronger hypotheses:

2.8 Theorem. $\forall n \geq 0$ there is a principal morphism in $L^{n+1.5} \Longleftrightarrow \forall n \geq 0$ E_L^n is representable in EN_L (i.e. E_L^n is the image (w.r.t. I) of a numbered set).

§.3 Generalized numbered sets and presheaves.

At last we return of the relations between GEN^L and $Set^{(EN^L)^{op}}$. First let us define the embedding functors I_{EN} and I_{GEN}.

3.1 Definition. i) $I_{EN} = \lambda \underline{A}.\lambda \underline{B}.EN^L(\underline{B},\underline{A})$ is the usual Yoneda embedding of EN^L in $Set^{(EN^L)^{op}}$,
ii) $I_{GEN} = \underline{X}.\lambda \underline{B}\ GEN^L(I\underline{B},\underline{X}) : GEN^L \to Set^{(EN^L)^{op}}$ □

3.2 Theorem. i) I_{EN} and I_{GEN} are full and faithful,
ii) preserve products and representations of morphisms,
iii) $I_{EN} = I_{GEN} \circ I$ □

(2.7) may be restated, using presheaves only, as follows.

3.3 Theorem. Let L be enumeration-acceptable, then there exist two presheaves F and G such that:
n) $L^n = E_L^n$, n+1.5) $L^{n+1.5} = E_L^{n+1\, E_L^n}$, where $E_L^0 = F$, $E_L^1 = G^F$ and $E_L^{n+1} = E_L^{n\, E_L^n}$ □

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