VIMADES

Viabilité, Marchés, Automatique et Décision

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Les microsystèmes que cèlent les équations d'Hamilton-Jacobi

Séminaire Philosophie et Mathématiques
École Normale Supérieure

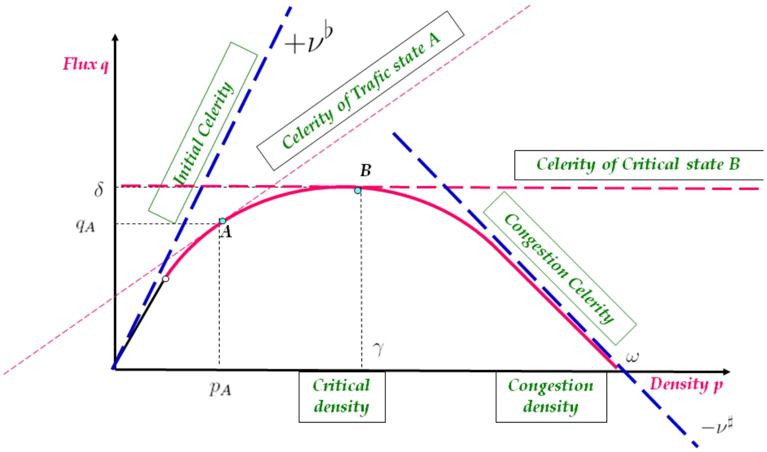
« Le meilleur des mondes possibles du finalisme au calcul des variations »

7 février 2011

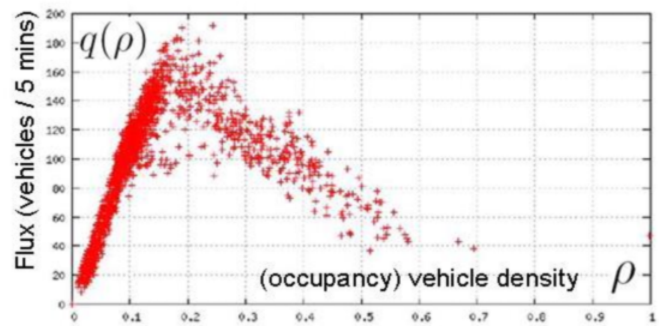
1 Introduction

2 The Best of the Worlds

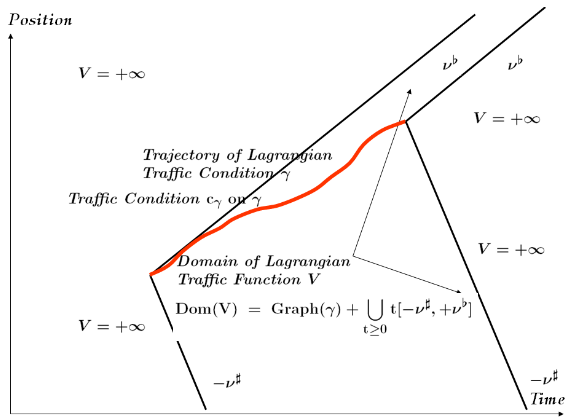
2.1 Macroscopic Traffic Management



Fundamental Diagram and Partial Differential Equation



$$\mathbf{h}\left(-\frac{\partial V(t,x)}{\partial x}\right) = \frac{\partial V(t,x)}{\partial t}$$



Traffic conditions : $\mathbf{c} : (t,x) \mapsto \mathbf{c}(t,x) \in \mathbb{R} \cup \{+\infty\}$**,**
for instance

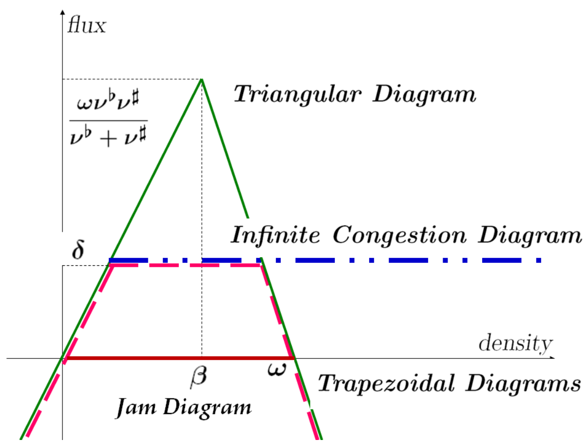
$$\mathbf{c}(t,\gamma(t)) \;:=\; \mathbf{c}(d,\gamma(d)) + \int_d^t \mathbf{l}(\gamma'(\tau)d\tau)$$

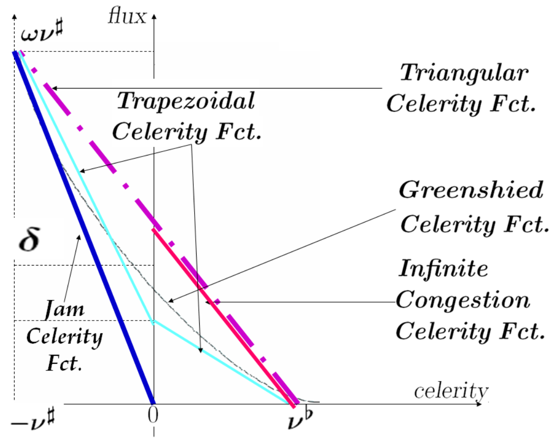
Traffic function V **:**

$$\left\{\begin{array}{ll} (i) & \forall\, t > 0, \forall\, x \;\neq\; \gamma(t), \;\; \mathbf{h}\left(-\dfrac{\partial V(t,x)}{\partial x}\right) \;=\; \dfrac{\partial V(t,x)}{\partial t} \\ (ii) & \forall\, t > 0,, \;\; V(t,\gamma(t)) \;=\; \mathbf{c}(t,\gamma(t)) \end{array}\right. \tag{1}$$

From Density p to Hidden Celerity u (Legendre-Fenchel Transform)

$$\left\{\begin{array}{lll}\forall\, u, & \mathbf{l}(u) & := \sup_{p\in \mathrm{Dom}(\mathbf{h})}[\mathbf{h}(p)-\langle p,u\rangle] \\ \forall\, p, & \mathbf{h}(p) & = \inf_{u\in \mathrm{Dom}(\mathbf{l})}[\mathbf{l}(u)+\langle p,u\rangle]\end{array}\right. \tag{2}$$





2.3 The Hidden Seer behind the Variational Principle

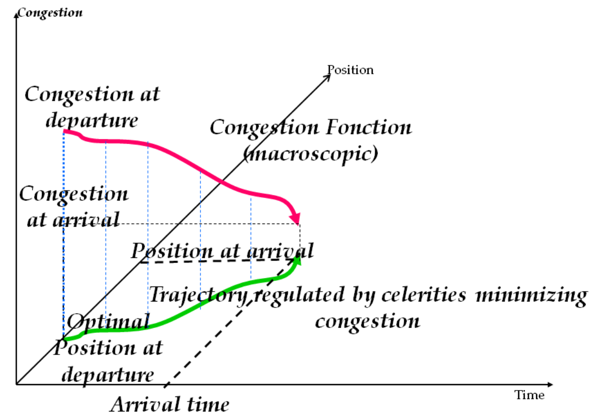
We regard $d \in [0,T]$ as a *departure time*, the associated *travel time* being equal to $s := T-d$. We consider the family $\mathcal{A}(d,T;x)$ of absolutely continuous traffic evolutions $\xi(\cdot)$ starting at departure time d at $\xi(d) \in \mathbf{C}(d)$ and arriving at time T at x. We assign to such a traffic evolution "traffic value" :

$$J(d,T;x) \ := \inf_{\xi(\cdot)\in\mathcal{A}(d,T;x)} \left(\int_d^T \mathbf{l}(\xi'(\tau))d\tau + \mathbf{c}(d,\xi(d)) \right)$$

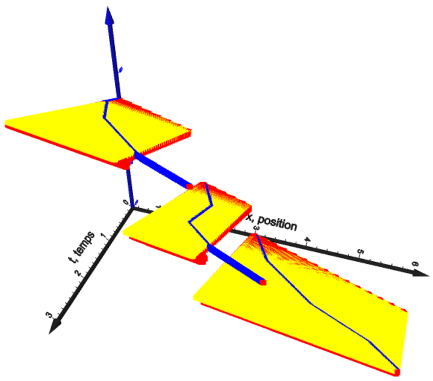
Variational Principle

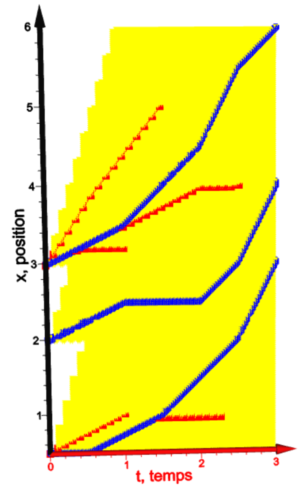
$$\left\{ \begin{array}{l} V(T,x) \ = \ \inf_{d\in[0,T]} J(d,T;x) \\ = \ \inf_{d\in[0,T]} \inf_{\xi(\cdot)\in\mathcal{A}(d,T;x)} \left(\int_d^T \mathbf{l}(\xi'(\tau))d\tau + \mathbf{c}(d,\xi(d)) \right) \end{array} \right. \quad (3)$$

2.4 Microscopic Traffic Management

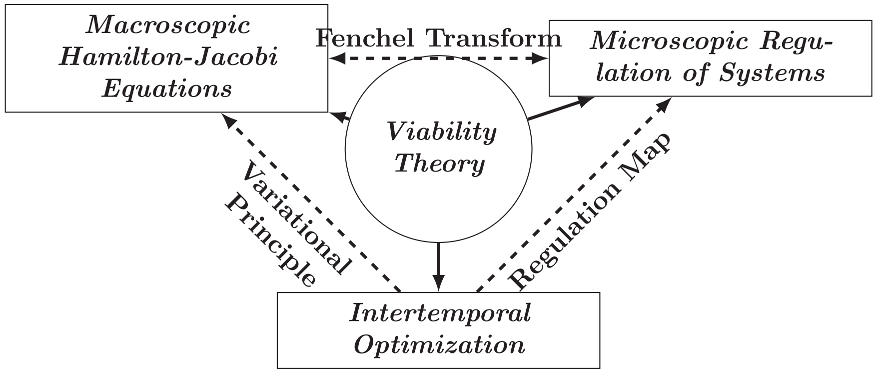


1 ***Retroaction Map.*** ***Viable (and thus) optimal evolutions are governed by a differential equation*** $\xi'(t) = r(t, \xi(t))$ ***where*** $\xi'(t) = u$ ***is the*** **advised celerity** ***at time*** t ***and position*** $\xi(t) = x$ ***for providing the velocity of the vehicle for minimizing congestion.*** ***The*** **retroaction map** $r : (t, x) \mapsto r(t, x)$ ***is computed from the Hamiltonian*** h ***and the viability solution.***





Traffic Function (Macroscopic) and Optimal Evolutions (Microscopic)
From Anya Désilles



The macroscopic approach through first-order partial differential equations, the microscopic version dealing with the regulation of an underlying control system and the intertemporal optimization problem are equivalent : ***Viability theory implies that the viability solution (defined below) solves these three problems at once*.**

3 From Frustrations to Motivations

The Death of the Seer, the Rise of the Demiurge

Intertemporal optimization requires

1. **the existence of an actor (agent, decision-maker, controller, etc., called the "*seer*"),**
2. **an optimality criterion,**
3. **that decisions are taken once and for all at the initial time,**
4. **a knowledge of the future (or of its anticipation).**

Paul Valéry (1871-1945) : Forecasting is a dream from which event wakes us up. (*La prévision est un rêve duquel l'événement nous tire.*)

Mathematical metaphors in life sciences and illustrations.

Viability theory started in 1976 by translating mathematically the title

$$\begin{array}{ccc} \textbf{\textit{Chance}} & \textbf{\textit{and}} & \textbf{\textit{Necessity}} \\ \Updownarrow & & \Updownarrow \\ x'(t) \in F(x(t)) & \& & x(t) \in K \end{array}$$

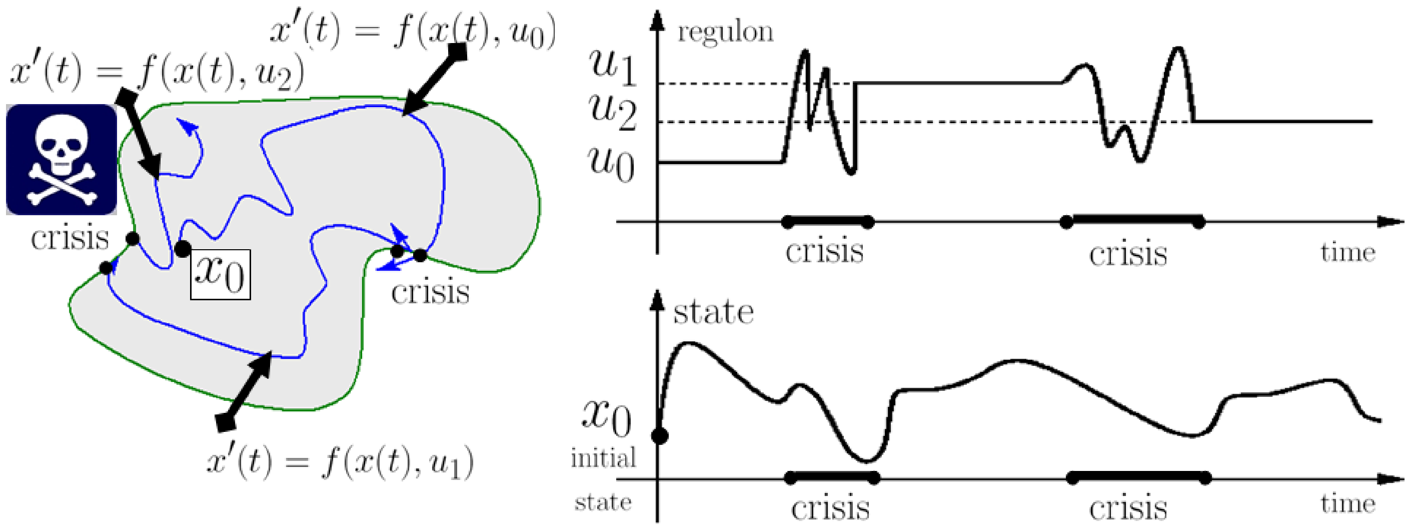
of the famous 1973 book by *Jacques Monod, Chance and Necessity*, taken from an (apocryphical ?) quotation of Democritus who held that *"the whole universe is but the fruit of two qualities, chance and necessity"*.

3.1 The Rise of the Demiurge : The Inertia Principle

After all, in biological evolution, intertemporal optimization can be traced back to Sumerian mythology which is at the origin of Genesis : one Decision-Maker, deciding what is good and bad and choosing the best (fortunately, on an intertemporal basis with infinite horizon, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week...

Darwin **(1809-1882) who had been working on his celebrated** ***Origin of Species*** **(1859) since 1844. Selection by viability and not by intertemporal optimization motivated viability theory.**

This is the reason why we shall assume that, in living system, there is no identified actor governing the evolution of the regulons, or that, if such an actor exists, he is ***myopic, explorer but lazy, opportunistic but conservative*****, and that the evolution of the controls, regarded in this case as regulons, obey the** ***inertia principle*****. This "actor" is called the "demiurge".**



$x'(t) = f(x(t), u_0)$
$x'(t) = f(x(t), u_2)$
crisis
x_0
crisis
$x'(t) = f(x(t), u_1)$
regulon
u_1
u_2
u_0
crisis
crisis
time
state
x_0
initial
state
crisis
crisis
time

Heavy Evolutions

We may assume instead that regulons evolve as "slowly" as possible because the change of regulons (or controls in engineering) is costly, even very costly. Hence we are led to assume that the regulons are constrained by some *inertia threshold* that can be estimated through some measure of their *velocities*.

Since evolutions under constant coefficients may not satisfy required properties, such as viability, capturability or optimality, the question arises to study *when, where and how* coefficients must cease to be constant and start to "evolve" in order to guarantee the viability property, for instance.

3.3 Hard and Soft Inertia Principles

The inertia principle edicts that are changed when the viability is at stakes.

1. **The *hard* version of the inertia principle requires that whenever the evolution reaches the boundary, then, and not before, the state has to switched instantaneously to a *new initial state* and a *myopic, explorer but lazy, opportunistic but conservative* new feedback has to be chosen,**
2. **The *soft* version of the inertia principle involves an inertia threshold determining *when*, at the right time, the *kairos*, *where*, in the critical zone, the regulon only has to *evolve and how*.**

3.4 Warning Time or “Kairos”

The concept of *warning time* is a mathematical translation of the anglo-saxon concept of *timing*, or the Italian concept of *tempismo*, modernizing the concept of *kairos* of classical Greece, meaning *propitious* or *opportune* moment. The ancient Greeks used this *qualitative* concept of time by opposition to *chronos*, the *quantitative*... chronological time, which can be measured by clocks.

2 ***Kairos*** ***Who and whence was the sculptor ? From Sikyon. And his name ? Lysippos. And who are you ? Time who subdues all things. Why do you stand on tip-toe ? I am ever running. And why you have a pair of wings on your feet ? I fly with the wind. And why do you hold a razor in your right hand ? As a sign to men that I am sharper than any sharp edge. And why does your hair hang over your face ? For him who meets me to take me by the forelock. And why, in Heaven's name, is the back of your head bald ? Because none whom I have once raced by on my winged feet will now, though he wishes it sore, take hold of me from behind. Why did the artist fashion you ? For your sake, stranger, and he set me up in the porch as a lesson.*** **Posidippos**

We leave the choice of the growth rates open, regarding them as *regulons* (regulation parameters) of the regulated system

$$x'(t) = u(t)x(t) \tag{4}$$

1. ***viability constraints* :**

$$\forall\, t \geq 0, \;\; x(t) \;\in\; [a, b]$$

2. ***inertia thresholds* imposing a *speed limit* on the evolutions of the regulons :**

$$\forall\, t \geq 0, \;\; u'(t) \;\in\; [-c, +c]$$

$\mathcal{P}(x,u)$ **is the set of solutions** $(x(\cdot),u(\cdot))$ **to system (4)** ***viable*** **in the interval** $[a,b]$ **starting at** (x,u)**.**

The ***inertia function*** **is defined by**

$$\alpha(x,u) := \inf_{(x(\cdot),u(\cdot))\in\mathcal{P}(x,u)} \sup_{t\geq 0} |u'(t)|$$

The inertia function is proved to be the ***viability solution*** **to the Hamilton-Jacobi-Bellman partial differential equation**

$$\begin{cases} \dfrac{\partial\alpha(x,u)}{\partial x}u - \alpha(x,u)\dfrac{\partial\alpha(x,u)}{\partial u} & \textbf{if } \; u \geq 0 \\ \dfrac{\partial\alpha(x,u)}{\partial x}u + \alpha(x,u)\dfrac{\partial\alpha(x,u)}{\partial u} & \textbf{if } \; u \leq 0 \end{cases}$$

The *metasystem* associated with initial control system is the auxiliary system

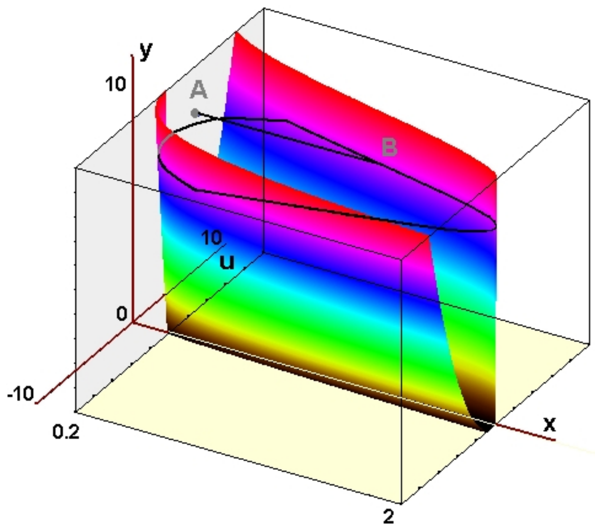
$$\left\{\begin{array}{lrcl} (i) & x'(t) & = & u(t)x(t) \\ (ii) & u'(t) & = & v(t) \\ (iii) & y'(t) & = & 0 \\ & \text{where } \|v(t)\| & \leq & y(t) \end{array}\right. \quad (5)$$

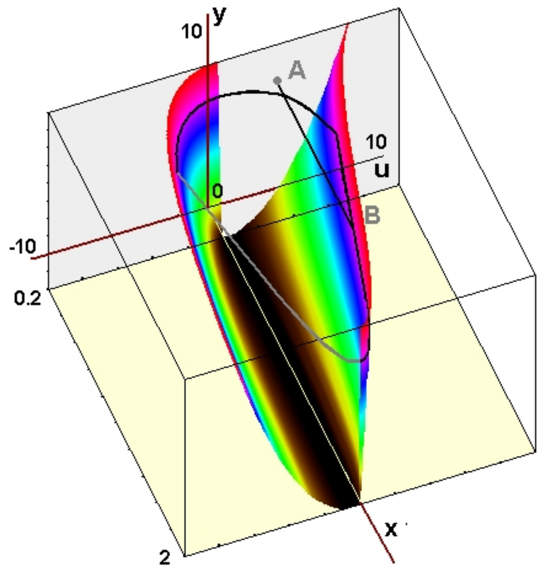
It is regulated by the velocities $v(t) = u'(t)$ of the controls of initial system, called *metacontrols*.

The inertia function is related to the viability kernel of $[a, b] \times \mathbb{R} \times \mathbb{R}_+$ under metasystem (5) by formula

$$\alpha(x, u) = \inf_{(x,u,y) \in \text{Viab}_{(5)}([a,b] \times \mathbb{R} \times \mathbb{R}_+)} y$$

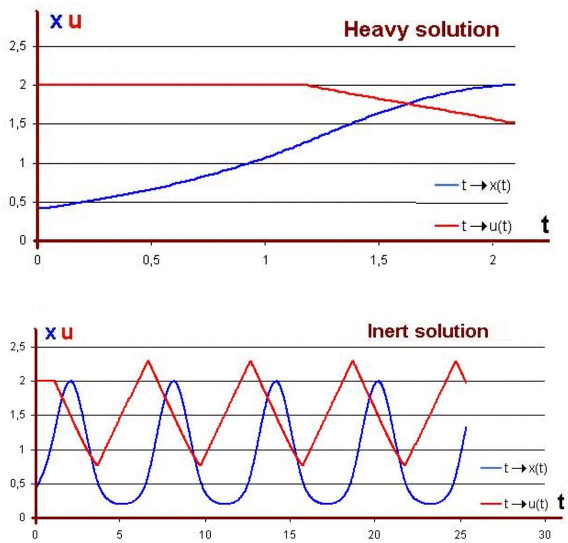
Graph of the Inertia Function

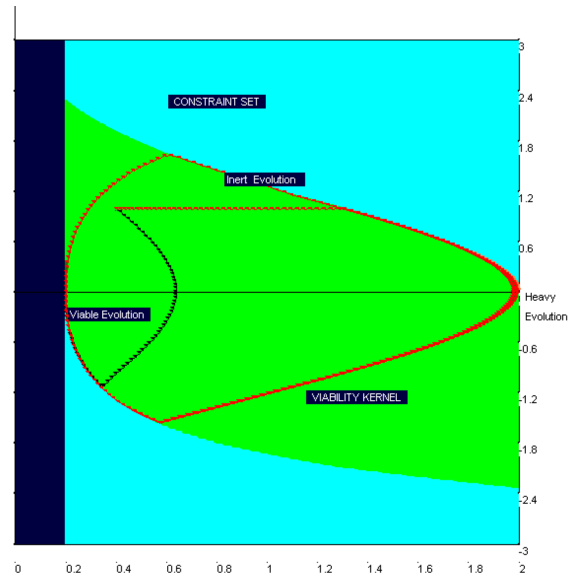




From Patrick Saint-Pierre

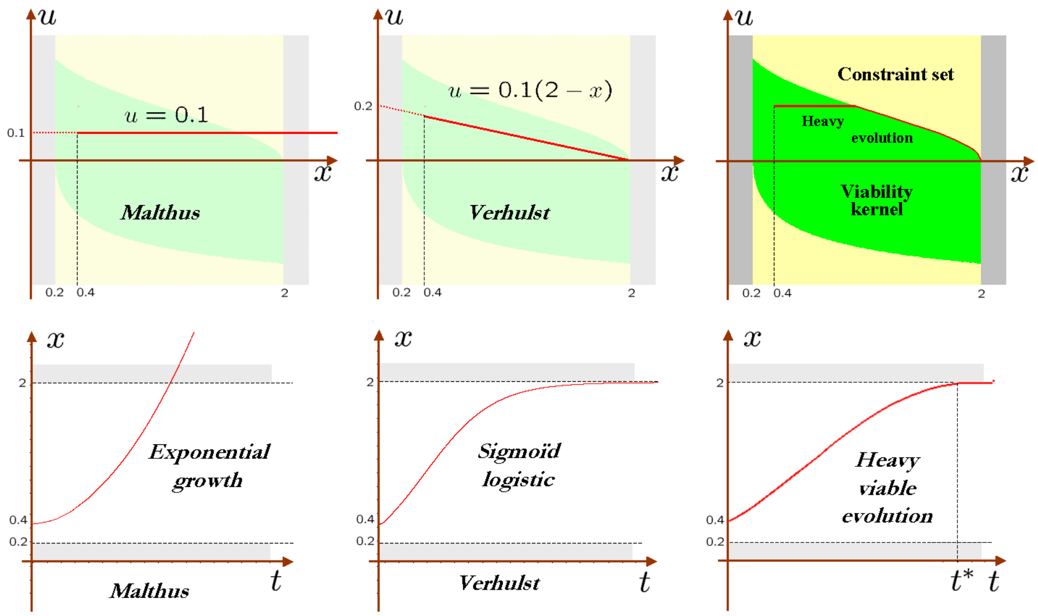
3.8 Heavy and Cyclic Evolutions ; Hysteresis





From Patrick Saint-Pierre

This very simple mathematical metaphor implies that two excitatory/inhibitory simple mechanism of a DNA site with bounds on the quantities and their accelerations are sufficient to explain the production of an isolated protein increasing up to a given viability bound and then, decreasing to disappear and being produced again according to a clock, the cyclicity of which is concealed in this very simple viability oscillator, triggering a biological clock.



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4 Uncertainties on Uncertainty

4.1 Tychastic Regulated Systems

Parameters in differential equations participate in different ways to the general concept of *uncertainty*. A given system can involve only controls, and is called a *controlled system*, or on regulons, and is called a *regulated system* or on tyches, and is called a *tychastic system*. It also can involve two or three of these parameters : for instance, if it involves controls and tyches, it is called a *tychastic controlled system*, and, in the case of regulons and tyches, or a *tychastic regulated system*.

Examples of States and Regulons.

Field	*State*	*Regulon*	*Viability*	*Actors*
Economics	physical goods	fiduciary goods	economic scarcity	economic agents
Genetics	phenotype	genotype	viability or homeostasis	bio-mechanical metabolism
sociology	psychological state	cultural codes	sociability	individual actors
cognitive sciences	sensorimotor states	conceptual codes	adaptiveness	organims

In living system, there is no identified actor governing the evolution of the regulons, or that, if such an actor exists, he is *myopic, explorer but lazy, opportunistic but conservative*

Tyche. **Uncertainty without statistical regularity can be translated mathematically by parameters on which actors, agents, decision makers, etc. have no controls. These parameters are often perturbations, disturbances (as in "robust control" or "differential games against nature") or more generally, *tyches* (meaning "chance" in classical Greek, from the Goddess Tyche) ranging over a state-dependent *tychastic map*. They could have be called "random variables" if this terminology were not already preempted in probability.**

Stochastic Uncertainty

Stochastic uncertainty is described by a space Ω, a filtration $\mathcal{F}$, a Brownian process $W(t)$, a drift $\gamma(x)$ and a volatility $\sigma(x)$: $dR(t) = \gamma(x(t))dt + \sigma(x(t))dW(t)$.

1. **The random events are not explicitly identified. One can always choose the space Ω of all evolutions. Only the drift and volatility are known ;**
2. **Stochastic uncertainty does not study the "package of evolutions" (depending on $\omega \in \Omega$), but *functionals over this package*, such as the different moments and their statistical consequences (averages, variance, etc.) used as evaluation of risk ;**
3. **Required properties are valid for "almost all" *constant* ω.**

Tychastic Uncertainty

Tyches **are parameters on which neither the Seer nor the Demiurge has any influence. The uncertainty is described by the tychastic map** $t \rightsquigarrow \mathcal{R}(t)$ **(instead of assuming that** $dR(t) = \gamma(x(t))dt + \sigma(x(t))dW(t)$**).**

1. **Tyche are usually identified (returns of a risky asset, as in finance) which can then *be used* when they are actually observed and known at each date during the evolution ;**
2. **For this reason, the results are computed in the worst case (*eradication of risk* instead of its *statistical evaluation*) ;**
3. **Required properties are valid for *"all" evolutions* of tyches** $t \mapsto R(t) \in \mathcal{R}(t)$**.**

The graph of the tychastic map can be assumed to be a fuzzy set (fuzzy tychastic uncertainty).

Western etymology

Stochastic refers to the Greek *stokhastikos* designating the draw of the rulers of the Athenian democracy. Random comes from the French verb "randon", from the verb *"randir"*, sharing the same root than the English "to run" and the German "rennen".

We borrow from Charles Peirce the use of tychastic evolution he introduced in 1893 in an article nicely entitled Evolutionary Love for describing the evolution of a system dependent on tyches arising without any statistical regularity.

The word tychastic comes from the Greek tyche , meaning luck, represented by the goddess Tyche , daughter of Oceanus and Tethys, whose goal was to disrupt the course of events either for good or for bad. Tyche became "Fortuna"in Latin, "rizikon" in Byzantine Greek, "rizq" in Arabic (with a positive connotation in these three cases).

Chinese etymology

The four ideograms follow, opportunity, reaction, change

随 机 应 变

are combined to express in Chinese :

1. **by the first half, "follow, opportunity", 随 机 , the concept of randomness or stochasticity,**
2. **while by the second half, "reaction, change", 应 变 , translates the concept of tychasticity (according to Shi Shuzhong)**
3. **and "no, necessary", 未 定 , translates contingent.**

Contingent Uncertainty

In fact, the guaranteed viability kernel decreases when the "tychastic reservoir increases.

The question arises whether contingent uncertainty "offsets" tychastic uncertainty.

The word contingent comes from the Latin verb contingere, to arrive unexpectedly or accidentally. Leibniz : "Contingency is a non-necessity, a characteristic attribute of freedom."

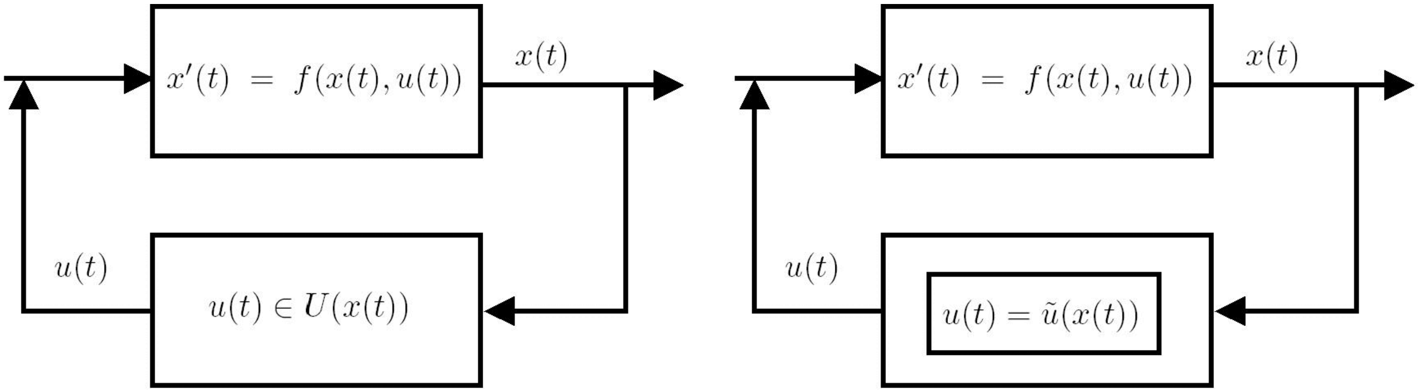
Impulse uncertainty

Since it may be difficult to determine the tychastic map $t \rightsquigarrow \mathcal{R}(t)$, the question arises to address the inverse problem. Instead of computing the minimum guaranteed viability kernel, we assume instead known a *reset map* $\Phi : K \rightsquigarrow K$, the domain $G := \mathrm{Dom}(\Phi) \subset K$ is the forbidden zone. It is the reservoir of *impulse feedbacks*. The *impulse evolutionary system* governs continuous time evolutions until the state reaches this forbidden set, and, then, the reset map uses a feedback to map the reset by an *impulse* (infinite velocity) the state in K from which the evolution $x(\cdot)$ continues to evolve.

Viability theory derives the tychastic map under which one can find a regulation map under such that the environment is viable.

Uncertainty can be insured or guaranteed by two inverse approaches :

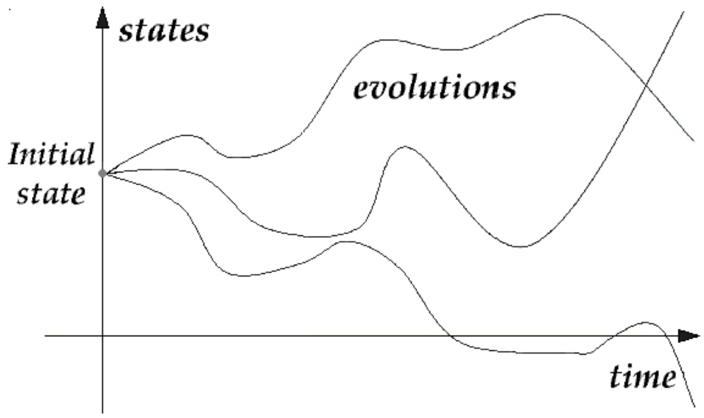
1. The predictive approach assumes the tychastic reservoir to be known and computes the guaranteed viability kernel ;
2. The impulse approach assumes the reset map known and provides the tychastic reservoir.

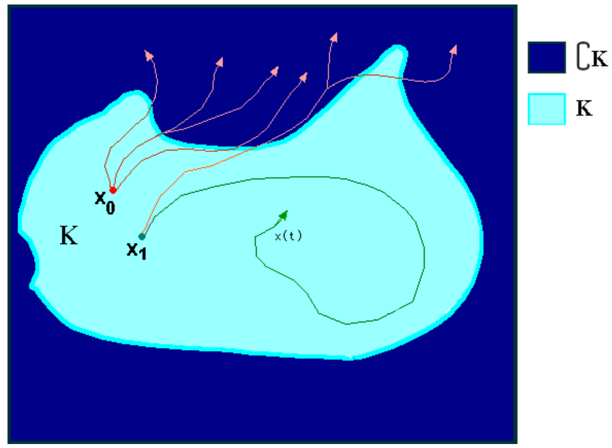


Evolutionary System

$\mathcal{S}$: it associates with any initial state $x_0 \in K$ the set $\mathcal{S}(x_0)$ of evolutions $x(\cdot)$ starting at x_0 governed by the regulated system.

Regulation maps governing the evolution of evolutions viable in one environment always or until they capture a target in finite time.





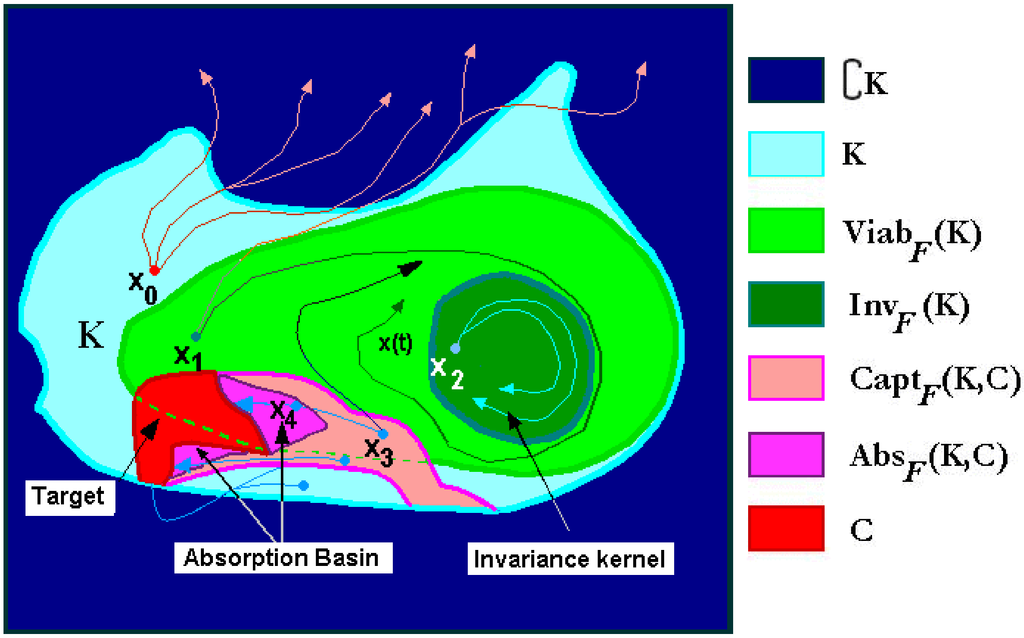
Let $K \subset X$ be a environment and $C \subset K$ be a target.

1. **The subset $\text{Viab}_{\mathcal{S}}(K, C)$ of initial states $x_0 \in K$ such that *at least one evolution* $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in *finite time* is called the *viability kernel of* K *with target* C under $\mathcal{S}$.**

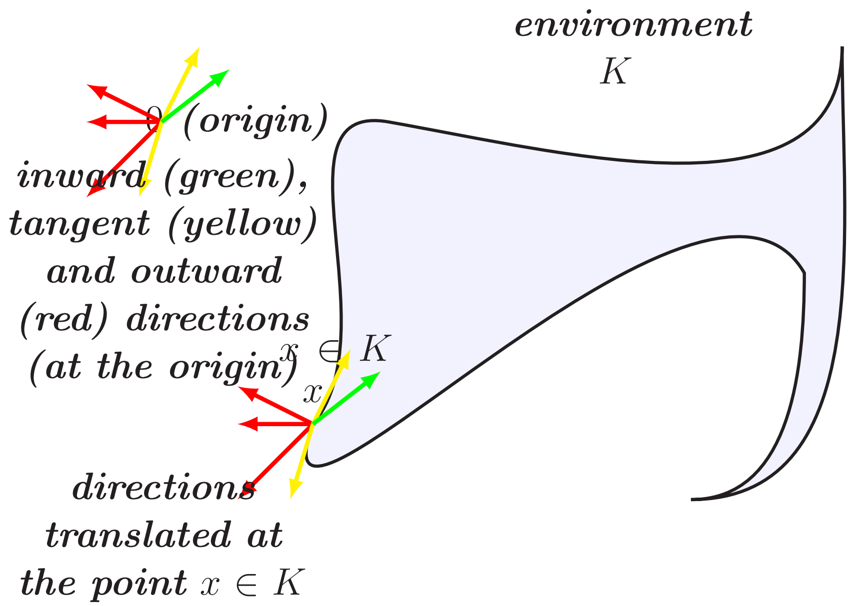
   When the target $C = \emptyset$ is the empty set, we say that $\text{Viab}_{\mathcal{S}}(K) := \text{Viab}_{\mathcal{S}}(K, \emptyset)$ is the *viability kernel* of K.

2. **The subset $\text{Capt}_{\mathcal{S}}(K, C)$ of initial states $x_0 \in K$ such that *at least one evolution* $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K until it reaches C in finite time is called the *capture basin of* C *viable in* K under $\mathcal{S}$. When $K = X$ is the whole space, we say that $\text{Capt}_{\mathcal{S}}(C) := \text{Capt}_{\mathcal{S}}(X, C)$ is the *capture basin* of C. We set $\text{Capt}_{\mathcal{S}}(K, \emptyset) = \emptyset$.**

Viability Kernel of an Environment with Target



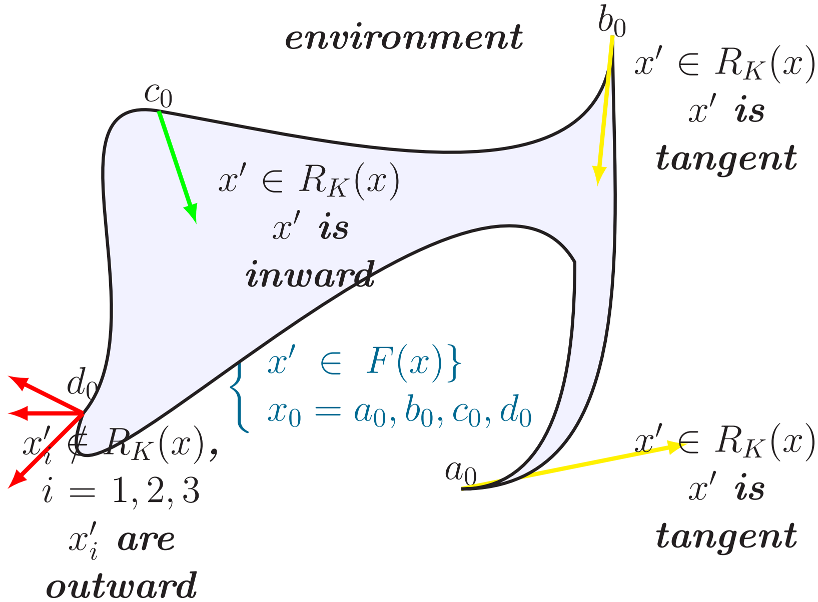
Let $K \subset X$ be a subset and $x \in K$ an element of K. A direction v is *contingent* (or, more simply, "tangent") to K at $x \in K$ if it is a limit of a sequence of directions v_n such that $x + h_n v_n$ belongs to K for some sequence $h_n \to 0+$. The collection of such contingent directions constitutes a closed cone $T_K(x)$, called the *contingent cone* to K at x, or more simply, *tangent cone*.


environment
K
0 (origin)
inward (green),
tangent (yellow)
and outward
(red) directions
(at the origin)
$x \in K$
x
directions
translated at
the point $x \in K$

5.5 Viable Regulation Map

We associate with the dynamical system described by (f, U) and with the constrained described by K the *(set-valued) regulation map* R_K : it maps any state $x \in K$ to the (possibly empty) subset $R_K(x)$ consisting of controls $u \in U(x)$ which are *viable* in the sense that $f(x, u)$ is tangent to K at x

$$R_K(x) := \{u \in U(x) \mid f(x, u) \ \in \ T_K(x)\}$$


environment
b_0
$x' \in R_K(x)$
x' is
tangent
c_0
$x' \in R_K(x)$
x' is
inward
$x' \in F(x)\}$
$x_0 = a_0, b_0, c_0, d_0$
d_0
$x'_i \notin R_K(x)$,
$i = 1, 2, 3$
x'_i are
outward
a_0
$x' \in R_K(x)$
x' is
tangent

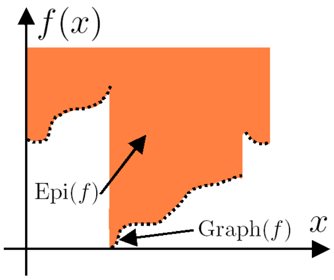
Under adequate assumptions, an evolution regulated by (f, U) is viable in the environment K forever or until it reaches the target C in finite if and only if it is governed by the viable regulated system $(f, R_{\mathrm{Viab}_{\mathcal{S}}(K,C)})$ on $\mathrm{Viab}_{\mathcal{S}}(K, C) \setminus C$ associated with the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K, C)$ of K with target C :

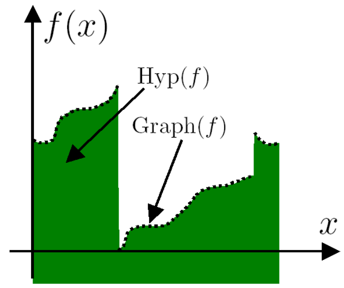
$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in R_{\mathrm{Viab}_{\mathcal{S}}(K,C)}(x(t)) \end{cases}$$

Epigraphical Approach

Let $V : X \mapsto \overline{\mathbb{R}}$ be an extended function.

1. **Its *epigraph* $\mathcal{E}p(V)$ is the set of pairs $(x, y) \in X \times \mathbb{R}$ satisfying $V(x) \leq y$.**
2. **Its *hypograph* $\mathcal{H}yp(V)$ is the set of pairs $(x, y) \in X \times \mathbb{R}$ satisfying $V(x) \geq y$,**
3. **Its *graph* $\mathrm{Graph}(V)$ is the set of pairs $(x, y) \in \mathrm{Dom}(V) \times \mathbb{R}$ satisfying $V(x) = y$.**





We introduce the Legendre-Fenchel Transform l of the Hamiltonian h defined by

$$\forall\ u, \quad \mathbf{l}(u) \ := \sup_{p \in \mathrm{Dom}(\mathbf{h})} [\mathbf{h}(p) - \langle p, u \rangle]$$

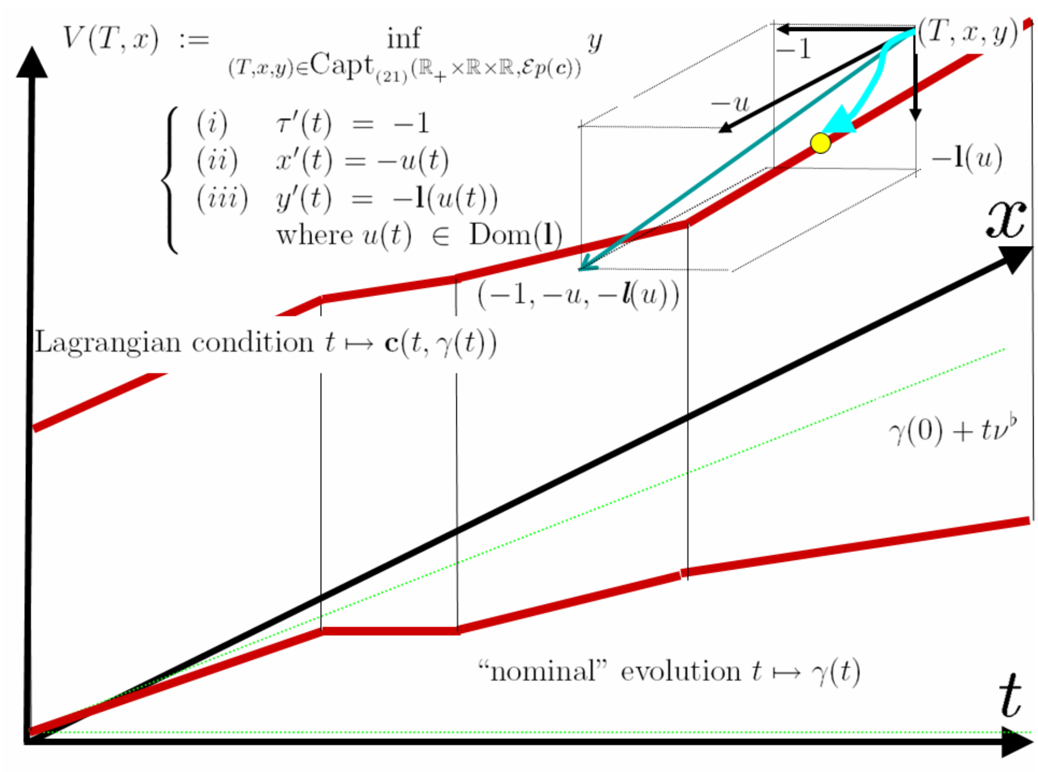
The characteristic system is defined by

$$\left\{ \begin{array}{lllll} (i) & \tau'(t) & = & -1 \\ (ii) & \xi'(t) & = & -u \\ (iii) & \eta'(t) & = & = -\mathbf{l}(u) \end{array} \right. \qquad (6)$$

$$V(T,x) := \inf_{(T,x,y)\in \mathrm{Viab}_{(6)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))} y \qquad (7)$$

The viability solution V is the valuation function of the underlying variational principle and viable and optimal evolutions coincide.

This definition may seem strange at first glance to solve a well known partial differential equation by a solution of an auxiliary and seemingly artificial viability problem. This allows us to use properties of viability kernels obtained at the simpler level of set-valued analysis, bypassing and avoiding the regularity requirements of pointwise version of classical analysis.

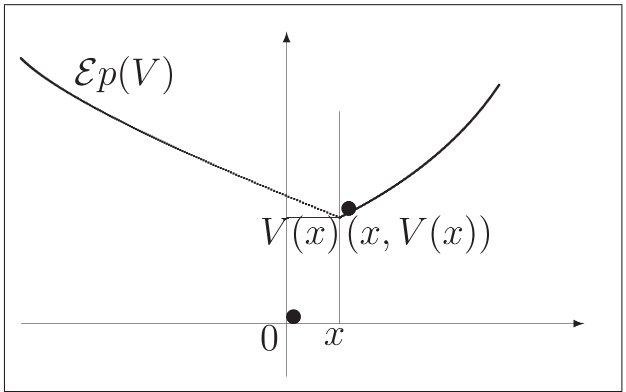

$V(T,x) := \inf_{(T,x,y)\in \mathrm{Capt}_{(21)}(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R},\mathcal{E}p(\mathbf{c}))} y$
(i) $\tau'(t) = -1$
(ii) $x'(t) = -u(t)$
(iii) $y'(t) = -\mathbf{l}(u(t))$
where $u(t) \in \mathrm{Dom}(\mathbf{l})$
(T,x,y)
-1
$-u$
$-\mathbf{l}(u)$
$(-1,-u,-\mathbf{l}(u))$
x
Lagrangian condition $t \mapsto \mathbf{c}(t,\gamma(t))$
$\gamma(0)+t\nu^\flat$
"nominal" evolution $t \mapsto \gamma(t)$
t

The viable regulation map

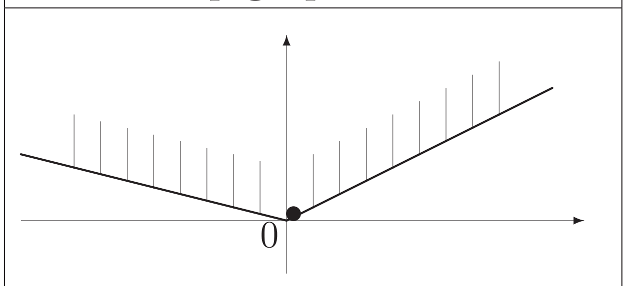
$$R_{\mathcal{E}p(V)}(t,x) \ := \ \left\{u \ \textbf{ such that } (-1,-u,-\mathbf{l}(t,x)) \ \in \ T_{\mathcal{E}p(V)}(t,x,V(t,x))\right\}$$

involving the tangent cone to the epigraph. Since Pierre de Fermat (1637), this is related to the *epiderivative* $D_\uparrow V(t,x)$ of V defined by

$$\mathcal{E}p(D_\uparrow V(t,x)) \ := \ T_{\mathcal{E}p(V)}(t,x,V(t,x))$$



Epigraph of V



Epigraph of $D_{\uparrow}V(x)$

The question asked is how to *regulate* viable (and thus, optimal) evolutions. The first answer is provided by

Theorem 1 ***Under adequate assumptions, viable evolutions*** $\xi(\cdot) \in \mathcal{A}(d,T;x)$ ***starting at optimal departure time*** $d := d(T,x)$ ***at*** $\xi(d) \in \mathbf{C}(d)$ ***and arriving at time*** T ***at*** x ***are regulated by differential inclusion***

$$\forall\, t \in [d,T[,\ \ \xi'(t) \ \in \ R_{\mathcal{E}p(V)}(t,\xi(t))$$

6.4 Viability Solutions are Viscosity Solutions to HJ Equation

The Viability Theorem and the links between epigraphs of epiderivatives and tangent cones to epigraphs imply that the viability solution generalizes the “viscosity solution” (Baron-Jensen/Franloska lower semicontinuous solution instead of continuous).

Merci pour votre attention

Thank You for Your Attention